Decentralized Detection in *Ad hoc* Sensor Networks With Low Data Rate Inter Sensor Communication

Lu Zheng, Yingwei Yao, *Member, IEEE*, Mo Deng, and Stephen S. T. Yau

***Abstract*—Decentralized binary detection problem in ad-hoc sensor networks where a link between two sensors is on with a certain probability is considered in this paper. We propose a consensus based detection scheme where sensors exchange their local decisions, update their own decisions based on the exchanges and finally reach a consensus about the state of nature. We analyze the error probability and convergence of this decision consensus scheme. We show that with our scheme, the detection performance in ad-hoc networks is asymptotically equivalent to that of a parallel sensor network where all the local decisions are processed by a central node (fusion center) in the sense that the error exponents are the same. The probability distribution of the consensus time is also studied. Simulation and numerical results are given to verify the theoretical results.**

***Index Terms*—*Ad hoc* network, consensus problems, decentralized detection, low data rate systems, Markov chain.**

I. Introduction

A decentralized detection environment is characterized by a set of sensors making observations about the state of nature H. The objective is to make an estimation about H based on the observations from all the sensors. Decentralized detection problem in sensor networks has been under extensive study in recent years[1]–[5]. In most previous works, despite of the variations in the network structures, it is assumed that there exists a central node (fusion center) which has stronger computational capability and direct or indirect access to all sensors across the network. The responsibility of this central node is to fuse all the data and produce a global decision about H. Decentralized detection problem in *ad hoc* sensor networks, on the other hand, has not been studied much. The major difficulty with the detection problem in *ad hoc* sensor networks is the lack of central control which makes it hard to aggregate the information over the network.

In this paper, we propose a consensus based decentralized detection scheme in *ad hoc* sensor networks. Our scheme is first motivated by the study of rational decision consensus in social and economic systems[8]. Similar procedure can be applied to the detection problem in ad hoc sensor networks: each sensor continuously collects information from its neighbors to update its local decision. In this way, sensors' decisions evolve over time and if the sensors unanimously agree on some hypothesis at some point, the global decision about H can be retrieved by querying any sensor. There are several problems we are interested in regarding a detection system like this:

- Can a consensus always be reached for this scheme?
- What is the limit distribution of the consensus?
- What is the speed of the consensus?
- What is the error probability of the detection system?

A. Related Work

The problem of distributed detection has been investigated extensively. Based on the earlier results from classical distributed detection problem[2], recent researchers study the distributed detection problem in sensor networks with bandwidth and power constraints [3]–[6]. A star topology is assumed in all these works.

In *ad hoc* networks, due to the lack of central control, information aggregation has to be conducted in a distributed manner. Decision consensus is a possible solution for the detection problem in ad hoc networks. Decision consensus among multiple agents is first investigated in the context of social system [8], [9]. Due to the similarity in the distributed nature of sensor networks and social networks, consensus procedure can also be applied to the sensor networks.

Consensus problem in sensor network with linear protocols, especially the average consensus, has attracted considerable attention in recent years. In [1], the authors give a thorough discussion about this problem for sensor networks with different topologies. It is shown that a consensus can always be reached under some mild constraint on the network connectivity. Besides, the authors also find that the convergence speed is directly related to the second largest eigenvalue of the connectivity matrix. These results are also given by Olshevsky and Tsitsiklis in [7]. The decision consensus considered in this paper cannot fit in this framework because it is inherently a consensus with nonlinear protocol.

There are several works regarding consensus problem with nonlinear protocols. In [14], the authors consider a binary consensus system employing analog communication over additive white Gaussian noise channels. They show that the steady state of such a system is independent of the initial state and hence the steady state error performance is unacceptable. In [15], the authors analyze the binary consensus system with three-state sensors and either binary or ternary signaling for information ex-

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L. Zheng, retired, is at 779 Lakehaven Dr., Sunnyvale, CA 94089 USA (e-mail: luzheng@andrew.cmu.edu).

Y. Yao, M. Deng, and S. S. T. Yau are with the University of Illinois at Chicago, Chicago, IL 60616 USA (e-mail: yyao@uic.edu; mdeng2@uic.edu; yau@uic.edu).

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change. A sensor updates its state after receiving the signal from a randomly chosen sensor. It was shown that, conditioned on initial sensor states, the error probability decays exponentially as the number of sensors grows. In [18], Acemoglu, Dahlen, Lobel, and Orzdaglar study the convergence rate of beliefs and decisions in social networks. They show the convergence to the correct action is faster than a polynomial rate when individuals observe the most recent action and is at a logarithmic rate when they sample a random action from the past. In [17], Swaszik and Willett did research about the "parleying" scheme on the sensor network detection system, showing that a consensus always occurs when sensors could update their local decisions based on the previous fused decision. In [16], Borkar and Varaiya model the process of consensus based distributed estimation as a martingale. They prove that the estimates of the decision agents asymptotically agree. In our work, we consider a different scenario where each indivisual sensor only get information from its neighbors. And in addition to showing the existence of a asymptotic consensus, we also analyze the "correctness" of the consensus.

Both [14] and [15] study binary consensus systems based on majority rule. In this paper, we study binary consensus systems that adopts a more general decision rule and thus can be applied to Neyman-Pearson detection problem and Bayesian detection with nonuniform priors. Each sensor updates its local decision after receiving the local decisions of a randomly chosen set of sensors. The information exchanges between sensors rely on digital communication. We analyze the error performance and convergence time of the consensus. Averaged over the initial sensor states, the error probability decays exponentially and the error exponent is identical to that of a decentralized detection system with a star topology.

B. Organization

The rest of this paper is organized as follows. In Section II, we are going to describe the system model and formulate the problem. In Section III, we give a brief review of mathematical tools used in this paper. Closed form and asymptotic analysis on the consensus algorithm will be presented in Section IV and Section V, respectively. In Section VI, we give numerical examples to verify the analytical results obtained in Section IV and Section V. Finally we reach our conclusion in Section VII.

II. Model

A. Detection Scheme

Consider the binary detection problem

$$H_0 : f_0 \; vs. \; H_1 : f_1 \tag{1}$$

where f_0 and f_1 are probability distributions associated with H_0 and H_1 respectively. Suppose there are N sensors in the network. Each sensor $i (i = 1, \ldots, N)$ makes an observation X_i which takes value from some observation set $\mathcal{X}$. X_i's are conditionally independent and identically distributed across all the sensors with distribution $f_j (j = 0, 1)$, given H_j.

To reduce the communication burden, each sensor processes its local information to obtain a binary local decision. A local decision rule γ is a mapping from the observation set $\mathcal{X}$ to the decision set $\mathcal{U} = \{0, 1\}$, i.e., $\gamma : \mathcal{X} \rightarrow \mathcal{U}$. We assume all the sensors use identical local decision rule to form their initial local decisions, i.e., $u_i(0) = \gamma(X_i)$.

To reach a consensus, sensors exchange local decisions and update their local decisions based on the exchange. We consider two communication models for the information exchange. In the asynchronous communication model, only one sensor retrieves information and updates its local decisions at each time slot. Under the synchronous communication model, in each time slot, all the sensors update their local decisions simultaneously.

