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FINITE-DIMENSIONAL FILTERS WITH NONLINEAR DRIFT VIII: CLASSIFICATION OF FINITE-DIMENSIONAL ESTIMATION ALGEBRAS OF MAXIMAL RANK WITH STATE-SPACE DIMENSION 4*

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Abstract. The idea of using estimation algebra to construct finite-dimensional nonlinear filters was first proposed independently by Brockett and Mitter. Estimation algebra turns out to be a useful concept in the investigation of finite-dimensional nonlinear filters. In his talk at the International Congress of Mathematics in 1983, Brockett proposed classifying all finite-dimensional estimation algebras. Chiou and the present authors classify all finite-dimensional estimation algebras of maximal rank with dimension of the state space less than or equal to three. In this paper, we succeed in classifying all finite-dimensional estimation algebras of maximal rank with state-space dimension equal to four. In fact our method gives classification of all finite-dimensional algebras of maximal rank with state-space dimension equal to or less than four.

Key words. finite-dimensional filter, estimation algebra of maximal rank, nonlinear drift, state-space dimension 4

AMS subject classifications. 17B30, 35J15, 60G35, 93E11

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1. Introduction.

In the 1960s and early 1970s, the basic approach to nonlinear filtering theory was via the “innovation methods” originally proposed by Kailath and subsequently rigorously developed by Fujisaki, Kallianpur, and Kunita [FKK] in 1972. As pointed out by Mitter [Mi], the difficulty with this approach is that the innovation process is not, in general, explicitly computable (except in the well-known Kalman–Bucy case). In the late 1970s, Brockett and Clark [BrCl], Brockett [Br], and Mitter [Mi] proposed the idea of using estimation algebras to construct a finite-dimensional nonlinear filter. The advantage of this finite-dimensional nonlinear filter is the same as the Kalman–Bucy filter. Moreover it avoids the disadvantages of the Kalman–Bucy filter such as the Gaussian initial condition as well as linearity assumption of the drift term. For more detail, we refer the readers to [TWY], [Ya], and the very interesting Ph.D. thesis by M. Cohen de Lara [La], in which the links between finite-dimensional estimation algebras and finite-dimensional filters were discussed. In [Ya], Yau has studied the general class of nonlinear filtering systems which included both Kalman–Bucy and Benes filtering systems as special cases. He gives necessary and sufficient conditions for an estimation algebra of such filtering systems to be finite dimensional. Using the Wei–Norman approach, he constructed explicitly finite-dimensional recursive filters for such a nonlinear filtering systems.

In his talk at the International Congress of Mathematics in 1983, Brockett proposed classifying all finite-dimensional estimation algebras. Since then, the concept of estimation algebra has been proven to be an invaluable tool in the study of nonlinear filtering problems. If the drift term of the nonlinear filtering system has a

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potential function (i.e., drift term is a gradient vector field), then the corresponding estimation algebra is called exact. In [TWY], Tam, Wong, and Yau have classified all finite-dimensional exact estimation algebras of maximal rank with arbitrary state-space dimension. In [ChYa], Chiou and Yau are able to classify all finite-dimensional estimation algebras of maximal rank with state-space dimension less than or equal to two. The novelty of their theorem is that there is no assumption on the drift term of the nonlinear filtering system. In [CLY], Chen, Leung, and Yau classify all finite-dimensional estimation algebras of maximal rank with state-space dimension equal to 3 (without any assumption on the drift term). This paper is a natural continuation of [ChYa] and [CLY]. The following is our main theorem.

MAIN THEOREM. *Suppose that the state space of the filtering system* (2.1) *is of dimension* $n \leq 4$. *If* E *is the finite-dimensional estimation algebra of maximal rank, then the drift term* f *must be a linear vector field (i.e., each component is a polynomial of degree one) plus a gradient vector field and* E *is a real vector space of dimension* $2n + 2$ *with basis given by* $1, x_1, \ldots, x_n, D_1, \ldots, D_n$ *and* L_0. *Moreover* η *is a degree 2 polynomial.*

This kind of nonlinear filtering system was studied by Yau [Ya]. Therefore, from Lie algebraic point of view, we have shown that the finite-dimensional filters considered in [Ya] are the most general finite-dimensional filters (cf. [Ch] for a nice review of the Yau filter).

