Finite-Dimensional Filters with Nonlinear Drift

III: Duncan–Mortensen–Zakai Equation with Arbitrary Initial Condition for the Linear Filtering System and the Beněs Filtering System

SHING-TUNG YAU
Harvard University

STEPHEN S.-T. YAU, Senior Member, IEEE
University of Illinois at Chicago

We consider the Duncan–Mortensen–Zakai (DMZ) equation for the Kalman–Bucy filtering system and Beněs filtering system. We show that this equation can be solved explicitly with an arbitrary initial condition by solving a system of ordinary differential equations and a Kolmogorov-type equation. Let n be the dimension of state space. We show that we need only n sufficient statistics in order to solve the DMZ equation.

Where to find parts I, II, and IV:

I: Finite dimensional filters with nonlinear drifts I: A class of filters containing both Kalman filters and Beněs filters, *Journal of Mathematical Systems, Estimation and Control*, **4** (1994), 181–203.

II: Finite dimensional filters with nonlinear drift II: Brockett's problem on classification of finite dimensional estimation algebra (with Wen-Lin Chiou), *SIAM Journal of Control and Optimization*, **32**, 1 (1994), 297–310.

IV: Finite dimensional filters with nonlinear drift IV: Classification of finite dimensional maximal rank estimation algebra with dimension of state space equal to 3 (with Jie Chen, Chi-Wah Leung), *SIAM Journal of Control and Optimization*, **34**, 1 (1996), 179–198.

Manuscript received April 1, 1996; revised September 2, 1996.

IEEE Log No. T-AES/33/4/06851.

This work was supported in part under U.S. Army Research Grant 93-G-0006.

Authors' addresses: S.-T. Yau, Dept. of Mathematics, Harvard University, Cambridge, MA 02138; Stephen S.-T. Yau, Control and Information Laboratory, Dept. of Mathematics, Statistics and Computer Sciences, University of Illinois at Chicago, 851 S. Morgon St., Chicago, IL 60607-7045.

I. INTRODUCTION

In [18], Mitter pointed out that the innovation approach to nonlinear filtering theory is not, in general, explicitly computable. It was first proposed by Brockett and Clark [5], Brockett [4], and Mitter [18] to use estimation algebras to construct finite-dimensional filters. The idea is to use the Lie algebraic method to solve the Duncan–Mortensen–Zakai (DMZ) equation, which is a stochastic partial differential equation. By working on the robust form of the DMZ equation, which is a stochastic partial differential equation, we can reduce the complexity of the problem to that of solving a time-variant partial differential equation.

In the past decade, the Lie algebraic viewpoint has been remarkably successful, and recent works [6, 7, 12, 16, 22, 25] have given us a deeper understanding of the DMZ equation, which was essential for progress in nonlinear filtering as well as in stochastic control. Nevertheless, it is extremely desirable to treat the DMZ equation by a direct method. In Sections III and IV we use a direct method to solve DMZ equation for linear filtering system and Beněs filtering system, respectively. For the linear filtering system (i.e., f, g, and h are linear functions), the explicit recursive filter was previously derived in a closed form for arbitrary initial conditions only for those systems (1) that are completely reachable and completely observable.

As discussed in [6], the estimation algebra E associated with (1) (i.e., the Lie algebra generated by L_0, $h_1(x),\ldots,h_m(x)$ as differential operators) is of maximal rank if $x_1,\ldots,x_n$ are in E. In the case where $m \geq n$, $h(x) = Hx$, and the matrix H is of maximal rank, E is of maximal rank. Notice that the explicit recursive filter for the Beněs filtering system was previously derived only for the maximal-rank case (cf. [2, 22]). The advantage of our approach is that one can write down the solution of DMZ equation with arbitrary initial conditions in terms of the solution of Kolmogorov equation and a finite system of ordinary differential equations (ODEs). Hence, we can write down the recursive universal filters in both cases. Moreover, our approach is very easy, and we no longer need the maximal-rank condition in our derivation. The significance of our results is that the estimation problem for linear or Beněs filtering systems has been factored into two parts: 1) the off-line calculation of the Kolmogorov-type equation, which does not depend on the observations, and 2) the on-line solution of a finite system of ordinary differential equations. We remark that an explicit closed-form solution of the Kolmogorov equation arising from linear filtering was constructed in [15].

For linear filtering with an arbitrary initial condition, Makowski [17] observed that the filtering question is genuinely one of nonlinear filtering and few results have been obtained before. Notable exceptions are the works of Beněs and Karatzas [3], Ocone [20], Makowski [17], and Haussmann and Pardoux [14]. Makowski [17] further remarked that it was shown in [3 and 20] that there always exists a set of sufficient statistics that can be recursively computed as outputs of a finite-dimensional dynamic system. In contrast with previous results, the sufficient statistics generated in [17] can be termed “universal” in the sense that they are independent of the initial state distribution. Furthermore, no assumptions on the moments of this initial state distribution or its absolute continuity are made in [17], as was the case in [3 and 20].

However, Makowski’s method has two major disadvantages. First, let n be the dimension of the state space. The number of sufficient statistics in order to compute the conditional expectation $\pi_t(\phi(x_t))$ of $\phi(x_t)$ in Makowski’s method is a polynomial of degree two in n, while in our method below (Theorem 2), the number of sufficient statistics is only n (see Remark 3.1). Second, the universal formula [17, (3.13)] for $\pi_t(\phi(x_t))$ is implicit and depends on ϕ. For a given ϕ, one has to do some computation before one can find the number of sufficient statistics.

Haussmann and Pardoux [14] considered a more general linear filtering problem than ours. However, even restricted to the classical Kalman–Bucy problem with a Gaussian initial condition, the number of sufficient statistics in order to compute the conditional probability density is again a polynomial of degree two in n in their method. Since the complexity of our method for linear filtering with an arbitrary initial condition is exactly n, the same as the classical Kalman–Bucy filter, and since our formulas are universal in the sense that they are independent of the initial state distribution, we can apply these results for the recursive numerical solution of filtering problems. These results make this work of practical importance.

The Beněs filtering problem is related to the linear filtering problem by the gauge transformation, as pointed out in [19]. The idea is also related to the concept of equivalence of parabolic equations, as discussed in [1]. However, no explicit solution has been written down. Notable exceptions are [2 and 22] in which the maximal rank condition of the estimation algebra is assumed. In [2] a special case of the maximal-rank condition is treated, while in [22] the general case of the maximal-rank condition is treated. Even in the maximal-rank case, the Wei–Norman approach to finding the solution of the robust DMZ equation is more complicated than our approach (Theorem 3). Not only must one solve a finite system of ODEs and a Kolmogorov equation, but also one must integrate n partial differential equations. More important, for the nonmaximal-rank case, since the basis of the estimation algebra is not explicitly known, the recursive algorithms cannot be written down explicitly. The novelty of our Theorem 3 is that our finite system of ordinary differential equations is explicitly written down and our algorithms are universal in the sense that they are independent of the initial state distribution. Moreover, we need only n sufficient statistics in order to compute the conditional probability density of the Beněs filtering with an arbitrary initial condition. So the result is of practical importance.

Finally, we mention that Daun [10] introduced a separation of variables method in solving the DMZ equation. His method is similar to ours in that he also made an a priori guess as to what the solution of the DMZ equation should be. However, his method is far more complicated than ours. Not only does he have to solve the Kolmogorov-type equation, a finite system of ordinary differential equations, but also he needs to solve a finite system of partial differential equations (cf., [10, eqns. (3) and (4)]. This finite system of overdetermined partial differential equations depends on the solution of the Kolmogorov-type equation mentioned before. Moreover, the time-varying parts of this finite system of overdetermined partial differential equations are unknown. Also (5) and (6) in his basic assumptions involve observation terms. Hence, it is not clear how his method can be implemented even for the Kalman–Bucy system.

II. FILTERING PROBLEM AND BASIC QUESTIONS CONSIDERED

The filtering problem considered here is based on the following signal observation model:

$$\begin{cases} dx(t) = f(x(t))dt + g(x(t))dv(t) & x(0) = x_0 \\ dy(t) = h(x(t))dt + dw(t) & y(0) = 0 \end{cases} \tag{1}$$

in which x, v, y, and w are, respectively, $\mathbf{R}^n$, $\mathbf{R}^p$, $\mathbf{R}^m$, and $\mathbf{R}^m$ valued processes and v and w have components that are independent, standard Brownian processes. We further assume that $n = p$; f,h are C^∞ smooth; and g is an orthogonal matrix. We refer to $y(t)$ as the observation at time t.

Let $\rho(t,x)$ denote the conditional probability density of the state given the observation $\{y(s) : 0 \le s \le t\}$. It is well known (see [11], for example) that $\rho(t,x)$ is given by normalizing a function $\sigma(t,x)$ that satisfies the following DMZ equation:

$$d\sigma(t,x) = L_0\sigma(t,x)dt + \sum_{i=1}^{m} L_i\sigma(t,x)dy_i(t),$$
$$\sigma(0,x) = \sigma_0 \tag{2}$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i} - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2$$

for $i = 1,\ldots,m$, L_i is the zero-degree differential operator, i.e., multiplication by h_i, and σ_0 is the probability density of the initial point x_0.

In [9], Davis introduced a new unnormalized density

$$u(t,x) = \exp\left(-\sum_{i=1}^{m} h_i(x)y_i(t)\right)\sigma(t,x).$$

It is easy to show that $u(t,x)$ satisfies the following time-varying partial differential equation:

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x) &= L_0 u(t,x) + \sum_{i=1}^{m} y_i(t)[L_0,L_i]u(t,x)\\ &\quad + \frac{1}{2}\sum_{i,j=1}^{m} y_i(t)y_j(t)[[L_0,L_i],L_j]u(t,x)\\ u(0,x) &= \sigma_0\end{aligned} \tag{3}$$

where $[\cdot,\cdot]$ is the Lie bracket.

If $h_i(x) = c_i$ = constant for $1 \le i \le m$, then (3) reduces to the Kolmogorov equation

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x) &= Lu(t,x)\\ &= \left(\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i(x)\frac{\partial}{\partial x_i}\right.\\ &\quad \left. - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) - \frac{1}{2}\sum_{i=1}^{m} c_i^2\right)u(t,x).\end{aligned} \tag{4}$$

In [26], the following Kolmogorov type equation (5) can be solved explicitly in a closed form.

THEOREM 1 *The equation*

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x) &= L_0(t,x)\\ &= \left(\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i(x)\frac{\partial}{\partial x_i}\right.\\ &\quad \left. - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) - \frac{1}{2}\sum_{i=1}^{m} h_i^2(x)\right)u(t,x)\end{aligned} \tag{5}$$

has a formal solution on $\mathbf{R}^n$ of the following form:

$$\begin{aligned}u(t,x) &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{1}{(\sqrt{2\pi})^n}t^{-n/2}\\ &\quad \times \exp\left(\frac{-1}{2t}\sum_{j=1}^{n}(x_j - y_j)^2\right)\\ &\quad \times b(t,x,y)\sigma_0(y)\,dy_1\cdots dy_n\end{aligned} \tag{6}$$

where $b(t,x,y) = \sum_0^\infty a_k(x,y)t^k$. Here $a_k(x,y)$ are described explicitly as follows. Let

$$a(x,y) = \int_0^1 \sum_{i=1}^{n}(x_i - y_i)f_i(y + t(x-y))\,dt. \tag{7}$$

Then

$$a_0(x,y) = e^{a(x,y)}. \tag{8}$$

Suppose that $a_{k-1}(x,y)$ is given. Let

$$\begin{aligned}g_k(x,y) &= \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 a_{k-1}}{\partial x_i^2}(x,y) - \sum_{i=1}^{n} f_i(x)\frac{\partial a_{k-1}}{\partial x_i}(x,y)\\ &\quad - \frac{1}{2}\left(\sum_{i=1}^{m} h_i^2(x)\right)a_{k-1}(x,y)\\ &\quad - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x)a_{k-1}(x,y).\end{aligned} \tag{9}$$

Then for $k \ge 1$

$$\begin{aligned}a_k(x,y) &= \exp(a(x,y))\int_0^1 t^{k-1}\exp(-a(y + t(x-y),y))\\ &\quad \times g_k(y + t(x-y),y)\,dt.\end{aligned}$$

Moreover, the procedure for obtaining the convergent solution of (5) from the formal solution is described in [26]. For the Kolmogorov equation (14) below arising from linear filtering, a simple closed-form solution was written down in [15].

