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FINITE-DIMENSIONAL FILTERS WITH NONLINEAR DRIFT VII: MITTER CONJECTURE AND STRUCTURE OF η*

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Abstract. The concept of estimation algebra introduced independently by Brockett and Mitter has been playing a fundamental role in the investigation of finite-dimensional nonlinear filters. Mitter conjectured that the observation terms $h_i(x)$ are polynomials of degree one if the corresponding estimation algebra is finite dimensional. Chiou, Leung, and the present authors classify all finite-dimensional estimation algebra of maximal rank with dimension of the state space less than or equal to three. In this paper, we prove the Mitter conjecture for finite-dimensional estimation algebra of maximal rank with arbitrary state space dimension. In the course of our proof, we show that the $\Omega = (\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j})$ matrix, where f denotes the drift term, has special linear structure which generalizes our previous result in [J. Chen and S. S.-T. Yau, *Math. Control Signals Systems*, 9 (1996), to appear]. We also give a structure theorem for $\eta = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^n f_i^2 + \sum_{i=1}^m h_i^2$.

Key words. finite-dimensional nonlinear filter, Mitter conjecture, estimation algebras

AMS subject classifications. 17B30, 35J15, 60G35, 93E11

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1. Introduction.

The idea of using estimation algebras to construct finite-dimensional nonlinear filters was first proposed in Brockett and Clark [Br–Cl], Brockett [Br], and Mitter [Mi]. The concept of estimation algebras has proved to be an invaluable tool in the study of nonlinear filtering problems. In 1983, Brockett proposed classifying all finite-dimensional estimation algebras. As a first step to attack this problem, Mitter conjectured that the observation terms $h_i(x)$ are affine polynomials. In [Ch2–Ya], Chiou and Yau first introduced the concept of estimation algebra of maximal rank. The purpose of this paper is to show that Mitter conjecture is true for all finite-dimensional estimation algebras of maximal rank. In [Wo], the concept of Ω is introduced, defined as the matrix whose (i, j) entry is $\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$, where f is the drift term of the state evolution equation. Recently, Yau [Ya] has studied a filtering system such that all entries of Ω are constants. He was able to classify all finite-dimensional estimation algebras of maximal rank and proved Mitter conjecture for such a filtering system. If the dimension of the state space is two or three, then Chiou and Yau [Ch2–Ya] and Chen, Leung, and Yau [C–L–Y] have shown, respectively, that all entries of Ω are constants as long as the estimation algebra is of maximal rank and finite dimensional. Thus finite-dimensional estimation algebra of maximal rank is completely classified if the dimension of the state space is at most three.

In [Ch1–Ya], we have shown that Ω is an affine matrix in the sense that every entry in Ω is an affine polynomial if the estimation algebra is of maximal rank and finite-dimensional. This is a fundamental step in classifying finite-dimensional estimation of maximal rank. In fact we proved that Ω has a special affine structure. The purpose of this paper is to give affirmative solution to Mitter conjecture for finite-dimensional estimation algebra of maximal rank. The following is our main theorem.

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MAIN THEOREM. *Let E be a finite-dimensional estimation algebra of maximal rank. Let k be the quadratic rank of E (cf. Definition* 2.2 *below). Then*

(1) *the observation terms $h_i(x)$, $1 \le i \le m$, are affine polynomials.*

(2) (a) *ω_{ij}, for $1 \le i \le k$ or $1 \le j \le k$, are constants*

(b) *ω_{ij}, for $k+1 \le i, j \le n$, are degree-one polynomials in $x_{k+1}, \dots, x_n$.*

(3) *$\eta = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^n f_i^2 + \sum_{i=1}^m h_i^2$ is a homogeneous polynomial of degree four. Moreover, η_4 (homogeneous polynomial of degree-four part of η) depends only on $x_{k+1}, \dots, x_n$ variables.*

Notice that in our previous paper, we proved only that ω_{ij} for $1 \le i \le k$ or $1 \le j \le k$ are affine polynomials in $x_1, \dots, x_k$. It is precisely the improvement of the result on ω_{ij} for $1 \le i \le k$ or $1 \le j \le k$ that allows us to solve the Mitter conjecture affirmatively. This paper is, in essence, a continuation of [Ch1–Ya], and we strongly recommend that readers familiarize themselves with the results in [Ch1–Ya]. However, every effort will be made to make this paper as self-contained as possible without too much duplication of the previous paper.

2. Basic concepts. The filtering problem here is based on the following signal observation model:

$$\begin{cases} dx(t) = f\big(x(t)\big)dt + g\big(x(t)\big)dv(t), & x(0) = x_0, \\ dy(t) = h\big(x(t)\big)dt + dw(t), & y(0) = 0, \end{cases} \tag{2.1}$$

in which x, v, y, and w are, respectively, $\mathbb{R}^n$-, $\mathbb{R}^p$-, $\mathbb{R}^m$-, and $\mathbb{R}^m$-valued processes and v and w have components which are independent, standard Brownian process. We further assume that $n = p, f, h$ are C^∞ smooth and that g is an orthogonal matrix. We shall refer to $x(t)$ as the state of the system at time t and $y(t)$ as the observation at time t. $\rho(t,x)$, the conditional probability density of the state, $x(t)$, given the observation $\big\{y(s) : 0 \le s \le t\big\}$ is determined by the Duncan–Mortensen–Zakai equation, which in the unnormalized form is given by (see [Da–Ma], for example)

$$\frac{d}{dt}\sigma(t,x) = L_0\sigma(t,x)dt + \sum_{i=1}^m L_i\sigma(t,x)dy_i(t), \quad \sigma(0,x) = \sigma_0, \tag{2.2}$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^n f_i\frac{\partial}{\partial x_i} - \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^m h_i^2$$

and for $i = 1, \dots, m, L_i$ is the zero-degree differential operator of multiplication by h_i. (If a is a vector, we use the notation a_i to represent the ith component of a.) σ_0 is the probability density of the initial point x_0. When the observation is absent—that is, $h = 0$—then (2.2) is simply the Kolmogorov equation.

It is important to find efficient ways to solve (2.2), which is the subject of many research studies in nonlinear filtering theory. For this purpose, we need to introduce the following definition.

DEFINITION 2.1. *The estimation algebra E of a filtering system* (2.1) *is defined to be the Lie algebra generated by $\{L_0, L_1, \dots, L_m\}$. E is said to be an estimation algebra of maximal rank if for every $1 \le i \le n$ there exists a constant c_i such that $x_i + c_i$ is in E.*

In [Wo], the concept of Ω is introduced, defined as the matrix whose (i,j) element ω_{ij} is $\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$.

Define

$$D_i = \frac{\partial}{\partial x_i} - f_i$$

and

$$\eta = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{m} h_i^2.$$

Then

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^{n} D_i^2 - \eta\right).$$

Recall here that the assumption of maximal rank on E implies that E contains all affine polynomial and the operations $D_1, \dots, D_n$. This follows from the fact that $D_j = [L_0, x_j + c_j]$ and $1 = [D_j, x_j + c_j]$.

The following theorem proved in [Ya] plays a fundamental roles in Mitter conjecture as well as the classification of finite-dimensional estimation algebra.

THEOREM 2.1. *Let E be a finite-dimensional estimation algebra of* (2.1) *such that $\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$ are constant functions. Then $h_i(x)$, $1 \le i \le m$, are affine polynomials. If in addition, E is of maximal rank, then E is a real vector space of dimension $2n+2$ with basis given by $1, x_1, x_2, \dots, x_n, D_1, \dots, D_n$ and L_0.*

We need the following basic result [Oc] for later discussion.