Let $\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$, which was first introduced by Wong [Wo3]. Our strategy is to prove ω_{ij} constant for all i, j. Then we can apply the result of [Ya] to finish the proof. This involves two steps. The first step is to prove that ω_{ij} is a degree-one polynomial. Let n be the dimension of the state space. In the case $n = 3$ there are three unknowns: ω_{12}, ω_{13}, and ω_{23}. It is easy to see that they are all degree-two polynomials in view of Ocone's theorem. In [YaLe], Leung and Yau showed that the coefficients of the quadratic parts of ω_{12}, ω_{13}, and ω_{23} have to satisfy 90 quadratic equations. It was shown in that paper that the 90 quadratic equations have only a trivial solution. Hence the proof of the first step is completed in this case. Obviously, this approach encounters difficulty when n is greater than 3. Fortunately, Chen and Yau [ChYa1] were able to prove that ω_{ij} is a degree-one polynomial for arbitrary n by means of their new algebraic technique. The second step is to prove that ω_{ij} is actually a constant. This is the hard part of the problem of classification of finite-dimensional estimation algebras of maximal rank. The purpose of this paper is to deal with the hard part of the problem by proving ω_{ij} constant for $n \leq 4$. We introduce a new matrix equation. The key point of this paper is to show that this matrix has no nontrivial solution. The advantage of our new technique is not only that we can obtain entirely new results for $n = 4$ but also that we have simple uniform proof of our previous results for $n \leq 3$.

The paper is in essence a continuation of [Ya], [ChYa], [CLY], [ChYa1], [ChYa2], and we strongly recommend that readers familiarize themselves with the results in [Ya], [ChYa1], [ChYa2]. However, every effort will be made to make this paper as self-contained as possible, with minimal duplication of the previous papers.

2. Basic concepts. In this section, we shall recall some basic concepts and results from [Ya] and [ChYa]. Consider a filtering problem based on the following observation model:

$$\begin{cases} dx(t) = f(x(t))dt + g(x(t))dv(t), & x(0) = x_0, \\ dy(t) = h(x(t))dt + dw(t), & y(0) = 0, \end{cases} \tag{2.1}$$

in which x, v, y, and w are, respectively, $\mathbb{R}^n$-, $\mathbb{R}^p$-, $\mathbb{R}^m$-, and $\mathbb{R}^m$-valued processes and v and w have components which are independent, standard Brownian processes. We further assume that $n = p, f, h$ are C^∞ smooth and that g is an orthogonal matrix. We shall refer to $x(t)$ as the state of the system at time t and to $y(t)$ as the observation at time t.

Let $\rho(t,x)$ denote the conditional density of the state given the observation $\{y(s) : 0 \le s \le t\}$. It is well known (see [DaMa], for example) that $\rho(t,x)$ is given by normalizing a function, $\sigma(t,x)$, which satisfies the Duncan–Mortensen–Zakai equation:

$$d\sigma(t,x) = L_0\sigma(t,x)dt + \sum_{i=1}^{m} L_i\sigma(t,x)dy_i(t), \qquad \sigma(0,x) = \sigma_0, \tag{2.2}$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i} - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2$$

and for $i = 1, \dots, m, L_i$ is the zero-degree differential operator of multiplication by h_i, σ_0 is the probability density of the initial point x_0. In this paper, we will assume σ_0 is a C^∞ function.

Equation (2.2) is a stochastic partial differential equation. The stochastic differential is a Stratonovich one and not an Ito one. In real applications, we are interested in constructing state estimators from observed sample paths with some property of robustness. Based on Rozovsky's transformation [Ro],

$$\xi(t,x) = \exp\left(-\sum_{i=1}^{m} h_i(x)y_i(t)\right)\sigma(t,x),$$

Davis [Da] proposed studying the following robust Duncan–Mortensen–Zakai equation:

$$\begin{aligned} &\frac{\partial\xi}{\partial t}(t,x) = L_0\xi(t,x) + \sum_{i=1}^{m} y_i(t)[L_0, L_i]\xi(t,x) + \frac{1}{2}\sum_{i,j=1}^{m} y_i(t)y_j(t)[[L_0, L_i], L_j]\xi(t,x), \\ &\xi(0,x) = \sigma_0, \end{aligned} \tag{2.3}$$

which is a time-varying partial differential equation. Here we have used the following notation.

