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EXISTENCE AND DECAY ESTIMATES FOR TIME DEPENDENT PARABOLIC EQUATION WITH APPLICATION TO DUNCAN-MORTENSEN-ZAKAI EQUATION*

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Abstract. In this paper, we provide existence and estimates of the equation

$$\frac{\partial u}{\partial t} = \triangle u + \sum_{i=1}^{n} f_i \frac{\partial u}{\partial x_i} - Vu.$$

0. Introduction. In control theory or in many branches of applied mathematics, we are interested in an evolution equation of the following type:

$$\frac{\partial u}{\partial t} = \triangle u + \sum_{i=1}^{n} f_i \frac{\partial u}{\partial x_i} - Vu,$$

where f_i and V are possibly time dependent functions.

Given an initial function at time zero, we would like to know existence of a positive solution of this equation. Furthermore if the initial function decay fast in spatial direction, we would like to know the spatial decay property of the solution for later time. In fact, in order for numerical calculation to be carried out effectively, we need to know quantitively this decay property. In this paper, we provide precise estimates of such an equation under reasonable assumptions on f and V. In applications f_i and V may not be smooth in time. We have therefore avoided any argument involving differentiation of f_i and V in time. A typical equation that can be treated are those arised in nonlinear filtering problem where the robust Duncan-Mortensen-Zakai equation has our form. We demonstrate existence and give decay estimate of this equation.

D. Strook pointed out that his paper with Norris (Heat flows with uniformly elliptic coefficients, Proceedings LMS (3), Vol.62, #2, (1991), 373–402) is closely related to section 1 of this paper where they treated the case with bounded coefficients. We were also informed that Fleming-Mitter, Sussmann, Baras-Blankenship-Hopkins have obtained important estimates on the DMZ equation. However the latter two papers are focused on one spatial dimension, while the former paper needs the boundedness of f and ∇f.

1. A priori estimations. To begin with, let us recall some well known formulas and inequality which will be used repeatedly throughout this paper.

Divergence Theorem Let Ω be a bounded domain with C^1-boundary $\partial\Omega$ and let ν denote the unit outward normal to $\partial\Omega$. For any vector field w in $C^0(\bar{\Omega}) \cap C^1(\Omega)$,

$$\int_{\Omega} \text{div } w \, dx = \int_{\partial\Omega} w \cdot \nu \, ds,$$

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where ds denotes that $(n-1)$-dimensional area element in $\partial\Omega$.

More generally for any scalar valued function $\alpha \in C^0(\bar{\Omega}) \cap C^1(\Omega)$,

$$\int_\Omega \alpha \text{ div } w \, ds = \int_{\partial\Omega} \alpha w \cdot \nu \, ds - \int_\Omega w \cdot \nabla\alpha \, dx.$$

Green's First Identity Let Ω be a domain for which the divergence theorem holds and let w and v be $C^1(\bar{\Omega}) \cap C^2(\Omega)$ functions, Then

$$\int_\Omega v\Delta w \, dx = -\int_\Omega \nabla v \cdot \nabla w \, dx + \int_{\partial\Omega} v \frac{\partial w}{\partial \nu} \, ds.$$

More generally for any $C^1(\bar{\Omega}) \cap C^2(\Omega)$ function α,

$$\int_\Omega \alpha \nabla v \cdot \nabla w \, dx = -\int_\Omega \alpha v \Delta w \, dx - \int_\Omega v \nabla\alpha \cdot \nabla w \, dx + \int_{\partial\Omega} \alpha v \frac{\partial w}{\partial \nu} \, ds.$$

The following inequality is true for any $a, b \in \mathbb{R}$ and any $\varepsilon > 0$,

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2.$$

For arbitrary real numbers $a_1, a_2, \cdots, a_n$, we have

$$(a_1 + \cdots + a_n)^2 \leq n a_1^2 + n a_2^2 + \cdots + n a_n^2.$$

We are now ready to prove the first theroem in this section.

THEOREM 1.1. *Let Ω be a compact domain in $\mathbb{R}^n$ with C^1-boundary $\partial\Omega$ and let ν denote the unit outward normal to $\partial\Omega$. Let $f_1(x,t), \cdots, f_n(x,t)$ and $V(x,t)$ be smooth functions in x-variable. Suppose that $f_1, \cdots, f_n$ vanish on $\partial\Omega$. Let u be a solution of the equation*

$$\frac{\partial u}{\partial t} = \Delta u + \sum_{i=1}^n f_i \frac{\partial u}{\partial x_i} - Vu \tag{1.1}$$

with boundary condition $\dfrac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$.

(i) If $\dfrac{\partial g}{\partial t} + \dfrac{1}{2}|\nabla g|^2 - \displaystyle\sum_{i=1}^n f_i \frac{\partial g}{\partial x_i} -$ div $f - 2V \leq 0$, for $0 \leq t \leq T$, then

$$\int_{\Omega\times\{T\}} e^g u^2 \leq \int_{\Omega\times\{0\}} e^g u^2. \tag{1.2}$$

(ii) If $\dfrac{\partial h}{\partial t} + |\nabla h + f|^2 - 2V + |\nabla V|^2 \leq 0$, for $0 \leq t \leq T$, then

$$\int_{\Omega\times\{T\}} e^h |\nabla u|^2 \leq \int_{\Omega\times\{0\}} e^h |\nabla u|^2 + \int_0^T \int_\Omega e^h u^2. \tag{1.3}$$

(iii) If $\dfrac{\partial \rho}{\partial t} + (\varepsilon_1 + \dfrac{1}{2\varepsilon_1^2})|\nabla\rho|^2 + \displaystyle\sum_{i=1}^n \rho_i f_i + \sum_{i=1}^n f_{i,i} \leq 0$ for $0 \leq t \leq T$, where $2 - 10\varepsilon_1^2 - \dfrac{2\varepsilon_1}{1-3\varepsilon_1} > 0$, then

$$\text{(1.4)} \quad \int_{\Omega\times\{T\}} e^\rho (\Delta u)^2 \leq \int_{\Omega\times\{0\}} e^\rho (\Delta u)^2 + O\Big(\int_{\Omega\times[0,T]} e^\rho |\nabla\rho|^2 |f|^2 |\nabla u|^2$$

$$+\int_{\Omega\times[0,T]} e^{\rho}|\nabla f|^2|\nabla u|^2 + \int_{\Omega\times[0,T]} e^{\rho}|f||\nabla u|^2|\Delta f| + \int_{\Omega\times[0,T]} e^{\rho}|f|^4|\nabla u|^2$$
$$+\int_{\Omega\times[0,T]} e^{\rho}|\nabla(Vu)|^2 + \int_{\Omega\times[0,T]} e^{\rho}|\nabla u|^2(\sum_{i=1}^{n} f_{i,i})^2\Big).$$

Proof. (i) Equation (1.1) implies

$$\int_{\Omega}\int_0^T e^g u(\frac{\partial u}{\partial t} - \Delta u - \sum_{i=1}^{n} f_i \frac{\partial u}{\partial x_i} + Vu) = 0. \tag{1.2}$$

Integrating by parts, we have

$$\begin{aligned} 0 = &\int_{\Omega\times\{T\}} \frac{e^g u^2}{2} - \int_{\Omega\times\{0\}} \frac{e^g u^2}{2} - \int_{\Omega}\int_0^T e^g \frac{u^2}{2}\frac{\partial g}{\partial t} \\ &+ \int_{\Omega}\int_0^T e^g u\nabla g\cdot\nabla u + \int_{\Omega}\int_0^T e^g|\nabla u|^2 \\ &+ \int_{\Omega}\int_0^T e^g\frac{u^2}{2}\sum_{i=1}^{n} f_i g_i + \int_{\Omega}\int_0^T e^g\frac{u^2}{2}\sum_{i=1}^{n} f_{i,i} \\ &+ \int_{\Omega}\int_0^T e^g u^2 V, \end{aligned} \tag{1.3}$$

where g_i denotes $\frac{\partial g}{\partial x_i}$ and $f_{i,i}$ denotes $\frac{\partial f_i}{\partial x_i}$. The boundary condition for (1.3)

$$\int_{\partial\Omega}\int_0^T u(\frac{\partial u}{\partial \nu} + \frac{1}{2}u\sum_{i=1}^{n} f_i\nu_i)$$

vanishes because $\frac{\partial u}{\partial \nu}, f_1\cdots, f_n$ vanish on $\partial\Omega$. But

$$\left|\int_{\Omega}\int_0^T e^g u\nabla g\cdot\nabla u\right| \le \frac{1}{4}\int_{\Omega}\int_0^T e^g u^2|\nabla g|^2 + \int_{\Omega}\int_0^T e^g|\nabla u|^2.$$

Put the above inequality in (1.3), we get

$$\begin{aligned} &\int_{\Omega\times\{T\}} e^g\frac{u^2}{2} \\ \le &\int_{\Omega\times\{0\}} e^g\frac{u^2}{2} + \int_{\Omega}\int_0^T e^g\frac{u^2}{2}\frac{\partial g}{\partial t} + \frac{1}{4}\int_{\Omega}\int_0^T e^g u^2|\nabla g|^2 \\ &+ \int_{\Omega}\int_0^T e^g|\nabla u|^2 - \int_{\Omega}\int_0^T e^g|\nabla u|^2 - \int_{\Omega}\int_0^T e^g\frac{u^2}{2}\sum_{i=1}^{n} f_i\frac{\partial g}{\partial x_i} \\ &- \int_{\Omega}\int_0^T e^g\frac{u^2}{2}\sum_{i=1}^{n} f_{i,i} - \int_{\Omega}\int_0^T e^g V u^2 \\ = &\int_{\Omega\times\{0\}} e^g\frac{u^2}{2} + \int_{\Omega}\int_0^T e^g\frac{u^2}{2}\Big[\frac{\partial g}{\partial t} + \frac{1}{2}|\nabla g|^2 - \sum_{i=1}^{n} f_i\frac{\partial g}{\partial x_i} - \sum_{i=1}^{n} f_{i,i} - 2V\Big]. \end{aligned} \tag{1.4}$$

Hence if

$$\frac{\partial g}{\partial t}-\frac{1}{2}|\nabla g|^2-\sum_{i=1}^{n} f_i\frac{\partial g}{\partial x_i}-\sum_{i=1}^{n} f_{i,i}-2V\leq 0 \text{ on } \Omega\times[0,T], \tag{1.5}$$

then we have

$$\int_{\Omega\times\{T\}}\frac{e^g u^2}{2}\leq\int_{\Omega\times\{0\}}\frac{e^g u^2}{2}. \tag{1.6}$$

(ii) By a similar argument, we obtain

$$\begin{aligned}
&\int_{\Omega\times\{T\}} e^h|\nabla u|^2-\int_{\Omega\times\{0\}} e^h|\nabla u|^2=\int_{\Omega\times[0,T]}\frac{\partial}{\partial t}(e^h|\nabla u|^2)\\
&=\int_{\Omega\times[0,T]} e^h\frac{\partial h}{\partial t}|\nabla u|^2+2\int_{\Omega\times[0,T]} e^h\nabla u\cdot\nabla u_t\\
&=\int_{\Omega\times[0,T]} e^h h_t|\nabla u|^2-2\int_{\Omega\times[0,T]} e^h(\nabla u\cdot\nabla h)u_t-2\int_{\Omega\times[0,T]} e^h(\Delta u)u_t
\end{aligned}$$

because of the vanishing of the boundary condition $u_t\frac{\partial u}{\partial\nu}$ on $\partial\Omega\times[0,T]$. So we have

$$\begin{aligned}
&\int_{\Omega\times\{T\}} e^h|\nabla u|^2-\int_{\Omega\times\{0\}} e^h|\nabla u|^2=\int_{\Omega\times[0,T]} e^h h_t|\nabla u|^2\\
&-2\int_{\Omega\times[0,T]} e^h(\nabla u\cdot\nabla h)[\Delta u+\sum_{j=1}^{n} f_j u_j-Vu]\\
&-2\int_{\Omega\times[0,T]} e^h(\Delta u)[\Delta u+\sum_{j=1}^{n} f_j u_j-Vu].
\end{aligned} \tag{1.7}$$

