Finite-Dimensional Filters with Nonlinear Drift

V: Solution to Kolmogorov Equation Arising From Linear Filtering with Non-Gaussian Initial Condition

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Despite its usefulness, the Kalman–Bucy filter is not perfect. One of its weaknesses is that it needs a Gaussian assumption on the initial data. Recently Yau and Yau introduced a new direct method to solve the estimation problem for linear filtering with non-Gaussian initial data. They factored the problem into two parts: 1) the on-line solution of a finite system of ordinary differential equations (ODEs), and 2) the off-line calculation of the Kolmogorov equation. Here we derive an explicit closed-form solution of the Kolmogorov equation. We also give some properties and conduct a numerical study of the solution.

Where to find parts I, II, and IV:

I: Finite dimensional filters with nonlinear drifts I: A class of filters containing both Kalman filters and Beněs filters, *Journal of Mathematical Systems, Estimation and Control*, **4** (1994), 181–203.

II: Finite dimensional filters with nonlinear drift II: Brockett's problem on classification of finite dimensional estimation algebra (with Wen-Lin Chiou), *SIAM Journal of Control and Optimization*, **32**, 1 (1994), 297–310.

IV: Finite dimensional filters with nonlinear drift IV: Classification of finite dimensional maximal rank estimation algebra with dimension of state space equal to 3 (with Jie Chen, Chi-Wah Leung), *SIAM Journal of Control and Optimization*, **34**, 1 (1996), 179–198.

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I. INTRODUCTION

Despite its usefulness, the Kalman–Bucy filter is not perfect. One of its weaknesses is that it needs a Gaussian assumption on the initial data. The situation is more complex when the statistics of the initial condition are modeled by an arbitrary distribution. As observed by Makowski [7], in that event, the filtering question is genuinely one in nonlinear filtering, and few results have been obtained before. Notable exceptions are the works of Benes and Karatzas [1], Ocone [8], and Makowski [7]. In [7], simple and direct probabilistic arguments are developed for evaluating the conditional expectation $\pi_t(\varphi(x_t))$ of $\varphi(x_t)$ given $\{y_s \mid 0 \le s \le t\}$. It was shown as in [1 and 8] that there always exists a set of sufficient statistics that can be recursively computed as outputs of a finite-dimensional dynamic system. In contrast with previous results, the sufficient statistics generated in [7] can be termed "universal" in the sense that they are independent of the initial state distribution. Furthermore, no assumptions on the moments of this initial state distribution or its absolute continuity are made in [7], as was the case in [1 and 8]. However, Makowski's method has a major disadvantage. Let n be the dimension of the state space. The number of sufficient statistics in order to compute the conditional expectation $\pi_t(\varphi(x_t))$ of $\varphi(x_t)$ in Makowski's method is a polynomial of degree two in n, while for the classical Kalman–Bucy filter, the number of sufficient statistics is only a polynomial of degree one in n.

In the case where the linear filter system (i.e., f, g, and h are linear functions in (1)) is completely reachable and completely observable, Hazewinkel observed in [6, pp. 115] that the estimation algebra E (see Section II for definition) is the $2n+2$ dimensional Lie algebra with an explicitly given basis. Even in this case, the Wei–Norman approach to finding an explicit filter is more complicated than the method of [10] (cf., Theorem 1 below). Not only must one solve a finite system of ordinary differential equations (ODEs) and a Kolmogorov equation, but one also has to integrate n partial differential equations corresponding to the operators $\partial/\partial x_1, \ldots, \partial/\partial x_n$. More important, if the Kalman–Bucy system is not completely reachable or completely observable, then the basis of the estimation algebra is not explicitly known (although it can be computed). As a result, there is an additional disadvantage of the Wei–Norman approach: one cannot write down the finite system of ODEs explicitly.

The novelty of the method of [10] is that their finite system of ODEs is explicitly written down and only n sufficient statistics are needed in order to compute the conditional expectation. They factored into two parts: 1) the on-line solution of a finite system of ODEs, 2) the off-line calculation of the Kolmogorov equation, which does not depend on observation. In this work, we derive an explicit closed-form solution to this Kolmogorov equation. We also give some properties of this solution in Section III and present a study of the equation in Section IV.

II. FILTERING PROBLEM AND BASIC QUESTIONS CONSIDERED

The filtering problem considered here is based on the following signal observation model:

$$\begin{cases} dx(t) = f(x(t))\,dt + g(x(t))\,dv(t) & x(0) = x_0 \\ dy(t) = h(x(t))\,dt + dw(t) & y(0) = 0 \end{cases} \tag{1}$$

in which x, v, y, and w are respectively $\mathbf{R}^n$-, $\mathbf{R}^p$-, $\mathbf{R}^m$-, and $\mathbf{R}^m$-valued processes and v and w have components that are independent, standard Brownian processes. We further assume that $n = p$; f, h are C^∞ smooth; and g is an orthogonal matrix. We refer to $x(t)$ as the state of the system at time t and $y(t)$ as the observation at time t.

Let $\rho(t,x)$ denote the conditional probability density of the state given the observation $\{y(s) : 0 \leq s \leq t\}$. It is well known (see [3], for example) that $\rho(t,x)$ is given by normalizing $\sigma(t,x)$, which satisfies the following Duncan–Mortensen–Zakai (DMZ) equation:

$$\begin{aligned} d\sigma(t,x) &= L_0\sigma(t,x)\,dt + \sum_{i=1}^m L_i\sigma(t,x)\,dy_i(t), \\ \sigma(0,x) &= \sigma_0 \end{aligned} \tag{2}$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^n f_i \frac{\partial}{\partial x_i} - \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^m h_i^2.$$

Equation (2) is a stochastic partial differential equation. In real applications, we are interested in constructing robust state estimators from observed sample paths with some property of robustness. From Rozovsky's transformation [9],

$$u(t,x) = \exp\left(-\sum_{i=1}^m h_i(x)y_i(t)\right)\sigma(t,x).$$

Davis [2] proposed to study the following robust DMZ equation:

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= L_0 u(t,x) + \sum_{i=1}^m y_i(t)[L_0,L_i]u(t,x) \\ &\quad + \frac{1}{2}\sum_{i=1}^m\sum_{j=1}^m y_i(t)y_j(t)[[L_0,L_i],L_j]u(t,x) \\ u(0,x) &= \sigma_0 \end{aligned} \tag{3}$$

which is a time-varying partial differential equation. Here we have used the following notation.

DEFINITION 1 If X and Y are differential operators, the Lie bracket of X and Y, $[X,Y]$, is defined by $[X,Y]\phi = X(Y\phi) - Y(X\phi)$ for any C^∞ function ϕ.

DEFINITION 2 The estimation algebra, E, of a filtering problem (1) is defined to be the Lie algebra generated by $\{L_0, L_1, \ldots, \delta_m\}$.

It is shown in [10] that (3) is equivalent to

$$\begin{aligned} \frac{\partial u}{\partial t}\langle t,x\rangle &= \frac{1}{2}\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}(t,x) \\ &\quad + \sum_{i=1}^n \left(-f_i(x) + \sum_{j=1}^n y_j(t)\frac{\partial h_j}{\partial x_i}(x)\right)\frac{\partial u}{\partial x_i}(t,x) \\ &\quad - \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}u(t,x) - \frac{1}{2}\sum_{i=1}^m h_i^2(x)u(t,x) \\ &\quad + \frac{1}{2}\sum_{i=1}^m y_i(t)\,\Delta h_i(x)u(t,x) \\ &\quad - \sum_{i=1}^m\sum_{j=1}^n y_i(t)f_j(x)\frac{\partial h_i}{\partial x_j}(x)u(t,x) \\ &\quad + \frac{1}{2}\sum_{i=1}^m\sum_{j=1}^n y_i(t)y_j(t)\sum_{k=1}^n \frac{\partial h_i}{\partial x_k}(x)\frac{\partial h_j}{\partial x_k}u(t,x). \end{aligned} \tag{4}$$

In [10], (4) is solved by means of a finite system of ODEs and a Kolmogorov equation.