DEFINITION 1. *If X and Y are differential operators, the Lie bracket of X and Y, $[X,Y]$, is defined by $[X,Y]\phi = X(Y\phi) - Y(X\phi)$ for any C^∞ function ϕ.*

DEFINITION 2. *The estimation algebra E of a filtering problem* (2.1) *is defined to be the Lie algebra generated by $\{L_0, L_1, \dots, L_m\}$. E is said to be an estimation algebra of maximal rank if for any $1 \le i \le n$ there exists a constant c_i such that $x_i + c_i$ is in E.*

Most of the known finite-dimensional estimation algebras are maximal. For example, if (2.1) is linear, i.e., $f(x) = Ax$, $g(x) = B$, and $h(x) = Cx$, and if (A, B, C)

also is minimal, then the corresponding estimation algebra is maximal [Ha]. We need the following basic result for later discussion.

THEOREM 2.1 (Ocone). *Let E be a finite-dimensional estimation algebra. If a function ξ is in E, then ξ is a polynomial of degree at most two.*

In [Wo3], the concept of Ω is introduced, defined as the matrix whose (i, j)-element ω_{ij} is $\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$. Define

$$D_i = \frac{\partial}{\partial x_i} - f_i$$

and

$$\eta = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{m} h_i^2.$$

Then

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^{n} D_i^2 - \eta\right).$$

The following theorem proved in [Ya] plays a fundamental role in the classification of finite-dimensional estimation algebras.

THEOREM 2.2 (Yau). *Let E be a finite-dimensional estimation algebra of* (2.1) *such that $\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$ are constant functions. If E is of maximal rank, then E is a real vector space of dimension $2n+2$ with basis given by $1, x_1, x_2, \dots, x_n, D_1, D_2, \dots, D_n$ and L_0.*

Recently, Chen and Yau [ChYa1] have made important progress in the program of classification of finite-dimensional estimation algebras of maximal rank. Namely, they have shown that Ω matrix is linear in the sense that all ω_{ij} are degree-one polynomials. More recently, in order to prove the Mitter conjecture for finite-dimensional estimation algebra of maximal rank, Chen and Yau [ChYa2] have sharpened the above result. To describe this new result, let us first recall some important concepts and notations introduced in [ChYa1].

Let Q be the space of quadratic forms in n variables, i.e., real vector space spanned by $x_i x_j$, with $1 \le i \le j \le n$. Let $X = (x_1, \dots, x_n)^T$. For any quadratic form $p \in Q$, there exists a symmetric matrix A such that $p(x) = X^T A X$. The rank of the quadratic form p is denoted by $\mathrm{rk}(p)$ and is defined to be the rank of the matrix A.

DEFINITION 3. *A fundamental quadratic form of the estimation algebra E is an element $p_0 \in E \cap Q$ with the greatest positive rank, i.e., $\mathrm{rk}(p_0) \ge \mathrm{rk}(p)$ for any $p \in E \cap Q$. The quadratic rank of the estimation algebra E is defined to be $\mathrm{rk}(p_0)$. The following Theorem* 2.3 *and Proposition* 2.4 *are proved in* [ChYa2].

THEOREM 2.3. *Let E be a finite-dimensional estimation algebra of maximal rank. Let k be the quadratic rank of E. Then*

(1) *the observation terms $h_i(x), 1 \le i \le m$, are affine polynomials.*

(2) (a) *ω_{ij}, for $1 \le i \le k$ or $1 \le j \le k$, are constants.*

(b) *ω_{ij}, for $k+1 \le i, j \le n$, are degree-one polynomials in $x_{k+1}, \dots, x_n$.*

(3) *$\eta = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{m} h_i^2$ is a homogeneous polynomial of degree four. Moreover, η_4 (homogeneous polynomial of degree-four part of η) depends only on $x_{k+1}, \dots, x_n$ variables.*

PROPOSITION 2.4. *Let E be a finite-dimensional estimation algebra of maximal rank. Let k be the quadratic rank of E. Then η is a polynomial of degree at most four, and any homogeneous polynomial of degree two in E depends only on $x_1, x_2, \dots, x_k$.*

Finally, we need to recall the following theorem proved in [Ya].

THEOREM 2.5. *Let $F(x_1, \dots, x_n)$ be a polynomial on $\mathbb{R}^n$. Suppose that there exists a polynomial path $c : \mathbb{R} \to \mathbb{R}^n$ such that $\lim_{t\to\infty} ||c(t)|| = \infty$ and $\lim_{t\to\infty} F(c(t)) = -\infty$. Then there are no C^∞ functions $f_1, \dots, f_n$ on $\mathbb{R}^n$ satisfying the equations*

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = F.$$

3. Proof of the main theorem. From Yau's theory [Ya], to classify all finite-dimensional estimation algebras of maximal rank, we need only to prove that the ω_{ij}'s corresponding to these estimation algebras of maximal rank are automatically constants.

