A Counterexample of "Comments on 'Stability Margin Evaluation for Uncertain Linear Systems' "

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Abstract— **In this paper, a counterexample of the above-mentioned paper[1] is presented, and it shows that the main result of this paper is not correct.**

Consider the uncertain linear dynamic system of order n

$$\dot{x}(t) = A(t)x(t) = [A_0 + \Delta A(t)]x(t) \tag{1}$$

where A_0 is the nominal stable system matrix and $\Delta A(t)$ is the time-varying uncertainty.

Recently, Gong and Thompson [1] have given a stability margin evaluation method for this unstructured matrix. The criterion in Theorem 2 of this paper is

$$\Delta A^T \Delta A < \tfrac{1}{4}\sigma_{\min}^2(U + U^T)A_0^T A_0.$$

It is also claimed in this paper that this bound is the tightest bound possible for all unstructured perturbations, such that (1) keeps its asymptotic stability. But this bound was shown not to be the tightest for all unstructured perturbations by the above-mentioned paper,[1] and a new bound also has been given; however, it is not right.

Let us consider the example of the paper[1] whose system matrix is

$$A_0 = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}.$$

In this paper, it is shown that the system is guaranteed to be asymptotically stable by Theorem 1 if

$$\Delta A^T \Delta A < \frac{0.707^2}{0.708^2} A_0^T A_0. \tag{2}$$

Let

$$\Delta A = \begin{bmatrix} 3.1 & 1.9 \\ 0 & 0 \end{bmatrix}.$$

It is obvious that $\Delta A^T \Delta A < (0.707^2/0.708^2)A_0^T A_0$. But

$$A_0 + \Delta A = \begin{bmatrix} 0.1 & -0.1 \\ 1 & 0 \end{bmatrix}$$

is not stable because its two eigenvalues are $0.05 \pm 0.3122i$.

References

[1] C. Gong and S. Thompson, "Stability margin evaluation for uncertain linear systems," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 548–550, 1994.

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Filtering Systems with Finite-Dimensional Estimation Algebras

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Abstract— **Estimation algebra turns out to be a useful concept in the investigation of finite-dimensional nonlinear filters. In this paper we study the natural question of classifying all filtering systems with finite-dimensional estimation algebras up to state-space diffeomorphism. In particular, we present some results on partial differential equations arising from the study of stochastic systems and nonlinear filtering problems.**

***Index Terms*—Estimation algebra, nonlinear filters, under-determined partial differential equation.**

I. Introduction

In many filtering systems and stochastic control problems, one has to deal with elliptic differential operators of a certain type. For example, consider a filtering problem based on the following signal observation model:

$$\begin{aligned} dx(t) &= f(x(t))\,dt + dv(t), \qquad x(0) = x_0 \\ dy(t) &= Hx(t)\,dt + dw(t), \qquad y(0) = 0 \end{aligned} \tag{1}$$

in which x, v, y, and w, are respectively, R^n, R^p, R^m, and R^m-valued processes, and v and w have components that are independent standard Brownian processes.

$\rho(t,x)$, the conditional probability density of the state $x(t)$, given the observation $\{y(s): 0 \le s \le t\}$, is determined by the Duncan–Mortensen–Zakai equation, which in the *un-normalized* form is given by (see [9] for example)

$$\begin{aligned} \frac{d}{dt}\sigma(t,x) &= L_0\sigma(t,x)\,dt + \sum_{i=1}^m L_i\sigma(t,x)\,dy_i(t) \\ \sigma(0,x) &= \sigma_0 \end{aligned} \tag{2}$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^n f_i\frac{\partial}{\partial x_i} - \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^m h_i^2$$

and for $i = 1, \cdots, m, L_i$ is the zero-degree differential operator of multiplication by h_i. (If p is a vector, we use the notation p_i to represent the ith component of p.) σ_0 is the probability density of the initial point x_0. When the observation is absent, that is $h = 0$, then (2) is simply the Kolgomorov equation.