In this paper, we assume that a sensor i is randomly and independently picked up for updating at any given time slot. It receives the local decision of any other sensor j with probability p_c. We say sensor j is a neighbor of sensor i at time t if sensor i has successfully retrieved the decision of sensor j at time t. Denote the set of all neighbors of sensor i and itself as $\mathcal{C}_i(t)$. We use $d_i(t)$ to denote the cardinality of $\mathcal{C}_i(t)$, i.e., $d_i(t) = |\mathcal{C}_i(t)|$. The model we have adopted can be applied to a wireless sensor network with sensors dispersed in a relatively small area. In such networks, the first-order communication signal strength effects are due to random fluctuations in the medium, such as Rayleigh and shadowing fading instead of distance, and communication between any two sensors succeeds with a certain probability[13]. This model can also be applied to sensor networks utilizing geographic gossiping where a sensor requests information from randomly chosen sensors that might be multiple hops away. It was shown that by utilizing this scheme, the information can spread much faster over the entire network [12].

Let $u_i(t)$ be the local decision of sensor i at time t, $f_j^\gamma(u) = \int_{x \in \gamma^{-1}(u)} f_j(x) dx$ be the distribution of local decisions under H_j. Each sensor of this system has an updating rule (sometimes also termed as dynamics) $\mathcal{L} : \mathcal{U}^{d_i(t)} \rightarrow \mathcal{U}$ to fuse all the retrieved data and form a new local decision. One natural choice of the updating rule is a test like this

$$u_i(t+1) = \begin{cases} 1 & \text{if } \Lambda_i(t) = \frac{1}{n_i^0(t) + n_i^1(t)} \sum\limits_{j \in \mathcal{C}_i(t)} L(u_j(t)) > \tau \\ 0 & \text{if } \Lambda_i(t) = \frac{1}{n_i^0(t) + n_i^1(t)} \sum\limits_{j \in \mathcal{C}_i(t)} L(u_j(t)) < \tau \\ u_i(t) & \text{otherwise} \end{cases} \tag{2}$$

where $n_i^0(t)$ and $n_i^1(t)$ are the numbers of zeros and ones among $u_j(t)$'s, $j \in \mathcal{C}_i(t)$. $L(\cdot)$ is the function defined as follows:

$$L(u) = \log \frac{f_1^\gamma(u)}{f_0^\gamma(u)}. \tag{3}$$

and τ is the threshold. One can verify that

$$\Lambda_i(t) \lesseqgtr \tau \iff n_i^1 \lesseqgtr \kappa n_i^0 \tag{4}$$

where

$$\kappa = \left(\log \frac{f_0^\gamma(u=0)}{f_1^\gamma(u=0)} + \tau \right) \left(\log \frac{f_1^\gamma(u=1)}{f_0^\gamma(u=1)} - \tau \right)^{-1}. \tag{5}$$

We say sensor i agrees with sensor j if and only if $u_i(t) = u_j(t)$. We say the network has reached a *consensus* if and only if $u_i(t) = u_j(t)$ for all $i \neq j$.

B. Markov Chain Model of the Asynchronous System

In this subsection, we establish a Markov Chain model for an asynchronous decision consensus system as a basis for our analysis. We choose the number of sensors whose current local decision is H_1 as the state of the system. We denote it as $S(t) \in \{0, 1, \ldots, N\}$. Now our task is to find out the state transition matrix $[P_{ij}]_{(N+1)\times(N+1)}$, where $P_{ij} \triangleq Pr(S(t+1) = j | S(t) = i)$.

Definition 1 [11]: A state i of a Markov Chain is an absorbing state if it is impossible to leave it, i.e., $P_{ii} = 1$.

Definition 2 [11]: A Markov Chain is absorbing if it has at least one absorbing state, and if from every state it is possible to go to an absorbing state(not necessarily in one step).

Obviously, $S(t) = 0$ and $S(t) = N$ are two absorbing states for this Markov chain since for $i = 0, N$

$$P_{ij} = \begin{cases} 1 & j = i \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Also, since at each time slot, there is only one sensor updating its local decision, $S(t)$ can at most change by 1 each time. Thus, $P_{ij} = 0$, for all $j \neq i-1, i, i+1$.

For the other entries of the transition matrix where $i \neq 0, N$, we can compute them as follows. Suppose current state $S(t) = i$ and it is sensor k's turn to retrieve other sensors' information and update its local decision. If $u_k(t) = 1$, given current state $S(t) = i$ $(i = 1, \ldots, N-1)$, $n_k^1(t) - 1$ and $n_k^0(t)$ are binomially distributed with parameter $B(i-1, p_c)$ and $B(N-i, p_c)$, respectively.

$$\begin{aligned} p^{10}(i) \triangleq\, & Pr(u_k(t+1) \\ &= 0 | u_k(t) = 1, S(t) = i) \\ &= Pr(n_k^1(t) < \kappa n_k^0(t) | S(t) = i) \\ &= \sum_{l=0}^{N-i} P(n_k^0(t) = l | S(t) = i) \\ &\times P(n_k^1(t) - 1 < \kappa l - 1 | S(t) = i) \\ &= \sum_{l=0}^{N-i} \binom{N-i}{l} p_c^l (1-p_c)^{N-i-l} \\ &\times I_{1-p_c}(i - 1 - \lfloor \kappa l - 1 \rfloor \wedge (i-1), \\ &\quad 1 + \lfloor \kappa l - 1 \rfloor \wedge (i-1)) \end{aligned} \tag{7}$$

where $I_{1-p_c}(n-k, k+1)$ is the regularized incomplete beta function

$$\begin{aligned} I_{1-p_c}(n-k, k+1) \triangleq (n-k)\binom{n}{k} \\ \times \int_0^{1-p_c} t^{n-k-1}(1-t)^{k+1} dt \end{aligned} \tag{8}$$

and $a \wedge b \triangleq \min\{a, b\}$.

Similarly if $u_k(t) = 0$, $n_k^1(t)$ and $n_k^0(t) - 1$ are binomially distributed with parameter $B(i, p_c)$ and $B(N-i-1, p_c)$, respectively. Then we have

$$\begin{aligned} p^{01}(i) &\triangleq Pr(u_k(t+1) = 1 | u_k(t) = 0, S(t) = i) \\ &= Pr(n_k^1(t) > \kappa n_k^0(t) | S(t) = i) \\ &= \sum_{l=0}^{N-i} P(n_k^1(t) = l | S(t) = i) \\ &\times P(n_k^0(t) - 1 < l/\kappa - 1 | S(t) = i) \\ &= \sum_{l=0}^{N-i} \binom{i-1}{l} p_c^l (1-p_c)^{i-1-l} \\ &\times I_{1-p_c}(i - 1 - \lfloor l/\kappa - 1 \rfloor \wedge (i-1), \\ &\quad 1 + \lfloor l/\kappa - 1 \rfloor \wedge (i-1)). \end{aligned} \tag{9}$$

Now we are ready to compute the entries of the transition matrix

$$\begin{aligned} P_{i,(i-1)} &= Pr(S(t+1) = i-1 | S(t) = i) \\ &= P(u_k(t+1) = 0 | u_k(t) = 1, S(t) = i) \\ &\times P(u_k(t) = 1 | S(t) = i) \\ &= \frac{i}{N} p^{10}(i) \end{aligned} \tag{10}$$

$$\begin{aligned} P_{i,(i+1)} &= Pr(S(t+1) = i+1 | S(t) = i) \\ &= P(u_k(t+1) = 1 | u_k(t) = 0, S(t) = i) \\ &\times P(u_k(t) = 0 | S(t) = i) \\ &= \frac{N-i}{N} p^{01}(i) \end{aligned} \tag{11}$$

and $P_{ii} = 1 - P_{i,(i-1)} - P_{i,(i+1)}$. And all the other entries are zero as we explained previously.