THEOREM 2.2 (Ocone). *Let E be a finite-dimensional estimation algebra. If φ is a function in E, then φ is a polynomial of degree at most two.*

In our previous paper [Ch1–Ya], we have introduced the following important concepts and notations. Let Q be the space of quadratic forms in n variables, i.e., real vector space spanned by $x_i x_j$, with $1 \le i \le j \le n$. Let $X = (x_1, \dots, x_n)^T$. For any quadratic form $p \in Q$, there exists a symmetric matrix A such that $p(x) = X^T A X$. The rank of the quadratic form p is denoted by $\mathrm{rk}(p)$ and is defined to be the rank of the matrix A.

DEFINITION 2.2. *A fundamental quadratic form of the estimation algebra E is an element $p_0 \in E \cap Q$ with the greatest positive rank, i.e.,* $\mathrm{rk}(p_0) \ge \mathrm{rk}(p)$ *for any $p \in E \cap Q$. The quadratic rank of the estimation algebra E is defined to be* $\mathrm{rk}(p_0)$.

After an orthogonal transformation on x, p_0 can be written as

$$p_0(x) = c_1 x_1^2 + c_2 x_2^2 + \cdots + c_k x_k^2, \tag{2.3}$$

where $c_i \ne 0$, $1 \le i \le k$, and k is the quadratic rank of E. From $p_0(x)$, we can construct a sequence of quadratic forms in $E \cap Q$ as follows:

$$q_0 = p_0,$$
$$q_j = \big[[L_0, q_{j-1}], q_0\big] = \sum_{i=1}^{k} 4^j c_i^{j+1} x_i^2.$$

In view of the invertibility of the Vandermonde matrix, we can assume that

$$p_0(x) = x_1^2 + x_2^2 + \cdots + x_k^2 \in E. \tag{2.4}$$

The following results were proven in [Ch1–Ya].

LEMMA 2.3. *Let k be the quadratic rank of the estimation algebra E with fundamental quadratic form $p_0(x) = x_1^2 + \cdots + x_k^2$. Then $p(x)$ is independent of $x_{k+1}, x_{k+2}, \ldots, x_n$ for any quadratic form $p(x)$ in E.*

Let $p_1(x)$ be a quadratic form in E with least positive rank, i.e., $\mathrm{rk}(p_1) \leq \mathrm{rk}(q)$ for any $q(x) \in E \cap Q$. After an orthogonal transform on X which fixes $x_{k+1}, \ldots, x_n$ (i.e., an orthogonal transform on $x_1, x_2, \ldots, x_k$) and the Vandermonde matrix procedure as before, we can assume

$$p_1(x) = \sum_{i=1}^{k_1} x_i^2 \in E, \quad 1 \leq k_1 \leq k. \tag{2.5}$$

Note that the orthogonal transform on $x_1, \ldots, x_k$ leaves $p_0(x)$ invariant. By definition, $p_0(x) = \sum_{i=1}^{k} x_i^2$ has the greatest positive rank and $p_1(x) = \sum_{i=1}^{k_1} x_i^2$ has the least positive rank. Define

$$S_1 = \{1, 2, \ldots, k_1\} \subseteq S = \{1, 2, \ldots, k\}, \tag{2.6}$$

$$Q_1 = \text{ real vector space spanned by } \{x_i x_j : k_1 + 1 \leq i \leq j \leq k\} \subseteq Q. \tag{2.7}$$

If $k_1 < k$, then $Q_1 \cap E$ is a nontrivial space since $p_1(x) - p_0(x) \in E \cap Q_1$. In a procedure similar to that above, there exists

$$p_2(x) = \sum_{i=k_1+1}^{k_2} x_i^2 \in E \cap Q_1 \tag{2.8}$$

with the least positive rank in $E \cap Q_1$. By induction, we can construct a series of S_i, Q_i, and $p_i(x)$ such that

$$S_i = \{k_{i-1} + 1, \ldots, k_i\}, \quad k_0 = 0, \quad k_i \leq k, \tag{2.9}$$

$$Q_i = \text{ linear span } \{x_i x_j : k_i + 1 \leq i \leq j \leq k\}, \tag{2.10}$$

$$p_i(x) = \sum_{j=k_{i-1}+1}^{k_i} x_j^2 = \sum_{j \in S_i} x_j^2, \quad i > 0, \tag{2.11}$$

and $p_i(x)$ has the least rank in $E \cap Q_{i-1}$.

LEMMA 2.4. *If $p(x) \in E \cap Q$, then*

$$p(0, \ldots, 0, x_{k_{i-1}+1}, \ldots, x_{k_i}, 0, \ldots, 0) = \lambda p_i(x) \quad \textit{for} \quad i > 0,$$
$$p(x_1, \ldots, x_{k_{i-1}}, 0, \ldots, 0, x_{k_i+1}, \ldots, x_k) \in E \quad \textit{for} \quad i > 0.$$

PROPOSITION 2.5. *Suppose that $p(x)$ is a quadratic form in E of the following form, which depends on $\{x_i \ i \in S_{\ell_1} \cup S_{\ell_2}\}$:*

$$p(x) = \left(X_{\ell_1}^T, X_{\ell_2}^T\right) \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} X_{\ell_1} \\ X_{\ell_2} \end{pmatrix},$$

where $X_i = \left(x_{k_{i-1}+1}, \ldots, x_{k_i}\right)^T$, i.e., $p(x) = \sum_{i \in S_{\ell_1}} \sum_{j \in S_{\ell_2}} 2a_{ij} x_i x_j$. Suppose that $\ell_1 < \ell_2$. Then $|S_{\ell_1}| = |S_{\ell_2}|$ and $A = bT$, where b is a constant and T is an orthogonal matrix.

THEOREM 2.6. *Suppose that E is a finite-dimensional estimation algebra of maximal rank. Then $\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$, $1 \le i, j \le n$, are polynomials of degree at most one.*

Notation. From now on we shall write $\omega_{ij} = \beta_{ij} + \gamma_{ij}$, where β_{ij} is the linear part of ω_{ij} while γ_{ij} is the constant part of ω_{ij}.

PROPOSITION 2.7. *Suppose that E is a finite-dimensional estimation algebra of maximal rank. With the same notation as before, if $\ell_1 \neq \ell_2$ and $i \in S_{\ell_1}$, $j \in S_{\ell_2}$, then $\beta_{ij} = 0$; i.e., ω_{ij} is a constant.*

THEOREM 2.8. *Suppose that E is a finite-dimensional estimation algebra of maximal rank. Let k be the quadratic rank of E. With the same notation as before, then*

(i) *for $j \in S_\ell$*

$$\begin{pmatrix} \beta_{jk_{\ell-1}+1} \\ \vdots \\ \beta_{jk_\ell} \end{pmatrix} = A_1^{j,\ell} X_\ell \qquad \text{with} \quad A_1^{j,\ell} = -(A_1^{j,\ell})^T,$$

where $X_\ell = \left(x_{k_{\ell-1}+1}, \dots, x_{k_\ell}\right)$ and $A_1^{j,\ell}$ is a $(k_\ell - k_{\ell-1}) \times (k_\ell - k_{\ell-1})$ matrix;

(ii) *for $j > k$*

$$\begin{pmatrix} \beta_{jk_{\ell-1}+1} \\ \vdots \\ \beta_{jk_\ell} \end{pmatrix} = \lambda_{j,\ell} X_\ell + A_2^{j,\ell} \widetilde{X_\ell},$$

where $\widetilde{X_\ell}$ denote the complementary variable vector of X_ℓ in $(x_1, \dots, x_k)^T$, i.e., $\widetilde{X_\ell} = \left(x_1, \dots, x_{k_{\ell-1}}, x_{k_\ell+1}, \dots, x_k\right)^T$, and $A_2^{j,\ell}$ is a $k_\ell \times (k - k_\ell)$ matrix.