It is shown in [26] that (3) is equivalent to

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x) &= \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}(t,x)\\ &\quad + \sum_{i=1}^{n}\left(-f_i(x) + \sum_{j=1}^{m} y_j(t)\frac{\partial h_j}{\partial x_i}(x)\right)\frac{\partial u}{\partial x_i}(t,x)\\ &\quad - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x)u(t,x) - \frac{1}{2}\sum_{i=1}^{m} h_i^2(x)u(t,x)\\ &\quad + \frac{1}{2}\sum_{i=1}^{m} y_i(t)\Delta h_i(x)u(t,x)\\ &\quad - \sum_{i=1}^{m}\sum_{j=1}^{n} y_i(t)f_j(x)\frac{\partial h_i}{\partial x_j}(x)u(t,x)\\ &\quad + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} y_i(t)y_j(t)\sum_{k=1}^{n}\frac{\partial h_i}{\partial x_k}(x)\frac{\partial h_j}{\partial x_k}(x)u(t,x).\end{aligned} \tag{10}$$

III. DMZ EQUATION FOR KALMAN–BUCY FILTERING SYSTEM WITH ARBITRARY INITIAL CONDITION

Despite its usefulness, the Kalman–Bucy filter is not perfect. One of its weaknesses is that it needs a Gaussian assumption on the initial data. The situation is more complex when the statistics of the initial condition are modeled by an arbitrary distribution. In the case where the linear filtering system (i.e., f, g, and h are linear functions in (1)) is completely reachable and completely observable, Hazewinkel observed [13, p. 115] that the estimation algebra E (i.e., a Lie algebra generated by differential operators L_0, $h_1(x),\ldots,h_m(x)$) is the $2n+2$ dimensional Lie algebra with basis $L_0, \partial/\partial x_1, \ldots, \partial/\partial x_n, x_1, \ldots, x_n, 1$. Even in this case, the Wei–Norman approach used to find the solution of (5) is more complicated than the procedure in Theorem 2 below because not only must one solve a finite system of ordinary differential equation and a Kolmogorov equation, but also one has to integrate n partial differential equations corresponding to the operators $\partial/\partial x_1, \ldots, \partial/\partial x_n$. More important, if the linear system is not completely reachable or completely observable, then the basis of the estimation algebra is not explicitly known (although it can be computed). As a result, there is an additional disadvantage of the Wei–Norman approach, namely, one cannot write down the finite system of ODEs explicitly. The novelty of Theorem 2 is that our finite system of ODEs is explicitly written down and our procedure to get the solution of (5) is simpler than the Lie algebra approach.

THEOREM 2 *Consider the linear filtering system* (1) *with*

$$h_i(x) = \sum_{j=1}^{n} c_{ij} x_j + c_i, \qquad 1 \le i \le m \tag{11}$$

where c_{ij} and c_i are constants, and

$$f_i(x) = \sum_{j=1}^{n} d_{ij} x_j + d_i, \qquad 1 \le i \le n \tag{12}$$

where d_{ij} and d_i are constants. Choose a homogeneous quadratic $F(x) = \frac{1}{2}\sum_{i,j=1}^{n} e_{ij} x_i x_j$ with $e_{ij} = e_{ji}$ such that

$$(E+D)^T (E+D) = C^T C + D^T D. \tag{13}$$

Here $E = (e_{ij})$, $D = (d_{ij})$ are an $n \times n$ matrix and $C = (c_{ij})$ is an $m \times n$ matrix. Then the solution $u(t,x)$ for the DMZ equation (10) *is reduced to the solution $\tilde{u}(t,x)$ for the Kolmogorov equation*

$$\frac{\partial \tilde{u}}{\partial t}(t,x) = \frac{1}{2}\Delta \tilde{u}(t,x) - \sum_{i=1}^{n}\left(f_i(x) + \frac{\partial F}{\partial x_i}(x)\right)\frac{\partial \tilde{u}}{\partial x_i}(t,x) - \sum_{i=1}^{n}\left(\frac{\partial f_i}{\partial x_i}(x) + \frac{\partial^2 F}{\partial x_i^2}(x)\right)\tilde{u}(t,x) \tag{14}$$

where

$$\tilde{u}(t,x) = \exp\left(F(x) + \sum_{i=1}^{n} a_i(t) x_i + c(t)\right) u(t, x + b(t)) \tag{15}$$

and $a_i(t)$, $b_i(t)$, and $c(t)$ satisfy the following system of ODEs:

$$a_i'(t) + \sum_{j=1}^{n} d_{ji} a_j(t) - \sum_{l=1}^{m}\sum_{j=1}^{n} c_{lj} b_j(t) c_{li} - \sum_{j=1}^{m}\sum_{k=1}^{n} y_j(t) d_{ki} c_{jk} + \sum_{j=1}^{n} d_j e_{ji} - \sum_{j=1}^{m} c_j c_{ji} = 0, \qquad 1 \le i \le n \tag{16}$$

$$\begin{aligned}
&\sum_{i=1}^{n} d_i a_i(t) + c'(t) - \frac{1}{2}\sum_{i=1}^{n} a_i^2(t) \\
&\quad - \frac{1}{2}\sum_{i=1}^{m}\left(\sum_{j=1}^{n} c_{ij} b_j(t)\right)^2 - \sum_{i=1}^{m} c_i \sum_{j=1}^{n} c_{ij} b_j(t) \\
&\quad - \sum_{j=1}^{m}\sum_{i=1}^{n} y_j(t) c_{ji}\left(\sum_{k=1}^{n} d_{ik} b_k(t) + d_i\right) \\
&\quad + \frac{1}{2}\sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=1}^{m} y_i(t) y_j(t) c_{ik} c_{jk} \\
&\quad + \frac{1}{2}\sum_{k=1}^{n} e_{kk} - \frac{1}{2}\sum_{i=1}^{m} c_i^2 = 0
\end{aligned} \tag{17}$$

$$b_i'(t) - a_i(t) + \sum_{j=1}^{m} c_{ji} y_j(t) - \sum_{j=1}^{n} d_{ij} b_j(t) = 0, \qquad 1 \le i \le n. \tag{18}$$

REMARK The initial conditions for $a_i(t)$, $b_i(t)$, and $c(t)$ can be arbitrary values. However, once $a_i(0)$, $b_i(0)$, and $c(0)$ are fixed, the initial condition of (14) is determined by the initial condition of (10).

PROOF We need only to show that if $u(t,x)$ satisfies (10), then $\tilde{u}(t,x)$ given by (15)–(18) will satisfy the Kolmogorov equation (14)

$$\begin{aligned}
\frac{\partial \tilde{u}}{\partial t}(t,x) &= \exp\left(F(x) + \sum_{i=1}^{n} a_i(t) x_i + c(t)\right) \\
&\quad \times \left[\left(\sum_{i=1}^{n} a_i'(t) x_i + c'(t)\right) u(t, x+b(t)) + \frac{\partial u}{\partial t}(t, x+b(t)) + \sum_{i=1}^{n} b_i'(t)\frac{\partial u}{\partial x_i}(t, x+b(t))\right]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \tilde{u}}{\partial x_i}(t,x) &= \exp\left(F(x) + \sum_{i=1}^{n} a_i(t) x_i + c(t)\right) \\
&\quad \times \left[\left(\frac{\partial F}{\partial x_i}(x) + a_i(t)\right) u(t, x+b(t)) + \frac{\partial u}{\partial x_i}(t, x+b(t))\right]
\end{aligned}$$

$$\frac{\partial^2 \tilde{u}}{\partial x_i^2}(t,x) = \exp\left(F(x) + \sum_{i=1}^n a_i(t)x_i + c(t)\right)$$
$$\times \left[\left(\frac{\partial F}{\partial x_i}(x) + a_i(t)\right)^2 u(t,x+b(t))\right.$$
$$+ 2\left(\frac{\partial F}{\partial x_i}(x) + a_i(t)\right)\frac{\partial u}{\partial x_i}(t,x+b(t))$$
$$\left.+ \frac{\partial^2 F}{\partial x_i^2}(x)u(t,x+b(t)) + \frac{\partial^2 u}{\partial x_i^2}(t,x+b(t))\right]$$

$$\frac{1}{2}\Delta\tilde{u}(t,x) = \exp\left(F(x) + \sum_{i=1}^n a_i(t)x_i + c(t)\right)$$
$$\times \left[\frac{1}{2}\sum_{i=1}^n \left(\frac{\partial F}{\partial x_i}(x) + a_i(t)\right)^2 u(t,x+b(t))\right.$$
$$+ \sum_{i=1}^n \left(\frac{\partial F}{\partial x_i}(x) + a_i(t)\right)\frac{\partial u}{\partial x_i}(t,x+b(t))$$
$$\left.+ \frac{1}{2}\Delta F(x)u(t,x+b(t)) + \frac{1}{2}\Delta u(t,x+b(t))\right]$$

$$\sum_{i=1}^n f_i(x)\frac{\partial \tilde{u}}{\partial x_i}(t,x)$$
$$= \exp\left(F(x) + \sum_{i=1}^n a_i(t)x_i + c(t)\right)$$
$$\times \left[\sum_{i=1}^n f_i(x)\left(\frac{\partial F}{\partial x_i}(x) + a_i(t)\right)u(t,x+b(t))\right.$$
$$\left.+ \sum_{i=1}^n f_i(x)\frac{\partial u}{\partial x_i}(t,x+b(t))\right].$$

Therefore we have

$$\frac{\partial \tilde{u}}{\partial t}(t,x) - \frac{1}{2}\Delta\tilde{u}(t,x) + \sum_{i=1}^n f_i(x)\frac{\partial \tilde{u}}{\partial x_i}(t,x)$$
$$= \exp\left(F(x) + \sum_{i=1}^n a_i(t)x_i + c(t)\right)$$
$$\times \left[\left(\sum_{i=1}^n a_i'(t)x_i + c'(t)\right)u(t,x+b(t))\right.$$
$$+ \frac{\partial u}{\partial t}(t,x+b(t)) + \sum_{i=1}^n b_i'(t)\frac{\partial u}{\partial x_i}(t,x+b(t))$$
$$- \frac{1}{2}\sum_{i=1}^n \left(\frac{\partial F}{\partial x_i}(x) + a_i(t)\right)^2 u(t,x+b(t))$$
$$- \sum_{i=1}^n \left(\frac{\partial F}{\partial x_i}(x) + a_i(t)\right)\frac{\partial u}{\partial x_i}(t,x+b(t))$$
$$- \frac{1}{2}\Delta F(x)u(t,x+b(t)) - \frac{1}{2}\Delta u(t,x+b(t))$$
$$+ \sum_{i=1}^n f_i(x)\left(\frac{\partial F}{\partial x_i}(x) + a_i(t)\right)u(t,x+b(t))$$
$$\left.+ \sum_{i=1}^n f_i(x)\frac{\partial u}{\partial x_i}(t,x+b(t))\right]$$

$$= \exp\left(F(x) + \sum_{i=1}^n a_i(t)x_i + c(t)\right)$$
$$\times \left[\left(\sum_{i=1}^n a_i'(t)x_i + c'(t)\right)u(t,x+b(t))\right.$$
$$+ \sum_{i=1}^n b_i'(t)\frac{\partial u}{\partial x_i}(t,x+b(t))$$
$$- \frac{1}{2}\sum_{i=1}^n \left(\frac{\partial F}{\partial x_i}(x) + a_i(t)\right)^2 u(t,x+b(t))$$
$$- \sum_{i=1}^n \left(\frac{\partial F}{\partial x_i}(x) + a_i(t)\right)\frac{\partial u}{\partial x_i}(t,x+b(t))$$
$$- \frac{1}{2}\Delta F(x)u(t,x+b(t))$$
$$+ \sum_{i=1}^n f_i(x)\left(\frac{\partial F}{\partial x_i}(x) + a_i(t)\right)u(t,x+b(t))$$
$$+ \sum_{i=1}^n\sum_{j=1}^m y_j(t)\frac{\partial h_j}{\partial x_i}(x+b(t))\frac{\partial u}{\partial x_i}(t,x+b(t))$$
$$- \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}(x+b(t))u(t,x+b(t))$$
$$- \frac{1}{2}\sum_{i=1}^m h_i^2(x+b(t))u(t,x+b(t))$$
$$+ \frac{1}{2}\sum_{i=1}^m y_i(t)\,\Delta h_i(x+b(t))u(t,x+b(t))$$
$$- \sum_{i=1}^m\sum_{j=1}^n y_i(t)f_j(x+b(t))$$
$$\times \frac{\partial h_i}{\partial x_j}(x+b(t))u(t,x+b(t))$$
$$+ \frac{1}{2}\sum_{i=1}^m\sum_{j=1}^m y_i(t)y_j(t)\sum_{k=1}^n \frac{\partial h_i}{\partial x_k}(x+b(t))$$
$$\times \frac{\partial h_j}{\partial x_k}(x+b(t))u(t,x+b(t))$$
$$\left.+ \sum_{i=1}^n (f_i(x) - f_i(x+b(t))\frac{\partial u}{\partial x_i}(t,x+b(t))\right].$$