But

$$\begin{aligned}
2\int_{\Omega\times[0,T]} e^h(\Delta u)Vu&=-2\int_{\Omega\times[0,T]} e^h(\nabla u\cdot\nabla h)Vu\\
&-2\int_{\Omega\times[0,T]} e^h[u\nabla u\cdot\nabla V+V|\nabla u|^2]
\end{aligned} \tag{1.8}$$

because of the vanishing of the boundary condition $Vu\frac{\partial u}{\partial\nu}$ on $\partial\Omega\times[0,T]$. Equations (1.8) and (1.9) imply

$$\begin{aligned}
&\int_{\Omega\times\{T\}} e^h|\nabla u|^2-\int_{\Omega\times\{0\}} e^h|\nabla u|^2\\
&=\int_{\Omega\times[0,T]} e^h h_t|\nabla u|^2-2\int_{\Omega\times[0,T]} e^h(\nabla u\cdot\nabla h)[\Delta u+\sum_{j=1}^{n} f_j u_j]\\
&-2\int_{\Omega\times[0,T]} e^h(\Delta u)^2-2\int_{\Omega\times[0,T]} e^h(\Delta u)(\sum_{j=1}^{n} f_j u_j)
\end{aligned} \tag{1.9}$$

$$-2\int_{\Omega\times[0,T]} e^h u\nabla u\cdot\nabla V - 2\int_{\Omega\times[0,T]} e^h V|\nabla u|^2.$$

By choosing coordinates, we can assume $u_i = 0$ for $i > 1$. Then

$$\begin{aligned}
(1.10)\qquad & 2\big((\nabla h + f)\cdot\nabla u\big)\Delta u + 2(\nabla u\cdot\nabla h)\Big(\sum_{i=1}^{n} f_i u_i\Big)\\
&= 2(h_1+f_1)u_1\Delta u + 2u_1^2 f_1 h_1\\
&\le h_1^2u_1^2 + (\Delta u)^2 + f_1^2u_1^2 + (\Delta u)^2 + 2u_1^2 f_1h_1\\
&= (h_1+f_1)^2u_1^2 + 2(\Delta u)^2\\
&\le |\nabla h + f|^2|\nabla u|^2 + 2(\Delta u)^2.
\end{aligned}$$

Equations (1.10) and (1.11) imply

$$\begin{aligned}
(1.11)\qquad & \int_{\Omega\times\{T\}} e^h|\nabla u|^2 - \int_{\Omega\times\{0\}} e^h|\nabla u|^2\\
&= \int_{\Omega\times[0,T]} e^h h_t|\nabla u|^2 - 2\int_{\Omega\times[0,T]} e^h[(\nabla h + f)\cdot\nabla u]\Delta u\\
&\quad -2\int_{\Omega\times[0,T]} e^h(\nabla u\cdot\nabla h)\sum_{j=1}^{n} f_ju_j - 2\int_{\Omega\times[0,T]} e^h(\Delta u)^2\\
&\quad -2\int_{\Omega\times[0,T]} e^h u\nabla u\cdot\nabla V - 2\int_{\Omega\times[0,T]} e^hV|\nabla u|^2\\
&\le \int_{\Omega\times[0,T]} e^h h_t|\nabla u|^2 + \int_{\Omega\times[0,T]} (e^h|\nabla h + f|^2|\nabla u|^2 + 2e^h(\Delta u)^2)\\
&\quad -2\int_{\Omega\times[0,T]} Ve^h|\nabla u|^2 - 2\int_{\Omega\times[0,T]} e^h(\Delta u)^2 - 2\int_{\Omega\times[0,T]} e^h u\nabla u\cdot\nabla V\\
&\le \int_{\Omega\times[0,T]} e^h|\nabla u|^2(h_t + |\nabla h + f|^2 - 2V) + \int_{\Omega\times[0,T]} e^h(u^2 + |\nabla u|^2|\nabla V|^2)\\
&= \int_{\Omega\times[0,T]} e^h|\nabla u|^2(h_t + |\nabla h + f|^2 - 2V + |\nabla V|^2 + \int_{\Omega\times[0,T]} e^hu^2.
\end{aligned}$$

Since

$$(1.12)\qquad h_t + |\nabla h + f|^2 - 2V + |\nabla V|^2 \le 0, \qquad \text{on } \Omega\times[0,T]$$

by assumption, we see that

$$(1.13)\qquad \int_{\Omega\times\{T\}} e^h|\nabla u|^2 \le \int_{\Omega\times\{0\}} e^h|\nabla u|^2 + \int_0^T\int_\Omega e^hu^2.$$

(iii) Similary we can estimate the higher order derivatives of u in the following way

$$\begin{aligned}
&\int_{\Omega\times\{T\}} e^\rho(\Delta u)^2 - \int_{\Omega\times\{0\}} e^\rho(\Delta u)^2\\
&= \int_{\Omega\times[0,T]} \frac{d}{dt}[e^\rho(\Delta u)^2]
\end{aligned}$$

$$=\int_{\Omega\times[0,T]} e^\rho \rho_t(\Delta u)^2 + 2\int_{\Omega\times[0,T]} e^\rho(\Delta u)\Delta u_t$$

$$=\int_{\Omega\times[0,T]} e^\rho \rho_t(\Delta u)^2 - 2\int_{\Omega\times[0,T]} e^\rho \Delta u\nabla\rho\cdot\nabla(\Delta u + \sum_{i=1}^n f_i u_i - Vu)$$

$$-2\int_{\Omega\times[0,T]} e^\rho\nabla(\Delta u)\cdot\nabla(\Delta u + \sum_{i=1}^n f_i u_i - Vu)$$

because of the vanishing of the boundary condition $\Delta u\frac{\partial u_t}{\partial\nu}$ on $\partial\Omega\times[0,T]$. Hence

$$(1.14)\quad \int_{\Omega\times\{T\}} e^\rho(\Delta u)^2 - \int_{\Omega\times\{0\}} e^\rho(\Delta u)^2$$

$$\le \int_{\Omega\times[0,T]} e^\rho\rho_t(\Delta u)^2 - 2\int_{\Omega\times[0,T]} e^\rho(\Delta u)\nabla\rho$$

$$+\frac{\varepsilon}{2}\int_{\Omega\times[0,T]} e^\rho(\Delta u)^2|\nabla\rho|^2 + \frac{2}{\varepsilon}\int_{\Omega\times[0,T]} e^\rho|\nabla(\sum_{i=1}^n f_i u_i)|^2$$

$$+\frac{\varepsilon}{2}\int_{\Omega\times[0,T]} e^\rho(\Delta u)^2|\nabla\rho|^2 + \frac{2}{\varepsilon}\int_{\Omega\times[0,T]} e^\rho|\nabla(Vu)|^2$$

$$-2\int_{\Omega\times[0,T]} e^\rho|\nabla(\Delta u)|^2 - 2\int_{\Omega\times[0,T]} e^\rho\nabla(\Delta u)\cdot\nabla(\sum_{i=1}^n f_i u_i)$$

$$+2\int_{\Omega\times[0,T]} e^\rho\nabla(\Delta u)\cdot\nabla(Vu)$$

$$=\int_{\Omega\times[0,T]} e^\rho\rho_t(\Delta u)^2 - 2\int_{\Omega\times[0,T]} e^\rho\Delta u\nabla\rho\cdot\nabla(\Delta u)$$

$$+\varepsilon\int_{\Omega\times[0,T]} e^\rho(\Delta u)^2|\nabla\rho|^2 + \frac{2}{\varepsilon}\int_{\Omega\times[0,T]} e^\rho|\nabla(\sum_{i=1}^n f_i u_i)|^2$$

$$+\frac{2}{\varepsilon}\int_{\Omega\times[0,T]} e^\rho|\nabla(Vu)|^2 - 2\int_{\Omega\times[0,T]} e^\rho|\nabla(\Delta u)|^2$$

$$-2\int_{\Omega\times[0,T]} e^\rho\nabla(\Delta u)\cdot\nabla(\sum_{i=1}^n f_i u_i) + 2\int_{\Omega\times[0,T]} e^\rho\nabla(\Delta u)\cdot\nabla(Vu).$$

But

$$(1.15)\quad -2\int_{\Omega\times[0,T]} e^\rho\nabla(\Delta u)\cdot\nabla(\sum_{i=1}^n f_i u_i)$$

$$= -2\int_{\Omega\times[0,T]} e^\rho\sum_{i,k=1}^n(\Delta u)_k u_i f_{i,k} - 2\int_{\Omega\times[0,T]} e^\rho\sum_{i,k=1}^n(\Delta u)_k f_i u_{ik}$$

$$(\text{where } f_{i,k} = \frac{\partial f_i}{\partial x_k})$$

$$= -2\int_{\Omega\times[0,T]} e^\rho\sum_{k=1}^n(\Delta u)_k u_i f_{i,k} + 2\int_{\Omega\times[0,T]} e^\rho\sum_{i,k=1}^n \rho_i(\Delta u)_k u_k f_i$$

$$+2\int_{\Omega\times[0,T]} e^{\rho}\sum_{i,k=1}^{n}(\Delta u)_{ki}f_i u_k + 2\int_{\Omega\times[0,T]} e^{\rho}\sum_{i,k=1}^{n}(\Delta u)_k u_k f_{i,i}$$

because of the vanishing of the boundary conditions $u_k(\Delta u)_k f\cdot\nu$, $1\le k\le n$, on $\partial\Omega\times[0,T]$. Observe that

$$(1.16)\qquad 2\int_{\Omega\times[0,T]} e^{\rho}\sum_{i,k=1}^{n}(\Delta u)_{ki}f_i u_k = 2\sum_{i=1}^{n}\int_{\Omega\times[0,T]} e^{\rho}f_i(\nabla(\Delta u)_i\cdot\nabla u)$$
$$= -2\sum_{i=1}^{n}\int_{\Omega\times[0,T]} e^{\rho}f_i(\Delta u)_i\nabla\rho\cdot\nabla u - 2\sum_{i=1}^{n}\int_{\Omega\times[0,T]} e^{\rho}(\Delta u)_i\nabla f_i\cdot\nabla u$$
$$-2\sum_{i=1}^{n}\int_{\Omega\times[0,T]} e^{\rho}f_i(\Delta u)_i\Delta u$$

because of the vanishing of the boundary conditions $f_i(\Delta u)_i\dfrac{\partial u}{\partial\nu}$, $1\le i\le n$, on $\partial\Omega\times[0,T]$. Now the last terms in (1.17) is

$$-2\int_{\Omega\times[0,T]}(e^{\rho}\Delta u)f\cdot\nabla(\Delta u) = 2\int_{\Omega\times[0,T]}\Delta u)\ \operatorname{div}[e^{\rho}(\Delta u)f]$$
$$(\text{since } (\Delta u)^2 f\cdot\nu = 0 \text{ on } \partial\Omega\times[0,T])$$
$$= 2\int_{\Omega\times[0,T]}(\Delta u)\nabla(e^{\rho}\Delta u)\cdot f + 2\int_{\Omega\times[0,T]}(\Delta u)e^{\rho}(\Delta u)(\operatorname{div} f)$$
$$= 2\int_{\Omega\times[0,T]}(\Delta u)^2 e^{\rho}\nabla\rho\cdot f + 2\int_{\Omega\times[0,T]}(\Delta u)e^{\rho}\nabla(\Delta u)\cdot f$$
$$+2\int_{\Omega\times[0,T]}(\Delta u)e^{\rho}(\Delta u)\ (\operatorname{div} f).$$

The above equation implies

$$(1.17)\ -2\int_{\Omega\times[0,T]}(e^{\rho}\Delta u)f\cdot\nabla(\Delta u) = \int_{\Omega\times[0,T]}(\Delta u)^2 e^{\rho}\nabla\rho\cdot f + \int_{\Omega\times[0,T]}(\Delta u)^2 e^{\rho}(\operatorname{div} f).$$

Putting (1.18) into (1.17), we get

$$(1.18)\qquad 2\int_{\Omega\times[0,T]} e^{\rho}\cdot\sum_{i,k=1}^{n}(\Delta u)_{ki}f_i u_k$$
$$= -2\int_{\Omega\times[0,T]}\sum_{i=1}^{n}e^{\rho}f_i(\Delta u)_i\nabla\rho\cdot\nabla u - 2\sum_{i=1}^{n}\int_{\Omega\times[0,T]} e^{\rho}(\Delta u)_i\nabla f_i\cdot\nabla u$$
$$+\int_{\Omega\times[0,T]}(\Delta u)^2 e^{\rho}\nabla\rho\cdot f + \int_{\Omega\times[0,T]}(\Delta u)^2\rho^2\ \operatorname{div} f.$$