THEOREM 2.9. *Suppose that E is a finite-dimensional estimation algebra of maximal rank. With the same notation as before, then $A_1^{j,\ell} = 0$ in* (i) *of Theorem* 2.8. *This means that $\beta_{ij} = 0$ for $i, j \in S_\ell = \left\{k_{\ell-1}+1, \dots, k_\ell\right\}$; i.e., $\omega_{ij} =$* constant *for $i, j \in S_\ell$.*

PROPOSITION 2.10. *Suppose that E is a finite-dimensional estimation algebra of maximal rank. Then*

(i) *ω_{ij} is a degree-one polynomial in $x_1, \dots, x_k$ for $1 \le i \le k$ or $1 \le j \le k$;*

(ii) *ω_{ij} is a degree-one polynomial in $x_{k+1}, \dots, x_n$ for $k+1 \le i, j \le n$.*

It follows from Theorem 2.6, Proposition 2.7, Theorem 2.9, and Proposition 2.10 that we have the following theorem.

THEOREM 2.11. *Let E be a finite-dimensional estimation algebra of maximal rank and k be the quadratic rank of E. Then all the entries $\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$ of Ω are degree-one polynomials. In fact, for $1 \le i, j \le k$, ω_{ij} are constants; for $1 \le i \le k$ or $1 \le j \le k$, ω_{ij} are degree-one polynomials in $x_1, \dots, x_k$; and for $k+1 \le i, j \le n$, ω_{ij} are degree-one polynomials in $x_{k+1}, \dots, x_n$.*

For the sake of convenience to the readers, we also provide the following lemma without proof. The proof can be found in [Ya].

LEMMA 2.12. (i) *$[XY, Z] = X[Y, Z] + [X, Z]Y$, where X, Y, and Z are differential operators.*

(ii) *$[gD_i, h] = g\frac{\partial h}{\partial x_i}$, where $D_i = \frac{\partial}{\partial x_i} - f_i$ and g and h are functions defined on $\mathbb{R}^n$.*

(iii) *$[gD_i, hD_j] = -gh\omega_{ij} + g\frac{\partial h}{\partial x_i} D_j - h\frac{\partial g}{\partial x_j} D_i$, where $\omega_{ji} = [D_i, D_j] = \frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i}$.*

(iv) *$[gD_i^2, h] = 2g\frac{\partial h}{\partial x_i} D_i + g\frac{\partial^2 h}{\partial x_i^2}$.*

(v) $[D_i^2, hD_j] = 2\frac{\partial h}{\partial x_i} D_i D_j - 2h\omega_{ij} D_i + \frac{\partial^2 h}{\partial x_i^2} D_j - h\frac{\partial \omega_{ij}}{\partial x_i}$.

(vi) $[D_i^2, D_j^2] = 4\omega_{ji} D_j D_i + 2\frac{\partial \omega_{ji}}{\partial x_j} D_i + 2\frac{\partial \omega_{ji}}{\partial x_i} D_j + \frac{\partial^2 \omega_{ji}}{\partial x_i \partial x_j} + 2\omega_{ji}{}^2$.

(vii) $[D_k^2, hD_iD_j] = 2\frac{\partial h}{\partial x_k} D_k D_i D_j + 2h\omega_{jk} D_i D_k + 2h\omega_{ik} D_k D_j + \frac{\partial^2 h}{\partial x_k^2} D_i D_j + 2h\frac{\partial \omega_{jk}}{\partial x_i} D_k + h\frac{\partial \omega_{jk}}{\partial x_k} D_i + h\frac{\partial \omega_{ik}}{\partial x_k} D_j + h\frac{\partial^2 \omega_{jk}}{\partial x_i \partial x_k}$.

(viii) $[gD_iD_j, hD_k] = g\frac{\partial h}{\partial x_j} D_i D_k + g\frac{\partial h}{\partial x_i} D_j D_k + gh\omega_{kj} D_i + gh\omega_{ki} D_j + g\frac{\partial^2 h}{\partial x_i \partial x_j} D_k + gh\frac{\partial \omega_{kj}}{\partial x_i} - h\frac{\partial g}{\partial x_k} D_i D_j$.

3. Structure of second-order operators and special linear structure of Ω. Building on our previous results in [Ch1–Ya], in this section, we shall prove that Ω has very special linear structure. It is exactly this result which allows us to prove Mitter conjecture for maximal rank finite-dimensional estimation algebra. To begin with, we need some results on second-order operator in E.

Consider the polynomial algebra $C^\infty(\mathbb{R}^n)[D_1, \dots, D_n]$ in variables $D_1, \dots, D_n$ with coefficients in $C^\infty(\mathbb{R}^n)$ modulo the relations $D_iD_j = D_jD_i + \omega_{ji}$ and $D_i a = aD_i + \frac{\partial a}{\partial x_i}$, where $C^\infty(\mathbb{R}^n)$ is the ring of all C^∞ functions on $\mathbb{R}^n$ and a is a C^∞ function. Every element $A \in C^\infty(\mathbb{R}^n)[D_1, \dots, D_n]$ has a representation in the following:

$$A = \sum a_{i_1 \cdots i_n}(x) D_1^{i_1} \cdots D_n^{i_n}, \tag{3.1}$$

i.e., a polynomial in D_i with coefficients in $C^\infty(\mathbb{R}^n)$. Since $D_iD_j = D_jD_i + \omega_{ji}$, $C^\infty(\mathbb{R}^n)[D_1, \dots, D_n]$ is not a commutative algebra. It is clear that every element of E has a representation of (3.1). For any $A \in E$, let P_A be the principal part of A, i.e., the highest homogeneous part of A in $D_1, \dots, D_n$. For example, for $L_0 \in E$, $P_{L_0} = \frac{1}{2}(D_1^2 + \cdots + D_n^2)$.

Lemma 3.1. *If $\frac{\partial a}{\partial x_j}(x) \neq 0$, then the principal part of $\left[D_j^\ell, a(x)D_1^{i_1} \cdots D_n^{i_n}\right]$ is given by*

$$P_{\left[D_j^\ell, a(x)D_1^{i_1} \cdots D_n^{i_n}\right]} = \ell \frac{\partial a}{\partial x_j}(x) D_1^{i_1} \cdots D_j^{i_j + \ell - 1} \cdots D_n^{i_n}. \tag{3.2}$$

Proof. If $\ell = 1$, then (3.2) is trivial. Suppose that (3.2) is true for $\ell - 1$; then

$$\begin{aligned} &\left[D_j^\ell, a(x)D_1^{i_1} \cdots D_n^{i_n}\right] \\ =&D_j\left[D_j^{\ell-1}, a(x)D_1^{i_1} \cdots D_n^{i_n}\right] + \left[D_j, a(x)D_1^{i_1} \cdots D_n^{i_n}\right] D_j^{\ell-1}. \end{aligned} \tag{3.3}$$

Hence

$$\begin{aligned} P_{\left[D_j^\ell, a(x)D_1^{i_1} \cdots D_n^{i_n}\right]} =& P_{D_j\left[D_j^{\ell-1}, a(x)D_1^{i_1} \cdots D_n^{i_n}\right]} + P_{\left[D_j, a(x)D_1^{i_1} \cdots D_n^{i_n}\right] D_j^{\ell-1}} \\ =&(\ell - 1)\frac{\partial a}{\partial x_j}(x) D_1^{i_1} \cdots D_j^{i_j + \ell - 1} \cdots D_n^{i_n} \\ &+ \frac{\partial a}{\partial x_j}(x) D_1^{i_1} \cdots D_j^{i_j + \ell - 1} \cdots D_n^{i_n} \\ =&\ell \frac{\partial a}{\partial x_j}(x) D_1^{i_1} \cdots D_j^{i_j + \ell - 1} \cdots D_n^{i_n}. \qquad \square \end{aligned}$$