C. Markov Chain Model for the Synchronous System

We still use the number of current ones in the network as the system state $S(t)$. Obviously, $S(t) = 0$ and $S(t) = N$ are still two absorbing states for this Markov chain, i.e., for $i = 0, N$

$$P_{ij} = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

where P_{ij} is the transition probability from state i to j. The other entries of the transition matrix where $i \neq 0, N$, can be computed as follows. Let $\mathcal{A}_t = \{k : u_k(t) = 1\}$ denote the set of sensors whose decisions at time t are 1. Note $S(t) = |\mathcal{A}_t|$. Naturally, $\mathcal{A}_t^c$ is the set of sensors whose decisions at time t are 0. Given current state $S(t) = i$ $(i = 1, \ldots, N-1)$, for all $k \in \mathcal{A}_t$, $n_k^1(t) - 1$, and $n_k^0(t)$ are binomially distributed with distribution $B(i-1, p_c)$ and $B(N-i, p_c)$ respectively. Hence

$$\begin{aligned} p^{11}(i) &= Pr(u_k(t+1) = 1 | u_k(t) = 1, S(t) = i) \\ &= Pr(n_k^1(t) > \kappa n_k^0(t) | S(t) = i) \\ &= \sum_{l=1}^{i} Pr(n_k^1(t) - 1 = l - 1) Pr\left(n_k^0(t) < l/\kappa\right) \\ &= \sum_{l=1}^{i} \binom{i-1}{l-1} p_c^{l-1} (1-p_c)^{i-l} \\ &\times I_{1-p_c}(N - i - \lfloor l/\kappa \rfloor \wedge (N-i) \\ &\quad 1 + \lfloor l/\kappa \rfloor \wedge (N-i)) \end{aligned} \tag{13}$$

where $I_{1-p_c}(i-l+1, l-1)$ is the regularized incomplete beta function as defined in (8). And we have already obtained p^{01} in (9).

We can divide the sensors in $\mathcal{A}_{t+1}$ into two subsets:

$$\begin{aligned}\mathcal{A}_{t+1}^{11} &= \{k : k \in \mathcal{A}_t, k \in \mathcal{A}_{t+1}\} \\ \mathcal{A}_{t+1}^{01} &= \{k : k \notin \mathcal{A}_t, k \in \mathcal{A}_{t+1}\}.\end{aligned} \tag{14}$$

Let $S_{11}(t+1)$ and $S_{01}(t+1)$ be the cardinality of $\mathcal{A}_{t+1}^{11}$ and $\mathcal{A}_{t+1}^{01}$ respectively. Thus we have $S(t+1) = S_{11}(t+1) + S_{01}(t+1)$. Given $S(t) = i$, we can see $S_{11}(t+1)$ and $S_{01}(t+1)$ are binomially distributed with distribution $B(i, p^{11}(i))$ and $B(N-i, p^{01}(i))$ respectively. Therefore, the transition probability $P_{ij}(i = 1, \ldots, N-1)$ is given by

$$\begin{aligned}P_{ij} &= Pr(S(t+1) = j | S(t) = i) \\ &= \sum_{l=\max\{0,i+j-N\}}^{\min\{i,j\}} Pr(S_{11}(t+1) = l) Pr(S_{01}(t+1) = j-l) \\ &= \sum_{l=\max\{0,i+j-N\}}^{\min\{i,j\}} \binom{i}{l}\binom{N-i}{j-l} (p^{11}(i))^l (1-p^{11}(i))^{i-l} \\ &\times (p^{01}(i))^{j-l} (1-p^{01}(i))^{N-i-j+l}.\end{aligned} \tag{15}$$

III. Performance Analysis

In this section, we present three propositions which give closed form expressions for the probability of decision consensus, the detection error probability and average consensus time of this system. These results hold for both the asynchronous and synchronous system.

Proposition 3.1 (Achievability of Consensus): In the system described in Section II, a global consensus about the state of nature will be reached with probability 1, i.e., $Pr(S(\infty) = 0 \text{ or } N) = 1$.

Proof: The conclusion comes directly from the fact that an absorbing Markov Chain will be absorbed by those absorbing states with probability one [10]. ∎

Now we start to derive the detection error probability under Bayesian rule. The distribution of the initial state of this system is determined by the hypothesis distribution and the local decision rule. Under $H_j (j = 0, 1)$, for any sensor k, the initial local decision $u_k(0)$ has the following distribution:

$$q_{j1} \triangleq Pr(u_k(0) = 1 | H_j) = \int_{x \in \gamma^{-1}(1)} f_j(x) dx \tag{16}$$

$$q_{j0} \triangleq Pr(u_k(0) = 0 | H_j) = \int_{x \in \gamma^{-1}(0)} f_j(x) dx. \tag{17}$$

Therefore, under H_j, the initial state of the system $S(0)$ is binomially distributed with $B(N, q_{j1})$, i.e.

$$\pi_j(i) \triangleq Pr(S(0) = i | H_j) = \binom{N}{i} q_{j1}^i (1-q_{j1})^{N-i}. \tag{18}$$

Re-arrange the transition matrix into the *canonicalform* [10]

$$\begin{aligned}\mathbf{P} &= \begin{pmatrix} \mathbf{Q} & \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \\ &= \begin{bmatrix} P_{11} & \cdots & P_{1(N-1)} & P_{10} & P_{1N} \\ P_{21} & \cdots & P_{2(N-1)} & P_{20} & P_{2N} \\ \vdots & & \ddots & & \vdots \\ P_{(N-1)1} & \cdots & P_{(N-1)(N-1)} & P_{(N-1)0} & P_{(N-1)N} \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix}\end{aligned} \tag{19}$$

where the entries P_{ij} is given in Section II, and $\mathbf{Q}$ and $\mathbf{R}$ are given by

$$\mathbf{Q} = \begin{bmatrix} P_{11} & \cdots & P_{1(N-1)} \\ \vdots & \ddots & \vdots \\ P_{(N-1)1} & \cdots & P_{(N-1)(N-1)} \end{bmatrix} \tag{20}$$

and

$$\mathbf{R} = \begin{bmatrix} P_{10} & P_{1N} \\ \vdots & \vdots \\ P_{(N-1)0} & P_{(N-1)N} \end{bmatrix}. \tag{21}$$

With all these, we have the following.