THEOREM 1 ([10]) *Consider the linear filtering system* (1) *with an arbitrary initial condition. Suppose that*

$$h_i(x) = \sum_{j=1}^n c_{ij}x_j + c_i, \qquad 1 \leq i \leq m \tag{5}$$

where c_{ij}, c_i are constants, and

$$f_i(x) = \sum_{j=1}^n d_{ij}x_j + d_i, \qquad 1 \leq i \leq m \tag{6}$$

where d_{ij}, d_i are constants. Choose a homogeneous quadratic $F(x) = \frac{1}{2}\sum_{i,j=1}^n e_{ij}x_i x_j$ with $e_{ij} = e_{ji}$ such that

$$(E + D)^T(E + D) = C^TC + D^TD. \tag{7}$$

Here $E = (e_{ij})$, $D = (d_{ij})$ are $n \times n$ matrices, and $C = (c_{ij})$ is an $m \times n$ matrix. Then the solution $u(t,x)$ for the DMZ equation (4) *is reduced to the solution $\tilde{u}(t,x)$ for the Kolmogorov equation*

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial t}(t,x) &= \tfrac{1}{2}\Delta\tilde{u}(t,x) - \sum_{i=1}^n \left(f_i(x) + \frac{\partial F}{\partial x_i}(x)\right)\frac{\partial \tilde{u}}{\partial x_i}(t,x) \\ &\quad - \sum_{i=1}^n \left(\frac{\partial f_i}{\partial x_i}(x) + \frac{\partial^2 F}{\partial x_i^2}(x)\right)\tilde{u}(t,x) \end{aligned} \tag{8}$$

where

$$\tilde{u}(t,x) = \exp\left(F(x) + \sum_{i=1}^{n} a_i(t)x_i + c(t)\right) u(t,x+b(t)) \tag{9}$$

and $a_i(t)$, $b_i(t)$, $1 \le i \le n$, and $c(t)$ satisfy the following system of ODEs

$$a_i'(t) + \sum_{j=1}^{n} d_{ji}a_i(t) - \sum_{l=1}^{m}\sum_{j=1}^{n} c_{lj}b_j(t)c_{li} - \sum_{j=1}^{m}\sum_{k=1}^{n} y_j(t)d_{ki}c_{jk} + \sum_{j=1}^{n} d_j e_{ji} - \sum_{j=1}^{m} c_j c_{ji} = 0, \qquad 1 \le i \le n \tag{10}$$

$$b_i'(t) - a_i(t) + \sum_{j=1}^{m} c_{ji}y_j(t) - \sum_{j=1}^{n} d_{ij}b_j(t) = 0 \qquad 1 \le i \le n \tag{11}$$

$$\begin{aligned}\sum_{i=1}^{n} d_i a_i(t) + c'(t) - \frac{1}{2}\sum_{i=1}^{n} a_i^2(t) - \frac{1}{2}\sum_{i=1}^{m}\left(\sum_{j=1}^{n} c_{ij}b_j(t)\right)^2 \\ - \sum_{i=1}^{m} c_i \sum_{j=1}^{n} c_{ij}b_j(t) - \sum_{j=1}^{m}\sum_{i=1}^{n} y_j(t)c_{ji} \\ \times \left(\sum_{k=1}^{n} d_{ik}b_k(t) + d_i\right) + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{n} y_i(t)y_j(t)c_{ik}c_{jk} \\ + \frac{1}{2}\sum_{k=1}^{n} e_{kk} - \frac{1}{2}\sum_{i=1}^{n} c_i^2 = 0.\end{aligned} \tag{12}$$

In Section III, we show how to solve (8) explicitly.

III. APPLICATION OF FOURIER TRANSFORM METHOD TO KOLMOGOROV EQUATION

In this section, we shall solve (8) by means of the Fourier transform method. Recall that $f_i(x) = \sum_{j=1}^{n} d_{ij}x_j + d_i$ and $F(x) = \frac{1}{2}\sum_{i,j=1}^{n} e_{ij}x_ix_j$ with $e_{ij} = e_{ji}$. It follows that

$$f_i(x) + \frac{\partial F}{\partial x_i}(x) = \sum_{j=1}^{n}(d_{ij} + e_{ij})x_j + d_i. \tag{13}$$

Putting (13) into (8), we have

$$\frac{\partial \tilde{u}}{\partial t}(t,x) = \tfrac{1}{2}\Delta\tilde{u}(t,x) - \sum_{i=1}^{n}\left(\sum_{j=1}^{n}(d_{ij} + e_{ij})x_j + d_i\right) \times \frac{\partial \tilde{u}}{\partial x_i}(t,x) - \sum_{j=1}^{n}(d_{ii} + e_{ii})\tilde{u}(t,x) \tag{14}$$

where

$$\Delta = \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}.$$

It remains to solve (14).

Let $\mathcal{S}$ be the Schwartz space of all C^∞ functions on $\mathbf{R}^n$ which, together with all their derivatives, die out faster than any power of x at infinity. That is, $\varphi \in \mathcal{S}$ if and only if $\varphi \in C^\infty$ and for all multi-indices α and β

$$\sup_{x\in\mathbf{R}^n} |x^\alpha \partial^\beta u(x)| < \infty, \qquad \text{where}$$

$$\partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha^1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha^n}.$$

In [4], it is shown that (14) has a solution $\tilde{u}(t,x)$ such that $\tilde{u}(t,x)$ is in $\mathcal{S}$ for any fixed t. We now solve (14) by means of a Fourier transform. Recall that if $\varphi \in L^1(\mathbf{R}^n)$, its Fourier transform $\hat{\varphi}$ is a bounded function on $\mathbf{R}^n$ defined by

$$\hat{\varphi}(\xi) = \int \exp(-2\pi\sqrt{-1}x\xi)\varphi(x)\,dx. \tag{15}$$

Clearly $\hat{\varphi}(\xi)$ is well defined for all ξ and $\|\hat{\varphi}\|_\infty \le \|\hat{\varphi}\|_1$. We recall the following properties of Fourier transform for convenience to the reader.

Property 1 If $\varphi, \psi \in L^1(\mathbf{R}^n)$, then $\widehat{(\varphi * \psi)} = \hat{\varphi}\hat{\psi}$.

Property 2 If $\varphi \in \mathcal{S}$, then $\hat{\varphi} \in C^\infty$ and $\partial^\beta\hat{\varphi} = [(-2\pi\sqrt{-1}x)^\beta\varphi]^\wedge$.

Property 3 If $\varphi \in \mathcal{S}$, then $(\widehat{\partial^\beta\varphi})(\xi) = (2\pi\sqrt{-1}\xi)^\beta\hat{\varphi}(\xi)$.

Property 4 If $\varphi \in \mathcal{S}$, then $\hat{\varphi} \in \mathcal{S}$.

Property 5 If $\varphi \in L^1$, then $\hat{\varphi}$ is continuous and tends to zero at infinity.

The Fourier transform is useful because we can recover the function from its Fourier transform.

DEFINITION 3 For $f \in L^1$, the function $\check{f}$ is defined by

$$\check{f}(x) := \int \exp(2\pi\sqrt{-1}x\xi)f(\xi)\,d\xi = \hat{f}(-x).$$

FOURIER INVERSION THEOREM. *If $f \in \mathcal{S}$, then* $(\hat{f})^\vee = f$.

We are now ready to solve (14). Let

$$\tilde{u}_1(t,x) = \exp\left(\sum_{i=1}^{n}(d_{ii} + e_{ii})t\right)\tilde{u}(t,x). \tag{16}$$

Then (14) becomes

$$\begin{aligned}\frac{\partial \tilde{u}_1}{\partial t}(t,x) &= \sum_{i=1}^{n}(d_{ii} + e_{ii})\exp\left(\sum_{i=1}^{n}(d_{ii} + e_{ii})t\right)\tilde{u}(t,x) \\ &\quad + \exp\left(\sum_{i=1}^{n}(d_{ii} + e_{ii})t\right) \\ &\quad \times \left[\frac{1}{2}\Delta\tilde{u}(t,x) - \sum_{i=1}^{n}\left(f_i(x) + \frac{\partial F}{\partial x_i}(x)\right)\frac{\partial \tilde{u}}{\partial x_i}(t,x) \right. \\ &\quad \left. - \sum_{i=1}^{n}(d_{ii} + e_{ii})\tilde{u}(t,x)\right]\end{aligned}$$

$$= \frac{1}{2}\exp\left(\sum_{i=1}^{n}(d_{ii}+e_{ii})t\right)\Delta\tilde{u}(t,x)$$

$$-\sum_{i=1}^{n}\left(f_i(x)+\frac{\partial F}{\partial x_i}(x)\right)$$

$$\times\exp\left(\sum_{i=1}^{n}(d_{ii}+e_{ii})t\right)\frac{\partial\tilde{u}}{\partial x_i}(t,x)$$

$$= \frac{1}{2}\Delta\tilde{u}_1(t,x)-\sum_{i=1}^{n}\left[\sum_{j=1}^{n}(d_{ij}+e_{ij})x_j+d_i\right]$$

$$\times\frac{\partial\tilde{u}_1}{\partial x_i}(t,x). \qquad (17)$$

Therefore it remains to solve (17). Let v denote the Fourier transform of $\tilde{u}_1$, that is,

$$v(t,\xi)=(\tilde{u}_1)^{\wedge}(t,\xi). \qquad (18)$$

In view of the properties (14) and Property 3 of the Fourier transform, we have

$$\left(x_j\frac{\partial\tilde{u}_1}{\partial x_i}\right)^{\wedge}=\frac{-1}{2\pi\sqrt{-1}}\left[(-2\pi\sqrt{-1}x_j)\frac{\partial\tilde{u}_1}{\partial x_i}\right]^{\wedge}$$

$$=\frac{-1}{2\pi\sqrt{-1}}\frac{\partial}{\partial\xi_j}\left[\left(\frac{\partial\tilde{u}_1}{\partial x_i}\right)^{\wedge}\right]$$

$$=\frac{-1}{2\pi\sqrt{-1}}\frac{\partial}{\partial\xi_j}[2\pi\sqrt{-1}\xi_i v(t,\xi)]$$

$$=-\delta_{ij}v(t,\xi)-\xi_i\frac{\partial v}{\partial\xi_j}(t,\xi). \qquad (19)$$