Proposition 3.2. *If the principal part of A is of the form*

$$P_A = \sum a_{i_1 \cdots i_n}(x) D_1^{i_1} \cdots D_n^{i_n},$$

where for some $(i_1, \dots, i_n)$ *and* j, $\frac{\partial a_{i_1\cdots i_n}}{\partial x_j}(x) \neq 0$, *then the principal part of* $[L_0, A]$ *is given by*

$$P_{[L_0,A]} = \sum_j \sum_{i_1\cdots i_n} \frac{\partial a_{i_1\cdots i_n}}{\partial x_j}(x) D_1^{i_1} \cdots D_j^{i_j+1} \cdots D_n^{i_n}.$$

Proof. The proposition follows immediately when Lemma 3.1 is applied repeatedly. ∎

THEOREM 3.3. *Let* E *be a finite-dimensional estimation algebra. Suppose that* A *is a second order in* E *with principal part* $P_A = \sum_{i\leq j} a_{ij}(x) D_i D_j$. *Then* $\frac{\partial a_{ii}}{\partial x_i}(x) = 0$.

Proof. Without loss of generality, we can assume $i = 1$. a_{11} must be a polynomial; otherwise E would be infinite dimensional. If $\frac{\partial a_{11}}{\partial x_1}(x) \neq 0$, then there exists a positive integer ℓ such that

$$\frac{\partial^{\ell+1} a_{11}}{\partial x_1^{\ell+1}}(x) = 0 \quad \text{and} \quad \frac{\partial^{\ell} a_{11}}{\partial x_1^{\ell}}(x) \neq 0.$$

This is true because $a_{11}(x)$ is a polynomial in view of a result of [Wo]. By Proposition 3.2, we have

$$P_{[L_0,A]} = \frac{\partial a_{11}}{\partial x_1}(x) D_1^3 + \text{lower-order term in } D_1.$$

Hence if we let $Ad_{L_0} A = [L_0, A]$ and $Ad_{L_0}^m A = \left[L_0, Ad_{L_0}^{m-1} A\right]$, then

$$P_{Ad_{L_0}^{\ell} A} = \frac{\partial^{\ell} a_{11}}{\partial x_1^{\ell}}(x) D_1^{\ell+2} + \text{lower-order term in } D_1.$$

Let $B = Ad_{L_0}^{\ell-1} A$ (for $\ell = 1$, take $B = A$). Then

$$\begin{aligned} P_{Ad_B^s L_0} =& (-1)^s (\ell+1)(\ell+2)(2\ell+2)(3\ell+2)\cdots((s-1)\ell+2) \left(\frac{\partial^{\ell} a_{11}}{\partial x_1^{\ell}}(x)\right)^s D_1^{s\ell+2} \\ &+ \text{lower-order term in } D_1. \end{aligned}$$

We have produced an infinite sequence of independent elements $\left\{Ad_B^s L_0 : s = 1, 2, \dots\right\}$ in E. This contradicts our assumption that E is finite dimensional. ∎

THEOREM 3.4. *Let* E *be a finite-dimensional estimation algebra and* A *be an element in* E *with principal part of* $P_A = \sum_{i\leq j} a_{ij}(x) D_i D_j$. *Suppose that* $\frac{\partial a_{ii}}{\partial x_j}(x) = \frac{\partial a_{jj}}{\partial x_i}(x) = 0$. *Then* $\frac{\partial a_{ij}}{\partial x_i}(x) = \frac{\partial a_{ij}}{\partial x_j}(x) = 0$.

Proof. Without loss of generality, we shall assume $i = 1$ and $j = 2$. Suppose to the contrary that either $\frac{\partial a_{12}}{\partial x_1}(x) \neq 0$ or $\frac{\partial a_{12}}{\partial x_2}(x) \neq 0$. Then

$$P_{[L_0,A]} = \frac{\partial a_{12}}{\partial x_1}(x) D_1^2 D_2 + \frac{\partial a_{12}}{\partial x_2}(x) D_1 D_2^2 + \text{ terms in degree } D_1, D_2 \text{ lower than 3.}$$

If $\frac{\partial a_{12}}{\partial x_1}(x) \neq 0$, then there exists a positive integer ℓ such that

$$\frac{\partial^{\ell} a_{12}}{\partial x_1^{\ell}}(x) \neq 0 \quad \text{and} \quad \frac{\partial^{\ell+1} a_{12}}{\partial x_1^{\ell+1}}(x) = 0.$$

Let $B = Ad_{L_0}^{\ell-1} A \in E$. Then

$$P_B = \frac{\partial^{\ell-1} a_{12}}{\partial x_1^{\ell-1}}(x) D_1^{\ell} D_2 + \text{other terms}.$$

Hence

$$P_{Ad_B^s L_0} = a\Big(\frac{\partial^\ell a_{12}}{\partial x_1^\ell}(x)\Big)^s D_1^{\ell+s} D_2 + \text{other terms},$$

where a is a nonzero constant. So we have produced an infinite sequence $\{Ad_B^s L_0 \in E : s = 1, 2, \dots\}$ of linearly independent elements in E. This contradicts to our hypothesis that $\dim E < \infty$. Therefore we conclude that $\frac{\partial a_{12}}{\partial x_1}(x) = 0$. Similarly, we can prove $\frac{\partial a_{12}}{\partial x_2}(x) = 0$. $\square$

We are now ready to prove the special linear structure of Ω.

THEOREM 3.5. *Suppose that E is a finite-dimensional estimation algebra of maximal rank. Let k be the quadratic rank of E. With the same notation as before let β_{ij} be the linear part of ω_{ij}. Then*

$$\begin{pmatrix} \beta_{jk_{\ell-1}+1} \\ \vdots \\ \beta_{jk_\ell} \end{pmatrix} = A_2^{j,\ell} \widetilde{X_\ell} \qquad \textit{for} \ \ j > k,$$

where $\widetilde{X_\ell} = \big(x_1, \dots, x_{k_{\ell-1}}, x_{k_\ell+1}, \dots, x_k\big)^T$ and $A_2^{j,\ell}$ is a $k_\ell \times (k - k_\ell)$ matrix.