Observe that

$$\sum_{i=1}^{n}\frac{\partial F}{\partial x_i}(x)\frac{\partial \tilde{u}}{\partial x_i}(t,x)$$

$$= \exp\left(F(x)+\sum_{i=1}^{n}a_i(t)x_i+c(t)\right)$$

$$\times\left[\sum_{i=1}^{n}\frac{\partial F}{\partial x_i}(x)\left(\frac{\partial F}{\partial x_i}(x)+a_i(t)\right)u(t,x+b(t))\right.$$

$$\left.+\sum_{i=1}^{n}\frac{\partial F}{\partial x_i}(x)\frac{\partial u}{\partial x_i}(t,x+b(t))\right].$$

It follows that

$$\frac{\partial \tilde{u}}{\partial t}(t,x)-\frac{1}{2}\Delta\tilde{u}(t,x)+\sum_{i=1}^{n}f_i(x)\frac{\partial \tilde{u}}{\partial x_i}(t,x)$$

$$+\sum_{i=1}^{n}\frac{\partial F}{\partial x_i}(x)\frac{\partial \tilde{u}}{\partial x_i}(t,x)$$

$$= \exp\left(F(x)+\sum_{i=1}^{n}a_i(t)x_i+c(t)\right)$$

$$\times\left[\left(\sum_{i=1}^{n}a_i'(t)x_i+c'(t)\right)u(t,x+b(t))\right.$$

$$+\sum_{i=1}^{n}b_i'(t)\frac{\partial u}{\partial x_i}(t,x+b(t))$$

$$-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial F}{\partial x_i}(x)+a_i(t)\right)^2 u(t,x+b(t))$$

$$+\sum_{i=1}^{n}\frac{\partial F}{\partial x_i}(x)\left(\frac{\partial F}{\partial x_i}(x)+a_i(t)\right)u(t,x+b(t))$$

$$-\sum_{i=1}^{n}a_i(t)\frac{\partial u}{\partial x_i}(t,x+b(t))$$

$$-\frac{1}{2}\Delta F(x)u(t,x+b(t))$$

$$+\sum_{i=1}^{n}f_i(x)\left(\frac{\partial F}{\partial x_i}(x)+a_i(t)\right)u(t,x+b(t))$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{m}y_j(t)\frac{\partial h_j}{\partial x_i}(x+b(t))\frac{\partial u}{\partial x_i}(t,x+b(t))$$

$$-\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x+b(t))u(t,x+b(t))$$

$$-\frac{1}{2}\sum_{i=1}^{m}h_i^2(x+b(t))u(t,x+b(t))$$

$$+\frac{1}{2}\sum_{i=1}^{m}y_i(t)\,\Delta h_i(x+b(t))u(t,x+b(t))$$

$$-\sum_{i=1}^{m}\sum_{j=1}^{n}y_i(t)f_j(x+b(t))$$

$$\times\frac{\partial h_i}{\partial x_j}(x+b(t))u(t,x+b(t))$$

$$+\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}y_i(t)y_j(t)\sum_{k=1}^{n}\frac{\partial h_i}{\partial x_k}(x+b(t))$$

$$\times\frac{\partial h_j}{\partial x_k}(x+b(t))u(t,x+b(t))$$

$$\left.+\sum_{i=1}^{n}(f_i(x)-f_i(x+b(t)))\frac{\partial u}{\partial x_i}(t,x+b(t))\right]$$

$$= \exp\left(F(x)+\sum_{i=1}^{n}a_i(t)x_i+c(t)\right)$$

$$\times\left\{\left[\sum_{i=1}^{n}a_i'(t)x_i+c'(t)-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial F}{\partial x_i}\right)^2(x)\right.\right.$$

$$-\sum_{i=1}^{n}\frac{\partial F}{\partial x_i}(x)a_i(t)-\frac{1}{2}\sum_{i=1}^{n}a_i^2(t)$$

$$+\sum_{i=1}^{n}\left(\frac{\partial F}{\partial x_i}\right)^2(x)+\sum_{i=1}^{n}\frac{\partial F}{\partial x_i}(x)a_i(t)$$

$$-\frac{1}{2}\Delta F(x)+\sum_{i=1}^{n}f_i(x)\frac{\partial F}{\partial x_i}(x)$$

$$+\sum_{i=1}^{n}f_i(x)a_i(t)-\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x+b(t))$$

$$-\frac{1}{2}\sum_{i=1}^{m}h_i^2(x+b(t))$$

$$+\frac{1}{2}\sum_{i=1}^{m}y_i(t)\,\Delta h_i(x+b(t))$$

$$-\sum_{i=1}^{m}\sum_{j=1}^{n}y_i(t)f_j(x+b(t))\frac{\partial h_i}{\partial x_j}(x+b(t))$$

$$+\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}y_i(t)y_j(t)\sum_{k=1}^{n}\frac{\partial h_i}{\partial x_k}(x+b(t))$$

$$\left.\times\frac{\partial h_j}{\partial x_k}(x+b(t))\right]u(t,x+b(t))$$

$$+\left[\sum_{i=1}^{n}(b_i'(t)-a_i(t))+\sum_{i=1}^{n}\sum_{j=1}^{m}y_j(t)\right.$$

$$\left.\times\frac{\partial h_j}{\partial x_i}(x+b(t))+\sum_{i=1}^{n}(f_i(x)-f_i(x+b(t)))\right]$$

$$\times\frac{\partial u}{\partial x_i}(t,x+b(t))$$

$$
\begin{aligned}
&= \exp\left(F(x) + \sum_{i=1}^{n} a_i(t)x_i + c(t)\right) \\
&\quad \times \Bigg\{ \Bigg[\sum_{i=1}^{n} a_i'(t)x_i + \sum_{i=1}^{n} f_i(x)a_i(t) + c'(t) \\
&\qquad - \frac{1}{2}\sum_{i=1}^{n} a_i^2(t) + \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial F}{\partial x_i}\right)^2(x) \\
&\qquad - \frac{1}{2}\Delta F(x) + \sum_{i=1}^{n} f_i(x)\frac{\partial F}{\partial x_i}(x) \\
&\qquad - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x+b(t)) - \frac{1}{2}\sum_{i=1}^{m} h_i^2(x+b(t)) \\
&\qquad + \frac{1}{2}\sum_{i=1}^{m} y_i(t)\,\Delta h_i(x+b(t)) \\
&\qquad - \sum_{i=1}^{m}\sum_{j=1}^{n} y_i(t)f_j(x+b(t))\frac{\partial h_i}{\partial x_j}(x+b(t)) \\
&\qquad + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} y_i(t)y_j(t)\sum_{k=1}^{n}\frac{\partial h_i}{\partial x_k}(x+b(t)) \\
&\qquad \times \frac{\partial h_j}{\partial x_k}(x+b(t))\Bigg] u(t,x+b(t)) \\
&\qquad + \sum_{i=1}^{n}\Bigg[b_i'(t) - a_i(t) + \sum_{j=1}^{m} y_j(t)\frac{\partial h_j}{\partial x_i}(x+b(t)) \\
&\qquad\qquad + f_i(x) - f_i(x+b(t))\Bigg]\frac{\partial u}{\partial x_i}(t,x+b(t))\Bigg\}.
\end{aligned}
$$