By taking the Fourier transform of (17), we get

$$\frac{\partial v}{\partial t}(t,\xi)=\frac{1}{2}\sum_{i=1}^{n}(2\pi\sqrt{-1}\xi_i)^2v(t,\xi)+\sum_{i=1}^{n}\sum_{j=1}^{n}(d_{ij}+e_{ij})$$

$$\times\left(\delta_{ij}v(t,\xi)+\xi_i\frac{\partial v}{\partial\xi_j}(t,\xi)\right)$$

$$-\sum_{i=1}^{n}d_i2\pi\sqrt{-1}\xi_i v(t,\xi)$$

$$=-2\pi\left(\pi\sum_{i=1}^{n}\xi_i^2+\sqrt{-1}\sum_{i=1}^{n}d_i\xi_i\right)v(t,\xi)$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}(d_{ij}+e_{ij})\xi_i\frac{\partial v}{\partial\xi_j}(t,\xi)+\sum_{i=1}^{n}(d_{ii}+e_{ii})v(t,\xi). \qquad (20)$$

We introduce a new function $\tilde{v}(t,z)$ such that $\tilde{v}(t,z)$ will satisfy a simpler differential equation than (20). Let

$$\tilde{v}(t,z)=v(t,B(t)z)$$

$$=v\left(t,\sum_{j=1}^{n}b_{1j}(t)z_j,\ldots,\sum_{j=1}^{n}b_{nj}(t)z_j\right). \qquad (21)$$

Then

$$\frac{\partial\tilde{v}}{\partial t}(t,z)=\frac{\partial v}{\partial t}(t,B(t)z)+\sum_{i=1}^{n}\sum_{j=1}^{n}b'_{ij}(t)z_j\frac{\partial v}{\partial\xi_i}(t,B(t)z)$$

$$=-2\pi\left[\pi\sum_{i=1}^{n}\left(\sum_{j=1}^{n}b_{ij}(t)z_j\right)^2\right.$$

$$\left.+\sqrt{-1}\sum_{i=1}^{n}d_i\sum_{j=1}^{n}b_{ij}(t)z_j\right]v(t,B(t)z)$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}(d_{ij}+e_{ij})\sum_{k=1}^{n}b_{ik}(t)z_k$$

$$\times\frac{\partial v}{\partial\xi_j}(t,B(t)z)+\sum_{i=1}^{n}(d_{ii}+e_{ii})v(t,B(t)z)$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}b'_{ij}(t)z_j\frac{\partial v}{\partial\xi_i}(t,B(t)z). \qquad (22)$$

Choose $b_{ij}(t)$ in such a way that for all $1\le i\le n$ we have

$$\sum_{j=1}^{n}b'_{ij}(t)z_j+\sum_{j=1}^{n}\sum_{k=1}^{n}(d_{ji}+e_{ji})b_{jk}(t)z_k=0. \qquad (23)$$

This means that we choose $b_{ij}(t)$ such that for all $1\le i,\ j\le n$, we have

$$b'_{ij}(t)+\sum_{k=1}^{n}(d_{ki}+e_{ki})b_{kj}(t)=0 \qquad (24)$$

that is,

$$B'(t)=-(D+E)^T B(t)$$
$$B(t)=\exp(-t(D+E)^T) \qquad (25)$$

where $D=(d_{ij})$ and $E=(e_{ij})$ are $n\times n$ matrices defined in Theorem 1. Then (22) becomes

$$\frac{\partial\tilde{v}}{\partial t}(t,z)=-2\pi\left[\pi\sum_{i=1}^{n}\left(\sum_{j=1}^{n}b_{ij}(t)z_j\right)^2\right.$$

$$\left.+\sqrt{-1}\sum_{i=1}^{n}d_i\sum_{j=1}^{n}b_{ij}(t)z_j\right]\tilde{v}(t,z)$$

$$+\sum_{i=1}^{n}(d_{ii}+e_{ii})\tilde{v}(t,z). \qquad (26)$$

Now (26) can be solved explicitly in the following form:

$$\tilde{v}(t,z)=\tilde{v}(0,z)\exp\left(-2\pi\int_0^t\left[\pi\sum_{i=1}^{n}\left(\sum_{j=1}^{n}b_{ij}(\tau)z_j\right)^2\right.\right.$$

$$\left.\left.+\sqrt{-1}\sum_{i=1}^{n}d_i\sum_{j=1}^{n}b_{ij}(\tau)z_j\right]d\tau+\sum_{i=1}^{n}(d_{ii}+e_{ii})t\right). \qquad (27)$$

So we have proved the following theorem.

THEOREM 2 *The Kolmogorov equation arising from linear filtering with arbitrary initial condition*

$$\frac{\partial \tilde{u}}{\partial t}(t,x) = \tfrac{1}{2}\Delta \tilde{u}(t,x) - \sum_{i=1}^{n}\left(\sum_{j=1}^{n}(d_{ij}+e_{ij})x_j + d_i\right)\frac{\partial \tilde{u}}{\partial x_i}(t,x) - \sum_{i=1}^{n}(d_{ii}+e_{ii})\tilde{u}(t,x)$$

can be solved in the following manner. Let

$$\tilde{u}(t,x) = \exp\left(-\sum_{i=1}^{n}(d_{ii}+e_{ii})t\right)\tilde{u}_1(t,x) \tag{28}$$

$$\begin{aligned} v(t,\xi) &= \textit{Fourier transform of } \tilde{u}_1 \textit{ in } x \textit{ variable} \\ &= \widehat{(\tilde{u}_1)}(t,\xi) \end{aligned} \tag{29}$$

and

$$\begin{aligned} \tilde{v}(t,z) &= v(t,B(t)z) \\ &= v\left(t, \sum_{j=1}^{n} b_{1j}(t)z_j, \ldots, \sum_{j=1}^{n} b_{nj}(t)z_j\right) \end{aligned} \tag{30}$$

where $B(t) = \exp(-t(D+E)^T)$. *Then* $\tilde{v}(t,z)$ *is given explicitly by the following equation:*

$$\tilde{v}(t,z) = \tilde{v}(0,z)\exp\left(-2\pi\int_0^t\left[\pi\sum_{i=1}^{n}\left(\sum_{j=1}^{n}b_{ij}(\tau)z_j\right)^2 + \sqrt{-1}\sum_{i=1}^{n}d_i\sum_{j=1}^{n}b_{ij}(\tau)z_j\right]d\tau + \sum_{i=1}^{n}(d_{ii}+e_{ii})t\right). \tag{31}$$

COROLLARY 1 *Let* $\tilde{u}(t,x)$ *be the solution of the Kolmogorov equation arising from linear filtering with an arbitrary initial condition:*

$$\frac{\partial \tilde{u}}{\partial t}(t,x) = \tfrac{1}{2}\Delta \tilde{u}(t,x) - \sum_{i=1}^{n}\left(\sum_{j=1}^{n}(d_{ij}+e_{ij})x_j + d_i\right) \times \frac{\partial \tilde{u}}{\partial x_i}(t,x) - \sum_{i=1}^{n}(d_{ii}+e_{ii})\tilde{u}(t,x).$$

Then for any t,

$$\int \tilde{u}(t,x)\,dx = \int \tilde{u}(0,x)\,dx. \tag{32}$$

PROOF The equation (31) implies

$$\tilde{v}(t,0) = \tilde{v}(0,0)\exp\left(\sum_{i=1}^{n}(d_{ii}+e_{ii})t\right). \tag{33}$$

On the other hand, recall that by (29), we have

$$v(t,\xi) = \int \exp\left(-2\pi\sqrt{-1}\xi x\right)\tilde{u}_1(t,x)\,dx.$$

Hence, (30) implies

$$\begin{aligned} \tilde{v}(t,z) &= v(t,B(t)z) \\ &= \int \exp\left(-2\pi\sqrt{-1}\sum_{i=1}^{n}x_i\left(\sum_{j=1}^{n}b_{ij}(t)z_j\right)\right) \\ &\quad \times \tilde{u}_1(t,x)\,dx. \end{aligned}$$

It follows that

$$\tilde{v}(t,0) = \int \tilde{u}_1(t,x)\,dx. \tag{34}$$

In particular,

$$\tilde{v}(0,0) = \int \tilde{u}_1(0,x)\,dx. \tag{35}$$

Putting (34) and (35) in (33), we get

$$\int \tilde{u}_1(t,x)\,dx = \left(\int \tilde{u}_1(0,x)\,dx\right)\exp\left(\sum_{i=1}^{n}(d_{ii}+e_{ii})t\right). \tag{36}$$

Combining (28) and (36), we get

$$\int \exp\left(\sum_{i=1}^{n}(d_{ii}+e_{ii})t\right)\tilde{u}(t,x)\,dx = \left(\int \tilde{u}(0,x)\,dx\right)\exp\left(\sum_{i=1}^{n}(d_{ii}+e_{ii})t\right).$$

This last equality implies

$$\int \tilde{u}(t,x)\,dx = \int \tilde{u}(0,x)\,dx$$

as required. Q.E.D.

REMARK Corollary 1 can also be seen in the following manner:

$$\begin{aligned} \frac{\partial}{\partial t}\int \tilde{u}(t,x)\,dx &= \frac{1}{2}\int \Delta\tilde{u}(t,x)\,dx \\ &\quad - \int \sum_{i=1}^{n}\left[\sum_{j=1}^{n}(d_{ij}+e_{ij})x_j + d_i\right]\frac{\partial \tilde{u}}{\partial x_i}(t,x)\,dx \\ &\quad - \int \sum_{i=1}^{n}(d_{ii}+e_{ii})\tilde{u}(t,x)\,dx. \end{aligned} \tag{37}$$

Observe that $\int \Delta\tilde{u}(t,x)\,dx = 0$ by the Divergence theorem (cf. [5]), while

$$\int \sum_{i=1}^{n} \left[\sum_{j=1}^{n}(d_{ij}+e_{ij})x_j + d_i\right] \frac{\partial \tilde{u}}{\partial x_i}(t,x)\,dx$$
$$= -\int \sum_{i=1}^{n}\sum_{j=1}^{n}(d_{ij}+e_{ij})\delta_{ij}\tilde{u}(t,x)\,dx$$
$$= -\int_{i=1}^{n}(d_{ii}+e_{ii})\tilde{u}(t,x)\,dx.$$

So $(\partial/\partial t)\int \tilde{u}(t,x)\,dx = 0$ as required.

