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Finite dimensional filters with non-linear drift IX Construction of finite dimensional estimation algebras of non-maximal rank[1]

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Abstract

The idea of using estimation algebra to construct finite-dimensional nonlinear filters was first proposed by Brockett and Mitter independently. It has proven to be an invaluable tool in the study of nonlinear filtering problem. In 1983, Brockett proposed to classify all finite-dimensional estimation algebras. In this paper, we give the construction of finite-dimensional estimation algebras of non-maximal rank. These non-maximal rank finite-dimensional estimation algebras play an important role in Brockett's classification problem. © 1997 Elsevier Science B.V.

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1. Introduction

In the late 1970s, Brockett and Clark [2], Brockett [1], and Mitter [11] proposed the idea of using estimation algebras to construct finite dimensional nonlinear filters. It has been proved that the Lie algebra approach plays a fundamental role in the study of nonlinear filtering problem. The motivation came from the Wei–Norman approach [13] of using Lie algebraic ideas to solve time varying linear differential equations. In spite of the importance of the concept of estimation algebra, very little was known about estimation algebra. It was only recently that the structure and classification of finite dimensional exact estimation algebras were studied in detail in [12, 10]. In [14], the concept of Ω is introduced, which is defined as the matrix whose (i,j) element is $\partial f_j/\partial x_i - \partial f_i/\partial x_j$, where f is the drift term of the state evolution equation. For the class of exact filtering systems, Ω is identically zero. More recently, Yau [15] has studied filtering systems in which all entries of Ω are constants. He was able to classify all finite dimensional estimation algebras of maximal rank in such filtering systems. If the dimension of the state space is two, three, or four, then Chiou–Yau [7] Chen–Leung–Yau [3, 4] have shown, respectively, that all entries of Ω are constants as long as the estimation algebra is of maximal rank (see Section 2 for definition) and finite dimensional. Thus finite dimensional estimation algebra of maximal rank is completely classified if the dimension of the state space is

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at most four. The novelty of their theorems is that there is no a priori assumption on the drift term of the nonlinear filtering system.

Yau's approach of complete classification of finite-dimensional estimation algebras of maximal rank consists of two steps. The first step is to prove that for such an estimation algebra, all the entries in the Ω-matrix are degree one polynomials. The second step is to prove that in fact all the entries in Ω are constants. Then we can apply the result of Yau [15] to give a complete classification of finite dimensional estimation algebras of maximal rank. Most recently, Chen–Yau [5] has completed the first step of this approach. Thus we have a pretty good picture of finite dimensional estimation algebras of maximal rank.

In this paper, we shall study finite dimensional estimation algebras of non-maximal rank. Specifically, we shall give general construction of finite dimensional estimation algebras of non-maximal rank. This construction gives rise to a new class of finite dimensional nonlinear filters which are not discussed previously. We suspect that all finite dimensional estimation algebras of non-maximal rank are essentially arising in this way. In Section 4, we shall show that the four-dimensional non-maximal rank estimation algebra of Wong [14] is isomorphic (as Lie algebra) to one of our finite dimensional estimation algebras constructed in Section 3.

2. Basic concepts

The filtering problem considered here is based on the following signal observation model:

$$\begin{aligned} \mathrm{d}x(t) &= f(x(t))\,\mathrm{d}t + g(x(t))\,\mathrm{d}v(t), \quad x(0) = x_0, \\ \mathrm{d}y(t) &= h(x(t))\,\mathrm{d}t + \mathrm{d}w(t), \quad y(0) = 0 \end{aligned} \tag{2.1}$$

in which x, v, y and w are, respectively, R^n, R^p, R^m, and R^m valued processes, and v and w have components which are independent, standard Brownian processes. We further assume that $n = p, f, h$ are C^∞ smooth, and that g is an orthogonal matrix. We shall refer to $x(t)$ as the state of the system at time t and $y(t)$ as the observation at time t.