Putting (11) and (12) into the above equation, we get

$$
\begin{aligned}
&\frac{\partial \tilde{u}}{\partial t}(t,x) - \frac{1}{2}\Delta\tilde{u}(t,x) + \sum_{i=1}^{n} f_i(x)\frac{\partial \tilde{u}}{\partial x_i}(t,x) \\
&\quad + \sum_{i=1}^{n}\frac{\partial F}{\partial x_i}(x)\frac{\partial \tilde{u}}{\partial x_i}(t,x) \\
&= \exp\left(F(x) + \sum_{i=1}^{n} a_i(t)x_i + c(t)\right) \\
&\quad \times \Bigg\{ \Bigg[\sum_{i=1}^{n} a_i'(t)x_i + \sum_{i=1}^{n}\left(\sum_{j=1}^{n} d_{ij}x_j + d_i\right) a_i(t) + c'(t) \\
&\qquad - \frac{1}{2}\sum_{i=1}^{n} a_i^2(t) + \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial F}{\partial x_i}\right)^2(x) - \frac{1}{2}\Delta F(x) \\
&\qquad + \sum_{i=1}^{n}\left(\sum_{j=1}^{n} d_{ij}x_j + d_i\right)\frac{\partial F}{\partial x_i}(x) - \sum_{i=1}^{n} d_{ii} \\
&\qquad - \frac{1}{2}\sum_{i=1}^{m}\left(\sum_{j=1}^{n} c_{ij}x_j + \sum_{j=1}^{n} c_{ij}b_j(t) + c_i\right)^2 \\
&\qquad - \sum_{j=1}^{m}\sum_{i=1}^{n} y_j(t)\left(\sum_{k=1}^{n} d_{ik}x_k + \sum_{k=1}^{n} d_{ik}b_k(t) + d_i\right)c_{ji} \\
&\qquad + \frac{1}{2}\sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=1}^{m} y_i(t)y_j(t)c_{ik}c_{jk}\Bigg] u(t,x+b(t)) \\
&\qquad + \sum_{i=1}^{n}\Bigg[b_i'(t) - a_i(t) + \sum_{j=1}^{m} c_{ji}y_j(t) \\
&\qquad\qquad - \sum_{j=1}^{n} d_{ij}b_j(t)\Bigg]\frac{\partial u}{\partial x_i}(t,x+b(t))\Bigg\} \\
&= \exp\left(F(x) + \sum_{i=1}^{n} a_i(t)x_i + c(t)\right) \\
&\quad \times \Bigg\{ \Bigg[\Bigg(\sum_{i=1}^{n} a_i'(t)x_i + \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ji}x_i a_j(t) \\
&\qquad - \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}b_j(t)\sum_{k=1}^{n} c_{ik}x_k \\
&\qquad - \sum_{j=1}^{m}\sum_{i=1}^{n} y_j(t)\sum_{k=1}^{n} d_{ik}x_k c_{ji}\Bigg) \\
&\qquad + \Bigg(\sum_{i=1}^{n} d_i a_i(t) + c'(t) - \frac{1}{2}\sum_{i=1}^{n} a_i^2(t) \\
&\qquad\quad - \frac{1}{2}\sum_{i=1}^{m}\left(\sum_{i=1}^{n} c_{ij}b_j(t)\right)^2 - \sum_{i=1}^{m} c_i\sum_{j=1}^{n} c_{ij}b_j(t) \\
&\qquad\quad - \sum_{j=1}^{m}\sum_{i=1}^{n} y_j(t)\left(\sum_{k=1}^{n} d_{ik}b_k(t) + d_i\right)c_{ji} \\
&\qquad\quad + \frac{1}{2}\sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=1}^{m} y_i(t)y_j(t)c_{ik}c_{jk}\Bigg) \\
&\qquad + \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial F}{\partial x_i}\right)^2(x) - \frac{1}{2}\Delta F(x) \\
&\qquad + \sum_{i=1}^{n}\left(\sum_{i=1}^{n} d_{ij}x_j + d_i\right)\frac{\partial F}{\partial x_i}(x) - \sum_{i=1}^{n} d_{ii} \\
&\qquad - \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}(c_{ij}x_j + c_i)^2\Bigg] u(t,x+b(t)) \\
&\qquad + \sum_{i=1}^{n}\Bigg[b_i'(t) - a_i(t) + \sum_{j=1}^{m} y_j(t)c_{ji} \\
&\qquad\qquad - \sum_{j=1}^{n} d_{ji}b_j(t)\Bigg]\frac{\partial u}{\partial x_i}(t,x+b(t))\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&= \exp\left(F(x) + \sum_{i=1}^{n} a_i(t)x_i + c(t)\right)\\
&\quad\times\left\{\left[\sum_{i=1}^{n} a_i'(t)x_i + \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ji}a_j(t)x_i\right.\right.\\
&\qquad - \sum_{l=1}^{m}\left(\sum_{j=1}^{n} c_{l_j}b_j(t)\right)\left(\sum_{i=1}^{n} c_{l_i}x_i\right)\\
&\qquad - \sum_{j=1}^{m}\sum_{k=1}^{n} y_j(t)\sum_{i=1}^{n} d_{ki}x_i c_{jk} + \sum_{i=1}^{n} d_i a_i(t) + c'(t)\\
&\qquad - \frac{1}{2}\sum_{i=1}^{n} a_i^2(t) - \frac{1}{2}\sum_{i=1}^{m}\left(\sum_{j=1}^{n} c_{ij}b_j(t)\right)^2\\
&\qquad - \sum_{i=1}^{m} c_i\sum_{j=1}^{n} c_{ij}b_j(t) - \sum_{j=1}^{m}\sum_{i=1}^{n} y_j(t)\\
&\qquad \times\left(\sum_{k=1}^{n} d_{ik}b_k(t) + d_i\right)c_{ji}\\
&\qquad + \frac{1}{2}\sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=1}^{m} y_i(t)y_j(t)c_{ik}c_{jk}\\
&\qquad + \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial F}{\partial x_i}\right)^2(x) - \frac{1}{2}\Delta F(x)\\
&\qquad + \sum_{i=1}^{n}\left(\sum_{j=1}^{n} d_{ij}x_j + d_i\right)\frac{\partial F}{\partial x_i}(x) - \sum_{i=1}^{n} d_{ii}\\
&\qquad \left. - \frac{1}{2}\sum_{i=1}^{m}\left(\sum_{j=1}^{n} c_{ij}x_j + c_i\right)^2\right]u(t,x+b(t))\\
&\quad + \sum_{i=1}^{n}\left[(b_i'(t) - a_i(t)) + \sum_{j=1}^{m} y_j(t)c_{ji}\right.\\
&\qquad\qquad \left.\left. - \sum_{j=1}^{n} d_{ij}b_j(t)\right]\frac{\partial u}{\partial x_i}(t,x+b(t))\right\}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\frac{\partial \tilde{u}}{\partial t}(t,x) - \frac{1}{2}\Delta\tilde{u}(t,x) + \sum_{i=1}^{n}\left(f_i(x) + \frac{\partial F}{\partial x_i}(x)\right)\frac{\partial \tilde{u}}{\partial x_i}(t,x)\\
&\qquad + \sum_{i=1}^{n}\left(\frac{\partial f_i}{\partial x_i} + \frac{\partial^2 F}{\partial x_i^2}(x)\right)\tilde{u}(t,x)\\
&\quad = \exp\left(F(x) + \sum_{i=1}^{n} a_i(t)x_i + c(t)\right)\\
&\qquad\times\left\{\left[\sum_{i=1}^{n} a_i'(t)x_i + \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ji}a_j(t)x_i\right.\right.\\
&\qquad\quad - \sum_{l=1}^{m}\left(\sum_{j=1}^{n} c_{lj}b_j(t)\right)\left(\sum_{i=1}^{n} c_{li}x_i\right)\\
&\qquad\quad - \sum_{j=1}^{m}\sum_{k=1}^{n} y_j(t)\sum_{i=1}^{n} d_{ki}x_i c_{jk}\\
&\qquad\quad + \sum_{i=1}^{n} d_i a_i(t) + c'(t) - \frac{1}{2}\sum_{i=1}^{n} a_i^2(t)\\
&\qquad\quad - \frac{1}{2}\sum_{i=1}^{m}\left(\sum_{j=1}^{n} c_{ij}b_j(t)\right)^2 - \sum_{i=1}^{m} c_i\sum_{j=1}^{n} c_{ij}b_j(t)\\
&\qquad\quad - \sum_{j=1}^{m}\sum_{i=1}^{n} y_j(t)\left(\sum_{k=1}^{n} d_{ik}b_k(t) + d_i\right)c_{ji}\\
&\qquad\quad + \frac{1}{2}\sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=1}^{m} y_i(t)y_j(t)c_{ik}c_{jk}\\
&\qquad\quad + \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial F}{\partial x_i}\right)^2(x) + \frac{1}{2}\Delta F(x)\\
&\qquad\quad + \sum_{i=1}^{n}\left(\sum_{j=1}^{n} d_{ij}x_j + d_i\right)\frac{\partial F}{\partial x_i}(x)\\
&\qquad\quad \left. - \frac{1}{2}\sum_{i=1}^{m}\left(\sum_{j=1}^{n} c_{ij}x_j + c_i\right)^2\right]u(t,x+b(t))\\
&\qquad + \sum_{i=1}^{n}\left[b_i'(t) - a_i(t) + \sum_{j=1}^{m} y_j(t)c_{ji}\right.\\
&\qquad\qquad\qquad \left.\left. - \sum_{j=1}^{n} d_{ij}b_j(t)\right]\frac{\partial u}{\partial x_i}(t,x+b(t))\right\}\\
&\quad = \exp\left(F(x) + \sum_{i=1}^{n} a_i(t)x_i + c(t)\right)\\
&\qquad\times\left\{\left[\sum_{i=1}^{n} a_i'(t)x_i + \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ji}a_j(t)x_i\right.\right.\\
&\qquad\quad - \sum_{l=1}^{m}\left(\sum_{j=1}^{n} c_{lj}b_j(t)\right)\left(\sum_{i=1}^{n} c_{li}x_i\right)\\
&\qquad\quad - \sum_{j=1}^{m}\sum_{k=1}^{n} y_j(t)\sum_{i=1}^{n} d_{ki}x_i c_{jk}\\
&\qquad\quad + \sum_{i=1}^{n}\left(\sum_{j=1}^{n} d_j e_{ji}\right)x_i - \sum_{i=1}^{n}\left(\sum_{j=1}^{m} c_j c_{ji}\right)x_i\\
&\qquad\quad + \sum_{i=1}^{n} d_i a_i(t) + c'(t) - \frac{1}{2}\sum_{i=1}^{n} a_i^2(t)\\
&\qquad\quad - \frac{1}{2}\sum_{i=1}^{m}\left(\sum_{j=1}^{n} c_{ij}b_j(t)\right)^2 - \sum_{i=1}^{m} c_i\sum_{j=1}^{n} c_{ji}b_j(t)
\end{aligned}
$$

$$- \sum_{j=1}^{m} \sum_{i=1}^{n} y_j(t) \left(\sum_{k=1}^{n} d_{ik} b_k(t) + d_i \right) c_{ji}$$

$$+ \frac{1}{2} \sum_{k=1}^{n} \sum_{i=1}^{m} \sum_{j=1}^{m} y_i(t) y_j(t) c_{ik} c_{jk}$$

$$+ \frac{1}{2} \sum_{k=1}^{n} e_{kk} - \frac{1}{2} \sum_{i=1}^{m} c_i^2 + \frac{1}{2} \sum_{i=1}^{n} \left(\frac{\partial F}{\partial x_i} \right)^2 (x)$$

$$+ \sum_{i=1}^{n} \left(\sum_{j=1}^{n} d_{ij} x_j \right) \frac{\partial F}{\partial x_i}(x)$$

$$\left. - \frac{1}{2} \sum_{i=1}^{m} \left(\sum_{j=1}^{n} c_{ij} x_j \right)^2 \right] u(t, x + b(t))$$

$$+ \sum_{i=1}^{n} \left[b_i'(t) - a_i(t) + \sum_{j=1}^{m} y_j(t) c_{ji} \right.$$

$$\left. \left. - \sum_{j=1}^{n} d_{ij} b_j(t) \right] \frac{\partial u}{\partial x_i}(t, x + b(t)) \right\}.$$

In view of (16), (17), and (18), the above equation becomes

$$\frac{\partial \tilde{u}}{\partial t}(t,x) - \frac{1}{2} \Delta \tilde{u}(t,x) + \sum_{i=1}^{n} \left(f_i(x) + \frac{\partial F}{\partial x_i}(x) \right) \frac{\partial \tilde{u}}{\partial x_i}(t,x)$$

$$+ \sum_{i=1}^{n} \left(\frac{\partial f_i}{\partial x_i} + \frac{\partial^2 F}{\partial x_i^2}(x) \right) \tilde{u}(t,x)$$

$$= \exp \left(F(x) + \sum_{i=1}^{n} a_i(t) x_i + c(t) \right)$$

$$\times \left[\frac{1}{2} \sum_{i=1}^{n} \left(\frac{\partial F}{\partial x_i} \right)^2 (x) + \sum_{i=1}^{n} \left(\sum_{j=1}^{n} d_{ij} x_j \right) \frac{\partial F}{\partial x_i}(x) \right.$$

$$\left. - \frac{1}{2} \sum_{i=1}^{m} \left(\sum_{j=1}^{n} c_{ij} x_j \right)^2 \right] u(t, x + b(t)). \quad (19)$$

Then (14) follows from (19) as long as we can show that

$$\frac{1}{2} \sum_{i=1}^{n} \left(\frac{\partial F}{\partial x_i} \right)^2 (x) + \sum_{i=1}^{n} \left(\sum_{j=1}^{n} d_{ij} x_j \right) \frac{\partial F}{\partial x_i}(x)$$

$$- \frac{1}{2} \sum_{i=1}^{m} \left(\sum_{j=1}^{n} c_{ij} x_j \right)^2 = 0 \quad (20)$$

is equivalent to (13). Notice that

$$\frac{\partial F}{\partial x_k} = \sum_{j=1}^{n} e_{kj} x_j.$$

So (20) becomes

$$\frac{1}{2} \sum_{k=1}^{n} \left(\sum_{j=1}^{n} e_{kj} x_j \right)^2 + \sum_{k=1}^{n} \left(\sum_{l=1}^{n} d_{kl} x_l \right) \left(\sum_{j=1}^{n} e_{kj} x_j \right)$$

$$- \frac{1}{2} \sum_{k=1}^{m} \left(\sum_{l=1}^{n} c_{kl} x_l \right)^2 = 0$$

which is equivalent to

$$\sum_{k=1}^{n} \sum_{j,l=1}^{n} e_{kj} e_{kl} x_j x_l + \sum_{k=1}^{n} \sum_{j,l=1}^{n} e_{kj} d_{kl} x_j x_l$$

$$+ \sum_{k=1}^{n} \sum_{j,l=1}^{n} e_{kl} d_{kj} x_j x_l = \sum_{k=1}^{m} \sum_{j,l=1}^{n} c_{kj} c_{kl} x_j x_l.$$

Thus, (20) is equivalent to

$$E^T E + E^T D + D^T E = C^T C$$

which is equivalent to (13). Q.E.D.