Proof. In view of part (ii) of Theorem 2.8, we have, for $j > k$,

$$\begin{pmatrix} \beta_{jk_{\ell-1}+1} \\ \vdots \\ \beta_{jk_\ell} \end{pmatrix} = \lambda_{j,\ell} X_\ell + A_2^{j,\ell} \widetilde{X_\ell},$$

where $X_\ell = \big(x_{k_{\ell-1}+1}, \dots, x_{k_\ell}\big)^T$. To prove the theorem, we need to prove $\lambda_{j,\ell} = 0$. For this purpose, we need only to prove that $\omega_{mk_{\ell-1}+1}$ does not depend on $x_{k_{\ell-1}+1}$. Since $p_\ell(x) = x_{k_{\ell-1}+1}^2 + \cdots + x_{k_\ell}^2 = \sum_{j\in S_\ell} x_j^2 \in E$, we have

$$\big[L_0, p_\ell\big] \in E \Longrightarrow \sum_{j\in S_\ell} x_j D_j \in E,$$

$$\begin{aligned}\Big[L_0, \sum_{j\in S_\ell} x_j D_j\Big] =& \frac{1}{2}\sum_{i=1}^n \sum_{j\in S_\ell} \big[D_i^2, x_j D_j\big] - \frac{1}{2} E_{k_\ell}(\eta) \\ =& \sum_{i=1}^n \sum_{j\in S_\ell} \Big(\delta_{ij} D_i D_j - x_j \omega_{ij} D_i - \frac{1}{2} x_j \frac{\partial \omega_{ij}}{\partial x_i}\Big) - \frac{1}{2} E_{k_\ell}(\eta),\end{aligned}$$

where $E_{k_\ell} = \sum_{j\in S_\ell} x_j \frac{\partial}{\partial x_j}$. Since E is of maximal rank and ω_{ij} is an affine function, we deduce that

$$Z_1 = \sum_{j\in S_\ell} D_j^2 - \sum_{i=1}^n \sum_{j\in S_\ell} x_j \omega_{ij} D_i - \frac{1}{2} E_{k_\ell}(\eta) \in E.$$

It follows from Lemma 2.12 that

$$\begin{aligned}P_{[L_0, Z_1]} =& P_{\left[\frac{1}{2}\sum_{i=1}^n D_i^2, \sum_{j\in S_\ell} D_j^2\right]} + P_{\left[\frac{1}{2}\sum_{i=1}^n D_i^2, -\sum_{r=1}^n \sum_{j\in S_\ell} x_j \omega_{rj} D_r\right]} \\ =& \sum_{i=1}^n \sum_{j\in S_\ell} 2\omega_{ji} D_j D_i - \sum_{i=1}^n \sum_{r=1}^n \sum_{j\in S_\ell} \frac{\partial (x_j \omega_{rj})}{\partial x_i} D_i D_r.\end{aligned}$$

Since $\omega_{k_{\ell-1}+1,k_{\ell-1}+1} = 0$, the coefficient of $D^2_{k_{\ell-1}+1}$ in $P_{[L_0,Z_1]}$ is

$$-\sum_{j\in S_\ell} \frac{\partial(x_j\omega_{k_{\ell-1}+1,j})}{\partial x_{k_{\ell-1}+1}} = 0$$

in view of Theorem 2.9. Similarly, for $m > k =$ quadratic rank of E, we can see that the coefficient of D^2_m is

$$-\sum_{j\in S_\ell} \frac{\partial(x_j\omega_{mj})}{\partial x_m} = 0$$

in view of Proposition 2.10. Now the coefficient of $D_{k_{\ell-1}+1}D_m$ is

$$\begin{aligned}
&2\omega_{k_{\ell-1}+1,m} - \sum_{j\in S_\ell}\Big[\frac{\partial(x_j\omega_{k_{\ell-1}+1,j})}{\partial x_m} + \frac{\partial(x_j\omega_{mj})}{\partial x_{k_{\ell-1}+1}}\Big]\\
&= 2\omega_{k_{\ell-1}+1,m} - \sum_{j\in S_\ell}\frac{\partial(x_j\omega_{mj})}{\partial x_{k_{\ell-1}+1}} \quad \text{in view of Theorem 2.11}\\
&= 3\omega_{k_{\ell-1}+1,m} - \sum_{j\in S_\ell} x_j\frac{\partial\omega_{mj}}{\partial x_{k_{\ell-1}+1}}\\
&= 3\omega_{k_{\ell-1}+1,m} - x_{k_{\ell-1}+1}\frac{\partial\omega_{mk_{\ell-1}+1}}{\partial x_{k_{\ell-1}+1}}
\end{aligned}$$

in view of part (ii) of Theorem 2.8. By Theorem 3.4, we know that the coefficient of $D_{k_{\ell-1}+1}D_m$ is independent of $x_{k_{\ell-1}+1}$. This simply means that

$$3\omega_{mk_{\ell-1}+1} + x_{k_{\ell-1}+1}\frac{\partial\omega_{mk_{\ell-1}+1}}{\partial x_{k_{\ell-1}+1}}$$

is independent of $x_{k_{\ell-1}+1}$. Hence $\omega_{mk_{\ell-1}+1}$ does not depend on $x_{k_{\ell-1}+1}$. ∎

THEOREM 3.6. *With the same hypothesis and notation as in Theorem* 3.5*, then* $A_2^{j,\ell} = 0$*, i.e.,* $\beta_{ij} = 0$ *for* $i \le k$ *(quadratic rank of* E*)* $< j$*.*

Proof. Since $[[L_0, p_\ell(x)], D_j] \in E$, we have $\sum_{i\in S_\ell} x_i\beta_{ji} \in E$. Theorem 3.5 says that

$$\begin{pmatrix}\beta_{jk_{\ell-1}+1}\\ \vdots\\ \beta_{jk_\ell}\end{pmatrix} = A_2^{j,\ell}\widetilde{X_\ell},$$

where $\widetilde{X_\ell} = \big(x_1,\dots,x_{k_{\ell-1}},x_{k_\ell+1},\dots,x_k\big)^T$. In view of Lemma 2.3, $X_\ell^T A_2^{j,\ell}\widetilde{X_\ell} = \sum_{i\in S_\ell} x_i\beta_{ji} \in E$ is independent of the $x_{k+1},\dots,x_n$ variable. Hence

$$A_2^{j,\ell} = \big(B_1^{j,\ell}, B_2^{j,\ell},\dots\big),$$

where $B_1^{j,\ell}, B_2^{j,\ell},\dots$ are constant multiples of some orthogonal matrices by Proposition 2.5. So we have

$$\begin{pmatrix}\beta_{jk_{\ell-1}+1}\\ \vdots\\ \beta_{jk_\ell}\end{pmatrix} = A_2^{j,\ell}\widetilde{X_\ell} = B_1^{j,\ell}X_1 + \cdots + B_{\ell-1}^{j,\ell}X_{\ell-1} + B_{\ell+1}^{j,\ell}X_{\ell+1} + \cdots.$$

In order to make clear what our strategy is, in the following we shall list several of these equations explicitly:

$$\begin{pmatrix}\beta_{j1}\\ \vdots\\ \beta_{jk_1}\end{pmatrix} = A_2^{j,1}\widetilde{X_1} = B_2^{j,1}X_2 + B_3^{j,1}X_3 + B_4^{j,1}X_4 + B_5^{j,1}X_5 + \cdots,$$

$$\begin{pmatrix}\beta_{jk_1+1}\\ \vdots\\ \beta_{jk_2}\end{pmatrix} = A_2^{j,2}\widetilde{X_2} = B_1^{j,2}X_1 + B_3^{j,2}X_3 + B_4^{j,2}X_4 + B_5^{j,2}X_5 + \cdots,$$

$$\begin{pmatrix}\beta_{jk_2+1}\\ \vdots\\ \beta_{jk_3}\end{pmatrix} = A_2^{j,3}\widetilde{X_3} = B_1^{j,3}X_1 + B_2^{j,3}X_2 + B_4^{j,3}X_4 + B_5^{j,3}X_5 + \cdots,$$

$$\begin{pmatrix}\beta_{jk_3+1}\\ \vdots\\ \beta_{jk_4}\end{pmatrix} = A_2^{j,4}\widetilde{X_4} = B_1^{j,4}X_1 + B_2^{j,4}X_2 + B_3^{j,4}X_3 + B_5^{j,4}X_5 + \cdots,$$

$$\vdots$$

We first claim that $B_m^{j,\ell} = \left(B_\ell^{j,m}\right)^T$. To see this, we observe that from the cyclic relation $\frac{\partial\omega_{j\ell}}{\partial x_m} + \frac{\partial\omega_{\ell m}}{\partial x_j} + \frac{\partial\omega_{mj}}{\partial x_\ell} = 0$ we deduce that

$$\frac{\partial\beta_{j\ell}}{\partial x_m} + \frac{\partial\beta_{\ell m}}{\partial x_j} + \frac{\partial\beta_{mj}}{\partial x_\ell} = 0.$$

If we take $m, \ell \le k =$ quadratic rank of $E < j$, then $\frac{\partial\beta_{\ell m}}{\partial x_j} = 0$ in view of Theorem 2.11. Therefore the above equation implies

$$\frac{\partial\beta_{j\ell}}{\partial x_m} = \frac{\partial\beta_{jm}}{\partial x_\ell},$$

from which we deduce easily that $B_m^{j,\ell} = \left(B_\ell^{j,m}\right)^T$.