We first introduce a new matrix equation which turns out to play an important role in classification of finite-dimensional estimation algebras of maximal rank.

THEOREM 3.1. *Suppose that η_4 is a homogeneous polynomial of degree four in n variables. If $n \le 4$ and $\triangle$ is an antisymmetric matrix with each entry a homogeneous polynomial of degree one such that*

$$\triangle \triangle^T = \frac{1}{2} H(\eta_4), \tag{3.1}$$

where $H(\eta_4) = (\frac{\partial^2 \eta_4}{\partial x_i \partial x_j})$ is the Hessian matrix of η_4, then $\triangle = 0$.

Proof. Write

$$\triangle = \sum_{i=1}^{n} A_i x_i, \tag{3.2}$$

$$\frac{1}{2} H(\eta_4) = \sum_{i \le j} H_{ij} x_i x_j, \tag{3.3}$$

where A_i's are real $n \times n$ antisymmetric matrices and H_{ij}'s are real $n \times n$ symmetric matrices, i.e.,

$$A_i = -A_i^T, \tag{3.4}$$

$$H_{ij} = H_{ij}^T. \tag{3.5}$$

Then (3.1) implies

$$A_i^2 = -H_{ii}, \tag{3.6}$$

$$A_i A_j + A_j A_i = -H_{ij}. \tag{3.7}$$

Denote the (i, j) entry of the matrix M by $M(i, j)$. For $i > j$, we let $H_{ij} = H_{ji}$. Note that

$$\frac{\partial^2 (x_i^2 x_j^2)}{\partial x_i^2} = 2x_j^2, \qquad \frac{\partial^2 (x_i^2 x_j^2)}{\partial x_j^2} = 2x_i^2, \quad \text{and} \qquad \frac{\partial^2 (x_i^2 x_j^2)}{\partial x_i \partial x_j} = 4x_i x_j. \tag{3.8}$$

From (3.3) and (3.8), we get

$$2H_{ii}(j, j) = 2H_{jj}(i, i) = H_{ij}(i, j) \quad \text{for } i \ne j. \tag{3.9}$$

Hence, for $i \neq j$, (3.6), (3.7), and (3.9) imply

$$\begin{aligned}\sum_l A_i(j,l)A_i(l,j) &= \sum_l A_j(i,l)A_j(l,i)\\ &= \frac{1}{2}\sum_l \left[A_i(i,l)A_j(l,j) + A_j(i,l)A_i(l,j)\right].\end{aligned} \tag{3.10}$$

Recall that each A_i is an antisymmetric matrix. So (3.10) is reduced to the following equation:

$$\begin{aligned}\sum_l \left[A_i(j,l)\right]^2 &= \sum_l \left[A_j(i,l)\right]^2\\ &= \frac{1}{2}\sum_l \left[A_i(i,l)A_j(j,l) + A_j(i,l)A_i(j,l)\right].\end{aligned} \tag{3.11}$$

In view of the Schwarz inequality, we have

$$\begin{aligned}2\sum_l \left[A_i(j,l)\right]^2 + 2\sum_l \left[A_j(i,l)\right]^2 &= 2\sum_l A_i(i,l)A_j(j,l) + 2\sum_l A_j(i,l)A_i(j,l)\\ &\leq \sum_{l\neq i,j} \left[A_i(i,l)\right]^2 + \sum_{l\neq i,j} \left[A_j(j,l)\right]^2 + \sum_{l\neq i,j} \left[A_j(i,l)\right]^2\\ &\quad + \sum_{l\neq i,j} \left[A_i(j,l)\right]^2.\end{aligned} \tag{3.12}$$

This implies

$$\begin{aligned}\sum_l \left[A_i(j,l)\right]^2 + \sum_l \left[A_j(i,l)\right]^2 + \sum_{l=i,j} \left[A_j(i,l)\right]^2 + \sum_{l=i,j} \left[A_i(j,l)\right]^2\\ \leq \sum_{l\neq i,j} \left[A_i(i,l)\right]^2 + \sum_{l\neq i,j} \left[A_j(j,l)\right]^2.\end{aligned} \tag{3.13}$$