Proposition 3.2 (Error Probability): Under Bayesian rule, the error probability is

$$\begin{aligned}P_e &= Pr(S(\infty) = N | H_0) P(H_0) + Pr(S(\infty) = 0 | H_1) P(H_1) \\ &= \Pi_0^{\mathrm{T}} \mathbf{b}_1 P(H_0) + \Pi_1^{\mathrm{T}} \mathbf{b}_2 P(H_1),\end{aligned} \tag{22}$$

where $\Pi_j (j = 0, 1)$ is the $(N-1)$-by-1 column vector whose ith entry is $\pi_j(i)$, $P(H_0)$ and $P(H_1)$ are the prior probabilities of H_0 and H_1 respectively, $\mathbf{b}_1$ and $\mathbf{b}_2$ are the first and second column of an $(N-1)$-by-2 matrix $\mathbf{B}$ which is defined as

$$\mathbf{B} = (\mathbf{I} - \mathbf{Q})^{-1} \mathbf{R}. \tag{23}$$

Proof: This proposition directly comes from the absorbing Markov chain property [10]. ∎

Proposition 3.3: Let T be the time required for consensus, then

$$\mathbb{E}(T) = (\Pi_0^{\mathrm{T}} P(H_0) + \Pi_1^{\mathrm{T}} P(H_1)) \mathbf{N} \mathbf{1} \tag{24}$$

where $\Pi_0, \Pi_1, P(H_0), P(H_1)$ are defined as in Proposition 3.2 and

$$\mathbf{N} = (\mathbf{I} - \mathbf{Q})^{-1}. \tag{25}$$

Proof: Proposition 3.3 is a direct result from absorbing Markov chain property [10]. ∎

IV. Asymptotic Performance

Though the results given in the last section fully characterize the ad hoc detection system we are interested in, we cannot get much insight about the system due to the complexity of those

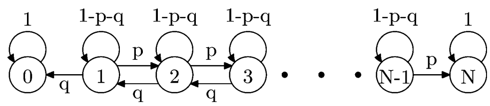

Fig. 1. Absorbing Markov chain.

formulas. To obtain more insight, we investigate the asymptotic performance of the system, in terms of error exponent. For this analysis, we are going to focus on the asynchronous system in this section. In parallel sensor networks, it is well known that detection error decays exponentially with respect to N, with the error exponent given in [3]. We here prove that the error probability of the ad hoc network also decays exponentially and the error exponent is the same as that of the parallel sensor networks, meaning that, an *ad hoc* network is asymptotically equivalent to a parallel sensor network in terms of error performance.

To obtain the overall error exponent of our system, we first obtain the distribution of the initial state; then we analyze the conditional consensus error probability given a certain initial state and finally, we will combine the two results to obtain the error exponent.

Lemma 4.1 (The Distribution of Initial State): In an ad hoc sensor network, given a local decision rule γ, the initial state of the system $S(0)$ satisfies

$$P\left(\frac{S(0)}{N} < t|H_1\right) \sim \exp\left\{-\lambda_1(t)N\right\}, \text{if } t < f_1^{\gamma}(1) \quad (26)$$

and

$$P\left(\frac{S(0)}{N} > t|H_0\right) \sim \exp\left\{-\lambda_0(t)N\right\}, \text{if } t > f_0^{\gamma}(1) \quad (27)$$

for some $\lambda_0(t), \lambda_1(t) > 0$.

Proof: Note that

$$S(0) = \sum_{k=1}^{N} 1_{\{u_k(0)=1\}}, \quad (28)$$

and

$$\mathbb{E}_j\left[1_{\{u_k(0)=1\}}\right] = f_j^{\gamma}(1), j = 0, 1. \quad (29)$$

Applying Cramer's Theory, we get (26) and (27). ■

Lemma 4.2: Consider the $N+1$-state absorbing Markov Chain shown in Fig. 1 with the transition probability

$$P(s(t+1)=j|s(t)=i) = \begin{cases} p & j=i+1, i\neq 0, N \\ q & j=i-1, i\neq 0, N \\ 1-p-q & j=i, i\neq 0, N \\ 1 & i=j=0 \text{ or } N \\ 0 & \text{otherwise} \end{cases} \quad (30)$$

where $0 < q, p < 1$, $0 < p+q \leq 1$. Given the starting state $s(0) = \lfloor aN \rfloor$, $(0 < a < 1)$, the limit distribution is

$$P(s(\infty) = N|S(0) = \lfloor aN \rfloor) = \frac{1 - \left(\frac{q}{p}\right)^{\lfloor aN \rfloor}}{1 - \left(\frac{q}{p}\right)^{N}} \quad (31)$$

$$P(s(\infty) = 0|s(0) = \lfloor aN \rfloor) = \frac{\left(\frac{q}{p}\right)^{\lfloor aN \rfloor} - \left(\frac{q}{p}\right)^{N}}{1 - \left(\frac{q}{p}\right)^{N}} \quad (32)$$

Proof: Let $z_i = P(s(\infty) = 0|s(0) = i)$. Use the first step analysis, we have

$$\begin{aligned} &P(s(\infty) = 0|s(0) = i) \\ &= P(s(\infty) = 0, s(1) = i+1|s(0) = i) \\ &+ P(s(\infty) = 0, s(1) = i-1|s(0) = i) \\ &+ P(s(\infty) = 0, s(1) = i|s(0) = i) \\ &= P(s(\infty) = 0|s(1) = i+1)P(s(1) = i+1|s(0) = i) \\ &+ P(s(\infty) = 0|s(1) = i-1)P(s(1) = i-1|s(0) = i) \\ &+ P(s(\infty) = 0|s(1) = i)P\left(s(1) = i|s(0) = i\right). \end{aligned} \quad (33)$$

Hence we have the recursive equation

$$(p+q)z_i = pz_{i+1} + qz_{i-1}, i = 1, \ldots, N-1 \quad (34)$$

with the boundary condition $z_0 = 0$ and $z_N = 1$. By solving the recursive (34), we can obtain (31) and (32). ■

Proposition 4.3 (Limit Distribution of the Consensus): Let z_b be the probability that the chain $S(t)$ is absorbed by state 0 starting in state $i = \lfloor bN \rfloor$ $(\kappa/(\kappa+1) < b < 1)$. Given a fixed b, z_b decays super-exponentially with N, i.e.,

$$z_b = o(\exp\{-cN\}) \text{ for all positive } c. \quad (35)$$

Similarly, let z_b' be the probability that the chain $S(t)$ is absorbed by state 1 starting in state $i = \lfloor bN \rfloor$ $(0 < b < \kappa/(\kappa+1)$. Given a fixed b, z_b' also decays super-exponentially with N, i.e.,

$$z_b' = o(\exp\{-cN\}) \text{ for all positive } c. \quad (36)$$

Proof: We only prove the super exponential decay of z_b. Fig. 2 illustrates the consensus Markov chain we describe in Section II. Consider an intermediate state $m = \lfloor aN \rfloor$ where $\kappa/(\kappa+1) < a < b$. If $m < i$, starting from state i, a necessary condition for the chain to be absorbed by state 0 is that it must first hit the state m before hitting N. Denote ϕ_{ba} as the probability that the chain starting in $i = \lfloor bN \rfloor$ hits $m = \lfloor aN \rfloor$ before hitting N. Hence we have

$$z_b = z_a \phi_{ba}. \quad (37)$$

We cut the original chain at state m to get a truncated chain consisting of state m, $m+1$, …, N as shown in Fig. 2. Let state m be an absorbing state of the truncated chain, then ϕ_{ba} is the absorbing probability by state m of the truncated Markov chain starting in state i. The transition probabilities are $p_m = P_{m,(m+1)}$ and $q_m = P_{m,(m-1)}$. We compare this truncated chain with a constructed Markov chain whose one step transition probabilities are given by p_{m+1} and q_{m+1} (also see Fig. 2). At each state j, the truncated chain has a higher probability to move to the right and a lower probability to move to the left