IV. ANALYTIC SOLUTIONS OF KOLMOGOROV EQUATION WITH SPECIAL INITIAL CONDITIONS

Recall from Section III that the solution $\tilde{u}(t,x)$ of (14) can be derived from

$$\tilde{v}(t,z) = \tilde{v}(0,z)\exp\left(-2\pi \int_0^t \left[\pi \sum_{i=1}^{n}\left(\sum_{j=1}^{n} b_{ij}(\tau)z_j\right)^2 + \sqrt{-1}\sum_{i=1}^{n} d_i \sum_{j=1}^{n} b_{ij}(\tau)z_j\right] d\tau + \sum_{i=1}^{n}(d_{ii}+e_{ii})t\right) \tag{38}$$

by the following procedure.

Let $B^{-1}(t) = (\tilde{b}_{ij})$ be the inverse of $B(t)$. Then

$$v(t,z) = v(t,B(t)(B^{-1}(t)z)) = \tilde{v}(t,B^{-1}(t)z)$$
$$= \tilde{v}(0,B^{-1}(t)z)\exp\left(\sum_{i=1}^{n}(d_{ii}+e_{ii})t\right)$$
$$\times \exp\left(-2\pi\int_0^t \left[\pi \sum_{i=1}^{n}\left(\sum_{j=1}^{n} b_{ij}(s)\left(\sum_{k=1}^{n}\tilde{b}_{jk}(t)z_k\right)\right)^2 + \sqrt{-1}\sum_{i=1}^{n} d_i \sum_{j=1}^{n} b_{ij}(s)\left(\sum_{k=1}^{n}\tilde{b}_{jk}(t)z_k\right)\right] ds\right)$$
$$= \tilde{v}(0,B^{-1}(t)z)\exp\left(\sum_{i=1}^{n}(d_{ii}+e_{ii})t\right)$$
$$\times \exp\left(-2\pi\int_0^t [\pi(B(s)\cdot B^{-1}(t)z)^T(B(s)\cdot B^{-1}(t)z) + \sqrt{-1}D^T\cdot B(s)\cdot B^{-1}(t)z]\,ds\right) \tag{39}$$

where

$$D = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}.$$

Let

$$\gamma_1(t,z) = \exp\left(-2\pi\int_0^t [\pi(B(s)\cdot B^{-1}(t)z)^T(B(s)\cdot B^{-1}(t)z) + \sqrt{-1}D^T\cdot B(s)\cdot B^{-1}(t)z]\,ds\right). \tag{40}$$

When we take the inverse Fourier transform of $v(t,z)$, we get to $\tilde{u}_1(t,x)$,

$$\tilde{u}_1(t,x) = [\tilde{v}(0,B^{-1}(t)z)\cdot\gamma_1(t,z)]^{\vee} \times \exp\left(\sum_{i=1}^{n}(d_{ii}+e_{ii})t\right). \tag{41}$$

Recall that $(F\cdot G)^{\vee} = F^{\vee} * G^{\vee}$. Let

$$\gamma(t,x) = [\gamma_1(t,z)]^{\vee} \tag{42}$$

$$\tilde{u}_1(t,x) = \exp\left(\sum_{i=1}^{n}(d_{ii}+e_{ii})t\right)\tilde{v}(0,B^{-1}(t)z)^{\vee} * \gamma(t,x). \tag{43}$$

Now for $\tilde{v}(0,B^{-1}(t)z)$,

$$\tilde{v}(0,B^{-1}(t)z) = v(0,B(0)B^{-1}(t)z)$$
$$= (\tilde{u}_1)^{\wedge}(0,B(0)B^{-1}(t)z)$$
$$= \int_{-\infty}^{\infty}\exp\left(-2\pi\sqrt{-1}\left[\sum_i x_i\left(\sum_j b_{ij}(0)\left(\sum_k \tilde{b}_{jk}(t)z_k\right)\right)\right]\right)\tilde{u}_1(0,x)\,dx$$
$$= \int_{-\infty}^{\infty}\exp\left(-2\pi\sqrt{-1}\left[\sum_k\left(\sum_i\left(\sum_j b_{ij}(0)\tilde{b}_{jk}(t)\right)x_i\right)z_k\right]\right)\tilde{u}_1(0,x)\,dx. \tag{44}$$

Let

$$\tilde{x} = [B(0)B^{-1}(t)]^T x. \tag{45}$$

Then

$$\begin{aligned}
&\tilde{v}(0,B^{-1}(t)z)\\
&\quad = \det(B^{-1}(0)^T B(t)^T)\int_{-\infty}^{\infty}\exp\left(-2\pi\sqrt{-1}\sum_k \tilde{x}_k z_k\right)\\
&\qquad \times \tilde{u}_1(0,B^{-1}(0)^T B(t)^T\tilde{x})\,d\tilde{x}\\
&\quad = \det(B^{-1}(0)^T B(t)^T)\int_{-\infty}^{\infty}\exp\left(-2\pi\sqrt{-1}\sum_k \tilde{x}_k z_k\right)\\
&\qquad \times \exp\left(\sum_{i=1}^n (d_{ii}+e_{ii})\cdot 0)\tilde{u}(0,B^{-1}(0)^T B(t)^T\tilde{x}\right)d\tilde{x}\\
&\quad = \det(B^{-1}(0)^T B(t)^T)[\tilde{u}(0,B^{-1}(0)^T B(t)^T x)]^{\wedge}.
\end{aligned} \tag{46}$$

So

$$\begin{aligned}
\tilde{u}_1(t,x) &= \exp\left(\sum_{i=1}^n (d_{ii}+e_{ii})t\right)\det(B^{-1}(0)^T B(t)^T)\\
&\qquad \times \tilde{u}(0,B^{-1}(0)^T B(t)^T x) * \gamma(t,x).
\end{aligned} \tag{47}$$

Replace $\tilde{u}_1(t,x)$ with $\exp(\sum_{i=1}^n (d_{ii}+e_{ii})t)\tilde{u}(t,x)$

$$\begin{aligned}
\tilde{u}(t,x) &= \det(B^{-1}(0)^T B(t)^T)\\
&\qquad \times \tilde{u}(0,B^{-1}(0)^T B(t)^T x) * \gamma(t,x).
\end{aligned} \tag{48}$$

Next we show how to get an analytic solution of $\tilde{u}(t,x)$ from (47) by two examples, one is one dimensional, another two dimensional.

EXAMPLE 1 Consider the equation

$$\begin{cases}
\dfrac{\partial\tilde{u}}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2\tilde{u}}{\partial x^2} - [(d_{11}+e_{11})x + d_1]\dfrac{\partial\tilde{u}}{\partial x} - (d_{11}+e_{11})\tilde{u}\\
\tilde{u}(0,x) = \alpha(x)
\end{cases} \tag{49}$$

where α is a function of x.

Then $B(t) = \exp(-(d_{11}+e_{11})t)$ and $B^{-1}(t) = \exp((d_{11}+e_{11})t)$.

From (42)