Taking the sum of left-hand side of (3.13) over $i < j$, we get

$$\begin{aligned}&\sum_{i<j}\sum_{l=1}^n \left[A_i(j,l)\right]^2 + \sum_{i<j}\sum_{l=1}^n \left[A_j(i,l)\right]^2 + \sum_{i<j} \left[A_j(i,j)\right]^2 + \sum_{i<j} \left[A_i(j,i)\right]^2\\ &\quad = \sum_{i<j}\sum_{l=1}^n \left[A_i(j,l)\right]^2 + \sum_{j<i}\sum_{l=1}^n \left[A_i(j,l)\right]^2 + \sum_{i<j} \left[A_j(i,j)\right]^2 + \sum_{j<i} \left[A_j(i,j)\right]^2\\ &\quad = \sum_{i\neq l, i\neq j} \left[A_i(j,l)\right]^2 + 2\sum_{i\neq l} \left[A_i(i,l)\right]^2.\end{aligned} \tag{3.14}$$

On the other hand, by taking the sum of right-hand side of (3.13) over $i < j$, we get

$$\begin{aligned}&\sum_{i<j}\sum_{l\neq i,j} \left[A_i(i,l)\right]^2 + \sum_{i<j}\sum_{l\neq i,j} \left[A_j(j,l)\right]^2\\ &\quad = \sum_{i<j}\sum_{l\neq i,j} \left[A_i(i,l)\right]^2 + \sum_{j<i}\sum_{l\neq i,j} \left[A_i(i,l)\right]^2\\ &\quad = (n-2)\sum_{i\neq l} \left[A_i(i,l)\right]^2.\end{aligned} \tag{3.15}$$

Comparing (3.14) and (3.15), we get

$$\sum_{i\neq l, i\neq j} \left[A_i(j,l)\right]^2 = (n-4)\sum_{i\neq l}\left[A_i(i,l)\right]^2. \tag{3.16}$$

When $n=4$, we see from (3.16) that

$$\sum_{i\neq l, i\neq j} \left[A_i(j,l)\right]^2 = 0;$$

hence

$$A_i(j,l) = 0 \quad \text{for } i\neq j \quad \text{and } i\neq l. \tag{3.17}$$

Using (3.17) in (3.10) gives

$$\left[A_i(i,j)\right]^2 = \left[A_j(j,i)\right]^2. \tag{3.18}$$

Note that for $i<j$

$$\frac{\partial^2 x_i^3 x_j}{\partial x_i \partial x_j} = 3x_i^2 \quad \text{and} \quad \frac{\partial^2 x_i^3 x_j}{\partial x_i^2} = 6x_i x_j. \tag{3.19}$$

From (3.3) and (3.19), we get

$$2H_{ij}(i,i) = H_{ii}(i,j). \tag{3.20}$$

In view of (3.6), (3.7), and (3.20), we have

$$2\sum_l A_i(i,l)A_j(l,i) = \sum_l A_i(i,l)A_i(l,j). \tag{3.21}$$

Using (3.17) in (3.21) gives

$$A_i(i,j)A_j(j,i) = 0. \tag{3.22}$$

In view of (3.18) and (3.22), we get

$$A_i(i,j) = 0 \quad \text{for all } i,j. \tag{3.23}$$

Therefore $A_1 = A_2 = A_3 = A_4 = 0$ by (3.17) and (3.23). This simply means that $\triangle = 0$. ∎

PROPOSITION 3.2. *Let E be a finite-dimensional estimation algebra of maximal rank associated with the filtering system* (2.1). *Then E contains the real vector space spanned by $1, x_1, \dots, x_n, D_1, \dots, D_n$ and L_0.*

Proof. Since E is a finite-dimensional estimation algebra with maximal rank, there are constants c_i's such that $x_i + c_i$ is in E for $1 \le i \le n$:

$$[L_0, x_j + c_j] = \frac{1}{2}\left[\sum_{i=1}^n D_i^2 - \eta, x_j\right] = \frac{1}{2}\sum_{i=1}^n \left[D_i^2, x_j\right] = D_j \in E,$$
$$[D_j, x_j + c_j] = 1 \in E.$$