In view of (1.19), (1.16) becomes

$$(1.19)\qquad -2\int_{\Omega\times[0,T]} e^{\rho}\nabla(\Delta u)\cdot\nabla(\sum_{i=1}^{n}f_i u_i)$$

$$= -2\int_{\Omega\times[0,T]} e^{\rho}\sum_{k=1}^{n}(\Delta u)_k u_i f_{i,k} + 2\int_{\Omega\times[0,T]} e^{\rho}\sum_{i,k=1}^{n}\rho_i(\Delta u)_k u_k f_i$$

$$+2\int_{\Omega\times[0,T]} e^{\rho}\sum_{i,k=1}^{n}(\Delta u)_k u_k f_{i,i} - 2\int_{\Omega\times[0,T]}\sum_{i,k=1}^{n} e^{\rho}\rho_k(\Delta u)_i f_i u_k$$

$$-2\int_{\Omega\times[0,T]}\sum_{i,k=1}^{n} e^{\rho}(\Delta u)_i u_k f_{i,k} + \int_{\Omega\times[0,T]}\sum_{i=1}^{n}\rho_i f_i(\Delta u)^2$$

$$+\int_{\Omega\times[0,T]} e^{\rho}(\sum_{i=1}^{n} f_{i,i})(\Delta u)^2$$

$$= -2\int_{\Omega\times[0,T]}\sum_{k=1}^{n} e^{\rho}(\Delta u)_k[\sum_{i=1}^{n} u_i(f_{i,k}+f_{k,i}) - u_k\sum_{i=1}^{n} f_i\rho_i + f_k\sum_{i=1}^{n} u_i\rho_i$$

$$-u_k\sum_{i=1}^{n} f_{i,i}] + \int_{\Omega\times[0,T]} e^{\rho}\sum_{i=1}^{n}\rho_i f_i(\Delta u)^2 + \int_{\Omega\times[0,T]} e^{\rho}(\sum_{i=1}^{n} f_{i,i})(\Delta u)^2.$$

We next look at

$$\text{(1.20)} \qquad \int_{\Omega\times[0,T]} e^{\rho}|\nabla\sum_{i=1}^{n} f_i u_i|^2$$

$$= -\int_{\Omega\times[0,T]} e^{\rho}\sum_{k=1}^{n}\rho_k(\sum_{i=1}^{n} f_i u_i)_k(\sum_{j=1}^{n} f_j u_j)$$

$$-\int_{\Omega\times[0,T]}\sum_{k=1}^{n} e^{\rho}(\sum_{i=1}^{n} f_{i,k}u_i + \sum_{i=1}^{n} f_i u_{i,k})_k(\sum_{j=1}^{n} f_j u_j)$$

$$\left(\text{since } (\sum_{i=1}^{n} f_i u_i)\frac{\partial}{\partial\nu}(\sum_{j=1}^{n} f_j u_j) = 0 \text{ on } \partial\Omega\times[0,T]\right)$$

$$\le \varepsilon\int_{\Omega\times[0,T]} e^{\rho}|\nabla\sum_{j=1}^{n} f_j u_j|^2 + \frac{1}{4\varepsilon}\int_{\Omega\times[0,T]} e^{\rho}|\nabla\rho|^2(\sum_{j=1}^{n} f_j u_j)^2$$

$$-\int_{\Omega\times[0,T]}\sum_{k=1}^{n} e^{\rho}(\sum_{i=1}^{n} f_{i,k}u_i)_k(\sum_{j=1}^{n} f_j u_j) - \int_{\Omega\times[0,T]}\sum_{k=1}^{n} e^{\rho}(\sum_{i=1}^{n} f_i u_{i,k})_k(\sum_{j=1}^{n} f_j u_j)$$

$$= \varepsilon\int_{\Omega\times[0,T]} e^{\rho}|\nabla\sum_{j=1}^{n} f_j u_j|^2 + \frac{1}{4\varepsilon}\int_{\Omega\times[0,T]} e^{\rho}|\nabla\rho|^2(\sum_{j=1}^{n} f_j u_j)^2$$

$$+\int_{\Omega\times[0,T]} e^{\rho}(\sum_{k=1}^{n}\sum_{i=1}^{n}\rho_k f_{i,k}u_i)(\sum_{j=1}^{n} f_j u_j) + \int_{\Omega\times[0,T]} e^{\rho}\sum_{k=1}^{n}(\sum_{i=1}^{n} f_{i,k}u_i)(\sum_{j=1}^{n} f_j u_j)_k$$

$$-\int_{\Omega\times[0,T]} e^{\rho}(\sum_{k=1}^{n}\sum_{i=1}^{n} f_{i,k}u_{i,k})(\sum_{j=1}^{n} f_j u_j) - \int_{\Omega\times[0,T]} e^{\rho}(\sum_{i=1}^{n} f_i(\Delta u)_i)(\sum_{j=1}^{n} f_j u_j)$$

$$\left(\text{since } (\sum_{j=1}^{n} f_j u_j)[\sum_{i=2}^{n}(\nabla f_i)u_i]\cdot\nu = 0 \quad \text{on } \partial\Omega\times[0,T]\right).$$

Let us estimate

$$(1.21)\qquad -\int_{\Omega\times[0,T]} e^{\rho}\Big(\sum_{k=1}^{n}\sum_{i=1}^{n} f_{i,k}u_{ik}\Big)\Big(\sum_{j=1}^{n} f_j u_j\Big)$$

$$= -\int_{\Omega\times[0,T]} e^{\rho}\Big(\sum_{j=1}^{n} f_j u_j\Big)\sum_{i=1}^{n}\nabla f_i\cdot\nabla u_i$$

$$= \int_{\Omega\times[0,T]} e^{\rho}\Big(\sum_{j=1}^{n} f_j u_j\Big)\sum_{i=1}^{n}(\triangle f_i)u_i + \int_{\Omega\times[0,T]}\sum_{i=1}^{n} u_i\nabla[e^{\rho}\sum_{j=1}^{n} f_j u_j]\cdot\nabla f_i$$

$$(\text{ since } (\sum_{j=1}^{n} f_j u_j)u_i\frac{\partial f_i}{\partial\nu} = 0, 1\le i\le n, \text{ on } \partial\Omega\times[0,T])$$

$$= \int_{\Omega\times[0,T]} e^{\rho}\Big(\sum_{i=1}^{n} u_i\triangle f_i\Big)\Big(\sum_{j=1}^{n} f_j u_j\Big) + \int_{\Omega\times[0,T]} e^{\rho}\sum_{i=1}^{n} u_i\Big(\sum_{k=1}^{n}\rho_k f_{i,k}\Big)\Big(\sum_{j=1}^{n} f_j u_j\Big)$$

$$+\int_{\Omega\times[0,T]} e^{\rho}\sum_{k=1}^{n}\Big(\sum_{i=1}^{n} u_i f_{i,k}\Big)\Big(\sum_{j=1}^{n} f_j u_j\Big)_k$$

$$\le \int_{\Omega\times[0,T]} e^{\rho}\Big(\sum_{i=1}^{n} u_i\triangle f_i\Big)\Big(\sum_{j=1}^{n} f_j u_j\Big) + \int_{\Omega\times[0,T]} e^{\rho}\sum_{i=1}^{n} u_i\Big(\sum_{k=1}^{n}\rho_k f_{i,k}\Big)\Big(\sum_{j=1}^{n} f_j u_j\Big)$$

$$+\varepsilon\int_{\Omega\times[0,T]} e^{\rho}\Big|\nabla\sum_{j=1}^{n} f_j u_j\Big|^2 + \frac{1}{4\varepsilon}\int_{\Omega\times[0,t]} e^{\rho}\sum_{k=1}^{n}\Big(\sum_{i=1}^{n} u_i f_{i,k}\Big)^2.$$

Put (1.22) into (1.21), we get

$$(1.22)\qquad \int_{\Omega\times[0,T]} e^{\rho}|\nabla\sum_{i=1}^{n} f_i u_i|^2$$

$$\le 2\varepsilon\int_{\Omega\times[0,T]} e^{\rho}|\nabla\sum_{j=1}^{n} f_j u_j|^2 + \frac{1}{4\varepsilon}\int_{\Omega\times[0,T]} e^{\rho}|\nabla\rho|^2(\sum_{j=1}^{n} f_j u_j)^2$$

$$+\frac{1}{4\varepsilon}\int_{\Omega\times[0,T]} e^{\rho}(\sum_{k=1}^{n}(\sum_{i=1}^{n} f_{i,k}u_i)^2 + 2\int_{\Omega\times[0,T]} e^{\rho}(\sum_{i,k=1}^{n}\rho_k f_{i,k}u_i)(\sum_{j=1}^{n} f_j u_j)$$

$$+\int_{\Omega\times[0,T]} e^{\rho}(\sum_{i=1}^{n}(\triangle f_i)u_i)(\sum_{j=1}^{n} f_j u_j) + \int_{\Omega\times[0,T]}\sum_{k=1}^{n} e^{\rho}(\sum_{i=1}^{n} f_{i,k}u_i)(\sum_{j=1}^{n} f_j u_j)_k$$

$$-\int_{\Omega\times[0,T]} e^{\rho}(\sum_{i=1}^{n} f_i(\triangle u)_i)(\sum_{j=1}^{n} f_j u_j)$$

$$\le 3\varepsilon\int_{\Omega\times[0,T]} e^{\rho}|\nabla\sum_{j=1}^{n} f_j u_j|^2 + \frac{1}{4\varepsilon}\int_{\Omega\times[0,T]} e^{\rho}|\nabla\rho|^2(\sum_{j=1}^{n} f_j u_j)^2$$

$$+\frac{1}{2\varepsilon}\int_{\Omega\times[0,T]} e^{\rho}\sum_{k=1}^{n}(\sum_{i=1}^{n} f_{i,k}u_i)^2 + 2\int_{\Omega\times[0,T]} e^{\rho}(\sum_{i,k=1}^{n}\rho_k f_{i,k}u_i)(\sum_{j=1}^{n} f_j u_j)$$

$$+\int_{\Omega\times[0,T]} e^{\rho}(\sum_{i=1}^{n}(\triangle f_i)u_i)(\sum_{j=1}^{n} f_j u_j) - \int_{\Omega\times[0,T]} e^{\rho}(\sum_{i=1}^{n} f_i(\triangle u)_i)(\sum_{j=1}^{n} f_j u_j).$$

Hence,

$$(1.23)\qquad (1-3\varepsilon)\int_{\Omega\times[0,T]} e^{\rho}|\nabla\sum_{j=1}^{n} f_j u_j|^2$$
$$\leq \frac{1}{4\varepsilon}\int_{\Omega\times[0,T]} e^{\rho}|\nabla\rho|^2(\sum_{j=1}^{n} f_j u_j)^2 + \frac{1}{2\varepsilon}\int_{\Omega\times[0,T]} e^{\rho}\sum_{k=1}^{n}(\sum_{i=1}^{n} f_{i,k}u_i)^2$$
$$+2\int_{\Omega\times[0,T]} e^{\rho}(\sum_{i,k=1}^{n}\rho_k f_{i,k}u_i)(\sum_{j=1}^{n} f_j u_j) + \int_{\Omega\times[0,T]} e^{\rho}(\sum_{i=1}^{n} u_i\Delta f_i)(\sum_{j=1}^{n} f_j u_j)$$
$$-\int_{\Omega\times[0,T]} e^{\rho}(\sum_{i=1}^{n} f_i(\Delta u)_i)(\sum_{j=1}^{n} f_j u_j).$$

Put (1.20) and (1.24) into (1.15), we get

$$(1.24)\qquad \int_{\Omega\times\{T\}} e^{\rho}(\Delta u)^2 - \int_{\Omega\times\{0\}} e^{\rho}(\Delta u)^2$$
$$\leq \int_{\Omega\times[0,T]} e^{\rho}\rho_t(\Delta u)^2 - 2\int_{\Omega\times[0,T]} e^{\rho}\Delta u\nabla\rho\cdot\nabla(\Delta u) + \varepsilon\int_{\Omega\times[0,T]} e^{\rho}(\Delta u)^2|\nabla\rho|^2$$
$$+\frac{2}{\varepsilon}(1-3\varepsilon)^{-1}\frac{1}{4\varepsilon}\int_{\Omega\times[0,T]} e^{\rho}|\nabla\rho|^2(\sum_{j=1}^{n} f_j u_j)^2$$
$$+\frac{2}{\varepsilon}(1-3\varepsilon)^{-1}\frac{1}{2\varepsilon}\int_{\Omega\times[0,T]} e^{\rho}\sum_{k=1}^{n}(\sum_{i=1}^{n} f_{i,k}u_i)^2$$
$$+\frac{2}{\varepsilon}(1-3\varepsilon)^{-1}\cdot 2\int_{\Omega\times[0,T]} e^{\rho}(\sum_{i,k=1}^{n}\rho_k f_{i,k}u_i)(\sum_{j=1}^{n} f_j u_j)$$
$$+\frac{2}{\varepsilon}(1-3\varepsilon)^{-1}\int_{\Omega\times[0,T]} e^{\rho}(\sum_{i=1}^{n} u_i\Delta f_i)(\sum_{j=1}^{n} f_j u_j)$$
$$-\frac{2}{\varepsilon}(1-3\varepsilon)^{-1}\int_{\Omega\times[0,T]} e^{\rho}(\sum_{i=1}^{n} f_i(\Delta u)_i)(\sum_{j=1}^{n} f_j u_j)$$
$$+\frac{2}{\varepsilon}\int_{\Omega\times[0,T]} e^{\rho}|\nabla(Vu)|^2 - 2\int_{\Omega\times[0,T]} e^{\rho}|\nabla(\Delta u)^2|$$
$$-2\int_{\Omega\times[0,T]}\sum_{k=1}^{n} e^{\rho}(\Delta u)_k[\sum_{i=1}^{n} u_i(f_{i,k}+f_{k,i}) - u_k\sum_{i=1}^{n} f_i\rho_i + f_k\sum_{i=1}^{n} u_i\rho_i - u_k\sum_{i=1}^{n} f_{i,i}]$$
$$+\int_{\Omega\times[0,T]} e^{\rho}\sum_{i=1}^{n}\rho_i f_i(\Delta u)^2 + \int_{\Omega\times[0,T]} e^{\rho}(\sum_{i=1}^{n} f_{i,i})(\Delta u)^2$$
$$+2\int_{\Omega\times[0,T]} e^{\rho}\nabla(\Delta u)\cdot\nabla(Vu).$$