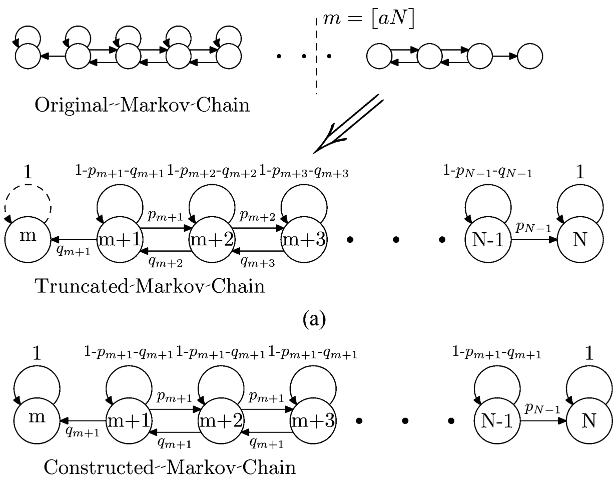

Fig. 2. Constructing a new Markov chain to compare to the original one. (a) Truncate the original Markov chain at state $m = \lfloor aN \rfloor$. (b) Constructed Markov chain.

than the constructed chain. Therefore, the truncated chain is less likely to end up in state m than the constructed chain. Denote the absorbing probability of state m of the constructed chain to be ϕ'_{ba}, we must have $\phi_{ba} < \phi'_{ba}$. Applying Lemma 4.1 to the constructed Markov chain, we have

$$\phi'_{ba} = \frac{\left(\frac{q_{m+1}}{p_{m+1}}\right)^{(\lfloor bN \rfloor - \lfloor aN \rfloor)} - \left(\frac{q_{m+1}}{p_{m+1}}\right)^{N-\lfloor aN \rfloor}}{1-\left(\frac{q_{m+1}}{p_{m+1}}\right)^{N-\lfloor aN \rfloor}} = \left(\frac{q_{m+1}}{p_{m+1}}\right)^{(\lfloor bN \rfloor - \lfloor aN \rfloor)} \frac{1-\left(\frac{q_{m+1}}{p_{m+1}}\right)^{N-\lfloor bN \rfloor}}{1-\left(\frac{q_{m+1}}{p_{m+1}}\right)^{N-\lfloor aN \rfloor}}. \quad (38)$$

To characterize the limit behavior of ϕ'_{ba}, we need to find out the limit of q_{m+1}/p_{m+1}. From (10), we have

$$q_{m+1} = \frac{m+1}{N} P(n_k^1(t) < \kappa n_k^0(t) | S(t) = m+1). \quad (39)$$

Without loss of generality, we can assume that $u_1(t) = \cdots = u_m(t) = 1$ and that the updating sensor is sensor $k \neq 1, \ldots, m$, then we have $n_k^1(t) = \sum_{j=1}^{\lfloor aN \rfloor} \mathbf{1}_{\{j \in \mathcal{C}_k(t)\}}$. By the Large Deviation theory, given $\nu < \mathbb{E}n_k^1(t)/N = \frac{\lfloor aN \rfloor}{N} p_c$

$$P\left(\frac{n_k^1(t)}{N} < \nu | S(t) = m+1\right) \sim \exp\{-c_1(\nu)N\}, \text{ for large enough } N \quad (40)$$

where $c_1(\nu)$ is a constant with respect to N. Similarly, for $\nu > \mathbb{E}n_k^0(t)/N = (1 - \frac{\lfloor aN \rfloor}{N})p_c$

$$P\left(\frac{n_k^0(t)}{N} > \nu | S(t) = m+1\right) \sim \exp\{-c_2(\nu)N\}, \text{ for large enough } N. \quad (41)$$

Hence for any ν s.t $(1 - \frac{\lfloor aN \rfloor}{N})p_c < \nu < \frac{\lfloor aN \rfloor}{\kappa N} p_c$, given $S(t) = m+1$ (we omit the condition in the equations below for expression simplicity), we have

$$\begin{aligned} &P(n_k^1(t) \le \kappa n_k^0(t)) \\ &\le P\left(\frac{n_k^0(t)}{N} > \nu\right) P(n_k^1(t) < \kappa n_k^0(t) | \frac{n_k^0(t)}{N} > \nu) \\ &+ P(\frac{n_k^0(t)}{N} < \nu, \frac{n_k^1(t)}{N} < \kappa\nu) \\ &\le P\left(\frac{n_k^0(t)}{N} > \nu\right) + P\left(\frac{n_k^1(t)}{N} < \kappa\nu\right) \\ &\sim \exp\{-c'(\nu)N\} \end{aligned} \quad (42)$$

where $c'(\nu) = \min\{c_1(\kappa\nu), c_2(\nu)\}$. Thus, $q_{m+1} \sim O(\exp\{-c'(\nu)N\})$. Similarly, for the same ν, we can prove that $p_{m+1} \sim 1 - O(\exp\{-c'(\nu)N\})$.

From (38)

$$\lim_{N \to \infty} -\frac{\log \phi'_{ba}}{N} = (a-b) \lim_{N \to \infty} \log \frac{q_{m+1}}{p_{m+1}} = +\infty. \quad (43)$$

That is, ϕ'_{ba} decays superexponentially. Since $\phi_{ba} < \phi'_{ba}$, ϕ_{ba} also decays superexponentially, i.e., $\phi_{ba} = o(\exp\{-cN\})$ for all positive c. So

$$z_b = z_a \phi_{ba} < \phi_{ba} = o(\exp\{-cN\}), \text{ for all positive } c. \quad (44)$$

■

Now we would like to compare the asymptotic performance of our system with that of the parallel sensor network. In parallel network, the fusion center has the access to local decisions from the sensors (we assume ideal communication channel between the fusion center and sensors) and makes the final decision using likelihood ratio test, i.e.

$$\Gamma(u_1, \ldots, u_N) = \begin{cases} 1 & N_1 > \kappa N_0 \\ 0 & o.w. \end{cases} \quad (45)$$

where N_1 and N_0 are the number of ones and zeros in u_k's $(k = 1, \ldots, N)$.

We say κ is *admissible* if

$$P(\gamma(X) = 1 | H_0) < \frac{\kappa}{1+\kappa} < P(\gamma(X) = 1 | H_1). \quad (46)$$

If the admissibility requirement is not satisfied, for instance, if $\frac{\kappa}{1+\kappa} < P(\gamma(X) = 0 | H_1)$, then, $\lim_{n\to\infty} P(error|H_0) > 0.5$. Similarly $\lim_{n\to\infty} P(error|H_1) > 0.5$ if $\frac{\kappa}{1+\kappa} > P(\gamma(X) = 1|H_1)$. Hence an optimal detection algorithm (Bayesian or Neyman-Pearson) must select an admissible threshold.

Corollary 4.4: The decision-consensus-based detection scheme in ad hoc sensor networks has the same asymptotic performance as the optimal detection in parallel sensor networks, i.e.,

$$E^{ad} = -\lim_{N \to \infty} \frac{1}{N} \log P_e^{ad} = E^p \quad (47)$$

where E^p is defined as follows:

$$E^p = -\lim_{n\longrightarrow\infty}\frac{1}{n}\log P_e^p = \begin{cases} CI(f_0^\gamma, f_1^\gamma) & \text{Bayesian} \\ D(f_0^\gamma, f_1^\gamma) & \text{Neyman-Pearson}\end{cases} \quad (48)$$

where the *ChernoffInformation* $CI(f_0^\gamma, f_1^\gamma)$ and the *Kullback-Leiblerdistance* $D(f_0^\gamma, f_1^\gamma)$ are defined as

$$CI(f_0^\gamma, f_1^\gamma) = \max_{0\leq s\leq 1} -\log \mathbb{E}_0\left[\frac{f_1^\gamma(X)}{f_0^\gamma(X)}\right]^s \quad (49)$$

$$D(f_0^\gamma, f_1^\gamma) = \mathbb{E}_1\left(\log\left(\frac{f_1^\gamma(X)}{f_0^\gamma(X)}\right)\right). \quad (50)$$

Proof: We consider the one-sided error probability. For parallel sensor networks, we know

$$\begin{aligned} E_0^p &= -\lim_{N\longrightarrow\infty}\frac{1}{n}\log Pr(\text{error}|H_0) \\ &= -\lim_{N\longrightarrow\infty}\frac{1}{n}\log Pr(S(0) > \kappa(N - S(0))|H_0) \\ &= \lim_{N\to\infty}\frac{1}{N}\log P\left(\frac{S(0)}{N} > \frac{\kappa}{1+\kappa}\middle| H_0\right) \\ &= \lambda_0\left(\frac{\kappa}{1+\kappa}\right) \end{aligned} \quad (51)$$

where we have utilized (46) and Lemma 5.1 in the last step. Similarly, we can obtain that $E_1^p = \lambda_1(\kappa/(1+\kappa))$.