It is important to find efficient ways to solve (2), which is the subject of many research studies in nonlinear filtering theory. A particularly useful concept is that of an estimation algebra, which was introduced in [3], [4], and [12]. The survey paper [11] provides a good introduction and many useful references to the concept. It is defined to be the Lie algebra of differential operators generated

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by $\{L_0, L_1, \cdots, L_m\}$. The elliptic differential operator can be more compactly represented if one defines $D_i = (\partial/\partial x_i) - f_i$. Then

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^{n} D_i^2 - \eta\right) \tag{3}$$

where

$$\eta = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{m} h_i^2. \tag{4}$$

This compact representation of L_0 was exploited in [17] and [18] to derive necessary conditions and sufficient conditions for *estimation algebras* to be finite dimensional. More directly, it is easier to understand the importance of this particular representation by noting that

$$D_i = \frac{\partial}{\partial x_i} - f_i = e^{F_i}\frac{\partial}{\partial x_i}e^{-F_i} \tag{5}$$

where $F_i = F_i(x) = \int_0^x f_i(t)\,dt$. Hence

$$D_i^2 = e^{F_i}\frac{\partial^2}{\partial x_i^2}e^{-F_i} \tag{6}$$

and

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^{n} e^{F_i}\frac{\partial^2}{\partial x_i^2}e^{-F_i} - \eta\right). \tag{7}$$

In the special case where f is the vector field of a potential function ϕ, (7) can be simplified as

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^{n} e^{\phi}\frac{\partial^2}{\partial x_i^2}e^{-\phi} - \eta\right). \tag{8}$$

By defining $\xi = e^{-\phi}\sigma$, (2) can be reformulated as

$$\begin{aligned} \frac{d}{dt}\xi(t,x) &= \frac{1}{2}(\Delta - \eta)\xi(t,x)\,dt + \sum_{i=1}^{m} L_i\xi(t,x)\,dy_i(t) \\ \xi(0,x) &= e^{-\phi}\sigma_0 \end{aligned} \tag{9}$$

where Δ denotes the Laplacian operator. If η is a quadratic polynomial in x, then the semigroup generated by the differential operator $\Delta - \eta$ is well known and can be used to explicitly derive solutions to the equation when h_i's are linear in x. Thus, there is a connection between this representation and the *gauge transformation*, as pointed out in [13]. This idea is also related with the concept of *equivalence* of parabolic equations as discussed in [1].

In [15], a finite-dimensional filter was explicitly derived for this case (f is a gradient vector field and h_i's are linear in x), using the so-called Wei–Norman–Brockett–Mitter approach. A crucial argument of this approach is that the estimation algebra is finite, which in turns depends on whether η is a quadratic polynomial if the estimation algebra is of *maximal rank*, that is, it contains elements of the form $x_i + c_i$ for $1 \le i \le n$.

All these results were extended later on by [19] to the case where f is a sum of a linear and a gradient vector field. In particular, the following extension of the main theorems in [15] holds.

Theorem 1.1: Let E be an estimation algebra of (1) satisfying $(\partial f_j/\partial x_i) - (\partial f_i/\partial x_j) = c_{ij}$, where c_{ij} are constants for all $1 \le i, j \le n$. Assume also that h is linear in x.

1) If η is a polynomial of degree at most two, then E is finite dimensional.
2) Conversely, if E is finite-dimensional and *maximal rank*, then η is a polynomial of degree at most two.

Theorem 1.2: Let E be an estimation algebra of (1) satisfying $(\partial f_j/\partial x_i) - (\partial f_i/\partial x_j) = c_{ij}$ where c_{ij} are constants for all $1 \le i, j \le n$. Suppose E is finite dimensional of maximal rank. Then $\eta = \Sigma_{i,j=1}^n a_{ij}x_ix_j + \Sigma_{i=1}^n b_ix_i + d$ where $a_{ij}, b_i,$ and d are constants for all $1 \le i, j \le n$, and the robust form of the Duncan–Mortensen–Zakai equation

$$\begin{aligned} \frac{\partial \xi}{\partial t}(t,x) &= L_0\xi(t,x) + \sum_{i=1}^{m} y_i(t)[L_0, L_i]\xi(t,x) \\ &\quad + \frac{1}{2}\sum_{i,j=1}^{m} y_i(t)y_j(t)[[L_0, L_i], L_j]\xi(t,x) \end{aligned} \tag{10}$$

has a solution for all $t \ge 0$ of the form

$$\begin{aligned} \xi(t,x) &= e^{T(t)}e^{r_n(t)x_n}\cdots e^{r_1(t)x_1}e^{s_n(t)D_n}\cdots e^{s_1(t)D_1} \\ &\quad \cdot e^{tL_0}\sigma_0 \end{aligned} \tag{11}$$

where $T(t), r_1(t), \cdots, r_n(t), s_1(t), \cdots, s_n(t)$ satisfy the ordinary differential equations (12)–(14). For $1 \le i \le n$

$$\frac{ds_i}{dt}(t) = r_i(t) + \sum_{j=1}^{n} s_j(t)c_{ji} + \sum_{k=1}^{m} h_{ki}y_k(t) \tag{12}$$

where $h_k(x) = \Sigma_{j=1}^n h_{kj}x_j + e_k$ for $1 \le k \le m; h_{kj}$ and e_k are constants.