Let $\rho(t, x)$ denote the conditional probability density of the state given the observation $\{y(s)\colon 0 \leqslant s \leqslant t\}$. It is well known (see [9], for example) that $\rho(t, x)$ is given by normalizing a function, $\sigma(t, x)$, which satisfies the following Duncan–Mortensen–Zakai equation:

$$\mathrm{d}\sigma(t, x) = L_0\sigma(t, x)\,\mathrm{d}t + \sum_{i=1}^{m} L_i\sigma(t, x)\,\mathrm{d}y_i(t), \quad \sigma(0, x) = \sigma_0, \tag{2.2}$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i} - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2$$

and for $i = 1, \ldots, m$, L_i is the zero degree differential operator of multiplication by h_i. σ_0 is the probability density of the initial point x_0.

Before we proceed, we give the definition of a differential operator.

Definition 1. For any $I = (i_1, i_2, \ldots, i_n) \in \mathbb{Z}_+^n$ where $\mathbb{Z}_+^n$ denotes the set of nonnegative integers, we shall use the following standard notation:

$$D^I = D_1^{i_1} D_2^{i_2} \cdots D_n^{i_n}, \qquad |I| = i_1 + i_2 + \cdots + i_n.$$

By a differential operator in $x_1, \ldots, x_n$ variables, we mean an operator of the form $F = \sum_{|I| \leqslant r} a_I(x_1, \ldots, x_n) D^I$, where $a_I(x_1, \ldots, x_n)$'s are C^∞ functions. If one of the $a_I(x_1, \ldots, x_n)$, for $|I| = r$, is nonzero, we say that F is a differential operator of order r.

Eq. (2.2) is a stochastic partial differential equation. In real applications, we are interested in considering robust state estimators from observed sample paths with some properties of robustness. Davis [8] studied this problem and proposed some robust algorithms. In our case, his basic idea reduces to defining a new unnormalized density

$$\zeta(t,x) = \exp\left(-\sum_{i=1}^{m} h_i(x)y_i(t)\right)\sigma(t,x).$$

It is easy to show that $\zeta(t,x)$ satisfies the following time varying partial differential equation:

$$\frac{\partial\zeta}{\partial t}(t,x) = L_0\zeta(t,x) + \sum_{i=1}^{m} y_i(t)[L_0, L_i]\zeta(t,x) + \frac{1}{2}\sum_{i,j=1}^{m} y_i(t)y_j(t)[[L_0, L_i], L_j]\zeta(t,x),$$

$$\zeta(0,x) = \sigma_0,$$

where $[\cdot,\cdot]$ is the Lie bracket defined as follows.

Definition 2. If X and Y are differential operators, the Lie bracket of X and Y, $[X, Y]$, is defined by

$$[X, Y]\zeta = X(Y\zeta) - Y(X\zeta)$$

for any C^∞ function ζ.

Definition 3. The estimation algebra E, of the filtering system (2.1), is defined to be the Lie algebra generated by $\{L_0, L_1, \ldots, L_m\}$.

In [7], Chiou and Yau first introduced the concept of maximal rank estimation algebra.

Definition 4. The estimation algebra E of (2.1) is said to be an estimation algebra of maximal rank if for any $1 \leqslant i \leqslant n$, there exist constants c_i such that $x_i + c_i$ is in E.

In [14], the concept of Ω was introduced, which is defined as the matrix whose (i,j) component ω_{ij} is $\partial f_j/\partial x_i - \partial f_i/\partial x_j$. Define

$$D_i = \frac{\partial}{\partial x_i} - f_i \quad \text{and} \quad \eta = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{m} h_i^2.$$

Then

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^{n} D_i^2 - \eta\right).$$

The following theorem proved in [15] plays a fundamental role in the classification of finite dimensional estimation algebras.

Theorem 2.1. *Let E be a finite dimensional estimation algebra of* (2.1) *such that $\omega_{ij} = (\partial f_j/\partial x_i) - (\partial f_i/\partial x_j)$ are constants. If E is of maximal rank, then E is a real vector space of dimension $2n+2$ with basis given by $1, x_1, x_2, \ldots, x_n, D_1, D_2, \ldots, D_n$ and L_0.*

In [5], Chen and Yau have completed the first step of the program of classification of finite dimensional maximal rank estimation algebras. In fact they also proved the so-called Mitter conjecture [6].