Observe the following estimates

$$(1.25)\qquad -2\int_{\Omega\times[0,T]} e^{\rho}\Delta u\nabla\rho\cdot\nabla(\Delta u)$$

$$\leq 2\delta_1 \int_{\Omega\times[0,T]} e^\rho |\nabla(\Delta u)|^2 + \frac{1}{2\delta_1}\int_{\Omega\times[0,T]} e^\rho (\Delta u)^2 |\nabla\rho|^2$$

$$\int_{\Omega\times[0,T]} e^\rho |\nabla\rho|^2 (\sum_{j=1}^n f_j u_j)^2 \leq \int_{\Omega\times[0,T]} e^\rho |\nabla\rho|^2 |f|^2 |\nabla u|^2 \tag{1.26}$$

$$\int_{\Omega\times[0,T]} e^\rho \sum_{k=1}^n (\sum_{i=1}^n f_{i,k} u_i)^2 \leq \int_{\Omega\times[0,T]} e^\rho \sum_{k=1}^n \sum_{i=1}^n (\frac{\partial f_i}{\partial x_k})^2 |\nabla u|^2 \tag{1.27}$$

$$= \int_{\Omega\times[0,T]} e^\rho |\nabla f|^2 |\nabla u|^2,$$

where $|\nabla f|^2 = \sum_{i=1}^n |\nabla f_i|^2$.

$$\int_{\Omega\times[0,T]} e^\rho (\sum_{i,k=1}^n \rho_k f_{i,k} u_i)(\sum_{j=1}^n f_j u_j) = \int_{\Omega\times[0,T]} e^\rho \sum_{k=1}^n (\sum_{j=1}^n f_j u_j)\rho_k \sum_{i=1}^n f_{i,k} u_i \tag{1.28}$$

$$\leq \frac{1}{2}\int_{\Omega\times[0,T]} e^\rho (\sum_{j=1}^n f_j u_j)^2 \sum_{k=1}^n \rho_k^2 + \frac{1}{2}\int_{\Omega\times[0,T]} e^\rho \sum_{k=1}^n (\sum_{i=1}^n f_{i,k} u_i)^2$$

$$= O(\int_{\Omega\times[0,T]} e^\rho |f|^2 |\nabla u|^2 |\nabla\rho|^2 + \int_{\Omega\times[0,T]} e^\rho |\nabla f|^2 |\nabla u|^2)$$

$$\int_{\Omega\times[0,T]} e^\rho (\sum_{i=1}^n u_i \Delta f_i)(\sum_{j=1}^n f_j u_j) \leq \int_{\Omega\times[0,T]} e^\rho |\nabla u|^2 |f| |\Delta f| \tag{1.29}$$

where $|\Delta f| = \sqrt{\sum_{i=1}^n (\Delta f_i)^2}$

$$\int_{\Omega\times[0,T]} e^\rho |f|^2 (\sum_{j=1}^n f_j u_j)^2 \leq \int_{\Omega\times[0,T]} e^\rho |f|^4 |\nabla u|^2 \tag{1.30}$$

$$-\int_{\Omega\times[0,T]} e^\rho (\sum_{i=1}^n f_i (\Delta u)_i)(\sum_{j=1}^n f_j u_j) = -\int_{\Omega\times[0,T]} e^\rho \sum_{i=1}^n [(\sum_{j=1}^n f_j u_j) f_i](\Delta u)_i \tag{1.31}$$

$$\leq \delta_2 \int_{\Omega\times[0,T]} e^\rho \sum_{i=1}^n [(\Delta u)_i]^2 + \frac{1}{4\delta_2}\int_{\Omega\times[0,T]} e^\rho \sum_{i=1}^n (\sum_{j=1}^n f_j u_j)^2 f_i^2$$

$$\leq \delta_2 \int_{\Omega\times[0,T]} e^\rho |\nabla(\Delta u)|^2 + \frac{1}{4\delta_2}\int_{\Omega\times[0,T]} e^\rho |f|^4 |\nabla u|^2$$

$$-\int_{\Omega\times[0,T]} e^\rho \sum_{k=1}^n (\Delta u)_k \sum_{i=1}^n u_i (f_{i,k} + f_{k,i}) \tag{1.32}$$

$$\leq \delta_3 \int_{\Omega\times[0,T]} e^\rho \sum_{k=1}^n [(\Delta u)_k]^2 + \frac{1}{4\delta_3}\int_{\Omega\times[0,T]} e^\rho \sum_{k=1}^n [\sum_{i=1}^n u_i (f_{i,k} + f_{k,i})]^2$$

$$= \delta_3 \int_{\Omega\times[0,T]} e^\rho |\nabla(\Delta u)|^2 + O(\int_{\Omega\times[0,T]} e^\rho |\nabla u|^2 |\nabla f|^2)$$

$$-\int_{\Omega\times[0,T]} e^\rho \sum_{k=1}^n (\Delta u)_k (-u_k \sum_{i=1}^n f_i \rho_i + f_k \sum_{i=1}^n u_i \rho_i) \tag{1.33}$$

$$\leq \delta_4 \int_{\Omega\times[0,T]} e^\rho \sum_{k=1}^n [(\Delta u)_k]^2 + \frac{1}{4\delta_4}\int_{\Omega\times[0,T]} e^\rho \sum_{k=1}^n (-u_k \sum_{i=1}^n f_i\rho_i + f_k \sum_{i=1}^n u_i\rho_i)^2$$

$$= \delta_4 \int_{\Omega\times[0,T]} e^\rho |\nabla(\Delta u)|^2 + O(\int_{\Omega\times[0,T]} e^\rho |\nabla u|^2 |f|^2 |\nabla\rho|^2)$$

$$(1.34) \quad -\int_{\Omega\times[0,T]} e^\rho \sum_{k=1}^n (\Delta u)_k u_k \sum_{i=1}^n f_{i,i}$$

$$\leq \delta_5 \int_{\Omega\times[0,T]} e^\rho \sum_{k=1}^n ((\Delta u)_k)^2 + \frac{1}{4\delta_5}\int_{\Omega\times[0,T]} e^\rho \sum_{k=1}^n (u_k \sum_{i=1}^n f_{i,i})^2$$

$$\leq \delta_5 \int_{\Omega\times[0,T]} e^\rho |\nabla(\Delta u)|^2 + \frac{1}{4\delta_5}\int_{\Omega\times[0,T]} e^\rho |\nabla u|^2 (\sum_{i=1}^n f_{i,i})^2$$

$$(1.35) \quad 2\int_{\Omega\times[0,T]} e^\rho \nabla(\Delta u)\cdot\nabla(Vu)$$

$$\leq 2\delta_6 \int_{\Omega\times[0,T]} e^\rho |\nabla(\Delta u)|^2 + \frac{1}{2\delta_6}\int_{\Omega\times[0,T]} e^\rho |\nabla(Vu)|^2$$

Put the estimates (1.26) – (1.36) into (1.25), we get

$$(1.36) \quad \int_{\Omega\times\{T\}} e^\rho (\Delta u)^2 - \int_{\Omega\times\{0\}} e^\rho (\Delta u)$$

$$\leq \int_{\Omega\times[0,T]} e^\rho \rho_t (\Delta u)^2 + 2\delta_1 \int_{\Omega\times[0,T]} e^\rho |\nabla(\Delta u)|^2 + \frac{1}{2\delta_1}\int_{\Omega\times[0,T]} e^\rho (\Delta u)|\nabla\rho|^2$$

$$+\varepsilon \int_{\Omega\times[0,T]} e^\rho (\Delta u)^2 |\nabla\rho|^2 + \frac{1}{2\varepsilon^2(1-3\varepsilon)}\int_{\Omega\times[0,T]} e^\rho |\nabla\rho|^2 |f|^2 |\nabla u|^2$$

$$+\frac{1}{\varepsilon^2(1-3\varepsilon)}\int_{\Omega\times[0,T]} e^\rho |\nabla f|^2 |\nabla u|^2 + \frac{4}{\varepsilon(1-3\varepsilon)} O(\int_{\Omega\times[0,T]} e^\rho |f|^2 |\nabla u|^2 |\nabla\rho|^2$$

$$+\int_{\Omega\times[0,T]} e^\rho |\nabla f|^2 |\nabla u|^2) + \frac{2}{\varepsilon(1-3\varepsilon)}\int_{\Omega\times[0,T]} e^\rho |\nabla u|^2 |f||\Delta f|$$

$$+\frac{2}{\varepsilon(1-3\varepsilon)}\delta_2 \int_{\Omega\times[0,T]} e^\rho |\nabla(\Delta u)|^2 + \frac{2}{\varepsilon(1-3\varepsilon)}\frac{1}{4\delta_2}\int_{\Omega\times[0,T]} e^\rho |f|^4 |\nabla u|^2$$

$$+\frac{2}{\varepsilon}\int_{\Omega\times[0,T]} e^\rho |\nabla(Vu)|^2 - 2\int_{\Omega\times[0,T]} e^\rho |\nabla(\Delta u)|^2$$

$$+2\delta_3 \int_{\Omega\times[0,T]} e^\rho |\nabla(\Delta u)|^2 + O(\int_{\Omega\times[0,T]} e^\rho |\nabla u|^2 |\nabla f|^2)$$

$$+2\delta_4 \int_{\Omega\times[0,T]} e^\rho |\nabla(\Delta u)|^2 + O(\int_{\Omega\times[0,T]} e^\rho |\nabla u|^2 |f|^2 |\nabla\rho|^2)$$

$$+2\delta_5 \int_{\Omega\times[0,T]} e^\rho |\nabla(\Delta u)|^2 + \frac{1}{2\delta_5}\int_{\Omega\times[0,T]} e^\rho |\nabla u|^2 (\sum_{i=1}^n f_{i,i})^2$$

$$+\int_{\Omega\times[0,T]} e^\rho \sum_{i=1}^n \rho_i f_i (\Delta u)^2 + \int_{\Omega\times[0,T]} e^\rho (\sum_{i=1}^n f_{i,i})(\Delta u)^2$$

$$
\begin{aligned}
&+ 2\delta_6 \int_{\Omega\times[0,T]} e^\rho|\nabla(\Delta u)|^2 + \frac{1}{2\delta_6}\int_{\Omega\times[0,T]} e^\rho|\nabla(Vu)|^2 \\
= &\int_{\Omega\times[0,T]} e^\rho(\Delta u)^2[\rho_t + (\varepsilon + \frac{1}{2\delta_1})|\nabla\rho|^2 + \sum_{i=1}^n \rho_i f_i + \sum_{i=1}^n f_{i,i}] \\
&+ (2\delta_1 + \frac{2\delta_2}{\varepsilon(1-3\varepsilon)} - 2 + 2\delta_3 + 2\delta_4 + 2\delta_5 + 2\delta_6)\int_{\Omega\times[0,T]} e^\rho|\nabla(\Delta u)|^2 \\
&+ O\Big(\int_{\Omega\times[0,T]} e^\rho|\nabla\rho|^2|f|^2|\nabla u|^2 + \int_{\Omega\times[0,T]} e^\rho|\nabla f|^2|\nabla u|^2 \\
&+ \int_{\Omega\times[0,T]} e^\rho|f||\nabla u||\Delta f| + \int_{\Omega\times[0,T]} e^\rho|f|^4|\nabla u|^2 \\
&+ \int_{\Omega\times[0,T]} e^\rho|\nabla(Vu)|^2 + \int_{\Omega\times[0,T]} e^\rho|\nabla u|^2(\sum_{i=1}^n f_{i,i})^2\Big).
\end{aligned}
$$

Let $\delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta_5 = \delta_6 = \varepsilon^2$ with $2 - 10\varepsilon^2 - \frac{2\varepsilon}{1-3\varepsilon} > 0$. Since $\frac{\partial\rho}{\partial t} + (\varepsilon + \frac{1}{2\varepsilon^2})|\nabla\rho|^2 + \sum_{i=1}^n \rho_i f_i + \sum_{i=1}^n f_{i,i} \le 0$, for $0 \le t \le T$, by assumption, (1.36) implies (1.34). □

In view of Theorem 1.1, we are interested in constructing functions g, h and ρ which satisfy the conditions (i), (ii) and (iii) in Theorem 1.1. The following theorem is very useful in this regard.