REMARK 1 Equations (13) and (14) can be computed off-line. The only on-line computations are (16), (17), and (18). Upon inspection, the parametrization of the unnormalized conditional probability density displayed in Theorem 2 is seen to involve $2n + 1$ statistics. As in the situation covered by the Kalman–Bucy filter, the essential feature lies in the recursive computability of these statistics as outputs of a finite-dimensional dynamic system. The dimensionality of this parametrization can easily be reduced to n by eliminating the nonindependent statistics via (16)–(18). By (17) and (18), we can solve $b_i(t)$ and $c(t)$ in terms of $a_i(t)$. The characterization given in (15) thus really requires no more than n statistics.

IV. DMZ EQUATION FOR BENĚS FILTERING SYSTEM WITH ARBITRARY INITIAL CONDITION

Let us observe that the elliptic differential operator L_0 in (2) can be more compactly represented if one defines $D_i = \partial/\partial x_i - f_i$. Then

$$L_0 = \frac{1}{2} \left(\sum_{i=1}^{n} D_i^2 - \eta \right) \quad (21)$$

where

$$\eta = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{m} h_i^2. \quad (22)$$

This compact representation of L_0 was exploited in [22–24] to derive necessary conditions and sufficient conditions for estimation algebras to be finite dimensional. (The concept of an estimation algebra was introduced in [4, 5, 18]. It is defined to be the Lie algebra of differential operators generated by $\{L_0, L_1, \ldots, L_m\}$.) Observe that if we let $F_i(x)$ such that $\partial F_i/\partial x_i = f_i$, then

$$D_i = \frac{\partial}{\partial x_i} - f_i = e^{F_i}\frac{\partial}{\partial x_i}e^{-F_i}. \tag{23}$$

Hence

$$D_i^2 = e^{F_i}\frac{\partial^2}{\partial x_i^2}e^{-F_i} \tag{24}$$

and

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^{n} e^{F_i}\frac{\partial^2}{\partial x_i^2}e^{-F_i} - \eta\right). \tag{25}$$

If f is the vector field of a potential function ϕ, i.e., $f = \nabla\phi$, then $F_i = \phi$ for all $1 \le i \le n$ and we have a simple expression

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^{n} e^{\phi}\frac{\partial^2}{\partial x_i^2}e^{-\phi} - \eta\right). \tag{26}$$

By defining $\xi = e^{-\phi}\sigma$, (2) can be reformulated as

$$\begin{aligned} d\xi(t,x) &= \tfrac{1}{2}(\Delta - \eta)\xi(t,x)\,dt + \sum_{i=1}^{m} L_i\xi(t,x)\,dy_i(t), \\ \xi(0,x) &= e^{-\phi}\sigma_0 \end{aligned} \tag{27}$$

where Δ denotes the Laplacian operator $\sum_{i=1}^{n}(\partial^2/\partial x_i^2)$. The connection between this representation and the gauge transformation was pointed out in [19]. This idea is also related to the concept of equivalence of parabolic equations, as discussed in [1]. If η is a quadratic polynomial in x, then it was usually thought that the semigroup generated by the differential operator $\Delta - \eta$ is well known as can be used to write down an explicit solution to the equation when h_is are linear in x. However, to our knowledge, no explicit solution has been written down. Notable exceptions are references [2 and 22], both of which assume the maximal-rank condition of the estimation algebra. Using the Rozovsky's transformation

$$u(t,x) = \exp\left(-\sum_{i=1}^{m} h_i(x)y_i(t)\right)\xi(t,x)$$

we can reduce (27) to the following time-varying partial differential equation:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \tilde{L}_0 u(t,x) + \displaystyle\sum_{i=1}^{m} y_i(t)[\tilde{L}_0, L_i]u(t,x) \\ \qquad\qquad + \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{m} y_i(t)y_j(t)[[\tilde{L}_0, L_i], L_j]u(t,x) \\ u(0,x) = e^{-\phi}\sigma_0 \end{cases} \tag{28}$$

where

$$\tilde{L}_0 = \frac{1}{2}\left(\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - \eta\right). \tag{29}$$

It is easy to see that (28) is equivalent to the following equation:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\displaystyle\sum_{j=1}^{n}\frac{\partial^2 u}{\partial x_j^2}(t,x) + \sum_{j=1}^{n}\sum_{i=1}^{m} y_i(t) \\ \qquad \times \dfrac{\partial h_i}{\partial x_j}(x)\dfrac{\partial u}{\partial x_j}(t,x) \\ \qquad + \left(-\dfrac{1}{2}\eta(x) + \dfrac{1}{2}\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} y_i(t)\frac{\partial^2 h_i}{\partial x_j^2}\right. \\ \qquad + \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{m}\sum_{k=1}^{n} y_i(t)y_j(t) \\ \qquad \left.\times \dfrac{\partial h_i}{\partial x_k}(x)\dfrac{\partial h_j}{\partial x_k}(x)\right)u(t,x) \\ u(0,x) = e^{-\phi}\sigma_0. \end{cases} \tag{30}$$

If η is a quadratic in x and h_is are linear in x, then the estimation algebra generated by $\tilde{L}_0, L_1, \ldots, L_m$ is finite dimensional. If in addition the matrix H, which is defined by $h(x) = Hx$, is of maximal rank and $m \ge n$, then the estimation algebra has a basis $L_0, \partial/\partial x_1, \ldots, \partial/\partial x_n, x_1, \ldots, x_n, 1$. Even in this case, using the Wei–Norman approach the procedure to find the solution of (30) is more complicated than the procedure in Theorem 3 below because not only must one solve a finite system of ODEs and a Kolmogorov equation, but also one has to integrate n partial differential equations corresponding to the operators $\partial/\partial x_1, \ldots, \partial/\partial x_n$. More important, if H is not of maximal rank, then the basis of the estimation algebra is not explicitly known (although it can be computed). As a result, one cannot write down recursive algorithms explicitly. Reference [14] considered a more general Beněs filtering problem than ours. However, even restricted to the classical Beněs problem, the number of sufficient statistics in order to compute the conditional probability density is a polynomial of degree two in n in their method. The novelty of Theorem 3 is that our finite system of ODEs is explicitly written down and our algorithms apply uniformly for any η, h_is, and any initial condition σ_0. Moreover, we need only n sufficient statistics in order to compute the conditional probability density of the Beněs filtering with arbitrary initial condition.

THEOREM 3 *Consider the Beněs filtering system* (1) *with*

$$h_i(x) = \sum_{j=1}^{n} c_{ij}x_j + c_i, \qquad 1 \le i \le m \tag{31}$$

where c_{ij} and c_i are constants;

$$f_i(x) = \frac{\partial F}{\partial x_i}(x), \qquad 1 \le i \le n \tag{32a}$$

where F is a C^∞ function; and

$$\Delta F(x) + |\nabla F(x)|^2(x) + \sum_{i=1}^{m} h_i^2(x) = \sum_{i,j=1}^{n} e_{ij}x_i x_j + \sum_{i=1}^{n} e_i x_i + e. \tag{32b}$$

Choose a C^∞ function $G(x)$ such that

$$\Delta G(x) + |\nabla G|^2(x) = \sum_{i,j=1}^{n} e_{ij}x_i x_j. \tag{33}$$

(*In view of* (32b) *and* [12, *Theorem* 12], *there exists a G satisfying* (33).) *Then the solution $u(t,x)$ for the DMZ equation* (5) *is reduced to the solution $\tilde{u}(t,x)$ for the Kolmogorov equation*

$$\frac{\partial \tilde{u}}{\partial t}(t,x) = \frac{1}{2}\Delta\tilde{u}(t,x) - \sum_{i=1}^{n} \frac{\partial G}{\partial x_i}(x)\frac{\partial \tilde{u}}{\partial x_i}(t,x) - \sum_{i=1}^{n} \frac{\partial^2 G}{\partial x_i^2}(x)\tilde{u}(t,x) \tag{34}$$

where

$$\tilde{u}(t,x) = \exp\left(- F(x+b(t)) + G(x) + \sum_{i=1}^{n} a_i(t)x_i + c(t)\right) u(t,x+b(t)) \tag{35}$$

and $a_i(t)$, $b_i(t)$, and $c(t)$ satisfy the following system of ODEs:

$$b_i'(t) - a_i(t) + \sum_{j=1}^{m} c_{ji}y_j(t) = 0, \qquad 1 \le i \le n \tag{36}$$

$$c'(t) - \frac{1}{2}\sum_{i=1}^{n} a_i^2(t) + \frac{1}{2}\sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=1}^{m} c_{ik}c_{jk}y_i(t)y_j(t) - \frac{1}{2}\sum_{i=1}^{n} e_i b_i(t) - \frac{1}{2}\sum_{i,j=1}^{n} e_{ij}b_i(t)b_j(t) - \frac{1}{2}e = 0 \tag{37}$$

$$a_i'(t) - \frac{1}{2}\sum_{j=1}^{n}(e_{ij}+e_{ji})b_j(t) - \frac{1}{2}e_i = 0, \qquad 1 \le i \le n. \tag{38}$$

PROOF See Appendix.

REMARK The initial conditions for $a_i(t)$, $b_i(t)$, and $c(t)$ can be arbitrary values. However, the initial condition of (34) is determined by the initial condition of (10).

APPENDIX

The purpose of this Appendix is to give a detail proof Theorem 3. We use the same notation as above.

PROOF OF THEOREM 3 We need only to show that if $u(t,x)$ satisfies (5), then $\tilde{u}(t,x)$ given by (35)–(38) will satisfy the Kolmogorov equation (34):

$$\begin{aligned}\frac{\partial \tilde{u}}{\partial t}(t,x) = \exp&\left(-F(x+b(t)) + G(x) + \sum_{i=1}^{n} a_i(t)x_i + c(t)\right)\\ &\times\Bigg\{\Bigg[-\sum_{i=1}^{n}\frac{\partial F}{\partial x_i}(x+b(t))b_i'(t) + \sum_{i=1}^{n} a_i'(t)x_i + c'(t)\Bigg]\\ &\times u(t,x+b(t)) + \frac{\partial u}{\partial t}(t,x+b(t)) + \sum_{i=1}^{n} b_i'(t)\frac{\partial u}{\partial x_i}(t,x+b(t))\Bigg\}\end{aligned}$$

$$\begin{aligned}\frac{\partial \tilde{u}}{\partial x_i}(t,x) = \exp&\left(-F(x+b(t)) + G(x) + \sum_{i=1}^{n} a_i(t)x_i + c(t)\right)\\ &\times\Bigg\{\Bigg[-\frac{\partial F}{\partial x_i}(x+b(t)) + \frac{\partial G}{\partial x_i}(x) + a_i(t)\Bigg]\\ &\times u(t,x+b(t)) + \frac{\partial u}{\partial x_i}(t,x+b(t))\Bigg\}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \tilde{u}}{\partial x_i^2}(t,x) = \exp&\left(-F(x+b(t)) + G(x) + \sum_{i=1}^{n} a_i(t)x_i + c(t)\right)\\ &\times\Bigg\{\left(-\frac{\partial F}{\partial x_i}(x+b(t)) + \frac{\partial G}{\partial x_i}(x) + a_i(t)\right)^2\\ &\times u(t,x+b(t))\\ &+ 2\left(-\frac{\partial F}{\partial x_i}(x+b(t)) + \frac{\partial G}{\partial x_i}(x) + a_i(t)\right)\\ &\times \frac{\partial u}{\partial x_i}(t,x+b(t))\\ &+ \left(-\frac{\partial^2 F}{\partial x_i^2}(x+b(t)) + \frac{\partial^2 G}{\partial x_i^2}(x)\right)\\ &\times u(t,x+b(t)) + \frac{\partial^2 u}{\partial x_i^2}(t,x+b(t))\Bigg\}\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\Delta\tilde{u}(t,x) = \exp&\left(-F(x+b(t)) + G(x) + \sum_{i=1}^{n} a_i(t)x_i + c(t)\right)\\ &\times\Bigg\{\frac{1}{2}\sum_{i=1}^{n}\left(-\frac{\partial F}{\partial x_i}(x+b(t)) + \frac{\partial G}{\partial x_i}(x) + a_i(t)\right)^2\\ &\times u(t,x+b(t))\\ &+ \sum_{i=1}^{n}\left(-\frac{\partial F}{\partial x_i}(x+b(t)) + \frac{\partial G}{\partial x_i}(x) + a_i(t)\right)\\ &\times \frac{\partial u}{\partial x_i}(t,x+b(t))\end{aligned}$$