Theorem 2.2 (Chen and Yau [6]). *Let E be a finite dimensional estimation algebra of* (2.1). *Let k be the maximal rank of those quadratic forms in E. Then*

1. *The observation terms $h_i(x)$, $1 \leqslant i \leqslant m$, are affine polynomials.*
2. (a) *ω_{ij}, for $1 \leqslant i \leqslant k$ or $1 \leqslant j \leqslant k$, are constants.*
   (b) *ω_{ij}, for $k+1 \leqslant i, j \leqslant n$, are degree one polynomials in $x_{k+1}, \dots, x_n$.*
3. *$\eta = \sum_{i=1}^{n} \partial f_i/\partial x_i + \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{m} h_i^2$ is a homogenous polynomial of degree 4. Moreover, η_4 (= homogenous polynomial of degree 4 part of η) depends only on the $x_{k+1}, \dots, x_n$ variables.*

The following theorem is proved in [5–7].

Theorem 2.3. *Suppose that the state space of the filtering system* (2.1) *is of dimension $n \leqslant 4$. If E is the finite dimensional estimation algebra of maximal rank, then the drift term f must be a linear vector field* (*i.e. each component is a polynomial of degree one*) *plus a gradient vector field and E is a real vector space of dimension $2n+2$ with basis given by $1, x_1, \dots, x_n, D_1, \dots, D_n$ and L_0. Moreover η is a degree 2 polynomial.*

In view of these theorems we have a pretty good picture of all finite dimensional estimation algebras of maximal rank.

3. Construction of a finite dimensional estimation algebra of non-maximal rank

Suppose that E is the finite dimensional estimation algebra associated to the filtering system (2.1). Consider the following enlarged filtering system:

$$\begin{aligned} d\tilde{x}(t) &= \tilde{f}(\tilde{x}(t))\,dt + \tilde{g}(\tilde{x}(t))\,d\tilde{v}(t), \quad \tilde{x}(0) = \tilde{x}_0, \\ dy(t) &= h(\tilde{x}(t))\,d(t) + dw(t), \quad y(0) = 0. \end{aligned} \tag{3.1}$$

Here $\tilde{x} = (x_1, \dots, x_n, x_{n+1}, \dots, x_{n+k})$, $\tilde{f}(\tilde{x}(t)) = (f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n), f_{n+1}(x_{n+1}, \dots, x_{n+k}), \dots, f_{n+k}(x_{n+1}, \dots, x_{n+k}))$, $\tilde{g}(\tilde{x}(t))$ = orthogonal matrix, $h(\tilde{x}(t)) = h(x_1, \dots, x_n)$, and $\tilde{v}$ and w have components which are independent, standard Brownian processes.

Let $\tilde{E}$ be the estimation algebra associated to (3.1). We shall show that $\tilde{E}$ is isomorphic to E as a Lie algebra. Observe that

$$\tilde{\omega}_{ij} = \frac{\partial \tilde{f}_j}{\partial x_i} - \frac{\partial \tilde{f}_i}{\partial x_j} = \begin{cases} \dfrac{\partial f_j}{\partial x_i}(x) - \dfrac{\partial f_i}{\partial x_j}(x) = \omega_{ij} & \text{if } 1 \leqslant i, j \leqslant n, \\ 0 & \text{if } i \geqslant n, j \leqslant n \text{ or } i \leqslant n, j \geqslant n, \\ \dfrac{\partial f_j}{\partial x_i}(x_{n+1}, \dots, x_{n+k}) - \dfrac{\partial f_i}{\partial x_j}(x_{n+1}, \dots, x_{n+k}), & i, j \geqslant n, \end{cases}$$

$$\tilde{L}_0 = \frac{1}{2}\left(\sum_{i=1}^{n+k} D_i^2 - \tilde{\eta}\right),$$

where

$$D_{n+i} = \frac{\partial}{\partial x_{n+i}} - f_{n+i}(x_{n+1}, \dots, x_{n+k}), \quad 1 \leqslant i \leqslant k,$$

$$\tilde{\eta}(\tilde{x}) = \eta(x) + \sum_{i=n+1}^{n+k} \frac{\partial f_j}{\partial x_i}(x_{n+1}, \dots, x_{n+k}) + \sum_{i=n+1}^{n+k} f_i^2(x_{n+1}, \dots, x_{n+k}).$$