THEOREM 1.2. *Let $k : \Omega\times\mathbb{R} \to \mathbb{R}^n$ and $F : \Omega\times\mathbb{R} \to \mathbb{R}$ be a possibly time dependent vector field and function respectively. Fix a point $(x_0, t_0) \in \Omega\times\mathbb{R}$. Let (x,t) be an arbitrary point in $\Omega\times\mathbb{R}$ with $t > t_0$. Let $\mathcal{P} = \{$ differentiable path $\sigma = (\sigma_1,\sigma_2) : [0,1] \to \Omega\times\mathbb{R}$ such that $\sigma(0) = (\sigma_1(0),\sigma_2(0)) = (x_0,t_0), \sigma(1) = (x,t), \sigma_2(s)$ is linear in s with $\sigma_2'(s) \ge 0$ and $\int_0^1 \sigma_1(s)\, ds = \frac{x_0+x}{2}\}$. Define*

$$
E_\varepsilon(\sigma) = \frac{1}{4}\int_0^1 \frac{|\dot\sigma_1|^2}{\frac{\partial\sigma_2}{\partial s} + \varepsilon} - \frac{1}{2}\int_0^1 \sum_{i=1}^n k_i(\sigma_1(s),\sigma_2(s))\frac{d\sigma_1^i}{ds} + \int_0^1 F(\sigma_1(s),\sigma_2(s))\frac{d\sigma_2}{ds}
$$

$$
d_\varepsilon(x,t) = \min_{\sigma\in\mathcal{P}} E_\varepsilon(\sigma)
$$

Let σ be the curve that minimizes the functional $E_\varepsilon(\sigma)$. Then the following equations hold for almost all (x,t).

$$
\nabla d_\varepsilon(x,t) = \frac{1}{2}\frac{\dot\sigma_1(1)}{\frac{\partial\sigma_2}{\partial s}(1) + \varepsilon} - \frac{1}{2}k(x,t) \tag{1.37}
$$

$$
\sum_{i=1}^n k_i(x,t)(d_\varepsilon)_i(x,t) = -\frac{1}{2}|k(x,t)|^2 + \frac{1}{2}\Big(\sum_{i=1}^n k_i(x,t)\frac{d\sigma_1^i}{ds}(1)\Big)\Big(\frac{d\sigma_2}{ds}(1) + \varepsilon\Big)^{-1} \tag{1.38}
$$

$$
\frac{\partial d_\varepsilon}{\partial t}(x,t) = -\frac{1}{4}\frac{|\dot\sigma_1(1)|^2}{(\frac{\partial\sigma_2}{\partial s}(1) + \varepsilon)^2} + F(x,t) \tag{1.39}
$$

In particular $d_\varepsilon(x,t)$ satisfies the following equation

$$
\frac{\partial d_\varepsilon}{\partial t}(x,t) + |\nabla d_\varepsilon(x,t)|^2 + \sum_{i=1}^n k_i(x,t)(d_\varepsilon)_i(x,t) = F(x,t) - \frac{|k(x,t)|^2}{4}. \tag{1.40}
$$

Proof. If $\sigma = (\sigma_1, \sigma_2) \in \mathcal{P}$ is the path that realizes the minimum of $E_\varepsilon(\sigma)$, then it has to satisfy the Euler-Lagrange equations which we shall derive explicity as follows. Consider an one parameter family of path in $\mathcal{P}$:

$$\begin{aligned}
&\alpha : (-\varepsilon, \varepsilon) \times [0,1] \longrightarrow \Omega \times \mathbb{R} \\
&\alpha(0,s) = \sigma(s), \quad \forall \ s \in [0,1] \\
&\alpha(v,0) = (x_0, t_0), \quad \forall \ v \in (-\varepsilon, \varepsilon) \\
&\alpha(v,1) = (x,t), \quad \forall \ v \in (-\varepsilon, \varepsilon)
\end{aligned}$$

Denote $\bar{\alpha}(v)(s) = (\bar{\alpha}_1(v)(s), \bar{\alpha}_2(v)(s)) := \alpha(v,s) = (\alpha_1(v,s), \alpha_2(v,s))$ and $\bar{\alpha}_1(v)^i :=$ i-th component of $\bar{\alpha}_1(v)$.

$$\begin{aligned}
\frac{d}{dv} E_\varepsilon(\bar{\alpha}(v))\Big|_{v=0} =& \frac{d}{dv}\Big\{\frac{1}{4}\int_0^1 \frac{|\dot{\overline{\alpha_1}}(v)(s)|^2}{\frac{\partial \bar{\alpha}_2(v)(s)}{\partial s} + \varepsilon} - \frac{1}{2}\int_0^1 \sum_{i=1}^n k_i(\bar{\alpha}(v)(s), \bar{\alpha}_2(v)(s)) \frac{d\bar{\alpha}_1(v)^i}{ds}(s) \\
&+ \int_0^1 F(\bar{\alpha}_1(v)(s), \bar{\alpha}_2(v)(s)) \frac{d\bar{\alpha}_2(v)}{ds}(s)\Big\}\Big|_{v=0} \\
=& \frac{1}{4}\int_0^1 \left[\frac{2\frac{\partial \alpha_1}{\partial s}(v,s)\frac{\partial^2 \alpha_1}{\partial v \partial s}(v,s)}{\frac{\partial \alpha_2}{\partial s}(v,s) + \varepsilon}\Big|_{v=0} - \frac{\left|\frac{\partial \alpha_1}{\partial s}(v,s)\right|^2 \frac{\partial^2 \alpha_2}{\partial v \partial s}(v,s)}{\left(\frac{\partial \alpha_2}{\partial s}(v,s) + \varepsilon\right)^2}\Big|_{v=0}\right] \\
&- \frac{1}{2}\int_0^1 \sum_{i=1}^n [\sum_{j=1}^n \frac{\partial k_i}{\partial x_j}(\bar{\alpha}_1(v)(s), \bar{\alpha}_2(v)(s)) \frac{\partial \alpha_1^j}{\partial v}(v,s) \\
&+ \frac{\partial k_i}{\partial t}(\bar{\alpha}_1(v)(s), \bar{\alpha}_2(v)(s)) \frac{\partial \alpha_2}{\partial v}(v,s)] \frac{d\bar{\alpha}_1(v)^i}{ds}(s)\Big|_{v=0} \\
&- \frac{1}{2}\int_0^1 \sum_{i=1}^n k_i(\bar{\alpha}_1(v)(s), \bar{\alpha}_2(v)(s)) \frac{\partial^2 \alpha_1^i}{\partial v \partial s}(v,s)\Big|_{v=0} \\
&+ \int_0^1 \Big[\sum_{j=1}^n \frac{\partial F}{\partial x_j}(\bar{\alpha}_1(v)(s), \bar{\alpha}_2(v)(s)) \frac{\partial \alpha_1^j}{\partial v}(v,s) \\
&+ \frac{\partial F}{\partial t}(\bar{\alpha}_1(v)(s), \bar{\alpha}_2(v)(s)) \frac{\partial \alpha_2}{\partial v}(v,s)\Big] \frac{\partial \alpha_2}{\partial s}(v,s)\Big|_{v=0} \\
&+ \int_0^1 F(\bar{\alpha}_1(v)(s), \bar{\alpha}_2(v)(s)) \frac{\partial^2 \alpha_2}{\partial v \partial s}(v,s)\Big|_{v=0} \\
=& \frac{1}{4}\int_0^1 \left[\frac{2\frac{\partial \alpha_1}{\partial s}(0,s)\frac{\partial^2 \alpha_1}{\partial v \partial s}(0,s)}{\frac{\partial \alpha_2}{\partial s}(0,s) + \varepsilon} - \frac{|\frac{\partial \alpha_1}{\partial s}(0,s)|^2 \ \frac{\partial^2 \alpha_2}{\partial v \partial s}(0,s)}{(\frac{\partial \alpha_2}{\partial s}(0,s) + \varepsilon)^2}\right] \\
&- \frac{1}{2}\int_0^1 \sum_{i=1}^n \Big[\sum_{j=1}^n \frac{\partial k_i}{\partial x_j}(\alpha_1(0,s), \alpha_2(0,s)) \frac{\partial \alpha_1^j}{\partial v}(0,s) \\
&+ \frac{\partial k_i}{\partial t}(\alpha_1(0,s), \alpha_2(0,s)) \frac{\partial \alpha_2}{\partial v}(0,s)] \frac{d\alpha_1^i(0,s)}{ds} \\
&- \frac{1}{2}\int_0^1 \sum_{i=1}^n k_i(\alpha_1(0,s), \alpha_2(0,s)) \frac{\partial^2 \alpha_1^i}{\partial v \partial s}(0,s) \\
&+ \int_0^1 [\sum_{j=1}^n \frac{\partial F}{\partial x_j}(\alpha_1(0,s), \alpha_2(0,s)) \frac{\partial \alpha_1^j}{\partial v}(0,s)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\partial F}{\partial t}(\alpha_1(0,s),\alpha_2(0,s))\frac{\partial \alpha_2}{\partial v}(0,s)]\frac{\partial \alpha_2}{\partial s}(0,s)\\
&+\int_0^1 F(\alpha_1(0,s),\alpha_2(0,s))\frac{\partial^2\alpha_2}{\partial v\partial s}(0,s)\\
=&\frac{1}{2}\int_0^1 \frac{\frac{\partial \alpha_1}{\partial s}(0,s)}{\frac{\partial \alpha_2}{\partial s}(0,s)+\varepsilon}\frac{d}{ds}(\frac{\partial \alpha_1}{\partial v}(0,s))\\
&-\frac{1}{4}\int_0^1 \frac{|\frac{\partial \alpha_1}{\partial s}(0,s)|^2}{(\frac{\partial \alpha_2}{\partial s}(0,s)+\varepsilon)^2}\frac{d}{ds}(\frac{\partial \alpha_2}{\partial v}(0,s))\\
&-\frac{1}{2}\int_0^1 \sum_{i=1}^n\sum_{j=1}^n \frac{\partial k_i}{\partial x_j}(\alpha_1(0,s)),\alpha_2(0,s))\frac{\partial \alpha_1^j(0,s)}{\partial v}\frac{\partial \alpha_1^i(0,s)}{\partial s}\\
&-\frac{1}{2}\int_0^1 \sum_{i=1}^n \frac{\partial k_i}{\partial t}(\alpha_1(0,s)),\alpha_2(0,s))\frac{\partial \alpha_2}{\partial v}(0,s)\frac{\partial \alpha_1^i(0,s)}{\partial s}\\
&-\frac{1}{2}\int_0^1 \sum_{i=1}^n k_i(\alpha_1(0,s),\alpha_2(0,s))\frac{d}{ds}(\frac{\partial \alpha_1^i(0,s)}{\partial v})\\
&+\int_0^1 \sum_{j=1}^n \frac{\partial F}{\partial x_j}(\alpha_1(0,s),\alpha_2(0,s))\ \frac{\partial \alpha_1^j(0,s)}{\partial v}\ \frac{\partial \alpha_2(0,s)}{\partial s}\\
&+\int_0^1 \frac{\partial F}{\partial t}(\alpha_1(0,s),\alpha_2(0,s))\frac{\partial \alpha_2}{\partial v}(0,s)\frac{\partial \alpha_2}{\partial s}(0,s)\\
&+\int_0^1 F(\alpha_1(0,s),\alpha_2(0,s))\frac{d}{ds}(\frac{\partial \alpha_2}{\partial v}(0,s))\\
=&-\frac{1}{2}\int_0^1 \frac{\partial}{\partial s}\left[\frac{\frac{\partial \alpha_1}{\partial s}(0,s)}{\frac{\partial \alpha_2}{\partial s}(0,s)+\varepsilon}\right]\cdot\frac{\partial \alpha_1}{\partial v}(0,s)+\frac{1}{2}\frac{\frac{\partial \alpha_1}{\partial s}(0,s)}{\frac{\partial \alpha_2}{\partial s}(0,s)+\varepsilon}\frac{\partial \alpha_1}{\partial v}(0,s)\Big|_{s=0}^{s=1}\\
&+\frac{1}{4}\int_0^1 \frac{\partial}{\partial s}\left[\frac{|\frac{\partial \alpha_1}{\partial s}(0,s)|^2}{(\frac{\partial \alpha_2}{\partial s}(0,s)+\varepsilon)^2}\right]\cdot\frac{\partial \alpha_2}{\partial v}(0,s)\\
&-\frac{1}{4}\frac{|\frac{\partial \alpha_1}{\partial s}(0,s)|^2}{(\frac{\partial \alpha_2}{\partial s}(0,s)+\varepsilon)^2}\frac{\partial \alpha_2}{\partial s}(0,s)\Big|_{s=0}^{s=1}\\
&-\frac{1}{2}\int_0^1 \sum_{i=1}^n\sum_{j=1}^n \frac{\partial k_i}{\partial x_j}(\alpha_1(0,s),\alpha_2(0,s))\frac{\partial \alpha_1^i}{\partial s}(0,s)\frac{\partial \alpha_1^j}{\partial v}(0,s)\\
&-\frac{1}{2}\int_0^1 \sum_{i=1}^n \frac{\partial k_i}{\partial t}(\alpha_1(0,s),\alpha_2(0,s))\frac{\partial \alpha_1^i}{\partial s}(0,s)\frac{\partial \alpha_2}{\partial v}(0,s)\\
&+\frac{1}{2}\int_0^1 \sum_{i=1}^n\sum_{j=1}^n \frac{\partial k_i}{\partial x_j}(\alpha_1(0,s),\alpha_2(0,s))\frac{\partial \alpha_1^j(0,s)}{\partial s}\frac{\partial \alpha_1^i}{\partial v}(0,s)\\
&+\frac{1}{2}\int_0^1 \sum_{i=1}^n \frac{\partial k_i}{\partial t}(\alpha_1(0,s),\alpha_2(0,s))\frac{\partial \alpha_2}{\partial s}(0,s)\frac{\partial \alpha_1^i}{\partial v}(0,s)\\
&-\frac{1}{2}\sum_{i=1}^n k_i(\alpha_1(0,s),\alpha_2(0,s))\frac{\partial \alpha_1^i(0,s)}{\partial v}\Big|_{s=0}^{s=1}
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^1\sum_{j=1}^n\frac{\partial F}{\partial x_j}(\alpha_1(0,s),\alpha_2(0,s))\frac{\partial\alpha_2(0,s)}{\partial s}\frac{\partial\alpha_1^j(0,s)}{\partial v}\\
&+\int_0^1\frac{\partial F}{\partial t}(\alpha_1(0,s),\alpha_2(0,s))\frac{\partial\alpha_2}{\partial s}(0,s)\frac{\partial\alpha_2}{\partial s}(0,s)\\
&-\int_0^1\sum_{j=1}^n\frac{\partial F}{\partial x_i}(\alpha_1(0,s),\alpha_2(0,s))\frac{\partial\alpha_1^i(0,s)}{\partial s}\frac{\partial\alpha_2}{\partial v}(0,s)\\
&-\int_0^1\frac{\partial F}{\partial t}(\alpha_1(0,s),\alpha_2(0,s))\frac{\partial\alpha_2}{\partial s}(0,s)\frac{\partial\alpha_2}{\partial v}(0,s)\\
&+F(\alpha_1(0,s),\alpha_2(0,s))\frac{\partial\alpha_2}{\partial v}(0,s)\Big|_{s=0}^{s=1}\\
=&\frac{1}{2}\int_0\frac{\partial}{\partial s}\left[\frac{\frac{\partial\alpha_1}{\partial s}(0,s)}{\frac{\partial\alpha_2}{\partial s}(0,s)+\varepsilon}\right]\frac{\partial\alpha_1}{\partial v}(0,s)\\
&-\frac{1}{2}\int_0^1\sum_{i=1}^n\sum_{j=1}^n\frac{\partial k_i}{\partial x_j}(\alpha_1(0,s),\alpha_2(0,s))\frac{\partial\alpha_1^i}{\partial s}(0,s)\frac{\partial\alpha_1^j}{\partial v}(0,s)\\
&+\frac{1}{2}\int_0^1\sum_{i=1}^n\sum_{j=1}^n\frac{\partial k_i}{\partial x_j}(\alpha_1(0,s),\alpha_2(0,s))\frac{\partial\alpha_1^j}{\partial s}(0,s)\frac{\partial\alpha_1^i}{\partial v}(0,s)\\
&+\frac{1}{2}\int_0^1\sum_{i=1}^n\frac{\partial k_i}{\partial t}(\alpha_1(0,s),\alpha_2(0,s))\frac{\partial\alpha_2}{\partial s}(0,s)\frac{\partial\alpha_1^i}{\partial v}(0,s)\\
&+\frac{1}{2}\int_0^1\sum_{j=1}^n\frac{\partial F}{\partial x_j}(\alpha_1(0,s),\alpha_2(0,s))\frac{\partial\alpha_2}{\partial s}(0,s))\frac{\partial\alpha_1^j}{\partial v}(0,s)\\
&+\frac{1}{4}\int_0^1\frac{\partial}{\partial s}\left[\frac{|\frac{\partial\alpha_1}{\partial s}(0,s)|^2}{(\frac{\partial\alpha_2}{\partial s}(0,s)+\varepsilon)^2}\right]\frac{\partial\alpha_2}{\partial v}(0,s)\\
&-\frac{1}{2}\int_0^1\sum_{i=1}^n\frac{\partial F}{\partial x_i}(\alpha_1(0,s),\alpha_2(0,s))\frac{\partial\alpha_1^i}{\partial s}(0,s)\frac{\partial\alpha_2}{\partial v}(0,s)\\
&-\frac{1}{2}\int_0^1\sum_{i=1}^n\frac{\partial k_i}{\partial t}(\alpha_1(0,s),\alpha_2(0,s))\frac{\partial\alpha_1^i}{\partial s}(0,s)\frac{\partial\alpha_2}{\partial v}(0,s)
\end{aligned}
$$

