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Finite Dimensional Filters with Nonlinear Drift, XI

Explicit Solution of the Generalized Kolmogorov Equation in Brockett–Mitter Program*

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Ever since the technique of the Kalman–Bucy filter was popularized, there has been an intense interest in finding new classes of finite-dimensional recursive filters. In the late 1970s the concept of the estimation algebra of a filtering system was introduced. Brockett, Clark, and Mitter proposed to use the Wei–Norman approach to solve the nonlinear filtering problem. In 1990, Tam, Wong, and Yau presented a rigorous proof of the Brocket–Mitter program which allows one to construct finite-dimensional recursive filters from finite–dimensional estimation algebras. Later Yau wrote down explicitly a system of ordinary differential equations and generalized Kolmogorov equation to which the robust Duncan–Mortenser–Zakai equation can be reduced. Thus there remains three fundamental problems in Brockett–Mitter program. The first is the problem of finding explicit solution to the generalized Kolmogorov equation. The second is the problem of finding real-time solution of a system of ODEs. The third is the Brockett's problem of classification of finite–dimensional estimation algebras. In this paper, we solve the first problem.

Key Words: finite dimensional filters; Duncan–Mortensen–Zakai equation; generalized Kolmogorov equation; nonlinear drift; Wei–Norman approach.

1. INTRODUCTION

Until the 1970s the basic approach to nonlinear filtering theory was via "innovation methods," originally proposed by Kailath in 1967 and

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rigorously developed by Fujisaki *et al.* ([FKK] 1972) in their seminal paper. As pointed out by Mitter, the difficulty with this approach is that the innovation process is not, in general, explicitly computable (except in the well-known Kalman-Bucy case). In the late 1970s, the concept of the estimation algebra of a filtering system was introduced by Brockett and Clark [Br-Cl], Brockett [Br], and Mitter [Mi]. The motivation came from the Wei–Norman approach [We-No] of using Lie-algebraic ideas to solve some time varying linear differential equations. The extension of Wei–Norman approach to the nonlinear filtering problem is much more complicated. Instead of an ordinary differential equation, we have to solve the Duncan–Mortensen–Zakai (DMZ) equation, which is a stochastic partial differential equation. By working on the robust form of the DMZ equation we can reduce the complexity of the problem to that of solving a time varying partial differential equation. Wong [Wo 1] constructed some new finite dimensional estimation algebras and used the Wei–Norman approach to synthesize finite dimensional filters. However, the systems considered in [Wo 1] are very specific and the question whether the Wei–Norman approach works for a general system with finite dimensional estimation algebra remains open. In Tam *et al.* [TWY] presented a rigorous proof of the Wei–Norman program which allows one to construct finite dimensional recursive filters from finite dimensional estimation algebras. They considered a class of filtering systems having the property that the drift-term f of the state evolution equation is a gradient vector field. In [Wo 2], the concept of Ω is introduced, which is defined as the matrix whose (i, j)-entry is $\partial f_j/\partial x_i - \partial f_i/\partial x_j$. For the class of filtering systems considered in [TWY], Ω is zero. Conversely, if $\Omega = 0$, then by the Poincaré Lemma, f is a gradient vector field. So the class of filtering systems considered in [TWY] is characterized by the fact that $\Omega = 0$.

Motivated by the results of [TWY], recently Yau [Ya] considered a more general class of filtering systems having the property that Ω is a skew symmetric matrix with constant coefficients. It was shown that Ω is a skew symmetric matrix with constant coefficients if and only if f is the sum of gradient vector field and affine vector field ($(\ell_1, ..., \ell_n)$ is an affine vector field if ℓ_i, $1 \leqslant i \leqslant n$, are affine functions). In [Ya], Yau derived a simple necessary and sufficient condition for an estimation algebra of the above filtering systems to be finite dimensional. He classified all finite dimensional estimation algebra of maximal rank for these filter systems. In fact, he constructed the corresponding finite dimensional filters explicitly.

After the work of [TWY] and [Ya], we know that the robust DMZ equation is reduced to a generalized Kolmogorov equation and a system of ODEs. It is clear that there remains three fundamental problems in Brockett–Mitter program. The first one is the problem of finding explicit solution to the generalized Kolmogorov equation. The second one is the

problem of finding real-time solution of a system of ODEs. The third problem is the Brockett's problem of classification of finite dimensional estimation algebras.

In the recent works of Chiou and Yau [Ch Ya], Chen *et al.* [CLY 1, CLY 2], it was shown that if the estimation algebra of maximal rank is finite dimensional and the state space dimension is less than five, then the filtering systems are necessarily those considered by [Ya], i.e., Ω is a skew symmetric matrix with constant coefficients. In view of the theorem of [Ya], all finite dimensional estimation algebras maximal rank with state dimension less than five are completely classified. It was conjectured by the second named author that in general the Brockett problem can be solved in this way.

The purpose of this paper is to solve the generalized Kolomogorov equation which is the first fundamental problem in Brockett–Mitter program. The advantage of our approach is that we do not need to make any assumption on the drift term f except some regularity assumption at infinity and therefore it applies to general class of non-linear filtering systems. We can write down the formal solution and give estimates of it. We can also construct a convergent solution explicitly from the truncated formal solution. Most strikingly we can actually estimate the time interval on which our solution converges. More precisely we have proven the following theorems.

THEOREM A [Ya-Ya 1]. *The genralized Kolmogorov equation*

$$\begin{aligned}\frac{\partial u}{\partial t}(t, x) &= \left\{\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial}{\partial x_i} - f_i(x)\right)^2 \right.\\ &\qquad \left. -\frac{1}{2}\left(\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2(x) + \sum_{i=1}^{n} h_i^2(x)\right)\right\} u(t, x) \qquad (1.1)\\ u(0, x) &= \sigma_0(x)\end{aligned}$$

has a formal solution on $\mathbb{R}^n$ *of the following form*

$$\begin{aligned}u(t, x) = &\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}(2\pi t)^{-n/2}\\ &\times \exp\left(-\frac{1}{2t}\sum_{j=1}^{n}(x_j - y_j)^2 + \int_0^1 \sum_{i=1}^{n}(x_i - y_i)\, f_i(y + t(x-y))\, dt\right)\\ &\times [1 + \tilde{a}_1(x, y)\, t + \tilde{a}_2(x, y)\, t_2 + \cdots + \tilde{a}_k(x, y)\, t^k + \cdots]\\ &\times \sigma_0(y)\, dy_1 \cdots dy_n,\end{aligned}$$

where $\tilde{a}_k(x, y) = \int_0^1 t^{k-1} \tilde{g}_k(y + t(x-y), y)\, dt$,

$$\tilde{g}_k(x, y) = \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 \tilde{a}_{k-1}}{\partial x_i^2}(x, y) + \sum_{i=1}^{n} \left(\frac{\partial a}{\partial x_i}(x, y) - f_i(x) \right) \frac{\partial \tilde{a}_{k-1}}{\partial x_i}(x, y)$$
$$+ \left[\frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 a}{\partial x_i^2}(x, y) + \frac{1}{2} \sum_{i=1}^{n} \left(\frac{\partial a}{\partial x_i}(x, y) \right)^2 \right.$$
$$\left. - \sum_{i=1}^{n} f_i(x) \frac{\partial a}{\partial x_i}(x, y) - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(x) - \frac{1}{2} \sum_{i=1}^{m} h_i^2(x) \right] \tilde{a}_{k-1}(x, y)$$

and

$$a(x, y) = \int_0^1 \sum_{i=1}^{n} (x_i - y_i)\, f_i(y + t(x-y))\, dt.$$

THEOREM B. *Let*

$$\tilde{\phi}_N(t, x, y) = (2\pi t)^{-n/2} \exp\left(a(x, y) - \frac{|x-y|^2}{2t} \right)$$
$$\times [1 + \tilde{a}_1(x, y)\, t + \cdots + \tilde{a}_N(x, y)\, t^N]$$
$$e_N(t, x, y) = \frac{\partial \tilde{\phi}_N}{\partial t}(t, x, y) - L_x \tilde{\phi}_N(t, x, y),$$

where $L_x = \frac{1}{2} \sum_{i=1}^{n} (\partial/\partial x_i - f_i(x))^2 = \frac{1}{2}(\sum_{i=1}^{n} \partial f_i/\partial x_i(x) + \sum_{i=1}^{n} f_i^2(x) + \sum_{i=1}^{m} h_i^2(x))$ *is the operator defined by the right hand side of* (1.1). *Assume that*

$$\sup |\nabla^j f_i| \leqslant C(j!) \qquad j = 1, ..., N$$
$$\sup |\nabla^j h_i| \leqslant C[j!] \qquad j = 1, ..., N,$$

where $C > \max(2, |f_1(0)|, |f_2(0)|, ..., |f_n(0)|, |h_1(0)|, |h_2(0)|, ..., |h_m(0)|)$ *and* ∇^j *denotes any jth order partial differentiation in x variables. Then for* $|t| \leqslant 1$ *and* $N \geqslant 3n\sqrt{m}\, C - 1$

(a) $$|\tilde{\phi}_N(t, x, y)| \leqslant 2(N+1)^{4N} (1 + \sqrt{t}\,|x|)^{2N} (1 + \sqrt{t}\,|y|)^{2N} (2\pi t)^{-n/2}$$
$$\times \exp\left(a(x, y) - \frac{|x-y|^2}{2t} \right)$$

(b) $$|e_N(t, x, y)| \leqslant (N+2)^{4N+4} (1+\sqrt{t}\,|x|)^{2N+2} (1+\sqrt{t}\,|y|)^{2N+2} \times (2\pi t)^{-n/2} \left[\exp\left(a(x, y) - \frac{|x-y|^2}{2t} \right) \right].$$

THEOREM C. *For $N \geqslant 3n\sqrt{m}\,C - 1$, let $\phi_N(t, x, y)$ and $e_N(t, x, y)$ be defined as in Theorem B. Let*

$$\phi(t, x, y) = \tilde{\phi}_N(t, x, y) + \sum_{k=1}^{\infty} (-1)^{k+1} \phi_k(t, x, y),$$

where

$$\phi_k(t, x, y) = \frac{1}{\sqrt{k+2}} \cdot \int_{\sum_{i=0}^{k+1} \tau_i = t} \int_{x_{k+1}} \int_{x_k} \cdots \int_{x_1} \tilde{\phi}_N(\tau_{k+1}, x, x_{k+1}) \times e_N(\tau_k, x_{k+1}, x_k)\, e_N(\tau_{k-1}, x_k, x_{k-1}) \cdots e_N(\tau_0, x_1, y).$$

If t is chosen small enough so that

$$0 \leqslant t \leqslant \min \left\{ \frac{\sqrt{n^2C^2 + 2(N+1)} - nC}{8(N+1)}, \frac{1}{8nC} \right\}$$

then the infinite series $\phi(t, x, y)$ converges and $\phi(t, x, y)$ is the fundamental solution to the generalized Kolmogorov equation, i.e.,

$$\begin{cases} \dfrac{\partial \phi}{\partial t}(t, x, y) = L_x \phi(t, x, y) \\ \lim\limits_{t \to 0} \phi(t, x, y) = \delta_x(y) \end{cases}$$

COROLLARY D. *The fundamental solution $\phi(t, x, y)$ in Theorem C is approximated by*

$$\tilde{\phi}_N(t, x, y) + \sum_{k=0}^{K} (-1)^{k+1} \phi_k(t, x, y)$$

which is readily computable. The error for such an approximation is given by

$$\sum_{k=K+1}^{\infty} (-1)^{k+1} \phi_k(t, x, y)$$

which can be estimated by

$$(1+\sqrt{t}\,|x|)^{2N+2}\,(1+\sqrt{t}\,|y|)^{2N+2}$$
$$\times \exp\left[(x-y)\cdot f(x)+4(N+1)\,t\,|y|^2-\frac{|x-y|^2}{4t}\right]$$
$$\times \sum_{k=K+1}^{\infty} 2(N+2)^{4(k+2)\,N+4(k+1)}\sqrt{k+2}\,(\sqrt{2\pi})^{-(k+2)(n/2)}\,2^{4(N+1)(k+1)}$$
$$\times (\sqrt{4\pi})^{(k+1)(n/2)}\,\frac{t^{-(n/2)+k+1}}{(k+1)!}$$

which clearly tends to zero rapidly if t is small and K is large.