Lemma 3.1. *If F is a differential operator in $x_1, \ldots, x_n$ variables of order r, then $[F, \tilde{\eta}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k})] = [F, \eta(x_1, \ldots, x_n)]$ is a differential operator of order $r-1$ in $x_1, \ldots, x_n$ variables.*

Proof. We shall prove this by using induction on the order of the differential operator. If the order of F is zero, then F is a function and hence $[F, \tilde{\eta}] = 0 = [F, \eta]$. If $F = \sum_{i=1}^n a_i(x_1, \ldots, x_n)D_i + b(x_1, \ldots, x_n)$ is a differential operator of order one in $x_1, \ldots, x_n$ variables, then

$$[F, \tilde{\eta}] = \sum_{i=1}^{n} a_i(x_1, \ldots, x_n)\frac{\partial \tilde{\eta}}{\partial x_i} = \sum_{i=1}^{n} a_i(x_1, \ldots, x_n)\frac{\partial \eta}{\partial x_i} = [F, \eta].$$

Suppose that the lemma is true for all differential operators of order r in $x_1, \ldots, x_n$ variables. Let $F = \sum_{|I| \leqslant r+1} a_I(x_1, \ldots, x_n)D^I$ be a differential operator of order $r+1$ in $x_1, \ldots, x_n$ variables. We shall show that $[F, \tilde{\eta}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k})] = [F, \eta(x_1, \ldots, x_n)]$ is a differential operator of order r in $x_1, \ldots, x_n$ variables:

$$\begin{aligned}
&[F, \tilde{\eta}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k})] \\
&\quad = \sum_{|I|=r+1} a_I(x_1, \ldots, x_n)[D_1^{i_1} \cdots D_n^{i_n}, \tilde{\eta}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k})] \\
&\qquad + \left[\sum_{|I| \leqslant r} a_I(x_1, \ldots, x_n)D_1^{i_1} \cdots D_n^{i_n}, \tilde{\eta}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k})\right] \\
&\quad = \sum_{|I|=r+1} a_I(x_1, \ldots, x_n)[D_1^{i_1} \cdots D_n^{i_n}, \tilde{\eta}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k})] \\
&\qquad + \left[\sum_{|I| \leqslant r} a_I(x_1, \ldots, x_n)D_1^{i_1} \cdots D_n^{i_n}, \eta(x_1, \ldots, x_n)\right].
\end{aligned}$$

The last equality is a consequence of the induction hypothesis. Hence Lemma 3.1 will follow if we can show that for $i_1 + \cdots + i_n = r+1$,

$$[D_1^{i_1} \cdots D_n^{i_n}, \tilde{\eta}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k})] = [D_1^{i_1} \cdots D_n^{i_n}, \eta(x_1, \ldots, x_n)]$$

and is a differential operator of order r in $x_1, \ldots, x_n$ variables. In general for any differential operators X, Y, and Z, we have the following formula:

$$[XY, Z] = X[Y, Z] + [X, Z]Y.$$

It follows that for $i_1 + \cdots + i_n = r+1$,

$$\begin{aligned}
&[D_1^{i_1} \cdots D_n^{i_n}, \tilde{\eta}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k})] \\
&\quad = D_1[D_1^{i_1-1}D_2^{i_2} \cdots D_n^{i_n}, \tilde{\eta}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k})] \\
&\qquad + [D_1, \tilde{\eta}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k})]D_1^{i-1}D_2^{i_2} \cdots D_n^{i_n} \\
&\quad = D_1[D_1^{i_1-1}D_2^{i_2} \cdots D_n^{i_n}, \eta(x_1, \ldots, x_n)] + [D_1, \eta(x_1, \ldots, x_n)]D_1^{i_1-1}D_2^{i_2} \cdots D_n^{i_n} \\
&\quad = [D_1^{i_1} \cdots D_n^{i_n}, \eta(x_1, \ldots, x_n)].
\end{aligned}$$