For $1 \le j \le n$

$$\frac{dr_j}{dt}(t) = \frac{1}{2}\sum_{i=1}^{n} s_i(t)(a_{ij} + a_{ji}) \tag{13}$$

and

$$\begin{aligned} \frac{dT}{dt}(t) &= -\frac{1}{2}\sum_{i=1}^{n} r_i^2(t) - \frac{1}{2}\sum_{i=1}^{n} s_i^2(t)\left(\sum_{j=1}^{n} c_{ij}^2 - a_{ii}\right) \\ &\quad + \sum_{1\le i<k\le n} s_i(t)s_k(t) \\ &\quad \cdot \left(\sum_{j=1}^{n} c_{ij}c_{jk} + \frac{1}{2}(a_{ik} + a_{ki})\right) \\ &\quad + \sum_{i=1}^{n} r_i(t) - \sum_{j=2}^{n}\sum_{i=1}^{j} s_j(t)c_{ij} + \frac{1}{2}\sum_{i=1}^{n} s_i(t)b_i \\ &\quad + \frac{1}{2}\sum_{i,j=1}^{m} y_i(t)y_j(t)\left(\sum_{k=1}^{n} h_{ik}h_{jk}\right) - \sum_{i,j=1}^{n} s_i(t) \\ &\quad \cdot s_j(t). \end{aligned} \tag{14}$$

Given the importance of the estimation algebra, a natural question arises as to whether we can classify all finite-dimensional estimation algebras up to Lie-algebraic isomorphism. Building on the work of [14], [15], and [7], the classification of estimation algebras of maximal rank was achieved by Chiou–Yau [6] and Chen–Leung–Yau [5] if state-space dimension is two and three, respectively. A second question that arises naturally is whether we can classify all filtering systems with finite-dimensional estimation algebras up to state-space diffeomorphism. This is apparently a very difficult problem and requires a careful study of partial differential equations of the type (4). A case of these types of equations and the nonlinear filtering problem was first noted by Benes (see [2]) and was studied in some detail in our previous paper [8]. In this paper, we are going to study the general under-determined system which has not been studied before. The class of nonlinear filtering systems with a finite-dimensional estimation algebra can be characterized by the solutions of this family of under-determined nonlinear partial differential equations. These equations are the focal point of this study. We will give a simple algebraic necessary and sufficient condition for the existence of solutions of these equations. We will also study the growth property of solutions of these equations. Although we cannot completely describe all the solutions of these equations, we will provide various ways of constructing solutions of these equations. Our result here is far from providing a reasonable classification theory of systems with finite-dimensional estimation algebras, but it may be viewed as a necessary step.

II. Classification Theorems

For any filtering system defined in (1) with an estimation algebra E, (4) assigns a characteristic η. Theorem 1.1 implies that if the estimation algebra is finite dimensional with maximal rank, then this maps the given system to a quadratic polynomial. In order to develop a classification of systems with finite-dimensional estimation algebras, we also need to understand the properties of the inverse of this mapping. In the following, we will provide some partial results to these questions. The key to these questions is a complete understanding of the existence and uniqueness properties of the following equation on R^n:

$$\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}+\sum_{i=1}^{n} f_i^2 = q \tag{15}$$

where q is a C^∞ function on R^n. Using the vector notation $f = (f_1, f_2, \cdots, f_n)$, (15) becomes

$$\nabla\cdot f + |f|^2 = q. \tag{16}$$

This is a Ricatti-type partial differential equation. When the estimation algebra is exact, i.e., $f_i = (\partial\phi/\partial x_i)$ for certain potential function ϕ, (15) reduces to

$$\Delta\phi + |\nabla\phi|^2 = q. \tag{17}$$

By letting $\zeta = \exp\phi$, (17) becomes

$$\Delta\zeta - q\zeta = 0. \tag{18}$$

This equation has been studied extensively (see [10]).

Theorem 2.1 [10]: Let q be a C^∞ function defined on R^n. There exists a positive function ζ satisfying (18) on R^n if and only if the first eigenvalue $\lambda_1(\Delta - q)$ is nonnegative on R^n, where $\lambda_1(\Delta - q)$ is defined by

$$\lambda_1(\Delta - q) = \inf_{\phi\in C_0^\infty}\frac{\displaystyle\int|\nabla\phi|^2 + q\phi^2}{\displaystyle\int\phi^2}. \tag{19}$$

The following theorem gives a simple necessary and sufficient condition for an under-determined system (15) to have a solution. It is rather surprising that the existence conditions for an under-determined system (15) and a scalar elliptic equation (18) turn out to be identical, although as we will see in the next section (15) has many more solutions than (18) does.