$$+\left(-\frac{1}{2}\Delta F(x+b(t))+\frac{1}{2}\Delta G(x)\right)u(t,x+b(t))$$

$$+\frac{1}{2}\Delta u(t,x+b(t))\Bigg\}$$

$$\sum_{i=1}^{n} f_i(x)\frac{\partial\tilde{u}}{\partial x_i}(t,x)$$

$$=\exp\left(-F(x+b(t))+G(x)+\sum_{i=1}^{n}a_i(t)x_i+c(t)\right)$$

$$\times\Bigg\{\Bigg[-\sum_{i=1}^{n}f_i(x)\frac{\partial F}{\partial x_i}(x+b(t))+\sum_{i=1}^{n}f_i(x)\frac{\partial G}{\partial x_i}(x)$$

$$+\sum_{i=1}^{n}f_i(x)a_i(t)\Bigg]u(t,x+b(t))$$

$$+f_i(x)\frac{\partial u}{\partial x_i}(t,x+b(t))\Bigg\}$$

$$\frac{\partial\tilde{u}}{\partial t}(t,x)-\frac{1}{2}\Delta\tilde{u}(t,x)+\sum_{i=1}^{n}f_i(x)\frac{\partial\tilde{u}}{\partial x_i}(t,x)$$

$$=\exp\left(-F(x+b(t))+G(x)+\sum_{i=1}^{n}a_i(t)x_i+c(t)\right)$$

$$\times\Bigg\{\Bigg[-\sum_{i=1}^{n}\frac{\partial F}{\partial x_i}(x+b(t))b_i'(t)+\sum_{i=1}^{n}a_i'(t)x_i+c'(t)\Bigg]$$

$$\times u(t,x+b(t))+\frac{\partial u}{\partial t}(t,x+b(t))$$

$$+\sum_{i=1}^{n}b_i'(t)\frac{\partial u}{\partial x_i}(t,x+b(t))$$

$$+\Bigg[-\sum_{i=1}^{n}f_i(x)\frac{\partial F}{\partial x_i}(x+b(t))$$

$$+\sum_{i=1}^{n}f_i(x)\frac{\partial G}{\partial x_i}(x)+\sum_{i=1}^{n}f_i(x)a_i(t)\Bigg]$$

$$\times u(t,x+b(t))+\sum_{i=1}^{n}f_i(x)\frac{\partial u}{\partial x_i}(t,x+b(t))$$

$$-\frac{1}{2}\sum_{i=1}^{n}\Bigg[\left(\frac{\partial F}{\partial x_i}(x+b(t))\right)^2+\left(\frac{\partial G}{\partial x_i}(x)\right)^2$$

$$+a_i^2(t)-2\frac{\partial F}{\partial x_i}(x+b(t))\frac{\partial G}{\partial x_i}(x)$$

$$-2a_i(t)\frac{\partial F}{\partial x_i}(x+b(t))$$

$$+2a_i(t)\frac{\partial G}{\partial x_i}(x)\Bigg]u(t,x+b(t))$$

$$-\sum_{i=1}^{n}\left(-\frac{\partial F}{\partial x_i}(x+b(t))+\frac{\partial G}{\partial x_i}(x)+a_i(t)\right)$$

$$\times\frac{\partial u}{\partial x_i}(t,x+b(t))$$

$$-\left(-\frac{1}{2}\Delta F(x+b(t))+\frac{1}{2}\Delta G(x)\right)u(t,x+b(t))$$

$$-\frac{1}{2}\Delta u(t,x+b(t))\Bigg\}$$

$$=\exp\left(-F(x+b(t))+G(x)+\sum_{i=1}^{n}a_i(t)x_i+c(t)\right)$$

$$\times\Bigg\{\Bigg[-\sum_{i=1}^{n}\frac{\partial F}{\partial x_i}(x+b(t))b_i'(t)+\sum_{i=1}^{n}a_i'(t)x_i+c'(t)\Bigg]$$

$$\times u(t,x+b(t))+\sum_{i=1}^{n}b_i'(t)\frac{\partial u}{\partial x_i}(t,x+b(t))$$

$$+\Bigg[-\sum_{i=1}^{n}f_i(x)\frac{\partial F}{\partial x_i}(x+b(t))+\sum_{i=1}^{n}f_i(x)\frac{\partial G}{\partial x_i}(x)$$

$$+\sum_{i=1}^{n}f_i(x)a_i(t)\Bigg]u(t,x+b(t))$$

$$-\frac{1}{2}\sum_{i=1}^{n}\Bigg[\left(\frac{\partial F}{\partial x_i}(x+b(t))\right)^2+\left(\frac{\partial G}{\partial x_i}(x)\right)^2$$

$$+a_i^2(t)-2\frac{\partial F}{\partial x_i}(x+b(t))\frac{\partial G}{\partial x_i}(x)$$

$$-2a_i(t)\frac{\partial F}{\partial x_i}(x+b(t))+2a_i(t)\frac{\partial G}{\partial x_i}(x)\Bigg]$$

$$\times u(t,x+b(t))$$

$$-\sum_{i=1}^{n}\left[-\frac{\partial F}{\partial x_i}(x+b(t))+\frac{\partial G}{\partial x_i}(x)+a_i(t)\right]$$

$$\times\frac{\partial u}{\partial x_i}(t,x+b(t))$$

$$-\left[-\frac{1}{2}\Delta F(x+b(t))+\frac{1}{2}\Delta G(x)\right]u(t,x+b(t))$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{m}y_j(t)\frac{\partial h_j}{\partial x_i}(x+b(t))$$

$$\times\frac{\partial u}{\partial x_i}(t,x+b(t))$$

$$-\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x+b(t))u(t,x+b(t))$$

$$-\frac{1}{2}\sum_{i=1}^{m}h_i^2(x+b(t))u(t,x+b(t))$$

$$+\frac{1}{2}\sum_{i=1}^{m}y_i(t)\Delta h_i(x+b(t))u(t,x+b(t))$$

$$-\sum_{i=1}^{m}\sum_{j=1}^{n}y_i(t)f_j(x+b(t))$$

$$\times \frac{\partial h_i}{\partial x_j}(x+b(t))u(t,x+b(t))$$

$$+\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} y_i(t)y_j(t)\sum_{k=1}^{n}\frac{\partial h_i}{\partial x_k}(x+b(t))$$

$$\times \frac{\partial h_j}{\partial x_k}(x+b(t))u(t,x+b(t))$$

$$\left.+\sum_{i=1}^{n}(f_i(x)-f_i(x+b(t)))\frac{\partial u}{\partial x_i}(t,x+b(t))\right\}.$$

Observe that

$$\sum_{i=1}^{n}\frac{\partial G}{\partial x_i}(x)\frac{\partial \tilde{u}}{\partial x_i}(x,t)$$

$$= \exp\left(-F(x+b(t))+G(x)+\sum_{i=1}^{n}a_i(t)x_i+c(t)\right)$$

$$\times\left\{\left[-\sum_{i=1}^{n}\frac{\partial G}{\partial x_i}(x)\frac{\partial F}{\partial x_i}(x+b(t))+\sum_{i=1}^{n}\left(\frac{\partial G}{\partial x_i}(x)\right)^2\right.\right.$$

$$\left.+\sum_{i=1}^{n}\frac{\partial G}{\partial x_i}(x)a_i(t)\right]u(t,x+b(t))$$

$$\left.+\sum_{i=1}^{n}\frac{\partial G}{\partial x_i}(x)\frac{\partial u}{\partial x_i}(t,x+b(t))\right\}.$$

Therefore,

$$\frac{\partial \tilde{u}}{\partial t}(t,x)-\frac{1}{2}\Delta\tilde{u}(t,x)+\sum_{i=1}^{n}f_i(x)\frac{\partial \tilde{u}}{\partial x_i}(t,x)$$

$$+\sum_{i=1}^{n}\frac{\partial G}{\partial x_i}(x)\frac{\partial \tilde{u}}{\partial x_i}(t,x)$$

$$= \exp\left(-F(x+b(t))+G(x)+\sum_{i=1}^{n}a_i(t)x_i+c(t)\right)$$

$$\times\left\{\left[-\sum_{i=1}^{n}\frac{\partial F}{\partial x_i}(x+b(t))b_i'(t)\right.\right.$$

$$\left.+\sum_{i=1}^{n}a_i'(t)x_i+c'(t)\right]u(t,x+b(t))$$

$$+\sum_{i=1}^{n}b_i'(t)\frac{\partial u}{\partial x_i}(t,x+b(t))$$

$$+\left[-\sum_{i=1}^{n}f_i(x)\frac{\partial F}{\partial x_i}(x+b(t))\right.$$

$$+\sum_{i=1}^{n}f_i(x)\frac{\partial G}{\partial x_i}(x)$$

$$\left.+\sum_{i=1}^{n}f_i(x)a_i(t)\right]u(t,x+b(t))$$

$$-\frac{1}{2}\sum_{i=1}^{n}\left[\left(\frac{\partial F}{\partial x_i}(x+b(t))\right)^2+\left(\frac{\partial G}{\partial x_i}(x)\right)^2\right.$$

$$+a_i^2(t)-2\frac{\partial F}{\partial x_i}(x+b(t))\frac{\partial G}{\partial x_i}(x)$$

$$-2a_i(t)\frac{\partial F}{\partial x_i}(x+b(t))$$

$$\left.+2a_i(t)\frac{\partial G}{\partial x_i}(x)\right]u(t,x+b(t))$$

$$-\sum_{i=1}^{n}\left(-\frac{\partial F}{\partial x_i}(x+b(t))+a_i(t)\right)$$

$$\times\frac{\partial u}{\partial x_i}(t,x+b(t))$$

$$+\left[-\sum_{i=1}^{n}\frac{\partial G}{\partial x_i}(x)\frac{\partial F}{\partial x_i}(x+b(t))\right.$$

$$+\sum_{i=1}^{n}\left(\frac{\partial G}{\partial x_i}(x)\right)^2$$

$$\left.+\sum_{i=1}^{n}\frac{\partial G}{\partial x_i}(x)a_i(t)\right]u(t,x+b(t))$$

$$-\left(-\tfrac{1}{2}\Delta F(x+b(t))+\tfrac{1}{2}\Delta G(x)\right)u(t,x+b(t))$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{m}y_j(t)\frac{\partial h_j}{\partial x_i}(x+b(t))\frac{\partial u}{\partial x_i}(t,x+b(t))$$

$$-\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x+b(t))u(t,x+b(t))$$

$$-\frac{1}{2}\sum_{i=1}^{m}h_i^2(x+b(t))u(t,x+b(t))$$

$$+\frac{1}{2}\sum_{i=1}^{m}y_i(t)\Delta h_i(x+b(t))u(t,x+b(t))$$

$$-\sum_{i=1}^{m}\sum_{j=1}^{n}y_i(t)f_j(x+b(t))$$

$$\times\frac{\partial h_i}{\partial x_j}(x+b(t))u(t,x+b(t))$$

$$+\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}y_i(t)y_j(t)\sum_{k=1}^{n}\frac{\partial h_i}{\partial x_k}(x+b(t))$$