Observe that $[D_1^{i_1-1}D_2^{i_2} \cdots D_n^{i_n}, \eta(x_1, \ldots, x_n)]$ is a differential operator of order $r-2$ in $x_1, \ldots, x_n$ variables. Therefore $[D_1^{i_1} \cdots D_n^{i_n}, \tilde{\eta}]$ is a differential operator of order r in $x_1, \ldots, x_n$ variables. □

Lemma 3.2. *Let F be a differential operator of order r in $x_1, \ldots, x_n$ variables. Then for $n+1 \leqslant j \leqslant n+k$, $[D_j^2, F] = 0$.*

Proof. Observe that

$$[D_j, a_I(x_1, \dots, x_n)] = \frac{\partial a_I}{\partial x_j}(x_1, \dots, x_n) = 0 \quad \text{for } n+1 \leqslant j \leqslant n+k$$

and

$$[D_j, D_i] = \frac{\partial f_i}{\partial x_j}(x_1, \dots, x_n) - \frac{\partial f_j}{\partial x_i}(x_{n+1}, \dots, x_{n+k}) = 0 \quad \text{for } 1 \leqslant i \leqslant n \text{ and } n+1 \leqslant j \leqslant n+k.$$

Therefore, for $n+1 \leqslant j \leqslant n+k$

$$\begin{aligned}[D_j^2, F] &= \left[D_j^2, \sum_{|I| \leqslant r} a_I(x_1, \dots, x_n) D_1^{i_1} \cdots D_n^{i_n} \right] \\ &= \sum_{|I| \leqslant r} D_j^2 a_I(x_1, \dots, x_n) D_1^{i_1} \cdots D_n^{i_n} - \sum_{|I| \leqslant r} a_I(x_1, \dots, x_n) D_1^{i_1} \cdots D_n^{i_n} D_j^2 \\ &= 0. \qquad \square\end{aligned}$$

Lemma 3.3. *For any differential operator F in $x_1, \dots, x_n$ variables, $[\tilde{L}_0, F] = [L_0, F]$.*

Proof.

$$\begin{aligned}[\tilde{L}_0, F] &= \left[\frac{1}{2}\left(\sum_{i=1}^{n+k} D_i^2 - \tilde{\eta} \right), F \right] = \left[\frac{1}{2} \sum_{i=1}^{n} D_i^2, F \right] + \left[\frac{1}{2} \sum_{j=n+1}^{n+k} D_j^2, F \right] + \tfrac{1}{2}[F, \tilde{\eta}] \\ &= \left[\frac{1}{2} \sum_{i=1}^{n} D_i^2, F \right] + \tfrac{1}{2}[F, \eta] = \left[\frac{1}{2}\left(\sum_{i=1}^{n} D_i^2 - \eta \right), F \right] = [L_0, F]. \qquad \square\end{aligned}$$

Theorem 3.4. *The estimation algebra $\tilde{E}$ associated to the filtering system (3.1) is isomorphic to the estimation algebra E associated to the filtering system (2.1), and $\tilde{E}$ consists of a basis such that all elements in this basis are differential operators in $x_1, \dots, x_n$ variables except $\tilde{L}_0$. Furthermore, $\tilde{\Omega} = (\partial \tilde{f}_j/\partial x_i - \partial \tilde{f}_i/\partial x_j)$ is given by*

$$\tilde{\Omega} = \begin{pmatrix} \Omega & 0 \\ 0 & A \end{pmatrix},$$

where Ω is the $n \times n$ matrix $(\partial f_j/\partial x_i(x) - \partial f_i/\partial x_j(x))$ associated to (2.1) and A is a $k \times k$ matrix with (i, j)-entry $(\partial f_{n+j}/\partial x_{n+i} - \partial f_{n+i}/\partial x_{n+j})(x_{n+1}, \dots, x_{n+k})$.