Theorem 2.2: There exists a smooth vector field f on R^n satisfying (16) if and only if the first eigenvalue $\lambda_1 := \lambda_1(\Delta - q)$ [defined in (19)] is nonnegative on R^n.

Proof: The proof follows from Theorem 2.1; $\lambda_1 \geq 0$ implies the existence of a smooth solution to the following equation:

$$\Delta\phi + |\nabla\phi|^2 = q.$$

It is easy to verify that $f = \nabla\phi$ solves (16). We shall call this a *gradient solution*.

Now, we need to prove that the condition is also necessary. Let $\phi \in C_0^\infty$ be any C^∞ function with compact support. Multiply (16) with ϕ^2 and integrate the equation over R^n. We get

$$\begin{aligned}\int\phi^2 q &= \int\phi^2\nabla\cdot f + \int\phi^2|f|^2\\ &= -\int 2\phi\nabla\phi\cdot f + \int\phi^2|f|^2.\end{aligned} \tag{20}$$

The Schwartz inequality gives us

$$\int\phi^2 q \geq -\int|\nabla\phi|^2 - \int\phi^2|f|^2 + \int\phi^2|f|^2$$

which is equivalent to

$$\int|\nabla\phi|^2 + q\phi^2 \geq 0.$$

This implies $\lambda_1 \geq 0$ according to our definition in (19). ∎

In view of Theorem 1.1, we are particularly interested in finding the precise condition for the under-determined partial differential equation (16) to have a solution if q is a quadratic polynomial. The following is the corollary of Theorem 2.2 and our previous result [8, Th. 8].

Let q be a quadratic polynomial in $x_1, \cdots, x_n$. After an orthogonal transformation and a translation, q can be written in the form $\Sigma_{i=1}^n a_i x_i^2 - c$, where a_i and c are constants.

Corollary 2.3: There exists a smooth vector field f on R^n satisfying $\nabla\cdot f + |f|^2 = \Sigma_{i=1}^n a_i x_i^2 - c$ if and only if $a_i \geq 0$ and $c \leq \Sigma_{i=1}^n \sqrt{a_i}$.

For the remainder of this section, we use C to denote any controllable constants.

In [8], we have shown that the solutions of (17) have at most linear growth. This means that gradient solutions of (16) have at most linear growth. We conjecture that the same holds for all solutions of (16). This is a very difficult question, as it would imply the classical gradient estimate, which is a hard subject by itself.

Conjecture II.1: Suppose f is a smooth vector field on R^n satisfying

$$\nabla\cdot f + |f|^2 = q$$

where q is quadratic polynomial. Then

$$|f| \leq C(1 + |x|).$$

However, in terms of the general finite-dimensional filters constructed in [19], we assume that the drift term f satisfies the additional condition $(\partial f_j/\partial x_i) - (\partial f_i/\partial x_j) = c_{ij}$, where c_{ij} are constants. Therefore, the following theorem is very interesting for application purposes.

Theorem 2.4: Suppose that $f = (f_1, f_2, \cdots, f_n)$ is a solution of (16) and in addition $(\partial f_i/\partial x_j) - (\partial f_j/\partial x_i) = c_{ij}$ are constants, then

$$|f| \leq \sqrt{q + C}$$

provides that q has quadratic growth in the sense that $\Delta q \leq \vartheta$ and $|\nabla q| \leq \gamma(1 + |x|)$, where C depends only on n, ϑ, and γ.

Remark: From the condition $|\nabla q| \le \gamma(1+|x|)$, we easily deduce that $q \le C(1+|x|^2)$. Thus, this theorem shows that f has at most linear growth.

Proof: Denote $F = |f|^2 - q$, and we have

$$\begin{aligned}\Delta F &= \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_j^2} f_i^2 - \Delta q \\ &\ge 2\sum_{i\cdot j=1}^{n}\left(\frac{\partial f_i}{\partial x_j}\right)^2 + 2\sum_{i,j=1}^{n} f_i \frac{\partial^2 f_i}{\partial x_j \partial x_j} - \vartheta \\ &\ge 2\sum_{i=1}^{n}\left(\frac{\partial f_i}{\partial x_i}\right)^2 + 2\sum_{i,j=1}^{n} f_i \frac{\partial}{\partial x_j}\left(\frac{\partial f_j}{\partial x_i} + c_{ij}\right) - \vartheta \\ &\ge \frac{2}{n}\left(\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}\right)^2 + 2\sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n}\frac{\partial f_j}{\partial x_j}\right) - \vartheta \\ &= \frac{2}{n}F^2 - 2f\cdot\nabla F - \vartheta. \end{aligned} \tag{21}$$