It should be noted that in the literature (see for example [Fr]), the parametrix method was used to prove existence of fundamental solution for bounded domain. The estimate here is more complicated as growth assumptions have to be made on a non-bounded domain, otherwise there is no reason to expect existence of solution even for small time.

Let $L=\frac{1}{2}\Delta - f(x)\cdot\nabla - V(x)$ be an operator defined on $\mathbf{R}^n$ where Δ is the Laplacian operator, ∇ is the gradient operator and $V(x)=\sum_{i=1}^n \partial f_i/\partial x_i + \frac{1}{2}\sum_{i=1}^n h_i^2(x)$. Let $P(t)=\exp(tL)$ denote the corresponding semigroup on $L^2(\mathbf{R}^n)$ with kernel $p(t, x, y)$. We find an asymptotic expansion for $p(t, x, y)$ then use it to construct $p(t, x, y)$ as an infinite series by the parametrix method.

For a fixed large enough N, let $q(t, x, y)$ denote the N-th partial sum of the asymptotic expansion for $p(t, x, y)$ and let $e(t, x, y)=(\partial/\partial t - L)\,q(t, x, y)$ be the error. Let $Q(t)$, $E(t)$ be the operators whose kernels are $q(t, x, y)$, $e(t, x, y)$. Then $Q'(t)=LQ(t)+E(t)$ which yields

$$Q(t)=P(t)+\int_0^t P(t-s)\,E(s)\,ds$$

since $P(0)=I=Q(0)$. Thus one has Q equal to P plus a linear compact perturbation applied to P. Inverting this, one has the Neumann series

$$P(t)=Q(t)-\int_0^t Q(t-s)\,E(s)\,ds+\int_0^t\int_0^s Q(t-s)\,E(s-r)\,E(r)\,dr\,ds+\cdots.$$

We prove that if $|\nabla^j f_i|\leqslant C(j!)$ for $j\geqslant 1$ and $|\nabla^j h_i|\leqslant C((j-1)!)$ for $j\geqslant 1$, our estimates on e are good enough so that the Neumann series converges and yields $p(t, x, y)$.

When the drift term $f(x)$ is a gradient vector field so that L is self-adjoint, the paramatrix method is more well-known although we cannot find it in literature. If $h_1, ..., h_m$ are constants, then the result can be found in [Ya-Ya 2]. However when the drift term $f(x)$ is not a gradient vector field and $h_1, ..., h_m$ are not constants, the result is new and has potential application to many different fields besides the application to non-linear filtering theory.

2. BASIC CONCEPTS

The filtering problem considered here is based on the following signal observation model

$$\begin{cases} dx(t) = f(x(t))\,dt + g(x(t))\,dv(t) & x(0) = x_0 \\ dy(t) = h(x(t))\,dt + dw(t) & y(0) = 0, \end{cases} \tag{2.1}$$

in which x, v, y and w, are, respectively, $\mathbf{R}^n$, $\mathbf{R}^p$, $\mathbf{R}^m$, and $\mathbf{R}^m$ valued processes, and v and w have components which are independent, standard Brownian processes. We further assume that $n = p$, f and h are C^∞ smooth, and that g is an orthogonal matrix. We will refer to $x(t)$ as the state of the system at time t and $y(t)$ as the observation at time t.

Let $\rho(t, x)$ denote the conditional density of the state given the observation $\{y(s): 0 \leqslant s \leqslant t\}$. It is well-known that $\rho(t, x)$ is given by normalizing a function $\sigma(t, x)$, which satisfies the Duncan–Mortensen–Zakai (DMZ) equation,

$$d\sigma(t, x) = L_0\sigma(t, x)\,dt + \sum_{i=1}^{m} L_i\sigma(t, x)\,dy_i(t), \qquad \sigma(0, x) = \sigma_0, \tag{2.2}$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{n} h_i^2$$

and for $i = 1, ..., m$, L_i is the zero-degree differential operator of multiplication by h_i. σ_0 is the probability density of the initial point x_0. Let

$$u(t, x) = \exp\left(-\sum_{i=1}^{m} h_i(x)\, y_i(t) \right) \sigma(t, x).$$

It is easy to show that $u(t, x)$ satisfies the following time varying partial differential equation

$$\begin{aligned}\frac{\partial u}{\partial t}(t, x) &= L_0 u(t, x) + \sum_{i=1}^{m} y_i(t)[L_0, L_i]\, u(t, x) \\ &\quad + \frac{1}{2}\sum_{i,\, j=1}^{m} y_i(t)\, y_j(t)[[L_0, L_i], L_j]\, u(t, x) \\ u(0, x) &= \sigma_0\end{aligned} \tag{2.3}$$

where $[\cdot, \cdot]$ is the Lie bracket.

DEFINITION. (a) The estimation algebra E of a filtering system (2.1) is defined to be the Lie algebra genered by $\{L_0, ..., L_m\}$.

(b) E is said to be the estimation algebra of maximal rank if $x_i + c_i$ is in E for $1 \leqslant i \leqslant n$ where c_i is a constant.

We remark that if $h_i(x) = \sum_{j=1}^{m} \alpha_{ij} x_j + \beta_i$, $1 \leqslant i \leqslant m$, α_{ij}, β_i are constants, and the rank of the matrix (α_{ij}) is n, then the estimation algebra E is of maximal rank. Define

$$D_i = \frac{\partial}{\partial x_i} - f_i, \qquad \eta = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{m} h_i^2.$$

Then

$$L_0 = \tfrac{1}{2}\left(\sum_{i=1}^{n} D_i^2 - \eta\right).$$

THEOREM 2.1. [Ya] *Let E be an estimation algebra of* (2.1) *satisfying $\partial f_j/\partial x_j - \partial f_i/\partial x_j = c_{ij}$ where c_{ij}, $1 \leqslant i, j \leqslant n$, are constants. If E is finite dimensional, then $h_1, ..., h_m$ are affine. Suppose further that E is of maximal rank. Then $\eta = \sum_{i,\, j=1}^{n} a_{ij} x_i x_j + \sum_{i=1}^{n} b_i x_i + d$ is a quadratic polynomial and E has a basis (as real vector space) given by $1, x_1, ..., x_n, D_1, ..., D_n$ and L_0. In fact the robust* DMZ *Eq.* (2.3) *has a solution for all time $t \geqslant 0$ of the form*

$$U(t, x) = e^{T(t)} e^{r_n(t) x_n} \cdots e^{r_1(t)\, x_1} e^{s_n(t)\, Dn} \cdots e^{s_1(t) D_1} e^{tL_0} \sigma_0, \tag{2.4}$$

where $T(t), r_1(t), ..., r_n(t), s_1(t), ..., s_n(t)$ satisfy the following ordinary differential equations (5), (6) *and* (7)

$$\frac{ds_i}{dt}(t) = r_i(t) + \sum_{j=1}^{n} s_j(t)\, c_{ji} + \sum_{k=1}^{m} h_{ki} y_k(t) \qquad 1 \leqslant i \leqslant n, \tag{2.5}$$

where $h_i(x) = \sum_{j=1}^n h_{kj} x_j + e_k$, $1 \leqslant k \leqslant m$; h_{kj} *and* e_k *are constants*

$$\frac{dr_j}{dt}(t) = \frac{1}{2} \sum_{i=1}^n s_i(t)(a_{ij} + a_{ji}) \qquad 1 \leqslant j \leqslant n \tag{2.6}$$

$$\begin{aligned}\frac{dT}{dt}(t) = &-\frac{1}{2} \sum_{i=1}^n r_i^2(t) - \frac{1}{2} \sum_{i=1}^n s_i^2(t) \left(\sum_{j=1}^n c_{ij}^2 - a_{ii} \right) \\ &+ \sum_{1 \leqslant i < k \leqslant n} s_i(t)\, s_k(t) \left(\sum_{j=1}^n c_{ij} c_{jk} + \frac{1}{2}(a_{ik} + a_{ki}) \right) \\ &+ \sum_{i=1}^n r_i(t) - \sum_{j=2}^n \sum_{i=1}^j s_j(t)\, c_{ij} + \frac{1}{2} \sum_{i=1}^n s_i(t)\, b_i \\ &+ \frac{1}{2} \sum_{i,\, j=1}^n y_i(t)\, y_j(t) \left(\sum_{k=1}^n h_{ik} h_{jk} \right) - \sum_{i,\, j=1}^n s_i(t)\, r_j(t)\, c_{ij}. \end{aligned} \tag{2.7}$$

It is clear from the above theorem that the robust DMA equation is reduced to a generalized Kolmogorov equation

$$\begin{aligned}\frac{\partial u}{\partial u}(t, x) &= L_0 u(t, x) \\ &= \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}(t, x) - \sum_{i=1}^n f_i(x) \frac{\partial u}{\partial x_i}(t, x) - \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}(x)\, u(t, x) \\ &\quad - \frac{1}{2} \sum_{i=1}^m h_i^2(x)\, u(t, x) \\ &= \left\{ \frac{1}{2} \sum_{i=1}^n \left(\frac{\partial}{\partial x_i} - f_i(x) \right)^2 \right. \\ &\quad \left. - \frac{1}{2} \left(\sum_{i=1}^n \frac{\partial f_i}{\partial x_i}(x) + \sum_{i=1}^n f_i^2(x) + \sum_{i=1}^m h_i^2(x) \right) \right\} u(t, x) \end{aligned} \tag{2.8}$$

$$u(0, x) = \sigma_0(x)$$

and a system of ODEs. Therefore there remain three fundamental problems in Brockett–Mitter program mentioned in Section 1.

3. FORMAL SOLUTION TO GENERALIZED KOLMOGOROV EQUATION

The formal solution of the generalized Kolmogorov equation (2.8) was already obtained in [Ya-Ya 1]. The material included here without

proof is for the sake of the convenience of the reader and for our later use.