Proof. Observe that for $1 \leqslant l \leqslant m$

$$\begin{aligned}[\tilde{L}_0, h_l(x_1, \dots, x_n)] &= \left[\frac{1}{2}\left(\sum_{i=1}^{n+k} D_i^2 - \tilde{\eta} \right), h_l(x_1, \dots, x_n) \right] \\ &= \left[\frac{1}{2} \sum_{i=1}^{n} D_i^2, h_l(x_1, \dots, x_n) \right] + \left[\frac{1}{2} \sum_{j=n+1}^{n+k} D_j^2, h_l(x_1, \dots, x_n) \right] \\ &= \left[\frac{1}{2}\left(\sum_{i=1}^{n} D_i^2 - \eta \right), h_l(x_1, \dots, x_n) \right] + \sum_{j=n+1}^{n+k} \left[\frac{\partial h_l}{\partial x_j} + \frac{1}{2}\frac{\partial^2 h_l}{\partial x_j^2}(x_1, \dots, x_n) \right] \\ &= [L_0, h_l(x_1, \dots, x_n)],\end{aligned}$$

where $[L_0, h_l(x_1, \dots, x_n)]$ is a differential operator of order one in $x_1, \dots, x_n$ variables. In view of Lemma 3.3, we have

$$[\tilde{L}_0, [\tilde{L}_0, h_l(x_1, \dots, x_n)]] = [\tilde{L}_0, [L_0, h_l(x_1, \dots, x_n)]] = [L_0, [L_0, h_l(x_1, \dots, x_n)]].$$

For two differential operators X, Y we define $Ad_X Y = [X, Y]$. By induction, we can show that for any positive integer q,

$$Ad^q_{\tilde{L}_0} h_l(x_1, \dots, x_n) = Ad^q_{L_0} h_l(x_1, \dots, x_n).$$

It follows that $\tilde{E} = E$. It is also clear from the proof that $\tilde{E}$ consists of a basis such that all elements in this basis is a differential operator in $x_1, \dots, x_n$ variables except $\tilde{L}_0$. This finishes the proof of the Main Theorem. □

Remarks. (1) The finite dimensional estimation algebra $\tilde{E}$ is of non-maximal rank if $k > 0$. Observe also that A is quite arbitrary.

(2) We would like to emphasize that the orthogonal matrix $\tilde{g}(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+k})$ is arbitrary and is not necessarily of the form

$$\begin{pmatrix} g(x_1, \dots, x_n) & 0 \\ 0 & g_1(x_{n+1}, \dots, x_{n+k}) \end{pmatrix}.$$

So (3.1) is not a direct sum of two filtering systems. This is a crucial point of our theorem.

4. Wong's four-dimensional estimation algebra

Example 4.1. In [14], Wong considered the following filtering system defined on $\mathbb{R}^3$

$$\begin{aligned} &\mathrm{d}x_1 = (x_1 + x_2 + x_3 + \gamma(x_1 + x_2 + x_3))\,\mathrm{d}t + \mathrm{d}w_1, \qquad \mathrm{d}x_2 = (x_1 + x_3)\,\mathrm{d}t + \mathrm{d}w_2, \\ &\mathrm{d}x_3 = (x_1 + x_2)\,\mathrm{d}t + \mathrm{d}w_3, \qquad \mathrm{d}y = (x_2 - x_3)\,\mathrm{d}t + \mathrm{d}v, \end{aligned} \tag{4.1}$$

where γ is a C^∞ function with a bounded, non-zero first derivative and $w = (w_1, w_2, w_3)$ and v are independent, standard Brownian processes. Then

$$\Omega = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \gamma'(x_1 + x_2 + x_3),$$

$$f_1(x) = x_1 + x_2 + x_3 + \gamma(x_1 + x_2 + x_3), \qquad f_2(x) = x_1 + x_3, \qquad f_3(x) = x_1 + x_2,$$

$$h(x) = x_2 - x_3, \qquad D_i = \frac{\partial}{\partial x_i} - f_i(x), \quad 1 \leqslant i \leqslant 3,$$