Choose a standard cutoff function $\varphi_R(x) \in C_0^\infty(R^n)$ such that

$$\begin{aligned}\varphi_R(x) &= 1 \quad \text{on} \quad B_R(0)\\ \varphi_R(x) &= 0 \quad \text{on} \quad B_{2R}^c(0)\end{aligned}$$

and $0 \le \varphi_R(x) \le 1, |\nabla\varphi_R(x)| \le (C/R), |\Delta\varphi_R(x)| \le (C/R^2)$.

$\varphi_R^2 F$ achieves its maximum at $x_0 \in B_{2R}(0)$. At that point

$$\nabla(\varphi_R^2 F) = 0$$

or

$$2\nabla\varphi_R F + \varphi_R \nabla F = 0 \tag{22}$$

and

$$\begin{aligned}0 &\ge \Delta(\varphi_R^2 F)\\ &= \varphi_R^2 \Delta F + 4\varphi_R \nabla\varphi_R \cdot \nabla F + F\Delta\varphi_R^2.\end{aligned}$$

Plug (22) and (21) into the above inequality, we get

$$\begin{aligned}0 &\ge \varphi_R^2\left(\frac{2}{n}F^2 - 2f\cdot\nabla F - \vartheta\right) + [\Delta\varphi_R^2 - 8|\nabla\varphi_R|^2]F\\ &= \frac{2}{n}\varphi_R^2 F^2 + 4\varphi_R(f\cdot\nabla\varphi_R)F - \vartheta\varphi_R^4\\ &\quad + (2\varphi_R\Delta\varphi_R - 6|\nabla\varphi_R|^2)F.\end{aligned} \tag{23}$$

Multiply (23) by φ_R^2 and notice that

$$\begin{aligned}|f\cdot\nabla\varphi_R| &\le \sqrt{|f|^2\cdot|\nabla\varphi_R|^2}\\ &\le \sqrt{(F+q)\cdot|\nabla\varphi_R|^2}\\ &\le \sqrt{F|\nabla\varphi_R|^2 + C\gamma}.\end{aligned}$$

We have

$$\begin{aligned}&\frac{2}{n}\varphi_R^4 F^2 - 4\varphi_R^3 F\sqrt{F|\nabla\varphi_R|^2 + C\gamma} - \vartheta\varphi_R^4\\ &\quad + (2\varphi_R\Delta\varphi_R - 6|\nabla\varphi_R|^2)\varphi_R^2 F \le 0\end{aligned}$$

and the Schwartz inequality

$$\sqrt{a} \le \varepsilon a + \frac{1}{4\varepsilon}$$

gives us

$$\begin{aligned}&\frac{2}{n}\varphi_R^4 F^2 - 4\varphi_R^2 F\left(\varepsilon\varphi_R^2 F|\nabla\varphi_R|^2 + C\gamma\varphi_R^2 + \frac{1}{4\varepsilon}\right)\\ &\quad - \vartheta\varphi_R^4 + (2\varphi_R\Delta\varphi_R - 6|\nabla\varphi_R|^2)\varphi_R^2 F \le 0.\end{aligned}$$

The left-hand side is a quadratic polynomial in M, where $M = \varphi_R^2 F$. Take ε to be sufficiently small to make the leading coefficient positive. Notice that $\varphi_R, (\nabla\varphi_R), \Delta\varphi_R$ are all bounded quantities, and M is therefore bounded. Hence

$$\max_{B_R(0)} F \le \max_{B_{2R}(0)} \varphi_R^2 F = M \le C.$$

Let $R \to \infty$, and we have

$$|f|^2 \le q + C.$$

This is equivalent to our conclusion. ■

Let $q = \Sigma_{i,j=1}^n a_{ij}x_i x_j - c$, where $c \in R$ and the constant matrix $A = (a_{ij})$ is positive semidefinite. Let $\{\lambda_1, \cdots, \lambda_n\}$ be the eigenvalues of A and $c_0 = \Sigma_{i=1}^n \sqrt{\lambda_i}$. In [8], we have shown that when $c = c_0$, there is a quadratic polynomial, uniquely determined up to a constant which satisfies

$$\Delta\phi + |\nabla\phi|^2 = q.$$

Moreover, this is the unique solution up to a constant if either one of the following conditions holds.