THEOREM 3.1. *The generalized Kolmogorov equation* (2.8) *has a formal asymptotic solution on* $\mathbf{R}^n$. *In fact, the solution is of the following form*

$$u(t, x) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(\sqrt{2\pi})^n} t^{n/2} \times \left[\exp\left(-\sum_{j=1}^{n} (x_j - y_j)^2/2t \right) \right] b(t, x, y)\, \sigma_0(y)\, dy_1 \cdots dy_n, \quad (3.1)$$

where $b(t, x, y) = \sum_{k=0}^{\infty} a_k(x, y)\, t^k$. *Here* $a_k(x, y)$ *are described explicitly as follows. Let*

$$a(x, y) = \int_0^1 \sum_{i=1}^{n} (x_i - y_i)\, f_i(y + t(x - y))\, dt. \quad (3.2)$$

Then, let

$$a_0(x, y) = e^{a(x, y)}. \quad (3.3)$$

Suppose that $a_{k-1}(x, y)$ *is given. Let*

$$g_k(x, y) = \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 a_{k-1}}{\partial x_i^2}(x, y) - \sum_{i=1}^{n} f_i(x) \frac{\partial a_{k-1}}{\partial x_i}(x, y) - \frac{1}{2}\left(\sum_{i=1}^{m} h_i^2(x) \right) a_{k-1}(x, y) - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(x)\, a_{k-1}(x, y). \quad (3.4)$$

Then, for $k \geqslant 1$

$$a_k(x, y) = e^{a(x, y)} \int_0^1 t^{k-1} e^{-a(y + t(x - y), y)} g_k(y + t(x - y), y)\, dt.$$

LEMMA 3.2. *Let* $\tilde{a}_0(x, y) = 1$ *and* $\tilde{a}_{k-1}(x, y) = e^{-a(x, y)} a_{k-1}(x, y)$. *Let* $\tilde{g}_k(x, y) = e^{-a(x, y)} g_k(x, y)$. *Then*

$$\tilde{a}_k(x, y) = \int_0^1 t^{k-1} \tilde{g}_k(y + t(x - y), y)\, dt,$$

where

$$\tilde{g}_k(x, y) = \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 \tilde{a}_{k-1}}{\partial x_i^2}(x, y) + \sum_{i=1}^{n} \left(\frac{\partial a}{\partial x_i}(x, y) - f_i(x) \right) \frac{\partial \tilde{a}_{k-1}}{\partial x_i}(x, y)$$
$$+ \left[\frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 a}{\partial x_i^2}(x, y) + \frac{1}{2}\sum_{i=1}^{n} \left(\frac{\partial a}{\partial x_i}(x, y) \right)^2 - \sum_{i=1}^{n} f_i(x) \frac{\partial a}{\partial x_i}(x, y) \right.$$
$$\left. - \frac{1}{2}\left(\sum_{i=1}^{m} h_i^2(x) \right) - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(x) \right] \tilde{a}_{k-1}(x, y).$$

THEOREM 3.3. *The generalized Kolmogorov equation* (2.8) *has a formal solution on* $\mathbf{R}^n$ *of the following form*

$$u(t, x) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(\sqrt{2\pi})^n} t^{-n/2} \left[\exp\left(-\frac{1}{2t}\sum_{j=1}^{n} (x_j - y_j)^2 \right.\right.$$
$$\left.\left. + \int_0^1 \sum_{i=1}^{n} (x_i - y_i) f_i(y + t(x - y))\, dt \right) \right]$$
$$\times [1 + \tilde{a}_1(x, y)\, t + \cdots + \tilde{a}_k(x, y)\, t^k + \cdots]\, \sigma_0(y)\, dy_1 \cdots dy_n, \tag{3.5}$$

where $\tilde{a}_k(x, y) = \int_0^1 t^{k-1} \tilde{g}_k(y + t(x - y), y)\, dt$ *and*

$$\tilde{g}_k(x, y) = \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 \tilde{a}_{k-1}}{\partial x_i^2}(x, y) + \sum_{i=1}^{n} \left(\frac{\partial a}{\partial x_i}(x, y) - f_i(x) \right) \frac{\partial \tilde{a}_{k-1}}{\partial x_i}(x, y)$$
$$+ \left[\frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 a}{\partial x_i^2}(x, y) + \frac{1}{2}\sum_{i=1}^{n} \left(\frac{\partial a}{\partial x_i}(x, y) \right)^2 - \sum_{i=1}^{n} f_i(x) \frac{\partial a}{\partial x_i}(x, y) \right.$$
$$\left. - \frac{1}{2}\left(\sum_{i=1}^{m} h_i^2(x) \right) - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(x) \right] \tilde{a}_{k-1}(x, y).$$

4. ESTIMATES OF THE FORMAL ASYMPTOTIC SOLUTION

In this section, we shall give estimates of the formal asymptotic solution (3.5). These estimates will be used in Section 5 to construct convergent solution of the generalized Kolmogorov Eq. (2.8).

LEMMA 4.1. *Let* $\alpha = (\alpha_1, ..., \alpha_n)$ *and* $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. *Then*

$$\frac{\partial^{\alpha_1 + \cdots + \alpha_n} a}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}(x, y)$$

$$= \alpha_1 \int_0^1 t^{|\alpha|-1} \frac{\partial^{|\alpha|-1} f_1}{\partial x_1^{\alpha_1 - 1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}(y + t(x-y))\, dt$$

$$+ \cdots + \alpha_i \int_0^1 t^{|\alpha|-1} \frac{\partial^{|\alpha|-1} f_1}{\partial x_1^{\alpha_1} \cdots \partial x_{i-1}^{\alpha_{i-1}} \partial x_i^{\alpha_i - 1} \partial x_{i+1}^{\alpha_{i+1}} \cdots \partial x_n^{\alpha_n}}(y + t(x-y))\, dt$$

$$+ \cdots + \alpha_n \int_0^1 t^{|\alpha|-1} \frac{\partial^{|\alpha|-1} f_1}{\partial x_1^{\alpha_1} \cdots \partial x_{n-1}^{\alpha_{n-1}} \partial x_n^{\alpha_n - 1}}(y + t(x-y))\, dt$$

$$+ \int_0^1 t^{|\alpha|} \sum_{j=1}^{n} (x_j - y_j) \frac{\partial^{|\alpha|} f_j}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}(y + t(x-y))\, dt,$$

where $a(x, y)$ *is defined in* (3.2).

Proof.

$$\frac{\partial a}{\partial x_i} = \int_0^1 f_i(y + t(x-y))\, dt + \int_0^1 t \sum_{j=1}^{n} (x_j - y_j) \frac{\partial f_j}{\partial x_i}(y + t(x-y))\, dt.$$

The proof follows from induction. Q.E.D.

Let

$$\tilde{\phi}_N(t, x, y) = (2\pi t)^{-n/2} \left[\exp\left(a(x, y) - \frac{|x-y|^2}{2t} \right) \right]$$
$$\times [1 + \tilde{a}_1(x, y)\, t + \cdots + \tilde{a}_N(x, y)\, t^N]. \tag{4.1}$$

We shall estimate $|\tilde{\phi}_N(t, x, y)|$. Our basic assumption is that

$$\sup |\nabla^j f_i(x)| \leqslant C(j!) \qquad j = 1, ..., N, \quad 1 \leqslant i \leqslant n \tag{4.2}$$

$$\sup |\nabla^j h_i(x)| \leqslant C[j!] \qquad j = 1, ..., N, \quad 1 \leqslant i \leqslant m \tag{4.3}$$

where $C > \max(2, |f_1(0)|, ..., |f_n(0)|, |h_1(0)|, ..., |h_m(0)|)$. Here ∇^j denotes partial differentiation of order j with respect to x variables. The following Lemma is given in [Ya-Ya 2].

LEMMA 4.2. (a) $|f_i(x)-f_i(y)| \leqslant \sqrt{n}\, C\, |x-y|$

(b) $|\partial a/\partial x_i\,(x, y)| \leqslant |f_i(y)| + \sqrt{n}\, C\, |x-y|$

(c) $|\nabla^j a(x, y)| \leqslant (j-1)!\,(C+\sqrt{n}\, C\, |x-y|)$ *for* $j \leqslant 2$

(d) $|\partial a/\partial x_i\,(x, y) - f_i(x)| \leqslant \sqrt{n}\, |x-y|\, C$

(e) $|\nabla^j(\partial a/\partial x_i\,(x, y) - f_i(x))| \leqslant j!(\sqrt{n}\, C\, |x-y| + 2C)$

LEMMA 4.3. (i) *Suppose* p, j *are non-negative integers and* $p \leqslant j$. *Then*

$$\frac{(p+2k)!}{p!} \leqslant \frac{(j+2k)!}{j!}.$$

(ii) *For positive integer* N, $(N+2)^{2N+2} \geqslant (2N+2)!$ *for any non-negative integer* k.

Proof. (i)

$$\begin{aligned} &\frac{(p+2k)!}{p!} \leqslant \frac{(j+2k)!}{j!} \\ &\qquad \Leftrightarrow j!\,(p+2k)! \leqslant p!\,(j+2k)! \\ &\qquad \Leftrightarrow j! \leqslant p!\,(p+1+2k)(p+2+2k)\cdots(j+2k). \end{aligned}$$

The last inequality is obvious.

(ii) For any $1 \leqslant j \leqslant N+1$, it is clear that

$$N+4 > 2 \geqslant 2 + j - j^2$$

which implies

$$(N+2)^2 = N^2+4N+4 > N^2+3N+2+j-j^2 = (N+2-j)(N+1+j),$$

i.e.,

$$\frac{N+2}{N+1+j} \geqslant \frac{N+2-j}{N+2} \qquad \text{for} \quad 1 \leqslant j \leqslant N+1.$$

It follows that

$$\prod_{j=1}^{N=1} \frac{N+2}{N+1+j} \geqslant \prod_{j=1}^{N+1} \frac{N+2-j}{N+2}$$

which is equivalent to $(N+2)^{2N+2} \geqslant (2N+2)!$ Q.E.D.

PROPOSITION 4.4. *Let*

$$A_i(x, y) = \frac{\partial a}{\partial x_i}(x, y) - f_i(x)$$

$$B(x, y) = \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 a}{\partial x_i^2}(x, y) + \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial a}{\partial x_i}(x, y)\right)^2 - \sum_{i=1}^{n} f_i(x)\frac{\partial a}{\partial x_i}(x, y) - \frac{1}{2}\sum_{i=1}^{m} h_i^2(x) - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x).$$

Then

(a) $|\nabla^j A_i(x, y)| \leqslant j!\,(\sqrt{n}\,C\,|x-y| + 2C)$

(b) $|\nabla^j B(x, y)| \leqslant (j+1)!\,nm(\sqrt{n}\,C\,|y| + 2C\sqrt{n}\,|x-y| + 2C)^2$

(c) $|\nabla^j \tilde{a}_k(x, y)| \leqslant (j+2k)!\,n^k m^k(\sqrt{n}\,C\,|y| + 2C\sqrt{n}\,|x-y| + 2C)^{2k}$

Proof. Part (a) follows from (d) and (e) of Lemma 4.2.