$$\begin{aligned}
\gamma(t,x) &= \int_{-\infty}^{\infty}\left[\exp\left(-2\pi^2\int_0^t (\exp(-(d_{11}+e_{11})s)\right.\right.\\
&\qquad \left.\left.\times \exp((d_{11}+e_{11})t)z)^2\,ds\right)\right]\\
&\quad \times \exp\left(-2\pi\sqrt{-1}\int_0^t d_1\exp(-(d_{11}+e_{11})s)\,ds\right.\\
&\qquad \left.\times \exp((d_{11}+e_{11})t)z\right)\exp(2\pi\sqrt{-1}zx)\,dz\\
&= \int_{-\infty}^{\infty}\exp\left(-2\pi^2\frac{(\exp(2(d_{11}+e_{11})t)-1)z^2}{2(d_{11}+e_{11})}\right)\\
&\quad \times \exp\left(-2\pi d_1\sqrt{-1}\frac{(\exp((d_{11}+e_{11})t)-1)z}{(d_{11}+e_{11})}\right)\\
&\quad \times \exp(2\pi\sqrt{-1}zx)\,dz
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\gamma}{\partial x} &= \int_{-\infty}^{\infty}2\pi z\sqrt{-1}\exp\left(-2\pi^2\frac{(\exp(2(d_{11}+e_{11})t)-1)z^2}{2(d_{11}+e_{11})}\right)\\
&\quad \times \exp\left(-2\pi d_1\sqrt{-1}\frac{(\exp((d_{11}+e_{11})t)-1)z}{d_{11}+e_{11}}\right)\\
&\quad \times \exp(2\pi\sqrt{-1}zx)\,dz\\
&= -\frac{2(d_{11}+e_{11})\sqrt{-1}}{2\pi(\exp(2(d_{11}+e_{11})t)-1)}\int_{-\infty}^{\infty}\\
&\quad \times \frac{\partial}{\partial z}\left[\exp\left(-2\pi^2\frac{(\exp(2(d_{11}+e_{11})t)-1)z^2}{2(d_{11}+e_{11})}\right)\right]\\
&\quad \times \exp\left(-2\pi d_1\sqrt{-1}\frac{(\exp((d_{11}+e_{11})t)-1)z}{d_{11}+e_{11}}\right)\\
&\quad \times \exp(2\pi\sqrt{-1}zx)\,dz\\
&= \frac{2(d_{11}+e_{11})\sqrt{-1}}{2\pi(\exp(2(d_{11}+e_{11})t)-1)}\int_{-\infty}^{\infty}\\
&\quad \times \exp\left(-2\pi^2\frac{(\exp(2(d_{11}+e_{11})t)-1)z^2}{2(d_{11}+e_{11})}\right)\\
&\quad \times \frac{\partial}{\partial z}\left[\exp\left(-2\pi d_1\sqrt{-1}\frac{(\exp((d_{11}+e_{11})t)-1)z}{d_{11}+e_{11}}\right)\right.\\
&\qquad \left.\times \exp(2\pi\sqrt{-1}zx)\right]dz\\
&= \frac{2(d_{11}+e_{11})\sqrt{-1}}{2\pi(\exp(2(d_{11}+e_{11})t)-1)}\int_{-\infty}^{\infty}\\
&\quad \times \exp\left(-2\pi^2\frac{(\exp(2(d_{11}+e_{11})t)-1)z^2}{2(d_{11}+e_{11})}\right)\\
&\quad \times \exp\left(-2\pi d_1\sqrt{-1}\frac{(\exp((d_{11}+e_{11})t)-1)z}{d_{11}+e_{11}}\right)\\
&\quad \times \exp(2\pi\sqrt{-1}zx)\\
&\quad \times \left[-2\pi d_1\sqrt{-1}\frac{\exp((d_{11}+e_{11})t)-1}{d_{11}+e_{11}} + 2\pi\sqrt{-1}x\right]dz\\
&= \left[\frac{2d_1}{\exp((d_{11}+e_{11})t)+1} - \frac{2(d_{11}+e_{11})x}{\exp(2(d_{11}+e_{11})t)-1}\right]\\
&\quad \times \int_{-\infty}^{\infty}\exp\left(-2\pi^2\frac{(\exp(2(d_{11}+e_{11})t)-1)z^2}{2(d_{11}+e_{11})}\right)\\
&\quad \times \exp\left(-2\pi d_1\sqrt{-1}\frac{(\exp((d_{11}+e_{11})t)-1)z}{d_{11}+e_{11}}\right)\\
&\quad \times \exp(2\pi\sqrt{-1}zx)\,dz\\
&= \left[\frac{2d_1}{\exp((d_{11}+e_{11})t)+1} - \frac{2(d_{11}+e_{11})x}{\exp(2(d_{11}+e_{11})t)-1}\right]\gamma(t,x).
\end{aligned}$$

So

$$\gamma(t,x)=\gamma(t,0)\exp\left(\frac{2d_1x}{\exp((d_{11}+e_{11})t)+1}-\frac{(d_{11}+e_{11})x^2}{\exp(2(d_{11}+e_{11})t)-1}\right). \tag{50}$$

But

$$\gamma(t,0)=\int_{-\infty}^{\infty}\exp\left(-2\pi^2\frac{(\exp(2(d_{11}+e_{11})t)-1)z^2}{2(d_{11}+e_{11})}\right)\times\exp\left(-2\pi d_1\sqrt{-1}\frac{(\exp((d_{11}+e_{11})t)-1)z}{d_{11}+e_{11}}\right)dz.$$

We can treat this $\gamma(t,0)$ as the function of d_1, $\beta(t,d_1)=\gamma(t,0)$. Then

$$\begin{aligned}\frac{\partial\beta}{\partial d_1}&=\int_{-\infty}^{\infty}-2\pi\sqrt{-1}\frac{\exp((d_{11}+e_{11})t)-1}{d_{11}+e_{11}}z\\&\quad\times\exp\left(-2\pi^2\frac{(\exp(2(d_{11}+e_{11})t)-1)z^2}{2(d_{11}+e_{11})}\right)\\&\quad\times\exp\left(-2\pi d_1\sqrt{-1}\frac{(\exp((d_{11}+e_{11})t)-1)z}{d_{11}+e_{11}}\right)dz\\&=\frac{2\sqrt{-1}}{2\pi(\exp((d_{11}+e_{11})t)+1)}\int_{-\infty}^{\infty}\frac{\partial}{\partial z}\\&\quad\times\left[\exp\left(-2\pi^2\frac{(\exp(2(d_{11}+e_{11})t)-1)z^2}{2(d_{11}+e_{11})}\right)\right]\\&\quad\times\exp\left(-2\pi d_1\sqrt{-1}\frac{(\exp((d_{11}+e_{11})t)-1)z}{d_{11}+e_{11}}\right)dz\\&=-\frac{2d_1(\exp((d_{11}+e_{11})t)-1)}{(\exp((d_{11}+e_{11})t)+1)(d_{11}+e_{11})}\beta(t,d_1).\end{aligned}$$

So

$$\beta(t,d_1)=\beta(t,0)\exp\left(-\frac{d_1^2(\exp((d_{11}+e_{11})t)-1)}{(\exp((d_{11}+e_{11})t)+1}\right)\times(d_{11}+e_{11}). \tag{51}$$

But

$$\beta(t,0)=\int_{-\infty}^{\infty}\exp\left(-2\pi^2\frac{(\exp(2(d_{11}+e_{11})t)-1)z^2}{2(d_{11}+e_{11})}\right)dz=\frac{\sqrt{d_{11}+e_{11}}}{\sqrt{\pi(\exp(2(d_{11}+e_{11})t)-1)}}. \tag{52}$$

So

$$\gamma(t,0):=\beta(t,d_1)=\frac{\sqrt{d_{11}+e_{11}}}{\sqrt{\pi(\exp(2(d_{11}+e_{11})t)-1)}}\times\exp\left(-\frac{d_1^2(\exp((d_{11}+e_{11})t)-1)}{(\exp((d_{11}+e_{11})t)+1)(d_{11}+e_{11})}\right),$$

and hence

$$\begin{aligned}\gamma(t,x)&=\frac{\sqrt{d_{11}+e_{11}}}{\sqrt{\pi(\exp(2(d_{11}+e_{11})t)-1)}}\\&\quad\times\exp\left(-\frac{d_1^2(\exp((d_{11}+e_{11})t)-1)}{(\exp((d_{11}+e_{11})t)+1)(d_{11}+e_{11})}\right)\\&\quad\times\exp\left(\frac{2d_1x}{\exp((d_{11}+e_{11})t)+1}-\frac{(d_{11}+e_{11})x^2}{\exp(2(d_{11}+e_{11})t)-1}\right).\end{aligned} \tag{53}$$

From (48),

$$\begin{aligned}\tilde{u}(t,x)&\\&=\exp(-(d_{11}+e_{11})t)\alpha(\exp(-(d_{11}+e_{11})t)x)\\&\quad\times\left\{\frac{\sqrt{d_{11}+e_{11}}}{\sqrt{\pi(\exp(2(d_{11}+e_{11})t)-1)}}\right.\\&\qquad\times\exp\left(-\frac{d_1^2(\exp((d_{11}+e_{11})t)-1)}{(\exp((d_{11}+e_{11})t)+1)(d_{11}+e_{11})}\right)\\&\qquad\left.\times\exp\left(\frac{2d_1x}{\exp((d_{11}+e_{11})t)+1}-\frac{(d_{11}+e_{11})x^2}{\exp(2(d_{11}+e_{11})t)-1}\right)\right\}\\&=\frac{\sqrt{d_{11}+e_{11}}\exp(-(d_{11}+e_{11})t)}{\sqrt{\pi(\exp(2(d_{11}+e_{11})t)-1)}}\\&\quad\times\exp\left(-\frac{d_1^2(\exp((d_{11}+e_{11})t)-1)}{(\exp((d_{11}+e_{11})t)+1)(d_{11}+e_{11})}\right)\\&\quad\times\int_{-\infty}^{\infty}\alpha(\exp(-(d_{11}+e_{11})t)s)\\&\quad\times\exp\left(\frac{2d_1(x-s)}{\exp((d_{11}+e_{11})t)+1}-\frac{(d_{11}+e_{11})(x-s)^2}{\exp(2(d_{11}+e_{11})t)-1}\right)ds.\end{aligned} \tag{54}$$