Now we prove $B_1^{j,\ell} = 0$ for $\ell \ge 2$. Since $\sum_{i\in S_\ell} x_i\beta_{ji} = X_\ell^T A_2^{j,\ell}\widetilde{X_\ell} = X_\ell^T B_1^{j,\ell}X_1 + \cdots + X_\ell^T B_{\ell-1}^{j,\ell}X_{\ell-1} + X_\ell B_{\ell+1}^{j,\ell}X_{\ell+1} + \cdots \in E$, we conclude from Lemma 2.4 that $X_\ell^T B_1^{j,\ell}X_1 \in E$. Let $B_1^{j\ell} = \left(b_{ir}^{j\ell}\right), k_{\ell-1}+1 \le i \le k_\ell, 1 \le r \le k_1$. Then

$$\begin{aligned}
\left[L_0, X_\ell^T B_1^{j\ell} X_1\right] &= \Big[\frac{1}{2}\sum_{m=1}^n D_m^2, \sum_{i\in S_\ell}\sum_{r\in S_1} x_i b_{ir}^{j\ell} x_r\Big] \\
&= \sum_{m=1}^n \sum_{i\in S_\ell}\sum_{r\in S_1}\left(b_{ir}^{j\ell}x_r\delta_{im}D_m + b_{ir}^{j\ell}x_i\delta_{mr}D_m + b_{ir}^{j\ell}\delta_{im}\delta_{rm}\right) \\
&= \sum_{i\in S_\ell}\sum_{r\in S_1} b_{ir}^{j\ell}x_r D_i + \sum_{i\in S_\ell}\sum_{r\in S_1} b_{ir}^{j\ell}x_i D_r \\
&= \sum_{i\in S_\ell}\sum_{r\in S_1} b_{ir}^{j\ell}\left(x_r D_i + x_i D_r\right) \in E,
\end{aligned}$$

$$
\begin{aligned}
W_1 :=& \Big[L_0, \big[L_0, X_\ell^T B_1^{j\ell} X_1\big]\Big] \\
=& \Big[\frac{1}{2}\sum_{m=1}^{n} D_m^2 - \frac{1}{2}\eta, \sum_{i\in S_\ell}\sum_{r\in S_1} b_{ir}^{j\ell}\big(x_r D_i + x_i D_r\big)\Big] \\
=& \frac{1}{2}\sum_{m=1}^{n}\sum_{i\in S_\ell}\sum_{r\in S_1} b_{ir}^{j\ell}\big[D_m^2, x_r D_i + x_i D_r\big] + \text{ function} \\
=& \sum_{m=1}^{n}\sum_{i\in S_\ell}\sum_{r\in S_1} b_{ir}^{j\ell}\big(\delta_{mr} D_m D_i - x_r\omega_{mi} D_m + \delta_{mi} D_m D_r - x_i \omega_{mr} D_m\big) \\
& + \text{ function} \\
=& 2\sum_{i\in S_\ell}\sum_{r\in S_1} b_{ir}^{j\ell} D_r D_i - \sum_{m=1}^{n}\sum_{i\in S_\ell}\sum_{r\in S_1} b_{ir}^{j\ell}\big(x_r\omega_{mi} + x_i\omega_{mr}\big) D_m \\
& + \text{ function}, \\
[L_0, W_1] =& \Big[\sum_{m=1}^{n} D_m^2, \sum_{i\in S_\ell}\sum_{r\in S_1} b_{ir}^{j\ell} D_i D_r\Big] \\
& - \Big[\frac{1}{2}\sum_{v=1}^{m} D_v^2, \sum_{m=1}^{n}\sum_{i\in S_\ell}\sum_{r\in S_1} b_{ir}^{j\ell}\big(x_r\omega_{mi} + x_i\omega_{mr}\big) D_m\Big] \\
& + \text{ first-order term} \\
=& 2\sum_{m=1}^{n}\sum_{i\in S_\ell}\sum_{r\in S_1} b_{ir}^{j\ell}\big(\omega_{rm} D_i D_m + \omega_{im} D_m D_r\big) \\
& - \sum_{v=1}^{n}\sum_{m=1}^{n}\sum_{i\in S_\ell}\sum_{r\in S_1} b_{ir}^{j\ell}\frac{\partial(x_r\omega_{mi} + x_i\omega_{mr})}{\partial x_v} D_v D_m \\
& + \text{ first-order term.}
\end{aligned}
$$

The coefficient of D_1^2 in $[L_0, W_1]$ is

$$
2\sum_{i\in S_\ell} b_{ir}^{j\ell}\omega_{i1} - \sum_{i\in S_\ell}\sum_{r\in S_1} b_{ir}^{j\ell}\frac{\partial(x_r\omega_{1i} + x_i\omega_{1r})}{\partial x_1} = \sum_{i\in S_\ell} b_{i1}^{j\ell}\omega_{i1},
$$

which is a constant in view of Theorem 2.11.

On the other hand, the coefficient of D_j^2 in $[L_0, W_1]$ for $j > k =$ quadratic rank of E is

$$
-\sum_{i\in S_\ell}\sum_{r\in S_1} b_{ir}^{j\ell}\frac{\partial(x_r\omega_{ji} + x_i\omega_{jr})}{\partial x_j},
$$

which is a zero in view of Theorem 2.11.

We deduce from Theorem 3.4 that the coefficient of $D_1 D_j$ in $[L_0, W_1]$ is independent of x_1 and x_j variables. However, the coefficient of $D_1 D_j$ in $[L_0, W_1]$ is given by