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^{n} D_i^2 - \eta\right),$$

where

$$\begin{aligned} \eta &= \sum_{i=1}^{3} \frac{\partial f_j}{\partial x_i} + \sum_{i=1}^{3} f_i^2 + h^2(x) \\ &= 1 + \gamma'(x_1 + x_2 + x_3) + [x_1 + x_2 + x_3 + \gamma(x_1 + x_2 + x_3)]^2 + (x_1 + x_3)^2 + (x_1 + x_2)^2 + (x_2 - x_3)^2. \end{aligned}$$

It is easy to see that

$$[L_0, h(x)] = [L_0, x_2] - [L_0, x_3] = D_2 - D_3$$

and

$$\begin{aligned}[L_0, D_2 - D_3] &= [L_0, D_2] - [L_0, D_3] \\ &= \sum_{i=1}^{3}\left(\omega_{2i}D_i + \frac{1}{2}\frac{\partial \omega_{2i}}{\partial x_i}\right) + \frac{1}{2}\frac{\partial \eta}{\partial x_2} - \sum_{i=1}^{3}\left(\omega_{3i}D_i + \frac{1}{2}\frac{\partial \omega_{3i}}{\partial x_i}\right) - \frac{1}{2}\frac{\partial \eta}{\partial x_3} \\ &= \omega_{21}D_1 + \frac{1}{2}\frac{\partial \omega_{21}}{\partial x_1} + \frac{1}{2}\frac{\partial \eta}{\partial x_2} - \omega_{31}D_1 - \frac{1}{2}\frac{\partial \omega_{31}}{\partial x_1} - \frac{1}{2}\frac{\partial \eta}{\partial x_3} \\ &= \frac{1}{2}\frac{\partial \eta}{\partial x_2} - \frac{1}{2}\frac{\partial \eta}{\partial x_3} = 3(x_2 - x_3).\end{aligned}$$

Therefore, E is a four-dimensional Lie algebra with basis given by $\langle 1, x_2 - x_3, D_2 - D_3, L_0\rangle$.

We now claim that Wong's four-dimensional estimation algebra E above is isomorphic to one of the estimation algebras in our Main Theorem. Consider $n = 1$ and $k = 2$ in our Main Theorem. Let us look at the following filtering system:

$$\begin{aligned}&\mathrm{d}x_1 = \left(\sqrt{\alpha - 1}\,x_1 + \frac{\beta}{\sqrt{\alpha - 1}}\right)\mathrm{d}t + \mathrm{d}w_1 \quad \alpha, \beta \in \mathbb{R} \text{ and } \alpha > 1, \\ &\mathrm{d}x_2 = f_2(x_2, x_3)\,\mathrm{d}t + \mathrm{d}w_2, \qquad \mathrm{d}x_3 = f_3(x_2, x_3)\,\mathrm{d}t + \mathrm{d}w_3, \qquad \mathrm{d}y = x_1\,\mathrm{d}t + \mathrm{d}v,\end{aligned} \tag{4.2}$$

where f_2, f_3 are C^∞ functions with a suitable growth rate so that (4.2) is well-defined for all time, $\mathrm{d}w_i$ and $\mathrm{d}v$ are independent Brownian motions. Then

$$\Omega = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega_{23} \\ 0 & -\omega_{23} & 0 \end{bmatrix},$$

where $\omega_{23} = \partial f_3/\partial x_2(x_2, x_3) - \partial f_2/\partial x_3(x_2, x_3)$. It is easy to see that

$$[\tilde{L}_0, x_1] = \tilde{D}_1, \qquad [\tilde{D}_1, x_1] = 1, \qquad [\tilde{L}_0, \tilde{D}_1] = \alpha x_1 + \beta.$$