If the initial condition is

$$\tilde{u}(0,x)=\alpha(x)=\begin{cases}x, & 0\le x\le 1/2\\ 1-x, & 1/2\le x\le 1\\ 0, & \text{otherwise}\end{cases},$$

then

$$\alpha(\exp(-(d_{11}+e_{11})t)x)=\begin{cases}\exp(-(d_{11}+e_{11})t)x, & 0\le x\le\frac{1}{2}\exp((d_{11}+e_{11})t)\\ 1-\exp(-(d_{11}+e_{11})t)x, & \frac{1}{2}\exp((d_{11}+e_{11})t)\le x\le\exp((d_{11}+e_{11})t)\\ 0, & \text{otherwise}\end{cases}.$$

$$\tilde{u}(t,x) = \frac{\sqrt{d_{11}+e_{11}}\exp(-(d_{11}+e_{11})t)}{\sqrt{\pi(\exp(2(d_{11}+e_{11})t)-1)}} \times \exp\left(-\frac{d_1^2(\exp((d_{11}+e_{11})t)-1)}{(\exp((d_{11}+e_{11})t)+1)(d_{11}+e_{11})}\right) \times \left\{\int_0^{\exp((d_{11}+e_{11})t)/2} \exp(-(d_{11}+e_{11})t)s \times \exp\left(\frac{2d_1(x-s)}{\exp((d_{11}+e_{11})t)+1} - \frac{(d_{11}+e_{11})(x-s)^2}{\exp(2(d_{11}+e_{11})t)-1}\right)ds + \int_{\exp((d_{11}+e_{11})t)/2}^{\exp((d_{11}+e_{11})t)} [1-\exp(-(d_{11}+e_{11})t)s] \times \exp\left(\frac{2d_1(x-s)}{\exp((d_{11}+e_{11})t)+1} - \frac{(d_{11}+e_{11})(x-s)^2}{\exp(2(d_{11}+e_{11})t)-1}\right)ds\right\} \quad (55)$$

or

$$\tilde{u}(t,x) = \frac{\sqrt{d_{11}+e_{11}}\exp(-(d_{11}+e_{11})t)}{\pi\sqrt{2(\exp(2(d_{11}+e_{11})t)-1)}} \times \exp\left(-\frac{d_1^2(\exp((d_{11}+e_{11})t)-1)}{(\exp((d_{11}+e_{11})t)+1)(d_{11}+e_{11})}\right) \times \exp\left(\frac{2d_1 x}{\exp((d_{11}+e_{11})t)+1} - \frac{(d_{11}+e_{11})x^2}{\exp(2(d_{11}+e_{11})t)-1}\right) \times \left\{\int_0^{\exp((d_{11}+e_{11})t)/2} \exp(-(d_{11}+e_{11})t)s \times \left[\exp\left(\frac{-2d_1 s}{\exp((d_{11}+e_{11})t)+1} + \frac{2(d_{11}+e_{11})xs}{\exp(2(d_{11}+e_{11})t)-1} - \frac{(d_{11}+e_{11})s^2}{\exp(2(d_{11}+e_{11})t)-1}\right)\right]ds + \int_{\exp((d_{11}+e_{11})t)/2}^{\exp((d_{11}+e_{11})t)} (1-\exp(-(d_{11}+e_{11})t)s) \times \left[\exp\left(\frac{-2d_1 s}{\exp((d_{11}+e_{11})t)+1}\right) \times \exp\left(\frac{2(d_{11}+e_{11})xs}{\exp(2(d_{11}+e_{11})t)-1}\right) \times \exp\left(-\frac{(d_{11}+e_{11})s^2}{\exp(2(d_{11}+e_{11})t)-1}\right)\right]ds\right\}. \quad (56)$$

We observe that

$$\begin{aligned}
&\int_0^1 s\exp(-as^2)\exp(-bs)\exp(cxs)\,ds \\
&= \int_{k_1}^{k_2} s\exp\left(-a\left[s+\frac{(b-cx)}{2a}\right]^2\right)ds\exp\left(\frac{(b-cx)^2}{4a}\right) \\
&= \exp\left(\frac{(b-cx)^2}{4a}\right)\int_{k_1}^{k_2}\left[s+\frac{(b-cx)}{2a}\right] \times \exp\left(-a\left[s+\frac{(b-cx)}{2a}\right]^2\right)ds \\
&\quad - \exp\left(\frac{(b-cx)^2}{4a}\right)\int_{k_1}^{k_2}\left[\frac{(b-cx)}{2a}\right] \times \exp\left(-a\left[s+\frac{(b-cx)}{2a}\right]^2\right)ds \\
&= \frac{1}{2a}\exp\left(\frac{(b-cx)^2}{4a}\right)\left[\exp\left(-a\left(k_1+\frac{b-cx}{2a}\right)^2\right) - \exp\left(-a\left(k_2+\frac{b-cx}{2a}\right)^2\right)\right] \\
&\quad - \exp\left(\frac{(b-cx)^2}{4a}\right)\left(\frac{b-cx}{2a}\right) \times \int_{k_1+((b-cx)/2a)}^{k_2+((b-cx)/2a)} \exp(-as^2)\,ds \\
&= \frac{1}{2a}[\exp(-ak_1^2-bk_1+ck_1x)-\exp(-ak_2^2-bk_2+ck_2x)] \\
&\quad - \exp\left(\frac{(b-cx)^2}{4a}\right)\left(\frac{b-cx}{2a}\right)\int_{k_1+((b-cx)/2a)}^{k_2+((b-cx)/2a)} \exp(-as^2)\,ds.
\end{aligned} \quad (57)$$

Applying (57) to (56) with

$$a = \frac{(d_{11}+e_{11})}{\exp(2(d_{11}+e_{11})t)-1},$$

$$b = \frac{2d_1}{\exp((d_{11}+e_{11})t)+1}, \qquad \text{and}$$

$$c = 2\frac{(d_{11}+e_{11})}{\exp(2(d_{11}+e_{11})t)-1},$$

and replacing the corresponding value of k_1 and k_2, we get

$$\tilde{u}(t,x) = \frac{\sqrt{d_{11}+e_{11}}\exp(-(d_{11}+e_{11})t)}{\sqrt{\pi(\exp(2(d_{11}+e_{11})t)-1)}} \times \exp\left(-\frac{d_1^2(\exp((d_{11}+e_{11})t)-1)}{(\exp((d_{11}+e_{11})t)+1)(d_{11}+e_{11})}\right) \times \exp\left(\frac{2d_1 x}{\exp((d_{11}+e_{11})t)+1} - \frac{(d_{11}+e_{11})x^2}{\exp(2(d_{11}+e_{11})t)-1}\right)$$

$$
\begin{aligned}
&\times \Bigg\{ \frac{(\exp(2(d_{11}+e_{11})t)-1)\exp(-(d_{11}+e_{11})t)}{2(d_{11}+e_{11})} \\
&\Bigg[1+\exp\Bigg(-\frac{(d_{11}+e_{11})\exp(2(d_{11}+e_{11})t)}{\exp(2(d_{11}+e_{11})t)-1} \\
&\qquad -\frac{2d_1\exp((d_{11}+e_{11})t)}{\exp((d_{11}+e_{11})t)+1} \\
&\qquad +2x\frac{(d_{11}+e_{11})\exp((d_{11}+e_{11})t)}{\exp(2(d_{11}+e_{11})t)-1}\Bigg) \\
&-2\exp\Bigg(-\frac{(d_{11}+e_{11})\exp(2(d_{11}+e_{11})t)}{4(\exp(2(d_{11}+e_{11})t)-1)} \\
&\qquad -\frac{d_1\exp((d_{11}+e_{11})t)}{\exp((d_{11}+e_{11})t)+1} \\
&\qquad +x\frac{(d_{11}+e_{11})\exp((d_{11}+e_{11})t)}{\exp(2(d_{11}+e_{11})t)-1}\Bigg)\Bigg] \\
&-\frac{d_1(\exp((d_{11}+e^{11})t)-1)-(d_{11}+e_{11})x}{(d_{11}+e_{11})} \\
&\times\exp\left(\frac{\left(\frac{2d_1}{\exp((d_{11}+e_{11})t)+1}-\frac{2(d_{11}+e_{11})x}{\exp(2(d_{11}+e_{11})t)-1}\right)^2}{4\frac{d_{11}+e_{11}}{\exp(2(d_{11}+e_{11})t)-1}}\right) \\
&\times\exp(-(d_{11}+e_{11})t) \\
&\times\Bigg[\int_{(d_1(\exp((d_{11}+e_{11})t)-1)-(d_{11}+e_{11})x)/(d_{11}+e_{11})}^{(\exp((d_{11}+e_{11})t)/2)+(d_1(\exp((d_{11}+e_{11})t)-1)-(d_{11}+e_{11})x)/(d_{11}+e_{11})} \\
&\qquad\times\exp\left(-\frac{(d_{11}+e_{11})}{\exp(2(d_{11}+e_{11})t)-1}s^2\right)ds \\
&-\int_{(\exp((d_{11}+e_{11})t)/2)+(d_1(\exp((d_{11}+e_{11})t)-1)-(d_{11}+e_{11})x)/(d_{11}+e_{11})}^{\exp((d_{11}+e_{11})t)+(d_1(\exp((d_{11}+e_{11})t)-1)-(d_{11}+e_{11})x)/(d_{11}+e_{11})} \\
&\qquad\times\exp\left(-\frac{(d_{11}+e_{11})}{\exp(2(d_{11}+e_{11})t)-1}s^2\right)ds\Bigg] \\
&+\exp\left(\frac{\left(\frac{2d_1}{\exp(d_{11}+e_{11})t)+1}-\frac{2(d_{11}+e_{11})x}{\exp(2(d_{11}+e_{11})t)-1}\right)^2}{4\frac{d_{11}+e_{11}}{\exp(2(d_{11}+e_{11})t)-1}}\right) \\
&\times\int_{(\exp((d_{11}+e_{11})t)/2)+(d_1(\exp((d_{11}+e_{11})t)-1)-(d_{11}+e_{11})x)/(d_{11}+e_{11})}^{\exp((d_{11}+e_{11})t)+(d_1(\exp((d_{11}+e_{11})t)-1)-(d_{11}+e_{11})x)/(d_{11}+e_{11})} \\
&\times\exp\left(-\frac{(d_{11}+e_{11})}{\exp(2(d_{11}+e_{11})t)-1}s^2\right)ds\Bigg\}.
\end{aligned}
\tag{58}
$$

We draw the graph of this solution in Fig. 1.