1) *rank* $A = 0$ (namely, $A = 0$).
2) *rank* $A \ge n-2$.

We can generalize this to the under-determined system (16) in the second case.

Theorem 2.5: Consider the following equation:

$$\nabla\cdot f + |f|^2 = q \tag{24}$$

where $q = \Sigma_{i,j=1}^n a_{ij}x_i x_j - c$. Equation (24) has a unique solution if

1) $\lambda_1(\Delta - q) = 0$;
2) *rank* $A \ge n-2, A = (a_{ij})$.

Proof: Without loss of generality, we can assume that A is a diagonal matrix with eigenvalues $a_1, a_2, \cdots, a_n$. $\lambda_1(\Delta - q) = 0$ is equivalent to $a_i \ge 0$ and $c = \Sigma_{i=1}^n \sqrt{a_i}$.

In this case, ∇u_0 is a solution of (24), where $u_0 = -(1/2)\Sigma_{i=1}^n \sqrt{a_i}x_i^2$.

Let $w = f - \nabla u_0$, and we have

$$\nabla\cdot w + 2\nabla u_0\cdot w + |w|^2 = 0.$$

Multiplying both sides by e^{2u_0} and integrating on $B_R(0)$, we get

$$\begin{aligned}0 &= \int_{B_R(0)} e^{2u_0}(\nabla w + 2\nabla u_0\cdot w) + \int_{R_R(0)} e^{2u_0}|w|^2\\ &= \int_{B_R(0)} \nabla\cdot(e^{2u_0}w) + \int_{B_R(0)} e^{2u_0}|w|^2\\ &= -\int_{\partial B_R(0)} e^{2u_0} w\cdot\vec{n} + \int_{B_R(0)} e^{2u_0}|w|^2.\end{aligned}$$

Denote $\phi(R) = \int_{B_R(0)} e^{2u_0}|w|^2, \psi(R) = \int_{B_R(0)} e^{2u_0}$ and by the Schwartz inequality

$$\begin{aligned}\int_{\partial B_R(0)} e^{2u_0} w\cdot\vec{n} &\le \int_{\partial B_R(0)} e^{2u_0}|w|\\ &\le \sqrt{\int_{\partial B_R(0)} e^{2u_0}|w|^2 \cdot \int_{\partial B_R(0)} e^{2u_0}}\\ &= \sqrt{\phi'(R)\cdot\psi'(R)}.\end{aligned}$$

Hence

$$(\phi(R))^2 \le \phi'(R)\cdot\psi'(R)$$

or

$$\frac{\phi'(R)}{\phi^2(R)} \ge \frac{1}{\psi'(R)}.$$

Integrate over (R_0, ∞), and we have

$$\infty > \frac{1}{\phi(R_0)} - \frac{1}{\phi(\infty)} = \int_{R_0}^{\infty}\frac{\phi'(R)}{\phi^2(R)}\,dR \ge \int_{R_0}^{\infty}\frac{dR}{\psi'(R)} = \infty.$$

The contradiction shows that $\phi(R) \equiv 0$, which is equivalent to $w = 0$. ∎

III. Construction Solutions to the P.D.E.

One can construct many interesting examples of stochastic systems with finite-dimensional filters by finding solutions to (16). If f is such a solution, the system defined by (1) will have a finite-dimensional filter as defined in Theorem 1.2. Of course, the best known solutions to (16) include the linear solutions and some of the solutions discussed in [2]. However, these previously known cases are by no means exhaustive. In this section, we will concentrate on the problem of how to construct solutions for (16) when q is a quadratic polynomial.

After a Euclidean motion, q can be written in the form $\Sigma_{i=1}^n a_i x_i^2 - c$, where a_i and c are constants. In Corollary 2.3, we have shown that (16) is solvable if and only if $a_i \geq 0$ and $c \leq \Sigma_{i=1}^n \sqrt{a_i}$.

Let

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{bmatrix}$$

$$\sqrt{A} = \begin{bmatrix} \sqrt{a_1} & & & \\ & \sqrt{a_2} & & \\ & & \ddots & \\ & & & \sqrt{a_n} \end{bmatrix}.$$

We can then rewrite (16) into

$$\nabla \cdot f + |f|^2 = x^T A x - c \tag{25}$$

where $A \geq 0$ and $c \leq \mathrm{Tr}\,\sqrt{A}$.