Therefore the estimation algebra $\tilde{E}$ of (4.2) is four-dimensional with basis given by $\langle 1, x_1, \tilde{D}_1, \tilde{L}_0\rangle$. The isomorphism from our estimation algebra $\tilde{E}$ to Wong's estimation algebra E is explicitly given by

$$\phi : \tilde{E} = \langle 1, x_1, \tilde{D}_1, \tilde{L}_0\rangle \to E = \langle 1, x_2 - x_3, D_2 - D_3, L_0\rangle,$$

$$\begin{aligned}&\phi(a + bx_1 + c\tilde{D}_1 + \mathrm{d}\tilde{L}_0) \\ &\quad = a\left(2\alpha\sqrt{\frac{\alpha}{3}}\right) + b\left[\sqrt{\alpha}(x_2 - x_3) - 2\sqrt{\frac{\alpha}{3}}\beta\right] + c\left(\sqrt{\frac{\alpha}{3}}(D_2 - D_3)\right) + d\,\frac{\alpha}{3}L_0.\end{aligned}$$

It is easy to check that this linear isomorphism actually preserves Lie bracket structures. So ϕ is a Lie algebra isomorphism. For the sake of convenience to the reader, we include the multiplication table of these two Lie algebras:

E	1	$x_2 - x_3$	$D_2 - D_3$	L_0
1	0	0	0	0
$x_2 - x_3$	0	0	-2	$-(D_2 - D_3)$
$D_2 - D_3$	0	2	0	$-3(x_2 - x_3)$
L_0	0	$D_2 - D_3$	$3(x_2 - x_3)$	0

$\tilde{E}$	1	x_1	$\tilde{D}_1$	$\tilde{L}_0$
1	0	0	0	0
x_1	0	0	-1	$-D_1$
$\tilde{D}_1$	0	1	0	$-\alpha x_1 - \beta$
$\tilde{L}_0$	0	D_1	$\alpha x_1 + \beta$	0

In fact (4.1) can be directly transformed by the change of variables

$$\begin{cases} z_1 = \dfrac{x_2 - x_3}{\sqrt{2}}, \\[2ex] z_2 = \dfrac{x_2 + x_3}{\sqrt{2}}, \\[2ex] z_3 = x_1, \end{cases} \qquad \begin{cases} x_1 = z_3, \\[2ex] x_2 = \dfrac{z_1 + z_2}{\sqrt{2}}, \\[2ex] x_3 = \dfrac{z_2 - z_1}{\sqrt{2}}, \end{cases}$$

into the system

$$\begin{aligned} &\mathrm{d}z_1 = -z_1\,\mathrm{d}t + \mathrm{d}\tilde{w}_1, \qquad \mathrm{d}z_2 = (\sqrt{2}z_3 + z_2)\,\mathrm{d}t + \mathrm{d}\tilde{w}_2, \\ &\mathrm{d}z_3 = (z_3 + \sqrt{2}z_2 + \gamma(z_3 + \sqrt{2}z_2))\,\mathrm{d}t + \mathrm{d}\tilde{w}_3, \end{aligned} \tag{4.3}$$

where $(\tilde{w}_1, \tilde{w}_2, \tilde{w}_3) = ((w_2 - w_3)/\sqrt{2}, (w_2 + w_3)/\sqrt{2}, w_1)$ is a new Brownian motion.

We present another example in which g is not of a special form as we mentioned in Remark (2) of Section 3.

Example 4.2.

$$\begin{cases} \begin{pmatrix} \mathrm{d}z_1 \\ \mathrm{d}z_2 \\ \mathrm{d}z_3 \end{pmatrix} = \begin{pmatrix} -z_1\,\mathrm{d}t \\ (\sqrt{2}z_3 + z_2)\,\mathrm{d}t \\ z_3 + \sqrt{2}z_2 + \gamma(z_3 + \sqrt{2}z_2))\,\mathrm{d}t \end{pmatrix} + \begin{pmatrix} 0 & \cos z_2 & -\sin z_2 \\ 0 & \sin z_2 & \cos z_2 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{d}\tilde{w}_1 \\ \mathrm{d}\tilde{w}_2 \\ \mathrm{d}\tilde{w}_3 \end{pmatrix} \\[2ex] \mathrm{d}y = \sqrt{2}z_1\,\mathrm{d}t + \mathrm{d}v \end{cases}$$

It is clear that the system cannot be split into two subsystems. We can apply the Lie algebra approach to find the conditional probability density.

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