Now, we only need to show that for detections in ad hoc networks, the following exponents can be achieved:

$$E_0^{ad} = \lambda_0\left(\kappa/(1+\kappa)\right) \quad (52)$$

and

$$E_1^{ad} = \lambda_1\left(\kappa/(1+\kappa)\right). \quad (53)$$

We use κ as the threshold of the ratio of n_k^1 and n_k^0 for the updating rule. Let ε be a small positive number such that $\varepsilon < P(u_K(0) = 1|H_1) - \frac{\kappa}{1+\kappa}$. Let $\nu = \frac{\kappa}{1+\kappa} + \varepsilon$. Thus $\frac{\kappa}{1+\kappa} < \nu < P(u_k(0) = 1|H_1)$. The existence of such ν is guaranteed by (46).

Consider the one-sided error probability

$$\begin{aligned} &P(\text{error}|H_1) = P(S(\infty) = 0|H_1) \\ &= P(S(\infty) = 0|S(0) < \nu N, H_1)P(S(0) < \nu N|H_1) \\ &+ P(S(\infty) = 0|S(0) \geq \nu N, H_1)P(S(0) \geq \nu N|H_1) \\ &\leq P(S(0) < \nu N|H_1) + P(S(\infty) = 0|S(0) \geq \nu N, H_1). \end{aligned} \quad (54)$$

Given a $S(0)$, $S(\infty)$ and H_1 are independent, hence $P(S(\infty) = 0|S(0) \geq \nu N, H_1) = P(S(\infty) = 0|S(t) \geq \nu N) < \sum_{i=\lfloor tN\rfloor}^{N} P(S(\infty) = 0|S(t) = i)$, which decays super-exponentially according to Proposition 4.3. Therefore, $P(S(0) < \nu N|H_1) \sim \exp\{-\lambda_1(\nu)N\}$ is the dominant term in (54). Hence

$$\begin{aligned} E_1^{ad} &= -\lim_{N\longrightarrow\infty}\frac{\log P(error|H_1)}{N} \\ &\geq -\lim_{N\longrightarrow\infty}\frac{\log P(S(0) < \nu N|H_1)}{N} \\ &= \lambda_1(\nu) = \lambda_1\left(\frac{\kappa}{1+\kappa} + \varepsilon\right). \end{aligned} \quad (55)$$

Since ε is a arbitrarily chosen, we have $E_1^{ad} \geq \lambda_1(\frac{\kappa}{1+\kappa})$. Together with the trivial bound $E_1^{ad} \leq E_1^p = \lambda_1(\frac{\kappa}{1+\kappa})$, we have $E_1^{ad} = \lambda_1(\frac{\kappa}{1+\kappa})$. Similarly, We can show that $E_0^{ad} = \lambda_0(\frac{\kappa}{1+\kappa})$.

Hence we prove detections in ad hoc sensor network and parallel sensor network have equivalent asymptotic performance, i.e., $E^{ad} = E^p$. ■

A. Consensus Speed Analysis

In this section, we are concerned about the tail distribution of the absorption time T. Let $\alpha_{[1]}, \alpha_{[2]}, \ldots, \alpha_{[N-1]}$ be the ordered eigenvalues of $\mathbf{Q}$ (See the definition of $\mathbf{Q}$ in (20)) such that $|\alpha_{[1]}| \geq |\alpha_{[2]}| \ldots |\alpha_{[N-1]}|$. Let $\boldsymbol{\eta}_i$ be the column eigenvector corresponding to the eigenvalues $\alpha_{[i]}$ in the sense that $\mathbf{Q}\boldsymbol{\eta}_i = \alpha_{[i]}\boldsymbol{\eta}_i$ for all $i = 1, \ldots, N-1$. Since $\mathbf{Q}$ is a substochastic matrix, it is known by the Perron-Frobenius Theorem[11] that the largest eigenvalue $\alpha_{[1]}$ is unique and a real number between zero and one $(0 < \alpha_{[1]} < 1)$. Since $\mathbf{1}$ can always be written as a linear combination of the eigenvectors $\boldsymbol{\eta}_i$[11], i.e.

$$\mathbf{1} = \sum_{i=1}^{N-1} a_i\boldsymbol{\eta}_i \quad (56)$$

we obtain the following proposition.

Proposition 4.5: The consensus time T satisfies

$$P(T \geq n) = \sum_{i=1}^{N-1} C_i\alpha_{[i]}^{n-1}, \quad (57)$$

where $C_i = a_i(\Pi_0^{\mathrm{T}}P(H_0) + \Pi_1^{\mathrm{T}}P(H_1))\boldsymbol{\eta}_i$ $(i = 1, \ldots, N-1)$.

Proof: It is not hard to write out the pmf of T as

$$P(T = n) = (\Pi_0^{\mathrm{T}}P(H_0) + \Pi_1^{\mathrm{T}}P(H_1))\mathbf{Q}^{n-1}(\mathbf{I} - \mathbf{Q})\mathbf{1}. \quad (58)$$

So

$$\begin{aligned} P(T \geq n) &= (\Pi_0^{\mathrm{T}}P(H_0) + \Pi_1^{\mathrm{T}}P(H_1))\mathbf{Q}^{n-1}\mathbf{1} \\ &= (\Pi_0^{\mathrm{T}}P(H_0) + \Pi_1^{\mathrm{T}}P(H_1))\mathbf{Q}^{n-1}\sum_{i=1}^{N-1} a_i\boldsymbol{\eta}_i \\ &= \sum_{i=1}^{N-1} a_i(\Pi_0^{\mathrm{T}}P(H_0) + \Pi_1^{\mathrm{T}}P(H_1))\mathbf{Q}^{n-1}\boldsymbol{\eta}_i \\ &= \sum_{i=1}^{N-1} C_i\alpha_{[i]}^{n-1}. \end{aligned} \quad (59)$$

■

A straightforward corollary from Proposition 4.5 is given later.

Corollary 4.6: The tail probability $P(T \geq n)$ converges to zero exponentially in the following sense:

$$\lim_{n\longrightarrow\infty}\frac{1}{n}\log P(T \geq n) = \beta \quad (60)$$

where $\beta = \log\alpha_{[1]}$.