Construction I (Gradient Solutions): Let $c = \Sigma_{i=1}^n c_i$ with $c_i < \sqrt{a_i}$. By [8, Th. 11], there is a 1-parameter family of solutions of

$$f_i'(x_i) + f_i^2(x_i) = a_i x_i^2 - c_i. \tag{26}$$

It is easy to see that $f(x) = (f_1(x_1), \cdots, f_n(x_n))$ satisfies (25). There are $(2n-1)$-parameter families of such solutions to (25). Note that $n-1$ parameters come from the different ways of decomposing c into c_i's so that $c = \Sigma_{i=1}^n c_i$, and n parameters come from $f_i(0)$ by [8, Th. 11].

Construction II (Linear Solutions):

Theorem 3.1: Assuming *rank* $A = n$, (25) has a linear solution if and only if $|c| \leq \Sigma_{i=1}^n \sqrt{a_i}$ where $n \geq 2$.

Lemma 3.1: Let $O(n)$ be the orthogonal group. Define ρ: $O(n) \to R$ to be $\rho(O) = \mathrm{Tr}\,(\sqrt{A}O)$. Then, $\rho(O(n)) = [-\gamma, \gamma]$, where $\gamma = \mathrm{Tr}\,\sqrt{A}$.

Proof: It is easy to see that $\rho(O(n)) \subset [-\gamma, \gamma]$. Noticing that the special orthogonal group $SO(n)$ is a compact connected Lie group, $\rho(SO(n))$ must be a closed interval $K \subset R$. We claim that $[0, \gamma] \subset K$. The identity matrix $I \in SO(n)$ implies that $\gamma \in K$. We still need to find a nonpositive element in K. When n is even, $-I \in SO(n)$. Therefore, $-\gamma \in K$. When n is odd, consider $O_1, O_2, \cdots, O_n \in SO(n)$, where O_i is a diagonal matrix, whose ith element is 1 and all other elements are -1. Therefore, $\Sigma_{i=1}^n O_i = -(n-2)I$. Hence, $\Sigma_{i=1}^n \rho(O_i) = -(n-2)\,\mathrm{Tr}\,\sqrt{A} \leq 0$. One of the numbers must be nonpositive.

$\rho(O(n))$ is symmetric about the origin. $K \supset [0, \gamma]$ implies that $\rho(O(n)) \supset [-\gamma, \gamma]$. ∎

Proof (Theorem 3.1): Let $f = x^T P + p$, where P is an $n \times n$ matrix and $p \in R^n$. Substitute into (25) and use the fact that A is nondegenerate, then we get

$$PP^T = A$$
$$\mathrm{Tr}\,P = -c$$
$$p = 0.$$

Therefore, $P = \sqrt{A}O$ where $O \in O(n)$. Using the above lemma, we conclude the proof. ∎

Remark: When *rank* $A < n$, the existence condition becomes $c \leq \Sigma_{i=1}^n \sqrt{a_i}$. We omit the proof here as it is quite similar.

Construction III (Sums of Gradient and Linear Solutions): In certain situations, there are solutions of (25) which are sums of gradient and linear solutions. We shall illustrate this by considering $n = 2$ and $a_1 = a_2 = a + \lambda^2$. Then (25) becomes

$$\begin{aligned} \sum_{i=1}^2 \left(\frac{\partial f_i}{\partial x_i} + f_i^2 \right) &= (a + \lambda^2)(x_1^2 + x_2^2) - c \\ &= a_1 r^2 - c. \end{aligned} \tag{27}$$

We shall show that (27) has a solution of the form $f = \nabla\phi + (\ell_1, \cdots, \ell_n)$, where $\ell_i, 1 \leq i \leq n$, are degree-one polynomials if $c < 2\sqrt{a}$.

Lemma 3.2: $\Delta\phi + |\nabla\phi|^2 = ar^2 - c$ has a radial solution if $c < 2\sqrt{a}$.

Proof: Take any solution g of the equation (such solution exists in view of [8, Th. 12]). It is easy to see that $u = e^g$ satisfies the following linear partial differential equation:

$$\Delta u = (ar^2 - c)u. \tag{28}$$

Let $G(x) = \int_{\tau \in O} e^{g(\tau \cdot x)}$ where the integral takes place over all orthogonal transformations. G is still a positive solution which depends only on r. The radial function $\varphi = \ln G$ solves the equation $\Delta\varphi + |\nabla\varphi|^2 = ar^2 - c$. ∎