$$\times\frac{\partial h_j}{\partial x_k}(x+b(t))u(t,x+b(t))$$

$$+\sum_{i=1}^{n}(f_i(x)-f_i(x+b(t)))$$

$$\left.\times\frac{\partial u}{\partial x_i}(t,x+b(t))\right\}.$$

Hence,

$$
\begin{aligned}
&\frac{\partial \tilde{u}}{\partial t}(t,x) - \frac{1}{2}\Delta\tilde{u}(t,x) + \sum_{i=1}^{n}\frac{\partial G}{\partial x_i}(x)\frac{\partial \tilde{u}}{\partial x_i}(t,x) \\
&\qquad + \left[\frac{1}{2}\Delta G(x) - \frac{1}{2}|\nabla G|^2(x) \right.\\
&\qquad\qquad \left. - \sum_{i=1}^{n} f_i(x)\frac{\partial G}{\partial x_i}(x)\right]\tilde{u}(t,x) \\
&\quad = \exp\left(-F(x+b(t)) + G(x) + \sum_{i=1}^{n} a_i(t)x_i + c(t)\right) \\
&\qquad \times \left\{\left[-\sum_{i=1}^{n}\frac{\partial F}{\partial x_i}(x+b(t))b_i'(t) \right.\right.\\
&\qquad\qquad \left. + \sum_{i=1}^{n} a_i'(t)x_i + c'(t)\right] u(t,x+b(t)) \\
&\qquad\qquad + \sum_{i=1}^{n} b_i'(t)\frac{\partial u}{\partial x_i}(t,x+b(t)) \\
&\qquad\qquad + \left[-\sum_{i=1}^{n} f_i(x)\frac{\partial F}{\partial x_i}(x+b(t)) \right.\\
&\qquad\qquad\qquad \left. + \sum_{i=1}^{n} f_i(x)a_i(t)\right] u(t,x+b(t)) \\
&\qquad\qquad - \frac{1}{2}\sum_{i=1}^{n}\left[\left(\frac{\partial F}{\partial x_i}(x+b(t))\right)^2 + a_i^2(t) \right.\\
&\qquad\qquad\qquad - 2a_i(t)\frac{\partial F}{\partial x_i}(x+b(t)) \\
&\qquad\qquad\qquad \left. + 2a_i(t)\frac{\partial G}{\partial x_i}(x)\right] u(t,x+b(t)) \\
&\qquad\qquad - \sum_{i=1}^{n}\left(-\frac{\partial F}{\partial x_i}(x+b(t)) + a_i(t)\right) \\
&\qquad\qquad \times \frac{\partial u}{\partial x_i}(t,x+b(t)) \\
&\qquad\qquad + \sum_{i=1}^{n}\frac{\partial G}{\partial x_i}(x)a_i(t)u(t,x+b(t)) \\
&\qquad\qquad + \frac{1}{2}\Delta F(x+b(t))u(t,x+b(t)) \\
&\qquad\qquad + \sum_{i=1}^{n}\sum_{j=1}^{m} y_j(t)\frac{\partial h_j}{\partial x_i}(x+b(t))\frac{\partial u}{\partial x_i}(t,x+b(t)) \\
&\qquad\qquad - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x+b(t))u(t,x+b(t)) \\
&\qquad\qquad - \frac{1}{2}\sum_{i=1}^{m} h_i^2(x+b(t))u(t,x+b(t)) \\
&\qquad\qquad + \frac{1}{2}\sum_{i=1}^{m} y_i(t)\,\Delta h_i(x+b(t))u(t,x+b(t)) \\
&\qquad\qquad - \sum_{i=1}^{m}\sum_{j=1}^{n} y_i(t)f_j(x+b(t)) \\
&\qquad\qquad \times \frac{\partial h_i}{\partial x_j}(x+b(t))u(t,x+b(t)) \\
&\qquad\qquad + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} y_i(t)y_j(t)\sum_{k=1}^{n}\frac{\partial h_i}{\partial x_k}(x+b(t)) \\
&\qquad\qquad \times \frac{\partial h_j}{\partial x_k}(x+b(t))u(t,x+b(t)) \\
&\qquad\qquad - \sum_{i=1}^{n} f_i(x+b(t))\frac{\partial u}{\partial x_i}(t,x+b(t)) \\
&\qquad\qquad - \left[-\sum_{i=1}^{n} f_i(x)\frac{\partial F}{\partial x_i}(x+b(t)) \right.\\
&\qquad\qquad\qquad + \sum_{i=1}^{n} f_i(x)\frac{\partial G}{\partial x_i}(x) \\
&\qquad\qquad\qquad \left.\left. + \sum_{i=1}^{n} f_i(x)a_i(t)\right] u(t,x+b(t))\right\} \\
&\quad = \exp\left(-F(x+b(t)) + G(x) + \sum_{i=1}^{n} a_i(t)x_i + c(t)\right) \\
&\qquad \times \left\{\left[\sum_{i=1}^{n}(b_i'(t) - a_i(t) + \sum_{j=1}^{m} y_j(t)\frac{\partial h_j}{\partial x_i}(x+b(t)) \right.\right.\\
&\qquad\qquad \left. + \frac{\partial F}{\partial x_i}(x+b(t)) - f_i(x+b(t))\right] \\
&\qquad\qquad \times \frac{\partial u}{\partial x_i}(t,x+b(t)) \\
&\qquad\qquad + \left[-\sum_{i=1}^{n}\frac{\partial F}{\partial x_i}(x+b(t))b_i'(t) + \sum_{i=1}^{n} a_i'(t)x_i \right.\\
&\qquad\qquad\qquad + c'(t) - \sum_{i=1}^{n} f_i(x)\frac{\partial F}{\partial x_i}(x+b(t)) \\
&\qquad\qquad\qquad + \sum_{i=1}^{n} f_i(x)a_i(t) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial F}{\partial x_i}\right)^2(x+b(t)) \\
&\qquad\qquad\qquad - \frac{1}{2}\sum_{i=1}^{n} a_i^2(t) + \sum_{i=1}^{n} a_i(t)\frac{\partial F}{\partial x_i}(x+b(t)) \\
&\qquad\qquad\qquad + \frac{1}{2}\Delta F(x+b(t)) - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x+b(t)) \\
&\qquad\qquad\qquad - \frac{1}{2}\sum_{i=1}^{m} h_i^2(x+b(t))
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{i=1}^{m} y_i(t)\,\Delta h_i(x+b(t))\\
&-\sum_{i=1}^{m}\sum_{j=1}^{n} y_i(t) f_j(x+b(t))\frac{\partial h_i}{\partial x_j}(x+b(t))\\
&+\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} y_i(t)y_j(t)\sum_{k=1}^{n}\frac{\partial h_i}{\partial x_k}(x+b(t))\\
&\times\frac{\partial h_j}{\partial x_k}(x+b(t))+\sum_{i=1}^{n} f_i(x)\frac{\partial F}{\partial x_i}(x+b(t))\\
&-\sum_{i=1}^{n} f_i(x)\frac{\partial G}{\partial x_i}(x)\\
&\left.\left.-\sum_{i=1}^{n} f_i(x)a_i(t)\right]u(t,x+b(t))\right\}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\frac{\partial\tilde{u}}{\partial t}(t,x)-\frac{1}{2}\Delta\tilde{u}(t,x)+\sum_{i=1}^{n}\frac{\partial G}{\partial x_i}(x)\frac{\partial\tilde{u}}{\partial x_i}(t,x)\\
&\qquad+\left(\frac{1}{2}\Delta G(x)-\frac{1}{2}|\nabla G|^2(x)\right)\tilde{u}(t,x)\\
&\quad=\exp\left(-F(x+b(t))+G(x)+\sum_{i=1}^{n}a_i(t)x_i+c(t)\right)\\
&\qquad\times\left\{\left[\sum_{i=1}^{n}(b_i'(t)-a_i(t))+\sum_{j=1}^{m}y_j(t)\frac{\partial h_j}{\partial x_i}(x+b(t))\right.\right.\\
&\qquad\qquad\left.+\frac{\partial F}{\partial x_i}(x+b(t))-f_i(x+b(t))\right]\\
&\qquad\quad\times\frac{\partial u}{\partial x_i}(t,x+b(t))\\
&\qquad\quad+\left[-\sum_{i=1}^{n}\frac{\partial F}{\partial x_i}(x+b(t))b_i'(t)\right.\\
&\qquad\qquad+\sum_{i=1}^{n}a_i'(t)x_i+c'(t)\\
&\qquad\qquad-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial F}{\partial x_i}\right)^2(x+b(t))-\frac{1}{2}\sum_{i=1}^{n}a_i^2(t)\\
&\qquad\qquad+\sum_{i=1}^{n}a_i(t)\frac{\partial F}{\partial x_i}(x+b(t))+\frac{1}{2}\Delta F(x+b(t))\\
&\qquad\qquad-\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x+b(t))-\frac{1}{2}\sum_{i=1}^{m}h_i^2(x+b(t))\\
&\qquad\qquad+\frac{1}{2}\sum_{i=1}^{m}y_i(t)\,\Delta h_i(x+b(t))\\
&\qquad\qquad-\sum_{i=1}^{m}\sum_{j=1}^{n}y_i(t)f_j(x+b(t))\frac{\partial h_i}{\partial x_j}(x+b(t))\\
&\qquad\qquad+\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}y_i(t)y_j(t)\sum_{k=1}^{n}\frac{\partial h_i}{\partial x_k}(x+b(t))\\
&\qquad\qquad\left.\left.\times\frac{\partial h_j}{\partial x_k}(x+b(t))\right]u(t,x+b(t))\right\}.
\end{aligned}
$$

Putting (31) and (32a) into the above equation, we have

$$
\begin{aligned}
&\frac{\partial\tilde{u}}{\partial t}(t,x)-\frac{1}{2}\Delta\tilde{u}(t,x)+\sum_{i=1}^{n}\frac{\partial G}{\partial x_i}(x)\frac{\partial\tilde{u}}{\partial x_i}(t,x)\\
&\qquad+\left(\frac{1}{2}\Delta G(x)-\frac{1}{2}|\nabla G|^2(x)\right)\tilde{u}(t,x)\\
&\quad=\exp\left(-F(x+b(t))+G(x)+\sum_{i=1}^{n}a_i(t)x_i+c(t)\right)\\
&\qquad\times\left\{\sum_{i=1}^{n}\left[b_i'(t)-a_i(t)+\sum_{j=1}^{m}c_{ji}y_j(t)\right]\right.\\
&\qquad\quad\times\frac{\partial u}{\partial x_i}(t,x+b(t))\\
&\qquad\quad+\left[-\sum_{i=1}^{n}\frac{\partial F}{\partial x_i}(x+b(t))b_i'(t)\right.\\
&\qquad\qquad+\sum_{i=1}^{n}a_i(t)\frac{\partial F}{\partial x_i}(x+b(t))\\
&\qquad\qquad-\sum_{i=1}^{m}\sum_{j=1}^{n}y_i(t)\left[\frac{\partial F}{\partial x_j}(x+b(t))\right]c_{ij}\\
&\qquad\qquad+\sum_{i=1}^{n}a_i'(t)x_i+c'(t)-\frac{1}{2}\sum_{i=1}^{n}a_i^2(t)\\
&\qquad\qquad+\frac{1}{2}\sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=1}^{m}c_{ik}c_{jk}y_i(t)y_j(t)\\
&\qquad\qquad-\frac{1}{2}\Delta F(x+b(t))-\frac{1}{2}|\nabla F|^2(x+b(t))\\
&\qquad\qquad\left.\left.-\frac{1}{2}\sum_{i=1}^{m}h2_i(x+b(t))\right]u(t,x+b(t))\right\}.
\end{aligned}
$$