Hence $x_1, \dots, x_n \in E$. ∎

We are now ready to prove the main theorem of this paper stated in section 1. By Theorem 2.3, we know that ω_{ij}'s are constants for $1 \le i \le k$ or $1 \le j \le k$, where k is the quadratic rank of E. We also know that ω_{ij}'s are degree-one polynomials in $x_{k+1}, \dots, x_n$ variables for $k+1 \le i, j \le n$. We are going to prove that ω_{ij}'s are indeed constants for $k+1 \le i, j \le n$ and $n \le 4$. Observe that

$$\begin{aligned}\big[[L_0, D_j], D_l\big] &= \left[\sum_{i=1}^{n}\left(\omega_{ji}D_i + \frac{1}{2}\frac{\partial \omega_{ji}}{\partial x_i}\right) + \frac{1}{2}\frac{\partial \eta}{\partial x_j}, D_l\right] \\ &= \sum_{i=1}^{n}\left(\omega_{ji}\omega_{li} - \frac{\partial \omega_{ji}}{\partial x_l}D_i\right) - \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 \omega_{ji}}{\partial x_l \partial x_i} - \frac{1}{2}\frac{\partial^2 \eta}{\partial x_l \partial x_j} \\ &\in E.\end{aligned}$$

In view of Theorem 2.3 and Proposition 3.2, we deduce that

$$\sum_{i=1}^{n}\omega_{ji}\omega_{li} - \frac{1}{2}\frac{\partial^2 \eta}{\partial x_l \partial x_j} \in E. \tag{3.24}$$

Let η_m be the homogeneous polynomial of the degree-m part of η and β_{ij} be the homogeneous polynomial of the degree-one part of ω_{ij}. Then in view of Theorem 2.3 and Proposition 3.2, we have

$$\sum_{i=k+1}^{n}\beta_{ji}\beta_{li} - \frac{1}{2}\frac{\partial^2 \eta_4}{\partial x_l \partial x_j} \in E \tag{3.25}$$

for $k+1 \le l, j \le n$. Observe that $\sum_{i=k+1}^{n}\beta_{ji}\beta_{li} - \frac{1}{2}\frac{\partial^2 \eta}{\partial x_l \partial x_j}$ is a homogeneous polynomial of degree two in E which depends only on $x_1, \dots, x_k$ variables because k is the quadratic rank of E. On the other hand η_4, β_{ji}, and β_{li}, for $k+1 \le i, j, l \le n$, depend only on $x_{k+1}, \dots, x_n$ variables by Theorem 2.3. So the left-hand side of (3.25) depends only on $x_{k+1}, \dots, x_n$. Therefore we deduce that

$$\sum_{i=k+1}^{n}\beta_{ji}\beta_{li} - \frac{1}{2}\frac{\partial^2 \eta}{\partial x_l \partial x_j} = 0 \qquad \text{for } k+1 \le j, l \le n. \tag{3.26}$$

Let $\triangle = (\beta_{ij})$, $k+1 \le i, j \le n$, be an $(n-k) \times (n-k)$ antisymmetric matrix. Then we have

$$\triangle\triangle^T = \frac{1}{2}H(\eta_4), \tag{3.27}$$

where $H(\eta_4) = (\frac{\partial^2 \eta_4}{\partial x_i \partial x_j})$, $k+1 \le i, j \le n$, stands for the Hessian matrix for η_4. In view of Theorem 3.1, we have $\triangle = 0$. So we have shown that ω_{ij}'s are constants for $1 \le i, j \le n$. By Theorem 2.2 of Yau, E is a real vector space of dimension $2n+2$ with basis given by $1, x_1, x_2, \dots, x_n, D_1, D_2, \dots, D_n$ and L_0.

Since $\triangle = 0$, (3.27) implies $\eta_4 = 0$. So

$$\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{m} h_i^2 = \eta_0 + \eta_1 + \eta_2 + \eta_3,$$

which implies

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = \eta_0 + \eta_1 + \eta_2 - \sum_{i=1}^{m} h_i^2 + \eta_3. \tag{3.28}$$

By Theorem 2.3, $\tilde{F} = \eta_0 + \eta_1 + \eta_2 - \sum_{i=1}^{m} h_i^2$ is at most a polynomial of degree two. If η_3 is not identically zero, then we can choose a polynomial path $c : \mathbb{R} \to \mathbb{R}^n$ such that $\lim_{t\to\infty} ||c(t)|| = \infty$ and $\lim_{t\to\infty} F(c(t)) = -\infty$. This is not possible in view of Theorem 2.5. So we conclude that $\eta_3 = 0$; i.e., η is a polynomial of degree two.

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