Observe that

$$|f_i(x)| \leqslant |f_i(0)| + |f_i(x) - f_i(0)| \leqslant C + \sqrt{n}\,C\,|x|$$

$$|h_i(x)| \leqslant |h_i(0)| + |h_i(x) - h_i(0)| \leqslant C + \sqrt{n}\,C\,|x| \leqslant C(1 + \sqrt{n}\,|y| + \sqrt{n}\,|x-y|)$$

$$\begin{aligned}|B(x, y)| \leqslant{}& \frac{1}{2}\sum_{i=1}^{n}\left|\frac{\partial^2 a}{\partial x_i^2}(x, y)\right| \\ &+ \frac{1}{2}\sum_{i=1}^{n}\left|\frac{\partial a}{\partial x_i}(x, y)\left(\frac{\partial a}{\partial x_i}(x, y) - f_i(x)\right)\right| \\ &+ \frac{1}{2}\sum_{i=1}^{n}\left|f_i(x)\frac{\partial a}{\partial x_i}(x, y)\right| + \frac{1}{2}\sum_{i=1}^{n}|h_i^2(x)| + \sum_{i=1}^{n}\left|\frac{\partial f_i}{\partial x_i}(x)\right| \\ \leqslant{}& \frac{n}{2}(C + \sqrt{n}\,C\,|x-y|) + \frac{1}{2}\sum_{i=1}^{n}(|f_i(y)| + \sqrt{n}\,C\,|x-y|) \\ &\times(\sqrt{n}\,C\,|x-y|) + \frac{1}{2}\sum_{i=1}^{n}(|f_i(y)| + |f_i(x) - f_i(y)|) \\ &\times(|f_i(y)| + \sqrt{n}\,C\,|x-y|) \\ &+ \frac{m}{2}C^2(1 + \sqrt{n}\,y + \sqrt{n}\,|x-y|)^2 + nC\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{n}{2}(C+\sqrt{n}\,C\,|x-y|)+\frac{n}{2}(C+\sqrt{n}\,C\,|y|+\sqrt{n}\,C\,|x-y|)\\
&\quad\times(\sqrt{n}\,C\,|x-y|)\\
&\quad+\frac{n}{2}(C+\sqrt{n}\,C\,|y|+\sqrt{n}\,C\,|x-y|)^2\\
&\quad+\frac{mC^2}{2}(1+\sqrt{n}\,|y|+\sqrt{n}\,|x-y|)^2+nC\\
&\leqslant \frac{n}{2}(C+\sqrt{n}\,C\,|y|+C\sqrt{n}\,|x-y|)^2\\
&\quad+\left[\frac{n}{2}(C+\sqrt{n}\,C\,|y|+\sqrt{n}\,C\,|x-y|)\right.\\
&\quad\left.\times(\sqrt{n}\,C\,|x-y|)+\frac{n}{2}C^2\right]\\
&\quad+\frac{n}{2}(C+\sqrt{n}\,C\,|x-y|)\\
&\quad+\left[\frac{m}{2}(C+\sqrt{n}\,C\,|y|+2C\sqrt{n}\,|x-y|)^2+\frac{nm}{2}C^2\right]\\
&\leqslant \frac{nm}{2}(C+C\sqrt{n}\,|y|+C\sqrt{n}\,|x-y|)^2\\
&\quad+nm(C+C\sqrt{n}\,|y|+C\sqrt{n}\,|x-y|)(C\sqrt{n}\,|x-y|+C)\\
&\quad+\frac{nm}{2}(C\sqrt{n}\,|x-y|+C)^2+\frac{nm}{2}\\
&\quad\times(2C+\sqrt{n}\,C\,|y|+2C\sqrt{n}\,|x-y|)^2\\
&=\frac{nm}{2}[(C+C\sqrt{n}\,|y|+C\sqrt{n}\,|x-y|)\\
&\quad+(C\sqrt{n}\,|x-y|+C)]^2\\
&\quad+\frac{nm}{2}(2C+\sqrt{n}\,C\,|y|+2C\sqrt{n}\,|x-y|)^2\\
&=nm(\sqrt{n}\,C\,|y|+2C\sqrt{n}\,|x-y|+2C)^2
\end{aligned}
$$

$$
\begin{aligned}
|\nabla^j B(x,y)| &= \left|\frac{1}{2}\sum_{i=1}^{n}\nabla^j\frac{\partial^2 a}{\partial x_i^2}(x,y)\right.\\
&\quad\left.+\frac{1}{2}\sum_{i=1}^{n}\nabla^j\left[\frac{\partial a}{\partial x_i}(x,y)\left(\frac{\partial a}{\partial x_i}(x,y)-f_i(x)\right)\right]\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\sum_{i=1}^{n}\nabla^{j}\left[f_i(x)\frac{\partial a}{\partial x_i}(x,\,y)\right]\\
&-\frac{1}{2}\sum_{i=1}^{m}\nabla^{j}[h_i^2(x)]-\sum_{i=1}^{n}\nabla^{j+1}f_i(x)\Bigg|\\
\leqslant&\frac{n}{2}(j+1)!\,(C+\sqrt{n}\,C\,|x-y|)\\
&+\frac{1}{2}\sum_{i=1}^{n}\sum_{p+q=j}\frac{j!}{p!\,q!}\,|\nabla^{P+1}a(x,\,y)|\\
&\times\left|\nabla^{q}\left(\frac{\partial a}{\partial x_i}(x,\,y)-f_i(x)\right)\right|\\
&+\frac{1}{2}\sum_{i=1}^{n}\sum_{p+q=j}\frac{j!}{p!\,q!}\,|\nabla^{p}f_i(x)|\,|\nabla^{q+1}a(x,\,y)|\\
&+\frac{1}{2}\sum_{i=1}^{m}\sum_{p+q=j}\frac{j!}{p!\,q!}\,|\nabla^{p}h_i(x)|\,|\nabla^{q}h_i(x)|+n[(j+1)!]\,C\\
\leqslant&\frac{n}{2}[(j+1)!](C+\sqrt{n}\,C\,|x-y|)+\frac{n}{2}\sum_{p+q=j}\frac{j!}{p!\,q!}(p!)\\
&\times(C+\sqrt{n}\,C\,|x-y|)\,q!\,(\sqrt{n}\,C\,|x-y|+2C)\\
&+\frac{n}{2}\sum_{p+q=j}\frac{j!}{p!\,q!}\,p!(C+C\sqrt{n}\,|y|+C\sqrt{n}\,|x-y|)\\
&\times q!\,(C+\sqrt{n}\,C\,|x-y|)+\frac{m}{2}\sum_{p+q=j}\frac{j!}{p!\,q!}\\
&\times(p!\,C)(q!\,C)+n((j+1)!)\,C\\
\leqslant&\frac{n}{2}[(j+1)!](C+\sqrt{n}\,C\,|x-y|)\\
&+\frac{n}{2}[(j+1)!](C+\sqrt{n}\,C\,|x-y|)(2C+\sqrt{n}\,C\,|x-y|)\\
&+\frac{n}{2}[(j+1)!](C+C\sqrt{n}\,|y|+C\sqrt{n}\,|x-y|)\\
&\times(C+\sqrt{n}\,C\,|x-y|)\\
&+\frac{m}{2}[(j+1)!]\,C^2+n[(j+1)!]\,C
\end{aligned}
$$

$$\leqslant (j+1)!\,\frac{nm}{2}\,(C+C\sqrt{n}\,|x-y|)$$

$$\times(\sqrt{n}\,C\,|y|+2C\sqrt{n}\,|x-y|+2C)$$

$$+(j+1)!\,\frac{nm}{2}\,(C+C\sqrt{n}\,|x-y|)$$

$$\times(\sqrt{n}\,C\,|y|+2C\sqrt{n}\,|x-y|+2C)$$

$$+(j+1)!\,\frac{nm}{2}+(j+1)!\,nm(\sqrt{n}\,C\,|y|+C+C\sqrt{n}\,|x-y|)$$

$$\times(C+C\sqrt{n}\,|x-y|)+(j+1)!\,\frac{nm}{2}\,C^2$$

$$\leqslant (j+1)!\,\frac{nm}{2}\,(\sqrt{n}\,C\,|y|+2C\sqrt{n}\,|x-y|+2C)^2$$

$$+(j+1)!\,\frac{nm}{2}\,(\sqrt{n}\,C\,|y|+C+C\sqrt{n}\,|x-y|)^2$$

$$+(j+1)!\,nm(\sqrt{n}\,C\,|y|+C+C\sqrt{n}\,|x-y|)$$

$$\times(C+C\sqrt{n}\,|x-y|)$$

$$+(j+1)!\,\frac{nm}{2}\,(C+C\sqrt{n}\,|x-y|)^2$$

$$+(j+1)!\,\frac{nm}{2}\,(C+C\sqrt{n}\,|x-y|)^2$$

$$\leqslant (j+1)!\,nm(\sqrt{n}\,C\,|y|+2C\sqrt{n}\,|x-y|+2C)^2$$

$$\tilde{a}_{k+1}(x,\,y)=\int_0^1 t^k\tilde{g}_{k+1}(y+t(x-y),\,y)\,dt$$

$$=\int_0^1 t^k\left\{\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2\tilde{a}_k}{\partial x_i^2}\,(y+t(x-y),\,y)\right.$$

$$+\sum_{i=1}^{n}A_i(y+t(x-y),\,y)\,\frac{\partial\tilde{a}_k}{\partial x_i}\,(y+t(x-y),\,y)$$