V. Numerical Results

So far we have analyzed the detection error probability and consensus speed. In this section, we are going to show the

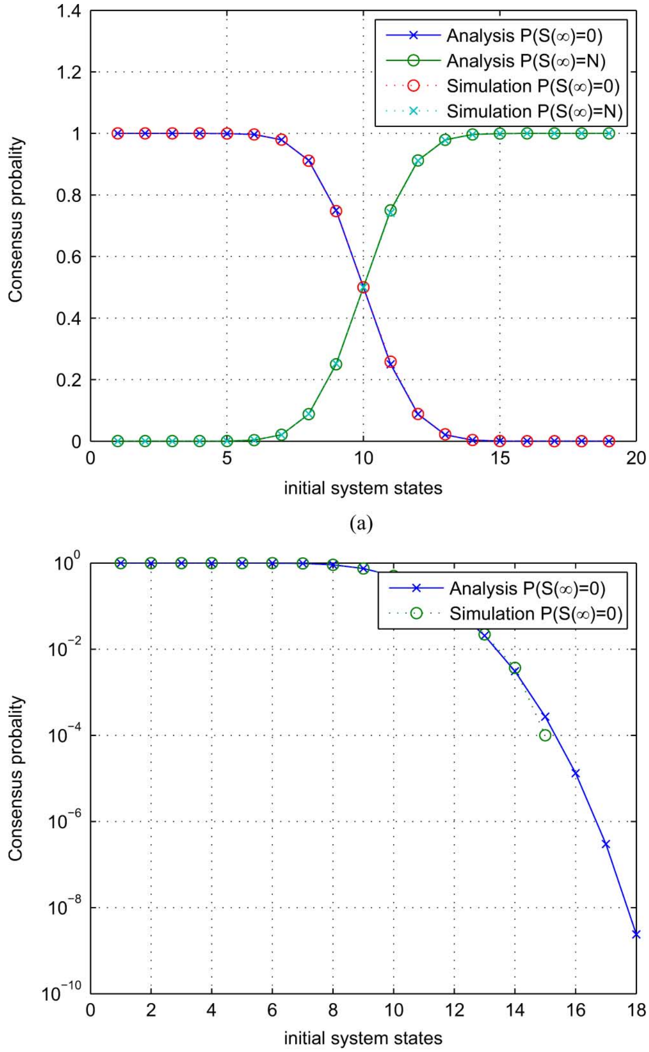

Fig. 3. Limit distribution of consensus in asynchronous system. (a) Limit distribution of consensus given initial state i. (b) A log-plot of $P(S(\infty)=0|S(0)=i)$.

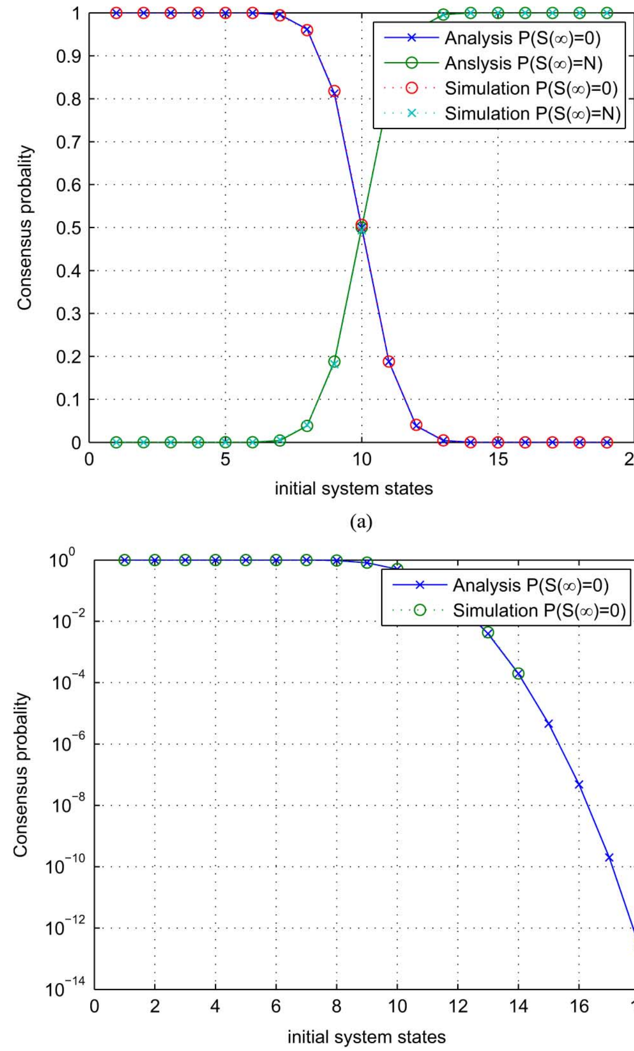

Fig. 4. Limit distribution of consensus in synchronous consensus scheme. (a) Limit distribution of consensus given initial state i. (b) A log plot of $P(S(\infty)=0|S(0)=i)$.

supporting numerical results about the asymptotic system behavior in previous sections. Let's consider the binary detection problem

$$H_0 : \mathcal{N}(-0.5,1)\ vs.\ H_1 : \mathcal{N}(0.5,1) \tag{61}$$

with prior probability $P(H_0)=0.3$ and $P(H_1)=0.7$. All sensors use identical local decision rule

$$\gamma(X)=\begin{cases}0 & \text{if } X<0\\ 1 & \text{otherwise.}\end{cases} \tag{62}$$

The communication between any two sensors succeed with probability $p_c=0.2$.

A. Distribution of Consensus Conditioning on Initial State

Fig. 3 gives an example of the limit distribution of consensus result in asymptotic system when sensor number $N=20$, given the initial state i. In this case, choose the updating threshold $\kappa=1$. Fig. 3(a) shows how $P(S(\infty)=0|S(0)=i)$ and $P(S(\infty)=1|S(0)=i)$ change with i. From Fig. 3(a), we can see generally with the increase of i, it becomes more likely to end up with all sensors agreeing on H_1 and vice versa. Fig. 3(b) is a log scale plot of $P(S(\infty)=0|S(0)=i)$. It shows the probability of consensus to state N drops fast with the initial state i when $i>N/2$.

Fig. 4 gives similar results for synchronized system. Here we use $N=10$.

B. Consensus Distribution Versus Number of Sensors

We first simuluate proposition 4.3. Given certain initial state $i=\lfloor bN\rfloor$, where $b=0.65$, the limit distribution of consensus result with increasing number of sensors is given by Figs. 5, 6. As shown in the figures, given a specified portion of sensors that make the right initial local decisions, the error probability of the consensus decision decreases superexponentially with the increase of the total sensor number. If we compare Figs. 5 and 6,

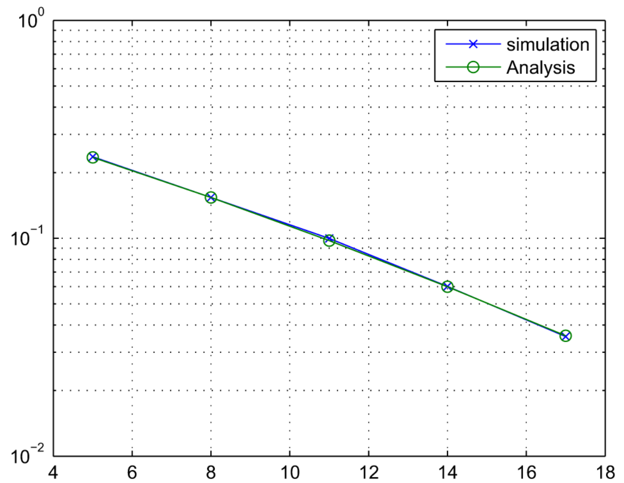

Fig. 5. Consensus distribution conditioned on initial state with increasing number of sensors in asynchronous consensus scheme.