$$
\begin{aligned}
&2\sum_{i\in S_\ell} b_{i1}^{j\ell}\omega_{ij} - \sum_{i\in S_\ell}\sum_{r\in S_1} b_{ir}^{j\ell}\Big(\frac{\partial(x_r\omega_{ji}+x_i\omega_{jr})}{\partial x_1} + \frac{\partial(x_r\omega_{1i}+x_i\omega_{1r})}{\partial x_j}\Big)\\
&= 2\sum_{i\in S_\ell} b_{i1}^{j\ell}\omega_{ij} - \sum_{i\in S_\ell}\sum_{r\in S_1} b_{ir}^{j\ell}\Big(\delta_{1r}\omega_{ji} + x_r\frac{\partial\omega_{ji}}{\partial x_1} + x_i\frac{\partial\omega_{jr}}{\partial x_1}\Big) \quad \text{in view of Theorem 2.11}\\
&= 2\sum_{i\in S_\ell} b_{i1}^{j\ell}\omega_{ij} - \sum_{i\in S_\ell} b_{i1}^{j\ell}\omega_{ji} - \sum_{i\in S_\ell}\sum_{r\in S_1} b_{ir}^{j\ell}\Big(x_r\frac{\partial\omega_{ji}}{\partial x_1} + x_i\frac{\partial\omega_{jr}}{\partial x_1}\Big)\\
&= 3\sum_{i\in S_\ell} b_{i1}^{j\ell}\omega_{ij} - \sum_{i\in S_\ell}\sum_{r\in S_1} b_{ir}^{j\ell}x_r\frac{\partial\omega_{ji}}{\partial x_1} \quad \text{by Theorem 3.5}\\
&= -3\sum_{i\in S_\ell} b_{i1}^{j\ell}\omega_{ji} - \sum_{i\in S_\ell} b_{i1}^{j\ell}x_1\frac{\partial\omega_{ji}}{\partial x_1} + \text{other terms not involving } \ x_1\\
&= -3\Big(b_{k_{\ell-1}+1,1}^{j\ell}\omega_{jk_{\ell-1}+1} + b_{k_{\ell-1}+2,1}^{j\ell}\omega_{jk_{\ell-1}+2} + \cdots + b_{k_\ell,1}^{j\ell}\omega_{jk_\ell}\Big)\\
&\quad - \Big(b_{k_{\ell-1}+1,1}^{j\ell}x_1\frac{\partial\omega_{jk_{\ell-1}+1}}{\partial x_1} + b_{k_{\ell-1}+2,1}^{j\ell}x_1\frac{\partial\omega_{jk_{\ell-1}+2}}{\partial x_1} + \cdots + b_{k_\ell,1}^{j\ell}x_1\frac{\partial\omega_{jk_\ell}}{\partial x_1}\Big)\\
&\quad + \ \text{other terms not involving } \ x_1\\
&= -3\Big\{\Big[\Big(b_{k_{\ell-1}+1,1}^{j\ell}\Big)^2x_1 + \cdots\Big] + \Big[\Big(b_{k_{\ell-1}+2,1}^{j\ell}\Big)^2x_1 + \cdots\Big] + \cdots + \Big[\Big(b_{k_\ell,1}^{j\ell}\Big)^2x_1 + \cdots\Big]\Big\}\\
&\quad - \Big[\Big(b_{k_{\ell-1}+1,1}^{j\ell}\Big)^2x_1 + \Big(b_{k_{\ell-1}+2,1}^{j\ell}\Big)^2x_1 + \cdots + \Big(b_{k_\ell,1}^{j\ell}\Big)^2x_1\Big]\\
&\quad + \ \text{other terms not involving } \ x_1\\
&= -4\sum_{i\in S_\ell}\big(b_{i1}^{j\ell}\big)^2x_1 + \text{other terms not involving } \ x_1.
\end{aligned}
$$

Therefore we conclude that

$$\sum_{i\in S_\ell}\big(b_{i1}^{j\ell}\big)^2 = 0.$$

This implies that the first column of the matrix $B_1^{j\ell} = \big(b_{ir}^{j\ell}\big)$, $k_{\ell-1}+1 \le i \le k_\ell$, $1 \le r \le k_1$, is a zero vector. Recall that $B_1^{j\ell}$ is either zero or nonsingular. We deduce that $B_1^{j\ell} = 0$ for any $j > k =$ quadratic rank of E and $\ell \ge 2$. By $B_m^{j\ell} = \big(B_\ell^{jm}\big)^T$, we conclude further that

$$
\begin{aligned}
&\begin{pmatrix}\beta_{j1}\\ \vdots\\ \beta_{jk_1}\end{pmatrix} = A_2^{j,1}\widetilde{X_1} = 0,\\
&\begin{pmatrix}\beta_{jk_1+1}\\ \vdots\\ \beta_{jk_2}\end{pmatrix} = A_2^{j,2}\widetilde{X_2} = B_3^{j2}X_3 + B_4^{j2}X_4 + B_5^{j2}X_5 + \cdots,\\
&\begin{pmatrix}\beta_{jk_2+1}\\ \vdots\\ \beta_{jk_3}\end{pmatrix} = A_2^{j,3}\widetilde{X_3} = B_2^{j3}X_2 + B_4^{j3}X_4 + B_5^{j3}X_5 + \cdots,
\end{aligned}
$$

$$\begin{pmatrix} \beta_{jk_3+1} \\ \vdots \\ \beta_{jk_4} \end{pmatrix} = A_2^{j,4}\widetilde{X_4} = B_2^{j4}X_2 + B_3^{j4}X_3 + B_5^{j4}X_5 + \cdots ,$$
$$\vdots$$

Similarly, by considering an element in E of the form $\sum_{i\in S_\ell} x_i\beta_{ji} = X_\ell^T A_2^{j,\ell}\widetilde{X_\ell}$ for $\ell \geq 3$, we can prove that $B_2^{j,\ell} = 0$ for $\ell \geq 3$. As before, we deduce $A_2^{j,2} = 0$ in view of $B_m^{j\ell} = \left(B_\ell^{jm}\right)^T$. Hence Theorem 3.6 follows easily by induction. □

As a consequence of Theorem 3.6 and Theorem 2.11, we have proved the following theorem.

THEOREM 3.7. *Let E be a finite-dimensional estimation algebra of maximal rank and k be the maximal rank of quadratic forms in E. Then (1) ω_{ij} are constants for $1 \leq i \leq k$ or $1 \leq j \leq k$; (2) ω_{ij} are affine polynomials in $x_{k+1}, \ldots, x_n$ for $k+1 \leq i, j \leq n$.*

4. Structure of $\boldsymbol{\eta}$. In this section, we shall study the possible structure of η, where

$$\eta = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^n f_i^2 + \sum_{i=1}^m h_i^2. \tag{4.1}$$

LEMMA 4.1. *Let E be a finite-dimensional estimation algebra of maximal rank. Then η is a polynomial of degree at most four.*

Proof. Since E is a finite-dimensional estimation algebra with maximal rank for any $1 \leq i \leq n$, there exists constant c_i such that $x_i + c_i$ is in E. Observe that

$$\begin{aligned} &[L_0, x_j + c_j] = D_j \in E, \\ &[D_j, x_j + c_j] = 1 \in E, \\ &[L_0, D_j] = \sum_{i=1}^n \left(\omega_{ji}D_i + \frac{1}{2}\frac{\partial \omega_{ji}}{\partial x_i}\right) + \frac{1}{2}\frac{\partial \eta}{\partial x_j} \in E. \end{aligned}$$

Since $\omega_{ji},\ 1 \leq i, j \leq n$, are polynomial of degree at most one, we deduce that

$$Y_j := \sum_{i=1}^n \omega_{ji}D_i + \frac{1}{2}\frac{\partial \eta}{\partial x_j} \in E. \tag{4.2}$$

As

$$[Y_j, D_m] = \sum_{i=1}^n \left(\omega_{ji}\omega_{mi} - \frac{\partial \omega_{ji}}{\partial x_m}D_i\right) - \frac{1}{2}\frac{\partial^2 \eta}{\partial x_m \partial x_j} \in E,$$

we deduce that

$$\sum_{\ell=1}^n \omega_{j\ell}\omega_{\ell m} + \frac{1}{2}\frac{\partial^2 \eta}{\partial x_m \partial x_j} \in E \tag{4.3}$$

for all $1 \leq j, m \leq n$. Hence $\frac{\partial^2 \eta}{\partial x_m \partial x_j}$ for $1 \leq j, m \leq n$ are polynomials of degree at most two in view of Ocone's theorem. It follows that η is a polynomial of degree at most four. □

Notation. η_4 is denoted the homogeneous part of degree four of η.