$$\left.+B(y+t(x-y),\,y)\,\tilde{a}_k(y+t(x-y),\,y)\right\}\,dt$$

$$
\begin{aligned}
|\nabla^j \tilde{a}_{k+1}(x, y)| \leqslant & \int_0^1 t^{j+k} \frac{n}{2} |\nabla^{j+2} \tilde{a}_k(y + t(x-y), y)| \, dt \\
& + \int_0^1 \sum_{i=1}^{n} t^{j+k} \sum_{p+q=j} \frac{j!}{p!\, q!} |\nabla^q A_i(y + t(x-y), y)| \\
& \times |\nabla^{p+1} \tilde{a}_k(y + t(x-y), y)| \, dt \\
& + \int_0^1 t^{j+k} \sum_{p+q=j} \frac{j!}{p!\, q!} |\nabla^q B(y + t(x-y), y)| \\
& \times |\nabla^p \tilde{a}_k(y + t(x-y), y)| \, dt \\
\leqslant & \frac{n}{2(j+k+1)} (j+2+2k)!\, n^k m^k \\
& \times (\sqrt{n}\, C\, |y| + 2C\sqrt{n}\, |x-y| + 2C)^{2k} \\
& + \frac{n}{(j+k+1)} \sum_{p+q=j} \frac{j!}{p!\, q!} q!\, (\sqrt{n}\, C\, |x-y| + 2C) \\
& \times n^k m^k (p+1+2k)!\, (\sqrt{n}\, C\, |y| + 2C\sqrt{n}\, |x-y| + 2C)^{2k} \\
& + \frac{1}{j+k+1} \sum_{p+q=j} \frac{j!}{p!\, q!} (q+1)! \\
& \times nm (\sqrt{n}\, C\, |y| + 2C\sqrt{n}\, |x-y| + 2C)^2 \cdot (p+2k)!\, n^k m^k \\
& \cdot (\sqrt{n}\, C\, |y| + 2C\sqrt{n}\, |x-y| + 2C)^{2k} \\
\leqslant & \frac{n^{k+1} m^k}{2(j+k+1)} (j+2+2k)! \\
& \times (\sqrt{n}\, C\, |y| + 2C\sqrt{n}\, |x-y| + 2C)^{2k} \\
& + \frac{n^{k+1} m^k}{j+k+1} \sum_{p+q=j} \frac{j!}{p!} (p+1+2k)! \\
& \times (\sqrt{n}\, C\, |x-y| + 2C)(\sqrt{n}\, C\, |y| + 2C\sqrt{n}\, |x-y| + 2C)^{2k} \\
& + \frac{n^{k+1} m^{k+1}}{j+k+1} \sum_{p+q=j} \frac{j!\, (q+1)}{p!} \cdot (p+2k)! \\
& \times (\sqrt{n}\, C\, |y| + 2C\sqrt{n}\, |x-y| + 2C)^{2k+2} \\
\leqslant & \frac{n^{k+1} m^{k+1}}{2(j+k+1)} (j+2(k+1))! \\
& \times (\sqrt{n}\, C\, |y| + 2C\sqrt{n}\, |x-y| + 2C)^{2k}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{n^{k+1}m^{k+1}}{j+k+1}(j+1)\cdot(j+1+2k)!\\
&\times(\sqrt{n}\,C|y|+2C\sqrt{n}\,|x-y|+2C)^{2k+1}\\
&\quad+\frac{n^{k+1}m^{k+1}}{j+k+1}(j+1)^2\cdot(j+2k)!\\
&\quad\times(\sqrt{n}\,C|y|+2C\sqrt{n}\,|x-y|+2C)^{2k+2}\\
&\leqslant n^{k+1}m^{k+1}(j+2(k+1))!\\
&\quad\times(\sqrt{n}\,C|y|+2C\sqrt{n}\,|x-y|+2C)^{2k+2}\\
&\quad\times\left[\frac{1}{2(j+k+1)(2C)^2}+\frac{j+1}{(j+k+1)(j+2k+2)\,2C}\right.\\
&\quad\left.+\frac{(j+1)^2}{(j+k+1)(j+2k+1)(j+2k+2)}\right]\\
&\leqslant n^{k+1}m^{k+1}[(j+2(k+1))!]\\
&\quad\times(\sqrt{n}\,C|y|+2C\sqrt{n}\,|x-y|+2C)^{2k+2}
\end{aligned}
$$

Q.E.D.

PROPOSITION 4.5.

(a) $\sqrt{n}\,C|y|+2C\sqrt{n}\,|x-y|+2C\leqslant 3\sqrt{n}\,C(1+|x|+|y|)$

(b) *For* $|t|\leqslant 1$ *and* $N\geqslant 3n\sqrt{m}\,C-1$

$$\left|\sum_{i=0}^{n}\tilde{a}_i(x,y)\,t^i\right|\leqslant 2(N+1)^{4N}(1+\sqrt{|t|}\,|x|)^{2N}(1+\sqrt{|t|}\,|y|)^{2N}$$

(c) *For* $|t|\leqslant 1$ *and* $N\geqslant 3n\sqrt{m}\,C-1$

$$
\begin{aligned}
|\tilde{\phi}_N(t,x,y)|&\leqslant 2(N+1)^{4N}(1+\sqrt{t}\,|x|)^{2N}(1+\sqrt{t}\,|y|)^{2N}(2\pi t)^{-n/2}\\
&\quad\times\exp\left(a(x,y)-\frac{|x-y|^2}{2t}\right).
\end{aligned}
$$

Proof.

(a)
$$
\begin{aligned}
&\sqrt{n}\,C|y|+2C\sqrt{n}\,|x-y|+2C\\
&\quad\leqslant\sqrt{n}\,C|y|+2C\sqrt{n}\,|x|+2C\sqrt{n}\,|y|+2C\\
&\quad\leqslant 3\sqrt{n}\,C(1+|x|+|y|)
\end{aligned}
$$

(b) $$\left|\sum_{i=0}^{N} \tilde{a}_i(x, y)\, t^i\right|$$

$$\leqslant \sum_{i=0}^{N} n^i m^i [(2i)!](3\sqrt{n}\, C)^{2i} (1+|x|+|y|)^{2i}\, t^i$$

$$\leqslant \sum_{i=0}^{N} n^i m^i [(2i)!]\, 3^{2i} n^i C^{2i} (=\sqrt{|t|}\,|x|)^{2i} (1+\sqrt{|t|}\,|y|)^{2i}$$

for $|t| \leqslant 1$

$$\leqslant \sum_{i=0}^{N} n^{2i} m^i (i+1)^{2i}\, 3^{2i} C^{2i} (1+\sqrt{|t|}\,|x|)^{2i} (1+\sqrt{t}\,|y|)^{2i}$$

$$\leqslant (N+1)^{2N} \sum_{i=0}^{N} [3n\sqrt{m}\, C(1+\sqrt{|t|}\,|x|)(1+\sqrt{|t|}\,|y|)]^{2i}$$

Observe that if $r \geqslant \sqrt{2}$, then $\sum_{i=0}^{N} r^{2i} \leqslant (r^{2N+2}-1)/(r^2-1) \leqslant 2r^{2N}$. So we have

$$\left|\sum_{i=0}^{N} \tilde{a}_i(x, y)\, t^i\right|$$

$$\leqslant (N+1)^{2N} 2[3n\sqrt{m}\, C(1+\sqrt{|t|}\,|x|)(1+\sqrt{|t|}\,|y|)]^{2N} \qquad \text{for} \quad |t| \leqslant 1$$

$$= 2(N+1)^{2N} 3^{2N} n^{2N} m^N C^{2N} (1+\sqrt{|t|}\,|x|)^{2N} (1+\sqrt{|t|}\,|y|)^{2N}.$$

If $N \geqslant 3n\sqrt{m}\, C - 1$, then we have

$$\left|\sum_{i=0}^{n} \tilde{a}_i(x, y)\, t^i\right| \leqslant 2(N+1)^{4N} (1+\sqrt{|t|}\,|x|)^{2N} (1+\sqrt{|t|}\,|y|)^{2N}.$$

(c) follows from (b) and (4.1). Q.E.D.

We now estimate

$$e_N := \frac{\partial \tilde{\phi}_N}{2t} - L_x \tilde{\phi}_N,$$

where

$$L_x = \frac{1}{2} \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} - f_i(x)\right)^2 - \frac{1}{2}\left(\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(x) + \sum_{i=1}^{n} f_i^2(x) + \sum_{i=1}^{m} h_i^2(x)\right) \tag{4.4}$$

is the operator defined in Theorem B.

PROPOSITION 4.6. *For* $N \geqslant 3n\sqrt{m}-2$ *and* $|t| \leqslant 1$, *let* $e_N(t, x, y) = \partial\tilde{\phi}_N/\partial t(t, x, y) - L_x\tilde{\phi}_N(t, x, y)$. *Then*

$$|e_N(t, x, y)| \leqslant (2\pi t)^{-n/2}\left[\exp\left(a(x, y) - \frac{|x-y|^2}{2t}\right)\right] \times (N+2)^{4N+4}(1+\sqrt{t}\,|x|)^{2N+2}(1+\sqrt{t}\,|y|)^{2N+2}.$$

Proof. In view of the computation in the proof of Theorem 2 of [Ya-Ya 1], we have

$$\begin{aligned} e_N(t, x, y) &= -(2\pi)^{-n/2}\,t^{-n/2}\left[\exp\left(-\frac{|x-y|^2}{2t}\right)\right] \\ &\quad\times\left\{\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 a_N}{\partial x_i^2}(x, y) - \sum_{i=1}^{n} f_i(x)\frac{\partial a_N}{\partial x_i}(x, y)\right. \\ &\quad\left.-\left(\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) + \frac{1}{2}\sum_{i=1}^{n} h_i^2(x)\right)a_N(x, y)\right\} t^{N+1} \\ &= -(2\pi)^{-n/2}\,t^{-n/2}\left[\exp\left(a(x, y) - \frac{|x-y|^2}{2t}\right)\right] \\ &\quad\times\left\{\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2\tilde{a}_N}{\partial x_i^2}(x, y) + \sum_{i=1}^{n}\left(\frac{\partial a}{\partial x_i}(x, y) - f_i(x)\right)\right. \\ &\quad\times\frac{\partial\tilde{a}_N}{\partial x_i} + \left[\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 a}{\partial x_i}(x, y) + \sum_{i=1}^{n}\frac{1}{2}\left(\frac{\partial a}{\partial x_i^2}(x, y)\right)^2\right. \\ &\quad - \sum_{i=1}^{n} f_i(x)\frac{\partial a}{\partial x_i}(x, y) \\ &\quad\left.\left.-\left(\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) + \frac{1}{2}\sum_{i=1}^{m} h_i^2(x)\right)\right]\tilde{a}_N(x, y)\right\} t^{N-1}. \end{aligned}$$