In order that σ minimizes $E_\varepsilon(\sigma)$, σ must make the above integrals vanish for every $\eta(s)=\left(\frac{\partial\alpha_1}{\partial v}(0,s),\frac{\partial\alpha_2}{\partial v}(0,s)\right)$ which vanishes at $s=0$ and $s=1$. So we have derived the following Euler-Lagrange equations

$$
\begin{aligned}
&-\frac{1}{2}\frac{d}{ds}\left[\frac{\frac{d\sigma_1^i}{ds}(s)}{\frac{d\sigma_2}{ds}(s)+\varepsilon}\right]+\frac{1}{2}\left(\sum_{j=1}^n\frac{\partial k_i}{\partial x_j}(\sigma_1(s),\sigma_2(s))\frac{\partial\sigma_1^j}{ds}(s)+\frac{\partial k_i}{\partial t}(\sigma_1(s),\sigma_2(s))\frac{\partial\sigma_2}{\partial s}(s)\right)\\
&\qquad-\frac{1}{2}\sum_{j=1}^n\frac{\partial k_j}{\partial x_i}(\sigma_1(s),\sigma_2(s))\frac{\partial\sigma_1^j}{ds}(s)+\frac{\partial F}{\partial x_i}(\sigma_1(s),\sigma_2(s))\frac{\partial\sigma_2}{\partial s}(s)=0 \qquad (1.41)\\
&\qquad\text{for } 1\le i\le n\\
&\frac{1}{4}\frac{d}{ds}\left[\frac{|\frac{d\sigma_1}{ds}(s)|^2}{(\frac{d\sigma_2}{ds}(s)+\varepsilon)^2}\right]-\frac{1}{2}\sum_{i=1}^n\frac{\partial k_i}{\partial t}(\sigma_1(s),\sigma_2(s))\frac{d\sigma_1^i}{ds}(s)
\end{aligned}
$$

$$(1.42) \qquad -\sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(\sigma_1(s), \sigma_2(s)\frac{d\sigma_1^i}{ds}(s) = 0.$$

Now suppose σ is the path in $\mathcal{P}$ which minimizes $E_\varepsilon(\sigma)$. Notices that σ depends on (x,t). Then

$$\begin{aligned}
\frac{\delta d_\varepsilon}{\delta x}(x,t) =& \frac{\delta}{\delta x}E_\varepsilon(\sigma)\\
=&\frac{1}{4}\int_0^1 \frac{\delta}{\delta x}\left[\frac{|\frac{d\sigma_1}{ds}(s)|^2}{\frac{d\sigma_2}{ds}(s)+\varepsilon}\right] - \frac{1}{2}\int_0^1 \frac{\delta}{\delta x}\left[\sum_{i=1}^n k_i(\sigma_1(s),\sigma_2(s))\frac{\partial\sigma_1^i}{\partial s}(s)\right]\\
&+\int_0^1 \frac{\delta}{\delta x}[F(\sigma_1(s),\sigma_2(s))\frac{\partial\sigma_2}{\partial s}]\\
=&\frac{1}{2}\int_0^1 \frac{< \frac{\partial\sigma_1}{\partial s}, \frac{\delta\partial\sigma_1}{\delta x\partial s} >}{\frac{\partial\sigma_2}{\partial s}+\varepsilon} - \frac{1}{4}\int_0^1 \frac{|\frac{\partial\sigma_1}{\partial s}|^2}{(\frac{\partial\sigma_2}{\partial s}+\varepsilon)^2}\frac{\delta\partial\sigma_2}{\delta x\partial s}\\
&-\frac{1}{2}\int_0^1 \sum_{i=1}^n\sum_{j=1}^n \frac{\partial k_i}{\partial x_j}(\sigma_1,\sigma_2)\frac{\delta\sigma_1^j}{\partial x}\frac{\partial\sigma_1^i}{\partial s} - \frac{1}{2}\int_0^1\sum_{i=1}^n \frac{\partial k_i}{\partial t}(\sigma_1,\sigma_2)\frac{\delta\sigma_2}{\delta x}\frac{\partial\sigma_1^i}{\partial s}\\
&-\frac{1}{2}\int_0^1\sum_{i=1}^n k_i(\sigma_1,\sigma_2)\frac{\delta\partial\sigma_1^i}{\delta x\partial s} + \int_0^1\sum_{i=1}^n\frac{\partial F}{\partial x_i}(\sigma_1,\sigma_2)\frac{\delta\sigma_1^i}{\delta x}\frac{\partial\sigma_2}{\partial s}\\
&+\int_0^1\frac{\partial F}{\partial t}(\sigma_1,\sigma_2)\frac{\delta\sigma_2}{\delta x}\frac{\partial\sigma_2}{\partial s} + \int_0^1 F(\sigma_1,\sigma_2)\frac{\delta\partial\sigma_2}{\delta x\partial s}\\
=&\frac{1}{2}\int_0^1\sum_{i=1}^n \frac{\frac{\partial\sigma_1^i}{\partial s}}{\frac{\partial\sigma_2}{\partial s}+\varepsilon}\frac{d}{ds}(\frac{\delta\sigma_1^i}{\delta x}) - \frac{1}{4}\int_0^1 \frac{|\frac{\partial\sigma_1}{\partial s}|^2}{(\frac{\partial\sigma_2}{\partial s}+\varepsilon)^2}\frac{d}{ds}(\frac{\delta\sigma_2}{\delta x})\\
&-\frac{1}{2}\int_0^1\sum_{i=1}^n\sum_{j=1}^n\frac{\partial k_i}{\partial x_j}(\sigma_1,\sigma_2)\frac{\partial\sigma_1^i}{\partial s}\frac{\delta\sigma_1^j}{\delta x} - \frac{1}{2}\int_0^1\sum_{i=1}^n\frac{\partial k_i}{\partial t}(\sigma_1,\sigma_2)\frac{\partial\sigma_1^i}{\partial s}\frac{\delta\sigma_2}{\delta x}\\
&-\frac{1}{2}\int_0^1\sum_{i=1}^n k_i(\sigma_1,\sigma_2)\frac{d}{ds}(\frac{\partial\sigma_1^i}{\delta x}) + \int_0^1\sum_{i=1}^n\frac{\partial F}{\partial x_i}(\sigma_1,\sigma_2)\frac{\partial\sigma_2}{\partial s}\frac{\delta\sigma_1^i}{\delta x}\\
&+\int_0^1\frac{\partial F}{\partial t}(\sigma_1,\sigma_2)\frac{\partial\sigma_2}{\partial s}\frac{\delta\sigma_2}{\delta x} + \int_0^1 F(\sigma_1,\sigma_2)\frac{d}{ds}(\frac{\delta\sigma_2}{\delta x})\\
=&-\frac{1}{2}\int_0^1\sum_{i=1}^n\frac{\partial}{\partial s}(\frac{\frac{\partial\sigma_1^i}{\partial s}}{\frac{\partial\sigma_2}{\partial s}+\varepsilon})\frac{\delta\sigma_1^i}{\delta x} + \frac{1}{2}\sum_{i=1}^n\frac{\frac{\partial\sigma_1^i}{\partial s}}{\frac{\partial\sigma_2}{\partial s}+\varepsilon}\frac{\delta\sigma_1^i}{\delta x}\Big|_{s=0}^{s=1}\\
&+\frac{1}{4}\int_0^1\frac{\partial}{\partial s}\left[\frac{|\frac{\partial\sigma_1}{\partial s}|^2}{(\frac{\partial\sigma_2}{\partial s}+\varepsilon)^2}\right]\frac{\delta\sigma_2}{\delta x} - \frac{1}{4}\frac{|\frac{\partial\sigma_1}{\partial s}|^2}{(\frac{\partial\sigma_2}{\partial s}+\varepsilon)^2}\frac{\delta\sigma_2}{\delta x}\Big|_{s=0}^{s=1}\\
&-\frac{1}{2}\int_0^1\sum_{j=1}^n\sum_{i=1}^n\frac{\partial k_j}{\partial x_i}(\sigma_1,\sigma_2)\frac{\partial\sigma_1^j}{\partial s}\frac{\delta\sigma_1^i}{\delta x} - \frac{1}{2}\int_0^1\sum_{i=1}^n\frac{\partial k_i}{\partial t}(\sigma_1,\sigma_2)\frac{\partial\sigma_1^i}{\partial s}\frac{\delta\sigma_2}{\delta x}\\
&+\frac{1}{2}\int_0^1\sum_{i=1}^n[\sum_{j=1}^n\frac{\partial k_i}{\partial x_j}(\sigma_1,\sigma_2)\frac{\partial\sigma_1^j}{\partial s} + \frac{\partial k_i}{\partial t}(\sigma_1,\sigma_2)\frac{\partial\sigma_2}{\partial s}]\frac{\delta\sigma_1^i}{\delta x}\\
&-\frac{1}{2}\sum_{i=1}^n k_i(\sigma_1,\sigma_2)\frac{\delta\sigma_1^i}{\delta x}\Big|_{s=0}^{s=1} + \int_0^1\sum_{i=1}^n\frac{\partial F}{\partial x_i}(\sigma_1,\sigma_2)\frac{\partial\sigma_2}{\partial s}\frac{\delta\sigma_1^i}{\delta x}
\end{aligned}$$