PROPOSITION 4.2. *Let E be a finite-dimensional estimation algebra of maximal rank. Then η_4 does not contain $x_{i_1}x_{i_2}x_jx_m$ for $i_1 \le k < i_2$, where k is the quadratic rank of E.*

Proof. Let us assume that this is false. From (4.3), we have

$$\sum_{\ell=1}^{n} \omega_{i_1\ell}\omega_{\ell m} + \frac{1}{2}\frac{\partial^2 \eta}{\partial x_m \partial x_{i_1}} \in E. \tag{4.4}$$

Recall that $\omega_{i_1\ell}$, $1 \le \ell \le n$, are constants, while $\omega_{\ell m}$, $1 \le \ell \le n$, are polynomials of degree one in $x_{k+1}, \dots, x_n$ by Theorem 3.7. Equation (4.4) says that E contains a quadratic form with term of the form $x_{i_2}x_j$. This contradicts Lemma 2.3. ∎

The following proposition also follows immediately from Theorem 3.7.

PROPOSITION 4.3. *Let E be a finite-dimensional estimation algebra of maximal rank with quadratic rank k. If $i \le k < j$, then $\sum_{\ell=1}^{n} \omega_{i\ell}\omega_{\ell j}$ is a degree-one polynomial in $x_{k+1}, \dots, x_n$.*

THEOREM 4.4. *Let E be a finite-dimensional estimation algebra of maximal rank with quadratic rank k. Then η_4 is an homogeneous polynomial of degree four depending only on $x_{k+1}, \dots, x_n$ variables.*

Proof. Since $p_0(x) = x_1^2 + \cdots + x_k^2 \in E$ by (2.4), we have

$$[L_0, p_0] \in E \Rightarrow \widetilde{E_k} := \sum_{i=1}^{k} x_i D_i \in E.$$

Let $E_k = \sum_{i=1}^{k} x_i \frac{\partial}{\partial x_i}$. Then in view of (4.2), we have

$$\begin{aligned}
[\widetilde{E_k}, Y_j] &= \Big[\sum_{i=1}^{k} x_i D_i, \sum_{i=1}^{k} \omega_{j\ell} D_\ell + \frac{1}{2}\frac{\partial \eta}{\partial x_j}\Big] \\
&= \sum_{i=1}^{k}\sum_{\ell=1}^{n} [x_i D_i, \omega_{j\ell} D_\ell] + \frac{1}{2} E_k \left(\frac{\partial \eta}{\partial x_j}\right) \\
&= \sum_{i=1}^{k}\sum_{\ell=1}^{n} \left(-x_i\omega_{j\ell}\omega_{i\ell} + x_i \frac{\partial \omega_{j\ell}}{\partial x_i} D_\ell - \omega_{j\ell}\delta_{i\ell} D_i\right) + \frac{1}{2} E_k \left(\frac{\partial \eta}{\partial x_j}\right) \\
&= \sum_{\ell=1}^{n} E_k(\omega_{j\ell}) D_\ell - \sum_{i=1}^{k} \omega_{ji} D_i + \frac{1}{2} E_k \left(\frac{\partial \eta}{\partial x_j}\right) - \sum_{\ell=1}^{n}\sum_{i=1}^{k} x_i \omega_{j\ell}\omega_{i\ell} \\
&\in E.
\end{aligned} \tag{4.5}$$

Recall that $\omega_{j\ell}$, $1 \le j, \ell \le n$, are polynomials depending only on $x_{k+1}, \dots, x_n$. So $E_k(\omega_{j\ell}) = 0$ for $1 \le j, \ell \le n$. Since ω_{ji}, $1 \le i \le k$, are constants and $D_i, 1 \le i \le k$, are in E, we deduce from (4.5) that

$$\frac{1}{2} E_k \left(\frac{\partial \eta}{\partial x_j}\right) - \sum_{i=1}^{k} x_i \Big(\sum_{\ell=1}^{n} \omega_{j\ell}\omega_{i\ell}\Big) \in E. \tag{4.6}$$

The second term in (4.6) is a polynomial of degree at most two by Proposition 4.3. This implies that

$$E_k \left(\frac{\partial \eta_4}{\partial x_j}\right) = 0, \qquad 1 \le j \le n, \tag{4.7}$$

because $E_k\big(\frac{\partial\eta_4}{\partial x_j}\big)$ is a polynomial of degree three. By Proposition 4.2, we have

$$\eta_4(x_1,\dots,x_n)=\eta_4^1(x_1,\dots,x_k)+\eta_4^2(x_{k+1},\dots,x_n),$$

where η_4^1 and η_4^2 are homogeneous polynomials of degree four. Equation (4.7) is equivalent to

$$E_k\left(\frac{\partial\eta_4^1}{\partial x_j}\right)=0,\qquad 1\le j\le k. \tag{4.8}$$

Since

$$E_k\left(\frac{\partial\eta_4^1}{\partial x_j}\right)=3\frac{\partial\eta_4^1}{\partial x_j},\qquad 1\le j\le k,$$

we deduce easily that $\eta_4^1(x_1,\dots,x_k)=0$. So η_4 depends only on $x_{k+1},\dots,x_n$ variables. ∎

5. Mitter conjecture. If E is a finite-dimensional estimation algebra, Ocone's theorem says that h_i, $1\le i\le m$, are polynomials of degree at most two. Mitter conjecture asserts that h_i has to be affine (i.e., degree-one polynomial). In this section, we shall prove the Mitter conjecture for finite-dimensional estimation algebras of maximal rank. For this purpose, let us first recall the following theorem proven in [Ya].

THEOREM 5.1. *Let $F(x_1,\dots,x_n)$ be a polynomial on $\mathbb{R}^n$. Suppose that there exists a polynomial path $c:\mathbb{R}\to\mathbb{R}^n$ such that $\lim_{t\to\infty}\|c(t)\|=\infty$ and $\lim_{t\to\infty}F\circ c(t)=-\infty$. Then there are no c^∞ functions $f_1,\dots,f_n$ on $\mathbb{R}^n$ satisfying the equation*

$$\sum_{i=1}^n\frac{\partial f_i}{\partial x_i}+\sum_{i=1}^n f_i^2=F.$$

THEOREM 5.2. *If E is a finite-dimensional estimation algebra of maximal rank, then h_i, $1\le i\le m$, are degree-one polynomials.*

Proof. For $1\le i\le m$, let $h_i(x)=q_i(x)+\ell_i(x)$, where $q_i(x)$ is a homogeneous degree-two polynomial, while $\ell_i(x)$ is a degree-one polynomial. Since $h_i\in E$ by definition, we deduce that $q_i(x)$ is also in E for all $1\le i\le m$ by the maximal rank condition of E. In view of Lemma 2.3, we conclude that q_i depends only on $x_1,x_2,\dots,x_k$ variables for all $1\le i\le m$. Hence

$$h_i(x)=q_i(x_1,\dots,x_k)+\ell_i(x),\qquad 1\le i\le m.$$

On the other hand, Theorem 4.4 tells us that

$$\sum_{i=1}^n\frac{\partial f_i}{\partial x_i}+\sum_{i=1}^n f_i^2+\sum_{i=1}^m h_i^2=\eta_4(x_{k+1},\dots,x_n)+\text{ polynomial of degree three},$$

which implies

$$\begin{aligned}&\sum_{i=1}^n\frac{\partial f_i}{\partial x_i}+\sum_{i=1}^n f_i^2\\&=-\sum_{i=1}^m\big(q_i(x_1,\dots,x_k)\big)^2+\eta_4(x_{k+1},\dots,x_n)+\text{ polynomial of degree three.}\end{aligned}$$

The above equation and Theorem 5.1 imply that $q_i(x_i, \dots, x_k) = 0$ for all $1 \leq i \leq m$; i.e., $h_i(x),\ 1 \leq i \leq m$, are degree-one polynomials. □

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