By applying the estimates in Lemma 4.2 and Proposition 4.4, we get

$$\begin{aligned} |e_N(t, x, y)| &\leqslant (2\pi)^{-n/2}\,t^{-n/2}\left[\exp\left(a(x, y) - \frac{|x-y|^2}{2t}\right)\right] \\ &\quad\times\left\{\frac{n}{2}[(2N+2)!]\,n^N m^N(\sqrt{n}\,C\,|y| + 2C\sqrt{n}\,|x-y| + 2C)^{2N}\right. \\ &\quad + n(\sqrt{n}\,|x-y|\,C)[(2N+1)!]\,n^N m^N \\ &\quad\times(\sqrt{n}\,C\,|y| + 2C\sqrt{n}\,|x-y| + 2C)^{2N} \\ &\quad + nm(\sqrt{n}\,C\,|y| + 2C\sqrt{n}\,|x-y| + 2C)^2\,[(2N)!]\,n^N m^N \\ &\quad\left.\times(\sqrt{n}\,C\,|y| + 2C\sqrt{n}\,|x-y| + 2C)^{2N}\right\} t^{N+1} \end{aligned}$$

$$\leqslant (2\pi t)^{-n/2}\left[\exp(a(x,\,y)-\frac{|x-y|^2}{2t}\right)]$$

$$\times\left\{\frac{1}{2}n^{N+1}m^N[(2N+2)!](\sqrt{n}\,C\,|y|\right.$$

$$+2C\sqrt{n}\,|x-y|+2C)^{2N}+n^{N+1}m^N[(2N+1)!]$$

$$\times(\sqrt{n}\,C\,|y|+2C\sqrt{n}\,|x-y|+2C)^{2N+1}$$

$$+n^{N+1}m^{N+1}[(2N)!](\sqrt{n}\,C\,|y|$$

$$\left.+2C\sqrt{n}\,|x-y|+2C)^{2N+2}\right\}t^{N+1}$$

$$\leqslant (2\pi t)^{-n/2}\left[\exp\left(a(x,\,y)-\frac{|x-y|^2}{2t}\right)\right]n^{N+1}m^{N+1}[(2N+2)!]$$

$$\times(\sqrt{n}\,C\,|y|+2C\sqrt{n}\,|x-y|+2C)^{2N+2}$$

$$\times\left\{\frac{1}{2}\frac{1}{4C^2}+\frac{1}{(2N+2)\,2C}+\frac{1}{(2N+1)(2N+2)}\right\}t^{N+1}$$

$$\leqslant (2\pi t)^{-n/2}\left[\exp\left(a(x,\,y)-\frac{|x-y|^2}{2t}\right)\right]$$

$$\times n^{N+1}m^{N+1}(N+2)^{2N+2}\,[3\sqrt{n}\,C(1+|x|+|y|)]^{2N+2}$$

$$\times\left\{\frac{1}{32}+\frac{1}{8}+\frac{1}{2}\right\}t^{N+1}$$

$$\leqslant (2\pi t)^{-n/2}\left[\exp\left(a(x,\,y)-\frac{|x-y|^2}{2t}\right)\right]$$

$$\times(N+2)^{2N+2}\,(3n\sqrt{m}\,C)^{2N+2}$$

$$\times(1+\sqrt{t}\,|x|)^{2N+2}\,(1+\sqrt{t}\,|y|)^{2N+2}.$$

Therefore if $N\geqslant 3n\sqrt{m}\,C-2$, then

$$|e_N(t,\,x,\,y)|\leqslant (2\pi t)^{-n/2}\left[\exp\left(a(x,\,y)-\frac{|x-y|^2}{2t}\right)\right]$$

$$\times(N+2)^{4N+4}\,(1+\sqrt{t}\,|x|)^{2N+2}\,(1+\sqrt{t}\,|y|)^{2N+2}.\quad \text{Q.E.D.}$$

5. CONSTRUCTION OF A CONVERGENT SOLUTION FROM FORMAL SOLUTION

We shall construct a convergent solution from the truncated formal solution

$$\tilde{\phi}_N(t, x, y) = (2\pi t)^{-n/2} \left[\exp\left(a(x, y) - \frac{|x-y|^2}{2t} \right) \right] \times [1 + \tilde{a}_1(x, y)\, t + \cdots + \tilde{a}_N(x, y)\, t^N]$$

which satisfies the following properties

$$|\tilde{\phi}_N(t, x, y)| \leqslant 2(N+1)^{4N} (1 + \sqrt{t}\,|x|)^{2N} (1 + \sqrt{t}\,|y|)^{2N} (2\pi t)^{-n/2} \times \exp\left(a(x, y) - \frac{|x-y|^2}{2t} \right)$$

and

$$\left| \frac{\partial \tilde{\phi}_N}{\partial t}(t, x, y) - L_x \tilde{\phi}_N(t, x, y) \right| = |e_N(t, x, y)| \leqslant (2\pi t)^{-n/2} \left[\exp\left(a(x, y) - \frac{|x-y|^2}{2t} \right) \right] (N+2)^{4N+4} \times (1 + \sqrt{t}\,|x|)^{2N+2} (1 + \sqrt{t}\,|y|)^{2N+2},$$

where $t \leqslant 1$ and $N \geqslant 3n\sqrt{m}\, C - 1$. In fact we claim that,

$$0 \leqslant t \leqslant \min\left\{ \frac{\sqrt{n^2C^2 + 2(N+1)} - nC}{8(N+1)}, \frac{1}{8nC} \right\}.$$

The following series converges,

$$\tilde{\phi}_N(t, x, y) + \sum_{k=0}^{\infty} (-1)^{k+1} \frac{1}{\sqrt{k+2}} \times \int_{\sum_{i=0}^{k+1} \tau_i = t} \int_{x_{k+1}} \int_{x_k} \cdots \int_{x_1} \tilde{\phi}_N(\tau_{k+1}, x, x_{k+1}) \times e_N(\tau_k, x_{k+1}, x_k)\, e_N(\tau_{k-1}, x_k, x_{k-1}) \cdots e_N(\tau_0, x_1, y), \quad (5.1)$$

and it represents a kernel $\phi(t, x, y)$ which satisfies

$$\frac{\partial \phi}{\partial t}(t, x, y) = L_x \phi(t, x, y)$$

$$\lim_{t \to 0} \phi(t, x, y) = \delta_x(y).$$

In this way, for $t \leqslant \epsilon/N$, we have found an explicit kernel for the equation. When time is equal to T which may be large, we can find the kernel up to time T by the formula

$$\phi(T, x, y) = \int_{x_1} \int_{x_2} \cdots \int_{x_K} \phi\left(\frac{T}{K}, x, x_1\right) \phi\left(\frac{T}{K}, x_1, x_2\right) \cdots \phi\left(\frac{T}{K}, x_K, y\right). \tag{5.2}$$

Here K is the smallest integer greater than TN/ε.

The following two lemmas can be found in [Ya-Ya 2].

LEMMA 5.1. (a) $|a(x, y) - \sum_{i=1}^n (x_i - y_i) f_i(x)| \leqslant n C |x - y|^2$

(b) $$\left| a(x, x_{k+1}) + \sum_{i=1}^{k} a(x_{i+1}, x_i) + a(x_1, y) - (x - y) f(x) \right|$$
$$\leqslant n C \left[\left(\frac{t}{\tau_0} + 1 \right) \cdot |y - x_1|^2 + \sum_{i=1}^{k} \left(\frac{t}{\tau_i} + 1 \right) |x_i - x_{i+1}|^2 + \left(\frac{t}{\tau_{k+1}} + 1 \right) |x_{k+1} - x|^2 \right],$$

where $f(x) = (f_1(x), ..., f_n(x))$ *and* $t = \sum_{i=1}^{k+1} \tau_i$.

LEMMA 5.2.

$$(1 + \sqrt{\tau_{k+1}}\, |x|)^{2N} \prod_{i=1}^{k+1} (1 + \sqrt{\tau_i}\, |x_i|)^{2N} \times \prod_{i=0}^{k} (1 + \sqrt{\tau_i}\, |x_{i+1}|)^{2N} (1 + \sqrt{\tau_0}\, |y|)^{2N}$$
$$\leqslant (1 + \sqrt{t}\, |x|)^{2N} (1 + \sqrt{t}\, |y|)^{2N} 2^{4N(k+1)} \times \exp\left[4Nt^2 \left(\sum_{j=1}^{k+1} \frac{|x_{j+1} - x_j|^2}{\tau_j} + \frac{|x_1 - y|^2}{\tau_0} \right) + 4Nt\, |y|^2 \right].$$

We are now ready to estimate the infinite series (5.1).

THEOREM 5.3. *If t is chosen small enough so that*

$$0 \leqslant t \leqslant \min\left\{\frac{\sqrt{n^2C^2+2(N+1)}-nC}{8(N+1)}, \frac{1}{8nC}\right\}, \tag{5.2}$$

then the general term in (5. 1) *has the estimate*

$$\begin{aligned}
&\int_{\sum_{i=0}^{k+1}\tau_i=t}\int_{x_{k+1}}\int_{x_k}\cdots\int_{x_1}|\tilde{\phi}_N(\tau_{k+1}, x, x_{i+1})\, e_N(\tau_k, x_{k+1}, x_k)\\
&\quad\times e_N(\tau_{k-1}, x_k, x_{k-1})\cdots e_N(\tau_0, x_1, y)|\\
&\leqslant 2(N+2)^{4(k+2)N+4(k+1)}(2\pi)^{-(k+2)(n/2)}\, 2^{4(N+1)(k+1)}\\
&\quad\times(1+\sqrt{t}\,|x|)^{2N+2}(1+\sqrt{t}\,|y|)^{2N+2}\\
&\quad\times(4\pi)^{(n/2)(k+1)}\sqrt{k+2}\,\frac{t^{-n/2+k+1}}{(k+1)!}\\
&\quad\times\exp\left[(x-y)\cdot f(x)+4(N+1)\,t\,|y|^2-\frac{|x-y|^2}{4t}\right].
\end{aligned}$$

Proof.