$$+\int_0^1 \frac{\partial F}{\partial t}(\sigma,\sigma_2)\frac{\partial\sigma_2}{\partial s}\frac{\delta\sigma_2}{\delta x}$$

$$-\int_0^1 \sum_{i=1}^n \left[\frac{\partial F}{\partial t}(\sigma_1,\sigma_2)\frac{\delta\sigma_2}{\delta x}+\frac{\partial F}{\partial t}(\sigma_1,\sigma_2)\frac{\partial\sigma_2}{\partial x}\right]\frac{\delta\sigma_2}{\delta s}+F(\sigma_1,\sigma_2)\frac{\delta\sigma_2}{\delta x}\Big|_{s=0}^{s=1}$$

$$=\int_0^1 \sum_{i=1}^n \Bigg\{-\frac{1}{2}\frac{d}{ds}\left[\frac{\frac{d\sigma_1^i}{ds}(s)}{\frac{d\sigma_2}{ds}(s)+\varepsilon}\right]$$

$$+\frac{1}{2}\left(\sum_{j=1}^n \frac{\partial k_i}{\partial x_j}(\sigma_1(s),\sigma_2(s))\frac{d\sigma_1^j}{ds}(s)+\frac{\partial k_i}{\partial t}(\sigma_1(s),\sigma_2(s))\frac{\partial\sigma_2}{\partial s}(s)\right)$$

$$-\frac{1}{2}\sum_{j=1}^n \frac{\partial k_j}{\partial x_i}(\sigma_1(s),\sigma_2(s))\frac{\partial\sigma_1^j}{\partial s}(s)+\frac{\partial F}{\partial x_i}(\sigma_1(s),\sigma_2(s))\frac{\partial\sigma_2}{\partial s}(s)\Bigg\}\frac{\delta\sigma_1^i}{\delta x}$$

$$+\int_0^1 \Bigg\{\frac{1}{4}\frac{d}{ds}\left[\frac{|\frac{d\sigma_1}{ds}(s)|^2}{(\frac{d\sigma_2}{ds}(s)+\varepsilon)^2}\right]-\frac{1}{2}\sum_{i=1}^n \frac{\partial k_i}{\partial t}(\sigma_1(s),\sigma_2(s))\frac{d\sigma_1^i}{ds}(s)-$$

$$\sum_{i=1}^n \frac{\partial F}{\partial x_i}(\sigma_1(s),\sigma_2(s)\frac{d\sigma_1^i}{ds}(s)\Bigg\}\frac{\delta\sigma_2}{\delta x}$$

$$+\frac{1}{2}\sum_{i=1}^n \frac{\frac{d\sigma_1^i}{ds}(s)}{\frac{d\sigma_2}{ds}(s)+\varepsilon}\frac{\delta\sigma_1^i}{\delta x}\Big|_{s=0}^{s=1}-\frac{1}{4}\frac{|\frac{d\sigma_1}{ds}(s)|^2}{(\frac{d\sigma_2}{ds}(s)+\varepsilon)^2}\frac{\delta\sigma_1^i}{\delta x}\Big|_{s=0}^{s=1}$$

$$-\frac{1}{2}\sum_{i=1}^n k_i(\sigma_1(s),\sigma_2(s))\frac{\delta\sigma_1^i}{\delta x}\Big|_{s=0}^{s=1}+F(\sigma_1(s),\sigma_2(s))\frac{\delta\sigma_2}{\delta x}\Big|_{s=0}^{s=1}$$

$$=\frac{1}{2}\sum_{i=1}^n \frac{\frac{d\sigma_1^i}{ds}(s)}{\frac{d\sigma_2}{ds}(s)+\varepsilon}\frac{\delta\sigma_1^i}{\delta x}\Big|_{s=0}^{s=1}-\frac{1}{4}\frac{|\frac{d\sigma_1}{ds}(s)|^2}{(\frac{d\sigma_2}{ds}(s)+\varepsilon)^2}\frac{\delta\sigma_2}{\delta x}\Big|_{s=0}^{s=1}$$

$$-\frac{1}{2}\sum_{i=1}^n k_i(\sigma_1(s),\sigma_2(s))\frac{\delta\sigma_1^i}{\delta x}\Big|_{s=0}^{s=1}+F(\sigma_1(s),\sigma_2(s))\frac{\delta\sigma_2}{\delta x}\Big|_{s=0}^{s=1}$$

because of (1.43) and (1.44). Observe that we do not move t in calculating $\frac{\delta d_\varepsilon}{\delta x}(x,t)$. Hence $\frac{\delta\sigma_2}{\delta x}=0$ and we obtain

$$\text{(1.43)}\quad \frac{\delta d_\varepsilon}{\delta x}(x,t)=\frac{1}{2}\sum_{i=1}^n \frac{\frac{d\sigma_1^i}{ds}}{\frac{d\sigma_2}{ds}+\varepsilon}\frac{\delta\sigma_1^i}{\delta x}\Big|_{s=0}^{s=1}-\frac{1}{2}\sum_{i=1}^n k_i(\sigma_1,\sigma_2)(\sigma_1,\sigma_2)\frac{\delta\sigma_1^i}{\delta x}\Big|_{s=0}^{s=1}$$

$$=\frac{1}{2}\sum_{i=1}^n \frac{\frac{d\sigma_1^i}{ds}(1)}{\frac{d\sigma_2}{ds}(1)+\varepsilon}\frac{\delta\sigma_1^i}{\delta x}(1)-\frac{1}{2}\sum_{i=1}^n k_i(\sigma_1(1),\sigma_2(1))\frac{\delta\sigma_1^i}{\delta x}(1)$$

It follows that

$$\nabla d_\varepsilon(x,t)=\frac{\delta d_\varepsilon}{\delta x}(x,t)=\frac{1}{2}\frac{\dot\sigma_1(1)}{\frac{\partial\sigma_2}{\partial s}(1)+\varepsilon}-\frac{1}{2}k(x,t)$$

Similar calculation as above will show that

$$\frac{\delta d_\varepsilon}{\delta t}(x,t)=\frac{1}{2}\sum_{i=1}^n \frac{|\frac{d\sigma_1^i}{ds}(s)|^2}{(\frac{d\sigma_2}{ds}(s)+\varepsilon)^2}\frac{\delta\sigma_1^i}{\delta t}\Big|_{s=0}^{s=1}-\frac{1}{4}\frac{|\frac{d\sigma_1}{ds}(s)|^2}{(\frac{d\sigma_2}{ds}(s)+\varepsilon)^2}\frac{\delta\sigma_2}{\delta t}\Big|_{s=0}^{s=1}$$

$$-\frac{1}{2}\sum_{i=1}^{n} k_i(\sigma_1(s),\sigma_2(s))\frac{\delta\sigma_1^i}{\delta t}\Big|_{s=0}^{s=1} + F(\sigma_1(s),\sigma_2(s))\frac{\delta\sigma_2}{\delta t}\Big|_{s=0}^{s=1}$$

Observe that we do not move x in calculating $\frac{\delta d_\varepsilon}{\delta t}(x,t)$. Hence $\frac{\delta\sigma_1^i}{\delta t} = 0$ for $1 \le i \le n$ and we obtain (1.41). Equation (1.42) follows immediately from (1.39), (1.40) and (1.41).

Notice that ∇d_ε exists only almost everywhere as only for almost every (x,t) there is a unique minimal curve σ joining (x,t) to (x_0,t_0). Hence (1.39), (1.40), (1.41) and (1.42) hold almost everywhere. □

COROLLARY 1.1. *In Theorem 1.2, let* $k_i(x,t) = -f_i(x,t)$ *and* $F(x,t) = \frac{1}{2}\sum_{n=1}^{n}\frac{\partial f_i}{\partial x_i}(x,t) + \frac{1}{4}|f(x,t)|^2 + V(x,t)$. *Let* $g_\varepsilon(x,t) = 2d_\varepsilon(x,t)$. *Then* $g(x,t)$ *satisfies condition (i) in Theorem 1.1.*

COROLLARY 1.2. *In Theorem 1.2, let* $k_i(x,t) = 2f_i(x,t)$ *and* $F(x,t) = 2V(x,t) - |\nabla V(x,t)|^2$. *Let* $h_\varepsilon(x,t) = d_\varepsilon(x,t)$. *Then* $h_\varepsilon(x,t)$ *satisfies condition (ii) in Theorem 1.1.*

COROLLARY 1.3. *In Theorem 1.2, let* $k_i(x,t) = f_i(x,t)$ *and* $F(x,t) = \frac{1}{4}|f|^2 - \frac{2\varepsilon_1^3+1}{2\varepsilon_1^2}\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}$. *Let* $\rho_\varepsilon(x,t) = \frac{2\varepsilon_1^2}{2\varepsilon_1^3+1}d_\varepsilon(s,t)$. *Then* $\rho_\varepsilon(x,t)$ *satisfies condition (iii) in Theorem 1.1.*

REMARK. Notice the difference between ε_1 in Theorem 1.1 and ε in Theorem 1.2.

LEMMA 1.1. *Let* $c \ge 4$ *be a constant. Assume that*

$$|f(x,t)| \le c(1+|x|) \tag{1.44}$$

$$|\nabla f(x,t)| = \sqrt{\sum_{i=1}^{n}|\nabla f_i|^2} \le c(1+|x|) \tag{1.45}$$

$$|V(x,t)| \le c(1+|x|^2) \tag{1.46}$$

$$|\nabla V(x,t)| \le c(1+|x|) \tag{1.47}$$

Then the following statements holds.

(a) $g_\varepsilon(x,t)$ *in Corollary 1.1 has lower bound in terms of*

$$\left[\frac{1}{2(t-t_0+\varepsilon)} - \frac{3c^2(\pi^2+1)}{\pi^2}(t-t_0) - \frac{c(5\pi^2+3)}{4\pi^2}\right]x^2 + \textit{lower order terms in } x.$$

(b) $h_\varepsilon(x,t)$ *in Corollary 1.2 has lower bound in terms of*

$$\left(\frac{1}{4(t-t_0+\varepsilon)} - \frac{c(5\pi^2+3)}{4\pi^2} - \frac{3c^2(t-t_0)(\pi^2+1)}{\pi^2}\right)x^2 + \textit{lower order terms in } x.$$

(c) $\rho_\varepsilon(x,t)$ *in Corollary 1.3 has linear bound in terms of* $\frac{2\varepsilon_1^2}{2\varepsilon_1^3+1}\Big[\big(\frac{1}{4(t-t_0+\varepsilon)} - (t-t_0)\ c\big(\frac{c}{16} + \frac{2\varepsilon_1^3+1}{8\varepsilon_1^2}\big) - \frac{3c}{4} - \frac{3c}{8\pi^2} - \frac{3c(\pi^2+1)}{8\pi^2}(t-t_0)\big(\frac{c}{2} + \frac{2\varepsilon_1^3+1}{\varepsilon_1^2}\big)\Big]x^2 +$ *lower order terms in* x.