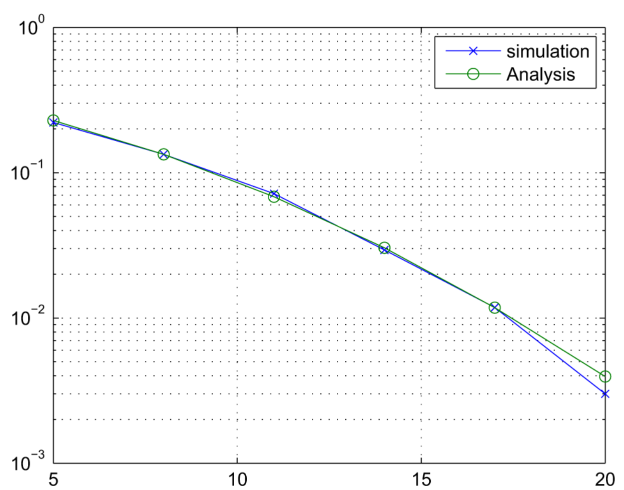

Fig. 6. Consensus distribution conditioned on initial state in synchronous consensus scheme.

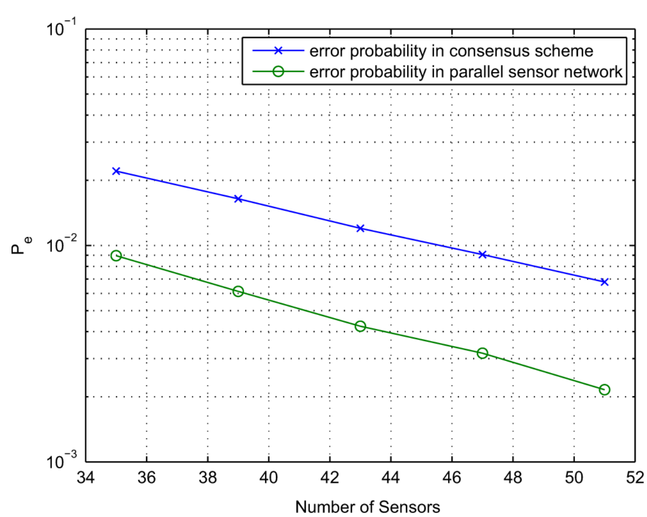

Fig. 7. Bayesian detection error probability in asynchronous consensus scheme.

we can see that the synchronized scheme slightly outperforms its asynchronized counterpart.

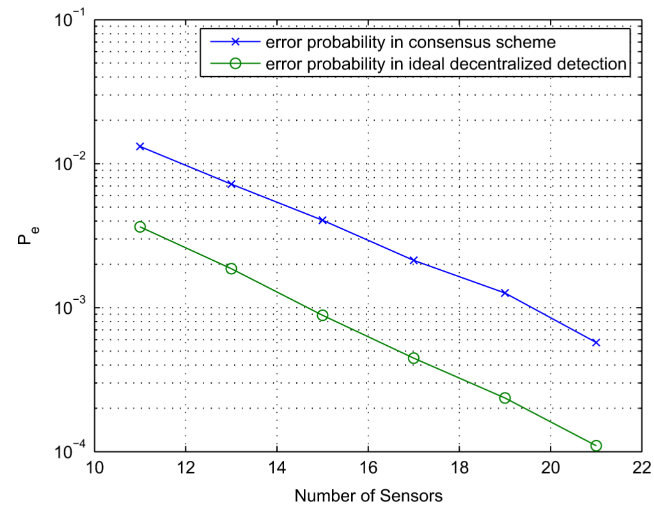

Fig. 8. Bayesian detection error probability in synchronous consensus scheme.

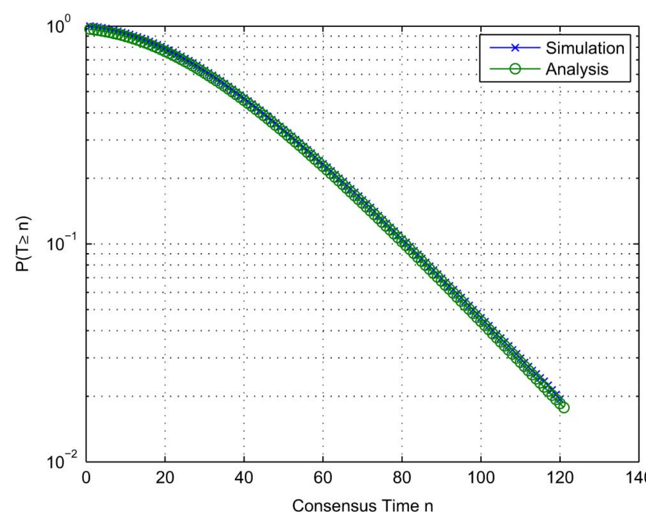

Fig. 9. Consensus speed of asynchronous consensus scheme.

We also simulate the performance of the consensus based detection scheme in Bayesian detection problem as shown in Figs. 7 and 8. For comparison, we plot the error probability curve for the parallel sensor network with perfect communication channels. From the figures, we can see that the ad hoc sensor network asymptotically performs as well as the parallel sensor network.

C. Consensus Speed

Now we come to give the simulation result for the Corollary 5.6 regarding the consensus speed as shown in Figs. 9 and 10. Here we let $N = 30$ (from above simulation, we know $N = 30$ gives us adequate accuracy in detection). From Fig. 10, we can see that the synchronized scheme reaches consensus really fast.

VI. Conclusion

In this paper, we propose a consensus based detection scheme for the detection problem in ad hoc sensor networks. We set up a Markov Chain model for this scheme and based on this model, we analyze the detection error probability of our scheme and show that it is asymptotically equivalent to that of a parallel

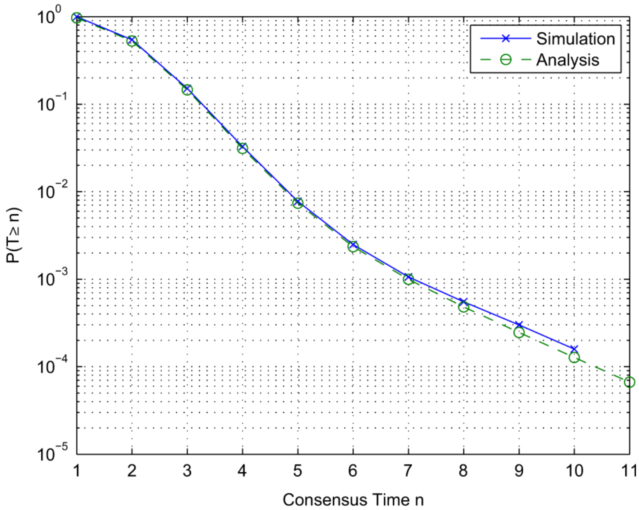

Fig. 10. Consensus speed of synchronous consensus scheme.

sensor networks in the sense that they have the same error exponent even though there is no central processing node in *ad hoc* sensor network. We also analyze the consensus speed of the this our proposed scheme. We show that it is not likely that the system takes a long time to reach consensus. the probability of consensus time follows an exponential decay manner and the exponent is associated with the eigenvalue of the transition matrix among the transient states.

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