$$\begin{aligned}
&\int_{\sum_{i=0}^{k+1}\tau_i=t}\int_{x_{k+1}}\int_{x_k}\cdots\int_{x_1}|\tilde{\phi}_N(\tau_{k+1}, x, x_{k+1})\, e_N(\tau_k, x_{k+1}, x_k)\\
&\quad\times e_N(\tau_{k-1}, x_k, x_{k-1})\cdots e_N(\tau_0, x_1, y)|\\
&\leqslant\int_{\sum_{i=0}^{k+1}\tau_i=t}\int_{x_{k+1}}\int_{x_k}\cdots\int_{x_1}2(N+1)^{4N}(2\pi\tau_{k+1})^{-n/2}(1+\sqrt{\tau_{k+1}}\,|x|)^{2N}\\
&\quad\times(1+\sqrt{\tau_{k+1}}\,|x_{k+1}|)^{2N}\left[\exp\left(a(x, x_{k+1})-\frac{|x-x_{k+1}|^2}{2\tau_{k+1}}\right)\right]\\
&\quad\times(2\pi\tau_k)^{-n/2}(N+2)^{4N+4}(1+\sqrt{\tau_k}\,|x_{k+1}|)^{2N+2}(1+\sqrt{\tau_k}\,|x_k|)^{2N+2}\\
&\quad\times\left[\exp\left(a(x_{k+1}, x_k)-\frac{|x_{k+1}-x_k|^2}{2\tau_k}\right)\right]\\
&\quad\times(2\pi\tau_{k-1})^{-n/2}(N+2)^{4N+4}(1+\sqrt{\tau_{k-1}}\,|x_k|)^{2N+2}\\
&\quad\times(1+\sqrt{\tau_{k-1}}\,|x_{k-1}|)^{2N+2}\left[\exp\left(a(x_k, x_{k-1})-\frac{|x_k-x_{k-1}|^2}{2\tau_{k-1}}\right)\right]\\
&\quad\cdots\cdots\cdots\cdots\cdots\\
&\quad\times(2\pi\tau_0)^{-n/2}(N+2)^{4N+4}(1+\sqrt{\tau_0}\,|x_1|)^{2N+2}(1+\sqrt{\tau_0}\,|y|)^{2N+2}\\
&\quad\times\exp\left(a(x_1, y)-\frac{|x_1-y|^2}{2\tau_0}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant 2(N+2)^{4(k+2)N+4(k+1)}(2\pi)^{-(k+2)n/2}\int_{\sum_{i=0}^{k+1}\tau_i=t}\int_{x_{k+1}}\int_{x_k}\cdots\int_{x_1}\\
&\quad\times(\tau_0\cdots\tau_{k+1})^{-n/2}(1+\sqrt{\tau_{k+1}}\,|x|)^{2N+2}\\
&\quad\times\prod_{i=1}^{k+1}(1+\sqrt{\tau_i}\,|x_i|)^{2N+2}\prod_{i=0}^{k}(1+\sqrt{\tau_i}\,|x_{i+1}|)^{2N+2}\\
&\quad\times(1+\sqrt{\tau_0}\,|y|)^{2N+2}\\
&\quad\times\left[\exp\left(a(x,x_{k+1})+\sum_{i=1}^{k}a(x_{i+1},x_i)+a(x,y)\right)\right]\\
&\quad\times\exp\left(-\frac{|x-x_{k+1}|^2}{2\tau_{k+1}}-\sum_{i=1}^{k}\frac{|x_{i+1}-x_i|^2}{2\tau_i}-\frac{|x_1-y|^2}{2\tau_0}\right)\\
&\leqslant 2(N+2)^{4(k+2)N+4(k+1)}(2\pi)^{-(k+2)n/2}\int_{\sum_{i=0}^{k+1}\tau_i=t}\int_{x_{k+1}}\int_{x_k}\cdots\int_{x_1}\\
&\quad\times(\tau_0\cdots\tau_{k+1})^{-n/2}(1+\sqrt{t}\,|x|)^{2N+2}\\
&\quad\times(1+\sqrt{t}\,|y|)^{2N+2}\,2^{4(N+1)(k+1)}\\
&\quad\times\exp\left[4(N+1)\,t^2\left(\sum_{j=1}^{k+1}\frac{|x_{j+1}-x_j|^2}{\tau_j}+\frac{|x_1-y|^2}{\tau_0}\right)+4(N+1)\,t\,|y|^2\right]\\
&\quad\times\exp\left\{(x-y)\,f(x)+nC\left[\left(\frac{t}{\tau_0}+1\right)|y-x_1|^2+\sum_{i=1}^{k}\left(\frac{t}{\tau_i}+1\right)\right.\right.\\
&\quad\left.\left.\times|x_i-x_{i+1}|^2+\left(\frac{t}{\tau_{k+1}}+1\right)|x_{k+1}-x|^2\right]\right\}\\
&\quad\times\exp\left(-\frac{|x-x_{k+1}|^2}{2\tau_{k+1}}-\sum_{i=1}^{k}\frac{|x_{k+1}-x_i|^2}{2\tau_i}-\frac{|x_1-y|^2}{2\tau_0}\right)\\
&=2(N+2)^{4(k+2)N+4(k+1)}(2\pi)^{-(k+2)n/2}\,2^{4(N+1)(k+1)}\\
&\quad\times(1+\sqrt{t}\,|x|)^{2N+2}(1+\sqrt{t}\,|y|)^{2N+2}\\
&\quad\times\exp[(x-y)\cdot f(x)]\int_{\sum_{i=0}^{k+1}\tau_i=t}\int_{x_{k+1}}\int_{x_k}\cdots\int_{x_1}(\tau_0\cdots\tau_{k+1})^{-n/2}\\
&\quad\times\exp\left\{\left[4(N+1)\frac{t^2}{\tau_0}+nC\left(\frac{t}{\tau_0}+1\right)-\frac{1}{2\tau_0}\right]|x_1-y|^2\right.\\
&\quad+\sum_{i=1}^{k}\left[4(N+1)\frac{t^2}{\tau_i}+nC\left(\frac{t}{\tau_i}+1\right)-\frac{1}{2\tau_i}\right]|x_{i+1}-x_i|^2\\
&\quad\left.+\left[nC\left(\frac{t}{\tau_{k+1}}+1\right)-\frac{1}{2\tau_{k+1}}\right]|x-x_{k+1}|^2+4(N+1)\,t\,|y|^2\right\}
\end{aligned}
$$

Since $0 \leqslant t \leqslant \min\{[\sqrt{n^2C^2+2(N+1)}-nC]/8(N+1), 1/8nC\}$, it is clear that

$$4(N+1)\,t^2+nC+ < \frac{1}{8} \qquad \text{and} \qquad nC < \frac{1}{8t}.$$

It follows that

$$\frac{4(N+1)\,t^2}{\tau_i}+nC\left(\frac{t}{\tau_i}+1\right)-\frac{1}{2\tau_i} < -\frac{1}{4\tau_i}$$

for all i. So the general term in our series is estimated by

$$\begin{aligned}
&2(N+2)^{4(k+2)N+4(k+1)}\,(2\pi)^{-(k+2)(n/2)}\,2^{4(N+1)(k+1)}\\
&\quad\times(1+\sqrt{t}\,|x|)^{2N+2}\,(1+\sqrt{t}\,|y|)^{2N+2}\\
&\quad\times\{\exp[(x-y)\cdot f(x)+4(N+1)\,t\,|y|^2]\}\\
&\quad\times\int_{\sum_{i=0}^{k+1}\tau_i=t}\int_{x_{k+1}}\int_{x_k}\cdots\int_{x_1}(\tau_0\cdots\tau_{k+1})^{-n/2}\\
&\quad\times\exp\left(\frac{|x-x_{k+1}|^2}{4\tau_{k+1}}-\frac{|x_{k+1}-x_k|^2}{4\tau_k}-\cdots-\frac{|x_1-y|^2}{4\tau_0}\right)\\
&=2(N+2)^{4(k+2)N+4(k+1)}\,(2\pi)^{-(k+2)(n/2)}\,2^{4(N+1)(k+1)}\\
&\quad\times(1+\sqrt{t}\,|x|)^{2N+2}\,(1+\sqrt{t}\,|y|)^{2N+2}\\
&\quad\times\{\exp[(x-y)\cdot f(x)+4(N+1)\,t\,|y|^2]\}\\
&\quad\times\int_{\sum_{i=0}^{k+1}\tau_i=t}\int_{x_{k+1}}\int_{x_k}\cdots\int_{x_1}(4\pi)^{(n/2)(k+2)}\\
&\quad\times H(\tau_{k+1},x-x_{k+1})*H(\tau_k,x_{k+1}-x_k)*\cdots*H(\tau_0,x_1-y),
\end{aligned}$$

where, for $1 \leqslant i \leqslant k$, $H_i(\tau_i, x)$ is the kernel $(4\pi\tau_i)^{-n/2}\exp(-|x|^2/4\tau_i)$, $H(\tau_0, x)$ is the kernel $(4\pi\tau_0)^{-n/2}\exp(-|x|^2/4\tau_0)$, and $H(\tau_{k+1}, x)$ is the kernel $(4\pi\tau_{k+1})^{-n/2}\exp(-|x|^2/4\tau_{k+1})$. Each $H(\tau_i, x)$ defines an integral operator acting on $L^2(\mathbf{R})$. In view of the semigroup property of H (see Theorem 3 on p. 33 of [Wi]), the convolution $H(\tau_{k+1}, x)*\cdot*H(\tau_0, x)$ is given by a kernel of the form

$$\begin{aligned}
&2(N+2)^{4(k+2)N+4(k+1)}\,(2\pi)^{-(k+2)(n/2)}\,2^{4(N+1)(k+1)}\\
&\quad\times(1+\sqrt{t}\,|x|)^{2N+2}\,(1+\sqrt{t}\,|y|)^{2N+2}\\
&\quad\times\exp[(x-y)\cdot f(x)+4(N+1)\,t\,|y|^2](4\pi)^{(n/2)(k+1)}\,t^{-n/2}\\
&\quad\times\exp\left(\frac{-|x-y|^2}{4t}\right)\int_{\sum_{i=0}^{k+1}\tau_i=t}1.
\end{aligned}$$

Notice that

$$\int_{\sum_{i=0}^{k+1}\tau_i=t} 1 = \mathrm{Vol}\left(\sum_{i=0}^{k+1}\tau_i = t,\ \tau_i \geqslant 0\right)$$

$$= \int_{\sum_{i=0}^{k}\tau_i \leqslant t,\ \tau_i \geqslant 0} \sqrt{1+|\mathrm{grad}\ \tau_{k+1}|^2}\, d\tau_k\, d\tau_{k-1} \cdots d\tau_0,$$

where $\tau_{k+1} = -(\tau_0 + \tau_1 + \cdots + \tau_k)$ is viewed as a function of $\tau_0, \tau_1, ..., \tau_k$.

Therefore

$$\mathrm{Vol}\left(\sum_{i=0}^{k+1}\tau_i = t\right) = \sqrt{k+2}\ \mathrm{Vol}\left(\sum_{i=0}^{k}\tau_i \leqslant t,\ \tau_i \geqslant 0\right)$$

$$= \sqrt{k+2}\,\frac{t^{k+1}}{(k+1)!}.$$

Therefore the general terrn in series (5.1) is estimated by

$$2(N+2)^{4(k+2)N+4(k+1)}(2\pi)^{-(k+2)(n/2)}\, 2^{4(N+1)(k+1)}$$
$$\times (1+\sqrt{t}\,|x|)^{2N+2}\,(1+\sqrt{t}\,|y|)^{2N+2}$$
$$\times \{\exp[(x-y)\cdot f(x) + 4(N+1)\,t\,|y|^2]\}(4\pi)^{(n/2)(k+1)}\sqrt{k+2}$$
$$\times \left[\exp\left(-\frac{|x-y|^2}{4t}\right)\right]\frac{t^{-n/2+k+1}}{(k+1)!}. \qquad \text{Q.E.D.}$$

THEOREM 5.4. *If t is chosen small enough so that*

$$0 \leqslant t \leqslant \min\left\{\frac{\sqrt{n^2C^2 + 2(N+1)} - nC}{8(N+1)}, \frac{1}{8nC}\right\}, \tag{5.3}$$

then the infinite series (5.1) *converges.*

Proof. This follows easily from the root test for convergent power series, Theorem 5.3 and the following Lemma 5.5. Q.E.D.

LEMMA 5.5. *For positive integer k, $k! \geqslant k^{k/2}$.*

Proof. We first assume that k is even. Observe that

$$(j-k)(j-1) \leqslant 0 \qquad \text{for} \quad 1 \leqslant j \leqslant k$$

which implies

$$(k-j+1)\,j \geqslant k \qquad \text{for} \quad 1 \leqslant j \leqslant k.$$

Hence

$$\prod_{j=1}^{k/2}[(k-j+1)\,j] \geqslant k^{k/2}.$$

The left hand side of the above inequality is exactly $k!$.