From (32b), we get

$$
\begin{aligned}
&\Delta F(t+b(t))+|\nabla F|^2(x+b(t))+\sum_{i=1}^{m}h_i^2(x+b(t))\\
&\quad=\sum_{i,j=1}^{n}e_{ij}(x_i+b_i(t))(x_j+b_j(t))+\sum_{i=1}^{n}e_i(x_i+b_i(t))+e\\
&\quad=\sum_{i,j=1}^{n}e_{ij}x_ix_j+\sum_{i=1}^{n}e_ix_i+e+\sum_{i,j=1}^{n}e_{ij}x_ib_j(t)\\
&\qquad+\sum_{i,j=1}^{n}e_{ij}b_i(t)x_j+\sum_{i=1}^{n}e_ib_i(t)+\sum_{i,j=1}^{n}e_{ij}b_i(t)b_j(t)
\end{aligned}
$$

and

$$\frac{\partial \tilde{u}}{\partial t}(t,x) - \frac{1}{2}\Delta\tilde{u}(t,x) + \sum_{i=1}^{n}\frac{\partial G}{\partial x_i}(x)\frac{\partial \tilde{u}}{\partial x_i}(t,x)$$
$$+ \left(\frac{1}{2}\Delta G(x) - \frac{1}{2}|\nabla G|^2(x)\right)\tilde{u}(t,x)$$
$$= \exp\left(-F(x+b(t)) + G(x) + \sum_{i=1}^{n} a_i(t)x_i + c(t)\right)$$
$$\times \left\{\sum_{i=1}^{n}\left[b_i'(t) - a_i(t) + \sum_{j=1}^{m} c_{ji}y_j(t)\right]\right.$$
$$\times \frac{\partial u}{\partial x_i}(t, x+b(t))$$
$$+ \left[\sum_{i=1}^{n}\left(-b_i'(t) + a_i(t) - \sum_{j=1}^{m} c_{ji}y_j(t)\right)\right.$$
$$\times \frac{\partial F}{\partial x_i}(x+b(t)) + c'(t) - \frac{1}{2}\sum_{i=1}^{n} a_i^2(t)$$
$$+ \frac{1}{2}\sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=1}^{m} c_{ik}c_{jk}y_i(t)y_j(t)$$
$$- \frac{1}{2}\sum_{i=1}^{n} e_i b_i(t) + \sum_{i=1}^{n} a_i'(t)x_i$$
$$- \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} e_{ij}b_j(t)x_i - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} e_{ji}b_j(t)x_i$$
$$- \frac{1}{2}\sum_{i,j=1}^{n} e_{ij}x_i x_j - \frac{1}{2}\sum_{i=1}^{n} e_i x_i$$
$$\left.\left. - \frac{1}{2}e - \frac{1}{2}\sum_{i,j=1}^{n} e_{ij}b_i(t)b_j(t)\right] u(t, x+b(t))\right\}.$$

In view of (35)–(38), the above equation becomes

$$\frac{\partial \tilde{u}}{\partial t}(t,x) = \frac{1}{2}\Delta\tilde{u}(t,x) - \sum_{i=1}^{n}\frac{\partial G}{\partial x_i}(x)\frac{\partial \tilde{u}}{\partial x_i}(t,x)$$
$$- \left(\frac{1}{2}\Delta G(x) - \frac{1}{2}|\nabla G|^2(x)\right)\tilde{u}(t,x)$$
$$+ \left(-\frac{1}{2}\sum_{i,j=1}^{n} e_{ij}x_i x_j\right)\tilde{u}(t,x)$$
$$= \frac{1}{2}\Delta\tilde{u}(t,x) - \sum_{i=1}^{n}\frac{\partial G}{\partial x_i}(x)\frac{\partial \tilde{u}}{\partial x_i}(t,x)$$
$$- \sum_{i=1}^{n}\frac{\partial^2 G}{\partial x_i^2}(x)\tilde{u}(t,x)$$
$$+ \left(\frac{1}{2}\Delta G(x) + \frac{1}{2}|\nabla G|^2(x) - \frac{1}{2}\sum_{i,j=1}^{n} e_{ij}x_i x_j\right)\tilde{u}(t,x).$$

Then (34) follows from (33) and the above equation.

Q.E.D.

REMARK Equations (33) and (34) can be computed off-line. The only on-line computations are (36), (37), and (38). Upon inspection, the parametrization of the unnormalized conditional probability density displayed in Theorem 3 is seen to involve $2n+1$ statistics. As in the situation covered by the Kalman–Bucy filter, the essential feature lies in the recursive computability of these statistics as outputs of a finite-dimensional dynamic system. Using (36)–(38), we can easily reduce dimensionality of this parametrization to n by eliminating the nonindependent statistics. By (37) and (38), we can solve $b_i(t)$ and $c(t)$ in terms of $a_i(t)$. The characterization given in (35) thus really requires no more than n statistics.

REFERENCES

[1] Baras, J. S. (1981)
Group invariance methods in nonlinear filtering of diffusion processes.
In M. Hazewinkel and J. C. Willems (Eds.), *Stochastic Systems: The Mathematics of Filtering and Identification and Applications*.
Dordrecht, The Netherlands: Reidel, 1981.

[2] Beněs, V. (1981)
Exact finite dimensional filters for certain diffusions with nonlinear drift.
Stochastics, **5** (1981), 65–92.

[3] Beněs, V. E., and Karatzas, I. (1983)
Estimation and control for linear partially observable systems with non-Gaussian initial distribution.
Stochastic Processes and Their Application, **14** (1983), 233–248.

[4] Brockett, R. W. (1981)
Nonlinear systems and nonlinear estimation theory.
In M. Hazewinkel and J. S. Willems (Eds.), *The Mathematics of Filtering and Identification and Application*.
Dordrecht: Reidel, 1981.

[5] Brockett, R. W., and Clark, J. M. C. (1980)
The geometry of the conditional density functions.
In O. L. R. Jacobs, et al. (Eds.), *Analysis and Optimization of Stochastic Systems*.
New York: Academic Press, 1980, 299–309.

[6] Chiou, W. L., and Yau, S. S.-T. (1994)
Finite dimensional filters with nonlinear drift II: Brockett's problem of classification of finite dimensional estimation algebras.
SIAM Journal of Control and Optimization, **32**, 1 (1994), 297–310.

[7] Chen, J., Leung, C. W., and Yau, S. S.-T.
Finite dimensional filters with nonlinear drift IV: Classification of finite dimensional estimation algebras of maximal rank with state space dimension three.
SIAM Journal of Control and Optimization, to be published.

[8] Collingwood, P. C. (1986)
Some remarks on estimation algebras.
Systems Control Letters, **7** (1986), 217–224.

[9] Davis, M. H. A. (1980)
On a multiplicative functional transformation arising in nonlinear filtering theory.
Z. Wahrsch. Verw. Gebiete, **54** (1980), 125–139.

[10] Daum, F. E. (1987)
Solution of the Zakai equation by separation of variables.
IEEE Transactions on Automatic Control, **AC-32**, 10 (1987), 941–943.

[11] Davis, M. H. A., and Marcus, S. I. (1981)
An introduction to nonlinear filtering.
In M. Hazewinkel and J. S. Willems (Eds.), *The Mathematics of Filtering and Identification and Applications*.
Dordrecht: Reidel, 1981.

[12] Dong, R. T., Tam, W. S., Wong, W. S., and Yau, S. S.-T. (1991)
Structure and classification theorems of finite dimensional exact estimation algebras.
SIAM Journal of Control and Optimization, **29** (1991), 866–877.

[13] Hazewinkel, M. (1988)
Lectures on linear and nonlinear filtering.
In W. Shiehlen and W. Wedig (Eds.), *Analysis and Estimation of Stochastic Mechanical Systems* (CISM Courses and Lectures 303).
New York: Springer, 1988, 103–135.

[14] Haussmann, U. G., and Pardouxx, E. (1988)
A conditionally almost linear filtering problem with non-Gaussian initial condition.
Stochastics, **23** (1988), 241–275.

[15] Liang, Z. G., Yau, S.-T., and Yau, S. S.-T.
Finite dimensional filters with nonlinear drift V: Solution to Kolmogorov equation arising from linear filtering with non-Gaussian initial condition.
To be published.

[16] Marcus, S. I. (1984)
Algebraic and geometric methods in nonlinear filtering.
SIAM Journal of Control and Optimization, **22** (1984), 817–844.

[17] Makowski, A. (1986)
Filtering formulae for partially observed linear systems with non-Gaussian initial conditions.
Stochastics, **16** (1986), 1–24.

[18] Mitter, S. K. (1979)
On the analogy between mathematical problems of nonlinear filtering and quantum physics.
Ricerche di automatica, **10**, 2 (1979), 163–216.

[19] Mitter, S. K. (1983)
Lectures on nonlinear filtering and stochastic control.
In S. K. Mitter and A. Moro (Eds.), *Nonlinear Filtering and Stochastic Control* (Springer Lecture Notes in Mathematics #972).
New York: Springer, 1983, 170–207.

[20] Ocone, D. (1980)
Topics in nonlinear filtering theory.
Ph.D. dissertation, Dept. of Mathematics, Massachusetts Institute of Technology, Cambridge, 1980.

[21] Rozovsky, B. L. (1972)
Stochastics partial differential equations arising in nonlinear filtering problems.
Uspekhi Math. Nauk, **27** (1972), 213–214 (in Russian).

[22] Tam, L. F., Wong, W. S., and Yau, S. S.-T. (1990)
On a necessary and sufficient condition for finite dimensionality of estimation algebras.
SIAM Journal of Control and Optimization, **28**, 1 (1990), 173–185.

[23] Wong, W. S. (1987)
On a new class of finite dimensional estimation algebras.
Systems and Control Letters, **9** (1987), 79–83.

[24] Wong, W. S. (1987)
Theorems on the structure of finite dimensional estimation algebras.
Systems and Control Letters, **9** (1987), 117–124.

[25] Yau, S. S.-T. (1994)
Finite dimensional filters with nonlinear drifts I: A class of filters containing both Kalman filters and Beněs filters.
Journal of Mathematical Systems, Estimation and Control, **4**, 2 (1994), 181–203.

[26] Yau, S.-T., and Yau, S. S.-T.
Explicit solution to a Kolmogorov equation.
Applied Mathematics and Optimization, to be published.

[27] Yau, S. S.-T., and Yau, S.-T. (1994)
New direct method for Kalman-Bucy filtering system with arbitrary initial condition.
In *Proceedings of the 33rd Conference on Decision and Control*, Lake Buena Vista, FL, Dec. 1994, 1221–1225.

Shing-Tung Yau was born April 4, 1949, in Swatow, China. He received his Ph.D. from the University of California at Berkley in 1971.

He was a professor at Stanford University from 1974–1979, at the Institute for Advanced Study from 1979–1984, at the University of California at San Diego from 1984–1987, and since 1987 he has been Professor of Mathematics at Harvard University.

Dr. Yau is a member of Academia Sinica, American Academy of Arts and Sciences, and the National Academy of Sciences. He received the Veblen Prize in 1981, the Carty Prize in 1981, the Fields Medal in 1982, and the Crafoord Prize in 1994.

Stephen S.-T. Yau (M'89—SM'94) received the M.S. and Ph.D. degrees from the State University of New York at Stony Brook in 1974 and 1976, respectively.

In 1976–1977, he was a member of the Institute for Advanced Study at Princeton University. From 1977 to 1980, he was a Benjamin Pierce Assistant Professor at Harvard University. He received the Sloan Fellowship from 1980 to 1982. In 1980, he joined the Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago (UIC) as Associate Professor. He was promoted to Professor at UIC in 1984. He has also held several visiting professorship positions at Princeton University (1981), Institute for Advanced Study (1981–1982), University of Southern California (1983–1984), Yale University (1984–1985), Institute Mittag-Leffler, Sweden (1987), The Johns Hopkins University (1989–1990), University of Pisa, Italy (1990). He was awarded the University Scholar (1987–1990) by University of Illinois.

Dr. Yau was the managing editor of the *Journal of Algebraic Geometry*. He has been the Director of the Control and Information Laboratory since 1993.