EXAMPLE 2 Consider the problem

$$
\begin{cases}
\dfrac{\partial\tilde{u}}{\partial t}=\dfrac{1}{2}\left(\dfrac{\partial^2\tilde{u}}{\partial x_1^2}+\dfrac{\partial^2\tilde{u}}{\partial x_2^2}\right) \\
\qquad -((d_{11}+e_{11})x_1+(d_{12}+e_{12})x_2+d_1)\dfrac{\partial\tilde{u}}{\partial x_1} \\
\qquad -((d_{21}+e_{21})x_1+(d_{22}+e_{22})x_2+d_2)\dfrac{\partial\tilde{u}}{\partial x_2} \\
\qquad -(d_{11}+e_{11}+d_{22}+e_{22})\tilde{u} \\
\tilde{u}(0,x)=\phi(x).
\end{cases}
\tag{59}
$$

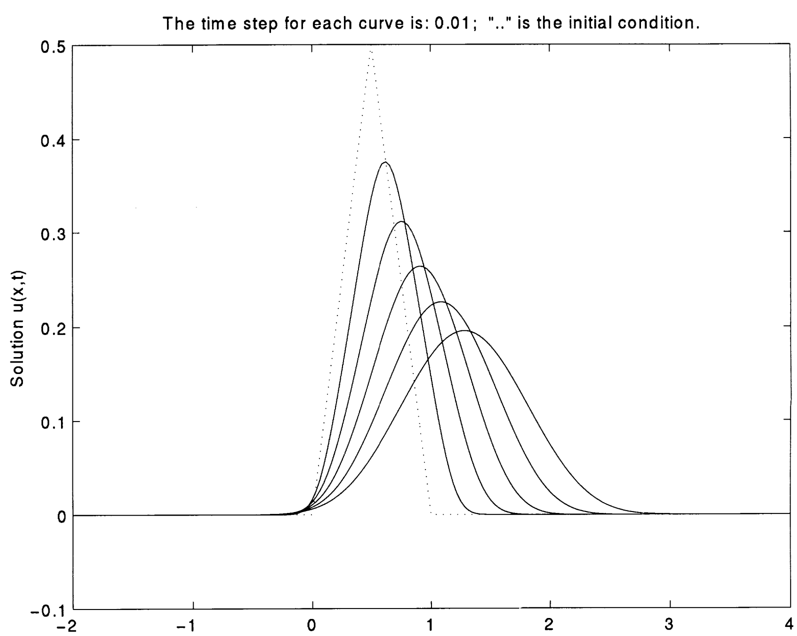

Fig. 1. Approximation to Kolmogorov equation with Fourier.

Let

$$A = \begin{pmatrix} d_{11}+e_{11} & d_{21}+e_{21} \\ d_{12}+e_{12} & d_{22}+e_{22} \end{pmatrix}.$$

Then $B(t) = e^{-At}$ and $B^{-1}(t) = e^{At}$. So $B^{-1}(0) = I \cdot \det(B^{-1}(0)B(t)^T) = \det(e^{-A^T t})$. To get a simple formula, we assume from now on that $A + A^T$ and A are nonsingular

$$\begin{aligned}
\gamma(t,x) &= \int_{\infty}^{\infty}\int_{\infty}^{\infty} \exp\left(-2\pi^2 \int_0^t (e^{-As}\cdot e^{At}z)^T \cdot (e^{-As}\cdot e^{At}z)\,ds\right) \\
&\quad \times \exp\left(\int_0^t -2\pi\sqrt{-1}D^T \cdot e^{-As}\cdot e^{At}\cdot z\,ds\right) \\
&\quad \times \exp(2\pi\sqrt{-1}x^T\cdot z)\,dz_1\,dz_2 \\
&= \int_{\infty}^{\infty}\int_{\infty}^{\infty} \exp\left(-2\pi^2\int_0^t z^T e^{(A+A^T)(t-s)}z\,ds\right) \\
&\quad \times \exp\left(\int_0^t -2\pi\sqrt{-1}D^T\cdot e^{A(t-s)}\cdot z\,ds\right) \\
&\quad \times \exp(2\pi\sqrt{-1}x^T\cdot z)\,dz_1\,dz_2 \\
&= \int_{\infty}^{\infty}\int_{\infty}^{\infty} \exp(-2\pi^2 z^T(A+A^T)^{-1}[e^{(A+A^T)t}-E]z) \\
&\quad \times \exp(-2\pi\sqrt{-1}D^T A^{-1}[e^{At}-E]z) \\
&\quad \times \exp(2\pi\sqrt{-1}x^T\cdot z)\,dz_1\,dz_2. \qquad (60)
\end{aligned}$$

Because $A + A^T$ is symmetric and nonsingular, we can find an orthogonal matrix

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

such that

$$A + A^T = P^T \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P. \qquad (61)$$

Then

$$(A+A^T)^{-1} = P^T \begin{bmatrix} 1/\lambda_1 & 0 \\ 0 & 1/\lambda_2 \end{bmatrix} P \qquad (62)$$

or

$$(A+A^T)^{-1}P^T = P^T \begin{bmatrix} 1/\lambda_1 & 0 \\ 0 & 1/\lambda_2 \end{bmatrix}. \qquad (63)$$

So

$$\begin{aligned}
z^T&(A+A^T)^{-1}[e^{(A+A^T)t}-E]z \\
&= z^T(A+A^T)^{-1}P^T \begin{bmatrix} e^{\lambda_1 t}-1 & 0 \\ 0 & e^{\lambda_2 t}-1 \end{bmatrix} Pz \\
&= z^T P^T \begin{bmatrix} 1/\lambda_1 & 0 \\ 0 & 1/\lambda_2 \end{bmatrix}\begin{bmatrix} e^{\lambda_1 t}-1 & 0 \\ 0 & e^{\lambda_2 t}-1 \end{bmatrix} Pz \\
&= (Pz)^T \begin{bmatrix} \dfrac{e^{\lambda_1 t}-1}{\lambda_1} & 0 \\ 0 & \dfrac{e^{\lambda_2 t}-1}{\lambda_2} \end{bmatrix} Pz. \qquad (64)
\end{aligned}$$

Make a variable transform $\omega = Pz$ in (60). Then we have

$$\begin{aligned}
\gamma(t,x) &= \det(P^T)\int_{\infty}^{\infty}\int_{\infty}^{\infty} \\
&\quad \times \exp\left(-2\pi^2\omega^T \begin{bmatrix} \dfrac{e^{\lambda_1 t}-1}{\lambda_1} & 0 \\ 0 & \dfrac{e^{\lambda_2 t}-1}{\lambda_2} \end{bmatrix}\omega\right) \\
&\quad \times \exp(-2\pi\sqrt{-1}D^T A^{-1}[e^{At}-E]P^T\omega) \\
&\quad \times \exp(2\pi\sqrt{-1}x^T P^T\omega)\,d\omega_1\,d\omega_2. \qquad (65)
\end{aligned}$$

Let

$$D^T A^{-1}[e^{At}-E]P^T = \Phi = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \qquad \text{and}$$

$$Px = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} p_{11}x_1 + p_{12}x_2 \\ p_{21}x_1 + p_{22}x_2 \end{bmatrix}.$$

Then

$$\begin{aligned}
\gamma(t,x) &= \int_{\infty}^{\infty}\int_{\infty}^{\infty} \exp\left(-2\pi^2\left(\frac{e^{\lambda_1 t}-1}{\lambda_1}\omega_1^2 + \frac{e^{\lambda_2 t}-1}{\lambda_2}\omega_2^2\right)\right) \\
&\quad \times \exp(-2\pi\sqrt{-1}(\mu_1\omega_1 + \mu_2\omega_2)) \\
&\quad \times \exp(2\pi\sqrt{-1}(\tilde{x}_1\omega_1 + \tilde{x}_2\omega_2))\,d\omega_1\,d\omega_2 \\
&= \int_{\infty}^{\infty} \exp\left(-2\pi^2\frac{e^{\lambda_1 t}-1}{\lambda_1}\omega_1^2\right)\exp(-2\pi\sqrt{-1}\mu_1\omega_1) \\
&\quad \times \exp(2\pi\sqrt{-1}\tilde{x}_1\omega_1)\,d\omega_1 \int_{\infty}^{\infty}\exp\left(-2\pi^2\frac{e^{\lambda_2 t}-1}{\lambda_2}\omega_2^2\right) \\
&\quad \times \exp(-2\pi\sqrt{-1}\mu_2\omega_2)\exp(2\pi\sqrt{-1}\tilde{x}_2\omega_2)\,d\omega_2. \qquad (66)
\end{aligned}$$

We need to calculate the integral

$$\int_{-\infty}^{\infty}\exp(-2\pi^2 h\omega^2)\exp(-2\pi\sqrt{-1}\mu\omega)\exp(2\pi\sqrt{-1}x\omega)\,d\omega.$$