We next assume that k is odd, say $k=2p+1$. Observe that

$$[j-(2p+1)](j-1)\leqslant 0 \qquad \text{for} \quad 1\leqslant j\leqslant 2p+1$$

which implies

$$(2p-j+2)\, j\geqslant 2p+1 \qquad \text{for} \quad 1\leqslant j\leqslant 2p+1.$$

Hence

$$\prod_{j=1}^{p} [(2p-j+2)\, j]\geqslant (2p+1)^p. \tag{5.4}$$

Clearly, we have

$$p+1\geqslant \sqrt{2p+1}. \tag{5.5}$$

Taking the product of (5.4) and (5.5), we have

$$(p+1)\prod_{j=1}^{p} [(2p-j+2)\, j]\geqslant \sqrt{2p+1}\,(2p+1)^p.$$

The left hand side of the above inequality is $k!$ while the right hand side is $k^{k/2}$. This finishes the proof. Q.E.D.

THEOREM 5.6. (i) $\lim_{t\to 0} \tilde{\phi}_N(t, x, y)=\delta_x(y)$.

(ii) $\lim_{t\to 0} \phi(t, x, y)=\delta_x(y)$ *where* $\phi(t, x, y)$ *denotes the infinite series* (5.1).

Proof. (i) For any differentiable function $\sigma(x)$ on $\mathbf{R}^n$, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \tilde{\phi}_N(t, x, y)\,\sigma(y)\, dy_1\cdots dy_n \\
&\quad=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \tilde{\phi}_N(t, x, x-y)\,\sigma(x-y)\, dy_1\cdots dy_n \\
&\quad=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} (2\pi t)^{-n/2}\left[\exp\left(a(x, x-y)-\frac{|y|^2}{2t}\right)\right] \\
&\qquad\times[1+\tilde{a}_1(x, x-y)\, t+\cdots+\tilde{a}_N(x, x-y)\, t^N]\,\sigma(x-y)\, dy_1\cdots dy_n.
\end{aligned}$$

Let $y = \sqrt{2tr}$ where $r = (r_1, ..., r_n)$. Then

$$a(x, x-y) = \int_0^1 \sum_{i=1}^n y_i f_i((x-y)+ty)\, dt$$
$$= \int_0^1 y \cdot f(x+(t-1)\, y)\, dt.$$

Hence

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \tilde{\phi}_N(t, x, y)\, \sigma(y)\, dy_1 \cdots dy_n$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \pi^{-n/2} \left[\exp\left(\int_0^1 \sqrt{2t}\, \gamma \cdot f(x+(t-1)\sqrt{2t}\, \gamma) - |r|^2 \right) \right]$$
$$\times [1 + \tilde{a}_1(x, x-\sqrt{2t}\, \gamma)\, t + \cdots + \tilde{a}_N(x, x-\sqrt{2t}\, \gamma)\, t^N]$$
$$\times \sigma(x-\sqrt{2t}\, \gamma)\, dr_1 \cdots dr_n.$$

It follows that

$$\lim_{t \to 0} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \tilde{\phi}_N(t, x, y)\, \sigma(y)\, dy_1 \cdots dy_n$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \pi^{-n/2} \left[\exp\left(-\sum_{i=1}^n r_i^2 \right) \right] \sigma(x)\, dr_1 \cdots dr_n = \sigma(x).$$

So (i) is proven.

(ii) Follows immediately from (i) and (5.8) below. Q.E.D.

THEOREM 5.7. *Let* $\phi(t, x, y)$ *denote the infinite series* (5.1). *Then* $\phi(t, x, y)$ *is the fundamental solution to the Kolmogorov equation, i.e.,*

$$\frac{\partial \phi}{\partial t}(t, x, y) = L_x \phi(t, x, y) \tag{5.6}$$

$$\lim_{t \to 0} \phi(t, x, y) = \delta_x(y), \tag{5.7}$$

where L_x *is defined by* (4.4).

Proof. In view of Theorem 5.4, there is no problem for convergence of the infinite series (5.1) and its derivatives. We can differentiate the series (5.1) term by term. We rewrite the integral

$$\int_{\sum_{i=0}^{k+1} \tau_i = t} \int_{x_{k+1}} \int_{x_k} \cdots \int_{x_1} \frac{1}{\sqrt{k+2}}\, \tilde{\phi}_N(\tau_{k+1}, x, x_{k+1})$$
$$\times e_N(\tau_k, x_{k+1}, x_k)\, e_N(\tau_{k-1}, x_k, x_{k-1}) \cdots e_N(\tau_0, x_1, y)$$

as

$$\begin{aligned}\phi_k(t, x, y) = &\int_0^t \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_k} \int_{x_{k+1}} \int_{x_k} \cdots \int_{x_1} \tilde{\phi}_N(t-t_1, x, x_{k+1}) \\ &\times e_N(t_1-t_2, x_{k+1}, x_k)\, e_N(t_2-t_3, x_k, x_{k-1}) \cdots \\ &\times e_N(t_k-t_{k+1}, x_2, x_1)\, e_N(t_{k+1}, x_1, y)\, dt_{k+1}\, dt_k \cdots dt_1.\end{aligned}$$

It follows easily that

$$\begin{aligned}\frac{\partial \phi_k}{\partial t}(t, x, y) = &\int_0^t \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_k} \int_{x_{k+1}} \int_{x_k} \cdots \int_{x_1} \\ &\times \frac{\partial \tilde{\phi}_N}{\partial t}(t-t_1, x, x_{k+1})\, e_N(t_1-t_2, x_{k+1}, x_k) \\ &\times e_N(t_2-t_3, x_k, x_{k-1}) \cdots e_N(t_k-t_{k+1}, x_2, x_1) \\ &\times e_N(t_{k+1}, x_1, y)\, dt_{k+1}\, dt_k \cdots dt_1 \\ &+ \lim_{t_1 \to t} \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_k} \int_{x_{k+1}} \int_{x_k} \cdots \int_{x_1} \tilde{\phi}_N(t-t_1, x, x_{k+1}) \\ &\times e_N(t_1-t_2, x_{k+1}, x_k)\, e_N(t_2, t_3, x_k, x_{k-1}) \cdots \\ &\times e_N(t_k-t_{k+1}, x_2, x_1)\, e_N(t_{k+1}, x_1, y)\, dt_{k+1}\, dt_k \cdots dt_2 \\ = &\int_0^t \int_0^{t_1} \cdots \int_0^{t_k} \int_{x_{k+1}} \int_{x_k} \cdots \int_{x_1} \frac{\partial \tilde{\phi}_N}{\partial t}(t-t_1, x, x_{k+1}) \\ &\times e_N(t_1-t_2, x_{k+1}, x_k)\, e_N(t_2-t_3, x_k, x_{k-1}) \cdots \\ &\times e_N(t_k-t_{k+1}, x_2, x_1)\, e_N(t_{k+1}, x_1, y)\, dt_{k+1}\, dt_k \cdots dt_1 \\ &+ \int_0^t \int_0^{t_2} \cdots \int_0^{t_k} \int_{x_k} \int_{x_{k-1}} \cdots \int_{x_1} e_N(t-t_2, x, x_k) \\ &\times e_N(t_2-t_3, x_k, x_{k-1}) \cdots e_N(t_k-t_{k+1}, x_2, x_1) \\ &\times e_N(t_{k+1}, x_1, y)\, dt_{k+1}\, dt_k \cdots dt_2.\end{aligned}$$

Hence

$$\begin{aligned}&\left(\frac{\partial}{\partial t} - L_x\right) \phi_k(t, x, y) \\ &\quad = \int_0^t \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_k} \int_{x_{k+1}} \int_{x_k} \cdots \int_{x_1} \left(\frac{\partial}{\partial t} - L_x\right) \tilde{\phi}_N(t-t_1, x, x_{k+1}) \\ &\qquad \times e_N(t_1-t_2, x_{k+1}, x_k)\, e_N(t_2-t_3, x_k, x_{k-1}) \cdots e_N(t_k-t_{k+1}, x_2, x_1) \\ &\qquad \times e_N(t_{k+1}, x_1, y)\, dt_{k+1}\, dt_k \cdots dt_1\end{aligned}$$

$$+\int_0^t\int_0^{t_2}\cdots\int_0^{t_k} e_N(t-t_2, x, x_k)\, e_N(t_2-t_3, x_k, x_{k-1})\cdots$$

$$\times e_N(t_k-t_{k+1}, x_2, x_1)\, e_N(t_{k+1}, x_1, y)\, dt_{k+1}\cdots dt_2$$

$$=\int_0^t\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_k}\int_{x_{k=1}}\int_{x_k}\cdots\int_{x_1} e_N(t-t_1, x, x_{k+1})$$

$$\times e_N(t_1-t_2, x_{k+1}, x_k)\, e_N(t_2-t_3, x_k, x_{k+1})\cdots$$

$$\times e_N(t_k-t_{k+1}, x_2, x_1)\, e_N(t_{k+1}, x_1, y)\, dt_{k+1}\cdots dt_2\, dt_1$$

$$+\int_0^t\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_k} e_N(t-t_2, x, x_k)\, e_N(t_2-t_3, x_k, x_{k-1})\cdots$$

$$\times e_N(t_k-t_{k+1}, x_2, x_1)\, e_N(t_{k+1}, x_1, y)\, dt_{k+1}\, dt_k\cdots dt_2.$$

Therefore

$$\left(\frac{\partial}{\partial t}-L_x\right)\left(\tilde{\phi}_N(t, x, y)+\sum_{k=0}^{k}(-1)^{k+1}\phi_k\right)$$

$$=(-1)^{k+1}\int_0^t\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_k}\int_{x_{k+1}}\int_{x_k}\cdots\int_{x_1} e_N(t-t_1, x, x_{k+1})$$

$$\times e_N(t_1-t_2, x_{k+1}, x_k)\, e_N(t_2-t_3, x_k, x_{k+1})\cdots e_N(t_{k+1}, x_1, y)$$

$$dt_{k+1}\cdots dt_2\, dt_1.$$

However, a similar estimate as in Theorem 5.3 shows that the above expression tends to zero uniformly as $k\to\infty$. Q.E.D.

COROLLARY 5.8. *The fundamental solution $\phi(t, x, y)$ in Theorem* 5.7 *is approximated by*

$$\tilde{\phi}_N(t, x, y)+\sum_{k=0}^{K}(-1)^{k+1}\phi_k(t, x, y)$$

which is readily computable. Here $\phi_k(t, x, y)$ is given by (5.9). *The error for such an approximation is given by*

$$\sum_{k=K+1}^{\infty}(-1)^{k+1}\phi_k(t, x, y)$$

which can be estimated by

$$(1+\sqrt{t}\,|x|)^{2N+2}\,(1+\sqrt{t}\,|y|)^{2N+2}\,\{\exp[(x-y)\cdot f(x)+4(N+1)\,t\,|y|^2]\}$$
$$\times\left[\exp\left(-\frac{|x-y|^2}{4t}\right)\right]\sum_{k=K+1}^{\infty}2(N+2)^{4(k+2)\,N+4(k+1)}$$
$$\times\sqrt{k+2}\,(\sqrt{2\pi})^{-(k+2)(n/2)}\,2^{4(N+1)(k+1)}(\sqrt{4\pi})^{(k+1)(n/2)}\,\frac{t^{-n/2+k+1}}{(k+1!)}$$

which clearly tends to zero rapidly if t is small and K is large.

Proof. This follows immediately from Theorem 5.3.

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