Now take

$$f_1 = \frac{\partial\varphi}{\partial x_1} + \lambda x_2$$

and

$$f_2 = \frac{\partial\varphi}{\partial x_2} - \lambda x_1$$

where φ is the radial solution of the equation

$$\Delta\varphi + |\nabla\varphi|^2 = ar^2 - c.$$

Then

$$\begin{aligned} &\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + f_1^2 + f_2^2 \\ &\quad = \Delta\varphi + |\nabla\varphi|^2 + \lambda^2(x_1^2 + x_2^2) + 2\lambda \\ &\quad\quad \cdot \left(x_2 \frac{\partial\varphi}{\partial x_1} - x_1 \frac{\partial\varphi}{\partial x_2} \right) \\ &\quad = (a + \lambda^2)(x_1^2 + x_2^2) - c. \end{aligned}$$

Construction IV (Degenerate Solution for Degenerate q): If q is degenerate, then there are degenerate solutions of (25). We shall illustrate this by considering $q(x) = ax_1^2 - c$ where $c \leq \sqrt{a/n}$. Then it is easy to see that

$$\sum_{i=1}^n \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^n f_i^2 = ax_1^2 - c$$

where $c \leq \sqrt{a/n}$ has a degenerate solution of the form

$$f(x) = (f_1(x), \cdots, f_n(x)) = (g(x_1), \cdots, g(x_1))$$

where $g(x_1)$ satisfies

$$\frac{dg_1(x_1)}{dx_1} + ng_1^2(x_1) = ax_1^2 - c. \tag{29}$$

Observe that (29) has a solution if and only if $nc \leq \sqrt{na}$, i.e., $c \leq \sqrt{a/n}$.

Construction V (Separation of Variables—The General Case): Let $c = \Sigma_{j=1}^{m} c_j$, then

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = \sum_{i=1}^{n} a_i x_i^2 - c \tag{30}$$

where $a_i \geq 0$ and $c_j \leq \sqrt{a_{n_{j-1}+1}} + \sqrt{a_{n_{j-1}+2}} + \cdots + \sqrt{a_{n_{j-1}+n_j}}$. Here $n_0 = 0$ and $n_1 + n_2 + \cdots + n_m = n$. It is clear that we have a solution of (30) of the form

$$\begin{aligned} f(x) =&(f_1(x_1, \cdots, x_{n_1}), \cdots, f_{n_1}(x_1, \cdots, x_{n_1}), \cdots, \\ &f_{n_1+\cdots+n_{m-1}+1}(x_{n_1+\cdots+n_{m-1}+1}, \cdots, x_n) \cdots, \\ &f_n(x_{n_1+\cdots+n_{m-1}+1}, \cdots, x_n)) \end{aligned}$$

where

$$\begin{aligned} &f_{n_1+\cdots n_{j-1}+1}(x_{n_1+\cdots n_{j-1}+1}, \cdots, x_{n_1+\cdots+n_j}), \cdots, \\ &\quad f_{n_1+\cdots+n_j}(x_{n_1+\cdots+n_{j-1}+1}, \cdots, x_{n_1+\cdots+n_j}) \end{aligned}$$

satisfies the equation

$$\sum_{i=n_1+\cdots n_{j-1}+1}^{n_1+\cdots n_j} \left(\frac{\partial f_i}{\partial x_i} + f_i^2 \right) = \sum_{i=n_1+\cdots+n_{j-1}+1}^{n_1+\cdots+n_j} a_i x_i^2 - c_j. \tag{31}$$

Construction VI (Sums of Gradient and Linear Solutions in Separable Variables): This is a special case of Construction V. In the above construction, we can take some of

$$\begin{aligned} &(f_{n_1+\cdots+n_{j-1}+1}(x_{n_1+\cdots+n_{j-1}+1}, \cdots, x_{n_1+\cdot+n_j}), \cdots, \\ &\ f_{n_1+\cdots+n_j}(x_{n_1+\cdots n_{j-1}+1}, \cdots, x_{n_1+\cdots+n_j})) \end{aligned}$$

to be gradient solutions of (30) and some of those to be linear solutions of (31).

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A Normalized Schur–Cohn Stability Test for the Delta-Operator-Based Polynomials

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***Abstract*—Two delta-operator-based stability tests, or more generally zero location tests, were recently proposed in a separate paper. Those tests establish two families of such tests, each spanning from the discrete-time to the continuous-time with the delta operator providing smooth transitions between the two domains. In this paper a third family is proposed. Specifically, the normalized Schur–Cohn test in the discrete-time domain is transformed into the delta-operator domain resulting in a new delta-operator test. The limit of this new test as the sampling interval vanishes is shown to be the recent Pham–Le Breton test in the continuous-time domain. Its relationships with the well-known Routh test and others are studied. A numerical example shows the advantage of the new test for fast sampling.**

***Index Terms*— Delta operator, inertia tests, Routh test, Schur–Cohn test, stability tests, zero locations.**

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