We can derive a general formula for

$$\begin{aligned}
\zeta(\omega,a,b,c) &= \int_{-\infty}^{\infty}\exp(-a\omega^2)\exp(-\sqrt{-1}b\omega) \\
&\quad \times \exp(\sqrt{-1}c\omega)\,d\omega. \\
\frac{\partial\zeta}{\partial c} &= \int_{-\infty}^{\infty}\sqrt{-1}\omega\exp(-a\omega^2) \\
&\quad \times \exp(-\sqrt{-1}b\omega)\exp(\sqrt{-1}c\omega)\,d\omega \\
&= \int_{-\infty}^{\infty} -\frac{\sqrt{-1}}{2a}\frac{\partial}{\partial\omega}(\exp(-a\omega^2)) \\
&\quad \times \exp(-\sqrt{-1}b\omega)\exp(\sqrt{-1}c\omega)\,d\omega \\
&= \int_{-\infty}^{\infty}\frac{\sqrt{-1}}{2a}(-\sqrt{-1}b + \sqrt{-1}c)\exp(-a\omega^2) \\
&\quad \times \exp(-\sqrt{-1}b\omega)\exp(\sqrt{-1}c\omega)\,d\omega \\
&= \left(\frac{b}{2a} - \frac{c}{2a}\right)\int_{-\infty}^{\infty}\exp(-a\omega^2) \\
&\quad \times \exp(-\sqrt{-1}b\omega)\exp(\sqrt{-1}c\omega)\,d\omega. \qquad (67)
\end{aligned}$$

So

$$\zeta(\omega,a,b,c) = \exp\left(\frac{bc}{2a} - \frac{c^2}{4a}\right)\zeta(\omega,a,b,0). \quad (68)$$

But

$$\frac{\partial\zeta(\omega,a,b,0)}{\partial b} = \int_{-\infty}^{\infty} -\sqrt{-1}\omega\exp(-a\omega^2)\exp(-\sqrt{-1}b\omega)\,d\omega$$

$$= \int_{-\infty}^{\infty} \frac{\sqrt{-1}}{2a}\frac{\partial}{\partial\omega}(\exp(-a\omega^2))\exp(-\sqrt{-1}b\omega)\,d\omega$$

$$= \int_{-\infty}^{\infty}\left(-\frac{\sqrt{-1}}{2a}\right)(-\sqrt{-1}b)$$

$$\times\exp(-a\omega^2)\exp(-\sqrt{-1}b\omega)\,d\omega$$

$$= -\frac{b}{2a}\int_{-\infty}^{\infty}\exp(-a\omega^2)\exp(-\sqrt{-1}b\omega)\,d\omega. \quad (69)$$

So

$$\zeta(\omega,a,b,0) = \exp\left(-\frac{b^2}{4a}\right)\zeta(\omega,a,0,0). \quad (70)$$

But

$$\zeta(\omega,a,0,0) = \int_{-\infty}^{\infty}\exp(-a\omega^2)\,d\omega = \sqrt{\frac{\pi}{a}}. \quad (71)$$

Hence,

$$\zeta(\omega,a,b,c) = \sqrt{\frac{\pi}{a}}\exp\left(\frac{bc}{2a} - \frac{c^2}{4a}\right)\exp\left(-\frac{b^2}{4a}\right). \quad (72)$$

From (72), let $a = 2\pi^2 h$, $b = 2\pi\mu$, and $c = 2\pi x$. We can calculate the integral as follows:

$$\int_{-\infty}^{\infty}\exp(-2\pi^2 h\omega^2)\exp(-2\pi\sqrt{-1}\mu\omega)\exp(2\pi\sqrt{-1}x\omega)\,d\omega$$

$$= \sqrt{\frac{\pi}{2\pi^2 h}}\exp\left(\frac{2\pi\mu 2\pi x}{4\pi^2 h} - \frac{c^2}{8\pi^2 h}\right)\exp\left(-\frac{(2\pi\mu)^2}{8\pi^2 h}\right)$$

$$= \frac{\exp\left(\frac{\mu x}{h} - \frac{x^2}{2h}\right)\exp\left(-\frac{\mu^2}{2h}\right)}{\sqrt{2\pi h}}. \quad (73)$$

So

$$\int_{-\infty}^{\infty}\exp\left(-2\pi^2\frac{e^{\lambda_1 t}-1}{\lambda_1}\omega_1^2\right)\exp(-2\pi\sqrt{-1}\mu_1\omega_1)$$

$$\times\exp(2\pi\sqrt{-1}\tilde{x}_1\omega_1)\,d\omega_1$$

$$= \frac{\sqrt{\lambda_1}}{\sqrt{2\pi(e^{\lambda_1 t}-1)}}\exp\left(\frac{\lambda_1\mu_1\tilde{x}_1}{e^{\lambda_1 t}-1} - \frac{\lambda_1\tilde{x}_1^2}{2(e^{\lambda_1 t}-1)}\right)$$

$$\times\exp\left(-\frac{\mu_1^2\lambda_1}{2(e^{\lambda_1 t}-1)}\right)\int_{\infty}^{\infty}\exp\left(-2\pi^2\frac{e^{\lambda_2 t}-1}{\lambda_2}\omega_2^2\right)$$

$$\times\exp(-2\pi\sqrt{-1}\mu_2\omega_2)\exp(2\pi\sqrt{-1}\tilde{x}_2\omega_2)\,d\omega_2 \quad (74)$$

$$= \frac{\sqrt{\lambda_2}}{\sqrt{2\pi(e^{\lambda_2 t}-1)}}\exp\left(\frac{\lambda_2\mu_2\tilde{x}_2}{e^{\lambda_2 t}-1} - \frac{\lambda_2\tilde{x}_2^2}{2(e^{\lambda_2 t}-1)}\right)$$

$$\times\exp\left(-\frac{\mu_2^2\lambda_2}{2(e^{\lambda_2 t}-1)}\right) \quad (75)$$

and hence

$$\gamma(t,x) = \frac{\sqrt{\lambda_1\lambda_2}}{2\pi\sqrt{(e^{\lambda_1 t}-1)(e^{\lambda_2 t}-1)}}\exp\left(\frac{\lambda_1\mu_1\tilde{x}_1}{e^{\lambda_1 t}-1} - \frac{\lambda_1\tilde{x}_1^2}{2(e^{\lambda_1 t}-1)}\right)$$

$$\times\exp\left(-\frac{\mu_1^2\lambda_1}{2(e^{\lambda_1 t}-1)}\right)\exp\left(\frac{\lambda_2\mu_2\tilde{x}_2}{e^{\lambda_2 t}-1} - \frac{\lambda_2\tilde{x}_2^2}{2(e^{\lambda_2 t}-1)}\right)$$

$$\times\exp\left(-\frac{\mu_2^2\lambda_2}{2(e^{\lambda_2 t}-1)}\right)$$

$$= \frac{\sqrt{\lambda_1\lambda_2}}{2\pi\sqrt{(e^{\lambda_1 t}-1)(e^{\lambda_2 t}-1)}}$$

$$\times\exp\left(\frac{\lambda_1\mu_1(p_{11}x_1+p_{12}x_2)}{e^{\lambda_1 t}-1} - \frac{\lambda_1(p_{11}x_1+p_{12}x_2)^2}{2(e^{\lambda_1 t}-1)}\right)$$

$$\times\exp\left(-\frac{\mu_1^2\lambda_1}{2(e^{\lambda_1 t}-1)}\right)$$

$$\times\exp\left(\frac{\lambda_2\mu_2(p_{21}x_1+p_{22}x_2)}{e^{\lambda_2 t}-1} - \frac{\lambda_2(p_{21}x_1+p_{22}x_2)^2}{2(e^{\lambda_2 t}-1)}\right)$$

$$\times\exp\left(-\frac{\mu_2^2\lambda_2}{2(e^{\lambda_2 t}-1)}\right). \quad (76)$$

We now apply (48):

$$\tilde{u}(t,x) = \det(e^{-A^T t})\phi(e^{-A^T t}x)*\gamma(t,x)$$

$$= \det(e^{-A^T t})\frac{\sqrt{\lambda_1\lambda_2}}{2\pi\sqrt{(e^{\lambda_1 t}-1)(e^{\lambda_2 t}-1)}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(e^{-A^T t}s)$$

$$\times\exp\left(-\frac{\mu_1^2\lambda_1}{2(e^{\lambda_1 t}-1)}\right)$$

$$\times\exp\left(\frac{\lambda_1\mu_1[p_{11}(x_1-s_1)+p_{12}(x_2-s_2)]}{e^{\lambda_1 t}-1}\right.$$

$$\left. - \frac{\lambda_1[p_{11}(x_1-s_1)+p_{12}(x_2-s_2)]^2}{2(e^{\lambda_1 t}-1)}\right)$$

$$\times\exp\left(\frac{\lambda_2\mu_2[p_{21}(x_1-s_1)+p_{22}(x_2-s_2)]}{e^{\lambda_2 t}-1}\right.$$

$$\left. - \frac{\lambda_2[p_{21}(x_1-s_1)+p_{22}(x_2-s_2)]^2}{2(e^{\lambda_2 t}-1)}\right)$$

$$\times\exp\left(-\frac{\mu_2^2\lambda_2}{2(e^{\lambda_2 t}-1)}\right)ds_1\,ds_2 \quad (77)$$

where

$$s = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}.$$

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