Proof.

(a) $$|F(x,t)| = |\frac{1}{2}\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x,t) + \frac{1}{4}|f(x,t)|^2 + V(x,t)|$$
$$\leq \frac{1}{2}\sum_{i=1}^{n}|\frac{\partial f_i}{\partial x_i}| + \frac{1}{4}|f(x,t)|^2 + |V(x,t)|$$
$$\leq \frac{1}{2}c(1+|x|) + \frac{c^2}{4}(1+|x|)^2 + c(1+|x|^2)$$
$$\leq c^2(1+|x|)^2$$

Recall that $\mathcal{P}$ consists of path $\sigma = (\sigma_1, \sigma_2)$ such that

$$\sigma(0) = (x_0, t_0), \sigma(1) = (x,t), \sigma_2(s) = (t-t_0)s + t_0, \int_0^1 \sigma_1(s)ds = \frac{x+x_0}{2}$$

$$g_\varepsilon(x,t) = 2d_\varepsilon(x,t) = 2\min_{\sigma\in\mathcal{P}} E_\varepsilon(\sigma)$$
$$= \min_{\sigma\in\mathcal{P}}\left\{\frac{1}{2}\int_0^1 \frac{|\dot\sigma(s)|^2}{\frac{\partial\sigma_2}{ds}+\varepsilon} - \int_0^1\sum_{i=1}^{n} k_i(\sigma_1(s),\sigma_2(s))\frac{d\sigma_1^i}{ds} + 2\int_0^1 F(\sigma_1(s),\sigma_2(s))\frac{d\sigma_2}{ds}\right\}$$
$$= \min_{\sigma\in\mathcal{P}}\left\{\frac{1}{2}\int_0^1 \frac{|\dot\sigma(s)|^2}{t-t_0+\varepsilon} + \int_0^1\sum_{i=1}^{n} f_i(\sigma_1(s),\sigma_2(s))\frac{d\sigma_1^i}{ds} + 2\int_0^1 F(\sigma_1(s),\sigma_2(s))\frac{d\sigma_2}{ds}\right\}$$
$$\geq \min_{\sigma\in\mathcal{P}}\left\{\frac{1}{2(t-t_0+\varepsilon)}\int_0^1 |\dot\sigma_1(s)|^2 - \int_0^1 |f(\sigma_1(s),\sigma_2(s))|\,|\frac{d\sigma_1}{ds}| - 2\int_0^1 c^2(1+|\sigma_1(s)|)^2\frac{d\sigma_2}{ds}\right\}$$
$$\geq \min_{\sigma\in\mathcal{P}}\left\{\frac{1}{2(t-t_0+\varepsilon)}\int_0^1 |\dot\sigma_1(s)|^2 - \int_0^1 c(1+|\sigma_1(s)|)|\frac{d\sigma_1}{ds}| - 2(t-t_0)\int_0^1 c^2(1+|\sigma_1(s)|)^2\right\}$$
$$\geq \min_{\sigma\in\mathcal{P}}\left\{(\frac{1}{2(t-t_0+\varepsilon)} - \frac{c}{2})\int_0^1 |\dot\sigma_1(s)|^2 - (2(t-t_0)c^2 + \frac{c}{2})\int_0^1 (1+|\sigma_1(s)|)^2\right\}$$
$$= \min_{\sigma\in\mathcal{P}}\left\{(\frac{1}{2(t-t_0+\varepsilon)} - \frac{c}{2})\int_0^1 |\dot\sigma_1(s)|^2 - (2(t-t_0)c^2 + \frac{c}{2}) - (4(t-t_0)c^2 + c)\int_0^1 |\sigma_1(s)| - (2(t-t_0)c^2 + \frac{c}{2})\int_0^1 |\sigma_1(s)|^2\right\}$$
$$\geq \min_{\sigma\in\mathcal{P}}\left\{(\frac{1}{2(t-t_0+\varepsilon)} - \frac{c}{2})\int_0^1 |\dot\sigma_1(s)|^2 - (2(t-t_0)c^2 + \frac{c}{2}) - [6(t-t_0)c^2 + \frac{3}{2}c]\int_0^1 (1+|\sigma_1(s)|)^2\right\}$$

Recall that the Poincare inequality states that

$$\int_0^1 |\sigma_1(s) - \int_0^1 \sigma_1(s)ds|^2 \leq D\int_0^1 |\dot\sigma_1|^2, \text{ where } D = \frac{1}{4\pi^2}$$

Since

$$|\sigma_1(s)|^2 \le 2|\sigma_1(s) - \int_0^1 \sigma_1|^2 + 2(\int_0^1 \sigma_1)^2,$$

we have

$$\int_0^1 |\sigma_1(s)|^2 \le 2\int_0^1 |\sigma_1(s) - \int_0^1 \sigma_1|^2 + 2(\int_0^1 \sigma_1)^2$$
$$\le 2D\int_0^1 |\dot\sigma_1|^2 + 2(\int_0^1 \sigma_1)^2$$

Hence

$$g_\varepsilon(x,t) \ge \min_{\sigma\in\mathcal{P}}\Big\{\big[\frac{1}{2(t-t_0)+\varepsilon)} - \frac{c}{2} - (6(t-t_0)c^2 + \frac{3}{2}c)2D\big]\int_0^1 |\dot\sigma(s)|^2$$
$$- \big(2(t-t_0)c^2 + \frac{c}{2}\big) - \big(12(t-t_0)c^2 + 3c\big)\big(\int_0^1 \sigma_1\big)^2\Big\}$$

Observe that $\int_0^1 |\dot\sigma_1|^2 \ge |\int_0^1 \dot\sigma_1|^2 = |x-x_0|^2$ by Schwartz inequality. It follows that

$$g_\varepsilon(x,t) \ge \left[\frac{1}{2(t-t_0+\varepsilon)} - \frac{c}{2} - (12(t-t_0)c^2+3c)D\right]|x-x_0|^2$$
$$- \big(2(t-t_0)c^2 + \frac{c}{2}\big) - \big(12(t-t_0)c^2+3c\big)|\frac{x+x_0}{2}|^2$$
$$= \left[\frac{1}{2(t-t_0+\varepsilon)} - \frac{c}{2} - 12(t-t_0)c^2D - 3cD - 3(t-t_0)c^2 - \frac{3}{4}c\right]x^2$$
$$+ \text{ lower order terms in } x.$$
$$= \left[\frac{1}{2(t-t_0+\varepsilon)} - \frac{3c^2(\pi^2+1)}{\pi^2}(t-t_0) - \frac{c(5\pi^2+3)}{4\pi^2}\right]x^2$$
$$+ \text{ lower order terms in } x.$$

(b)

$$\Big|F(x,t) = |2V(x,t) - |\nabla V(x,t)|\Big|$$
$$\le 2|V(x,t)| + |\nabla V(x,t)|$$
$$\le 2c(1+|x|^2) + c(1+|x|)$$
$$\le (c^2+2c)(1+|x|)^2$$
$$= 2c^2(1+|x|)^2$$

$$h_\varepsilon(x,t) = d_\varepsilon(x,t) = \min_{\sigma\in\mathcal{P}} E_\varepsilon(\sigma)$$
$$= \min_{\sigma\in\mathcal{P}}\left\{\frac{1}{4}\int_0^1 \frac{|\dot\sigma_1(s)|^2}{\frac{\partial\sigma_2}{\partial s}+\varepsilon} - \frac{1}{2}\int_0^1 \sum_{i=1}^n k_i(\sigma_1(s),\sigma_2(s))\frac{d\sigma_1^i}{ds} + \int_0^1 F(\sigma_1(s),\sigma_2(s))\frac{d\sigma_2}{ds}\right\}$$
$$\ge \min_{\sigma\in\mathcal{P}}\left\{\frac{1}{4(t-t_0+\varepsilon)}\int_0^1 |\dot\sigma_1(s)|^2 - \int_0^1 c(1+|\sigma_1(s)|)|\frac{d\sigma_1}{ds}|\right.$$
$$\left. -(t-t_0)\int_0^1 2c^2(1+|\sigma_1(s)|)^2\right\}$$

$$\geq \min_{\sigma\in\mathcal{P}}\left\{(\frac{1}{4(t-t_0+\varepsilon)}-\frac{c}{2})\int_0^1|\dot{\sigma}_1(s)|^2-(2(t-t_0)c^2+\frac{c}{2})\int_0^1(1+|\sigma_1(s)|)^2\right\}$$

$$=\min_{\sigma\in\mathcal{P}}\left\{(\frac{1}{4(t-t_0+\varepsilon)}-\frac{c}{2})\int_0^1|\dot{\sigma}_1(s)|^2-(2(t-t_0)c^2+\frac{c}{2})\right.$$
$$\left.-(4(t-t_0)c^2+c)\int_0^1|\sigma_1(s)|-(2(t-t_0)c^2+\frac{c}{2})\int_0^1|\sigma_1(s)|^2\right\}$$

$$\geq \min_{\sigma\in\mathcal{P}}\left\{(\frac{1}{4(t-t_0+\varepsilon)})-\frac{c}{2})\int_0^1|\dot{\sigma}_1(s)|^2-(2(t-t_0)c^2+\frac{c}{2})\right.$$
$$\left.-(6(t-t_0)c^2+\frac{3c}{2})\int_0^1|\sigma_1(s)|^2\right\}$$

$$\geq \min_{\sigma\in\mathcal{P}}\left\{[\frac{1}{4(t-t_0+\varepsilon)}-\frac{c}{2}-(12(t-t_0)c^2D+3cD)]\int_0^1|\dot{\sigma}_1(s)|^2\right.$$
$$\left.-(2(t-t_0)c^2+\frac{c}{2})-(12(t-t_0)c^2+3c)(\int_0^1\sigma_1)^2\right\}$$

$$\geq \min_{\sigma\in\mathcal{P}}\left\{[\frac{1}{4(t-t_0+\varepsilon)}-\frac{c}{2}-3cD-12(t-t_0)c^2D]|\int_0^1\dot{\sigma}_1|^2\right.$$
$$\left.-(2(t-t_0)c^2+\frac{c}{2})-(12(t-t_0)c^2+3c)(\int_0^1\sigma_1)^2\right\}$$

$$=\left(\frac{1}{4(t-t_0+\varepsilon)}-\frac{c(2\pi^2+3)}{4\pi^2}-\frac{3c^2(t-t_0)}{\pi^2}\right)|x-x_0|^2$$
$$(2(t-t_0)c^2+\frac{c}{2})-(12(t-t_0)c^2+3c)|\frac{x+x_0}{2}|^2$$

$$=\left(\frac{1}{4(t-t_0+\varepsilon)}-\frac{c(5\pi^2+3)}{4\pi^2}-\frac{3c^2(t-t_0)(\pi^2+1)}{\pi^2}\right)x^2+\text{lower order terms in } x.$$

(c) $|F(x,t)|=\left|\frac{1}{4}|f|^2-\frac{2\varepsilon_1^3+1}{\partial\varepsilon_1^2}\sum_{i=1}^n\frac{\partial f_i}{\partial x_i}\right|$

$$\leq \frac{1}{4}|f|^2+\frac{2\varepsilon_1^3+1}{2\varepsilon_1^2}|\sum_{i=1}^n\frac{\partial f_i}{\partial x_i}|$$
$$\leq \frac{1}{4}c^2(1+|x|)^2+\frac{2\varepsilon^3+1}{2\varepsilon^2}c(1+|x|)$$
$$\leq c(\frac{c}{4}+\frac{2\varepsilon_1^3+1}{2\varepsilon_1^2})(1+|x|)^2$$

$$\rho_\varepsilon(x,t)=\frac{2\varepsilon_1^2}{2\varepsilon_1^3+1}d_\varepsilon(x,t)$$
$$=\frac{2\varepsilon_1^2}{2\varepsilon_1^3+1}\min_{\sigma\in\mathcal{P}}\left\{\frac{1}{4}\int_0^1\frac{|\dot{\sigma}_1(s)|^2}{\frac{\partial\sigma_2}{\partial s}+\varepsilon}-\frac{1}{2}\int_0^1\sum_{i=1}^n k_i(\sigma_1(s),\sigma_2(s))\frac{d\sigma_1^i}{ds}\right.$$
$$\left.+\int_0^1F(\sigma_1(s),\sigma_2(s)\frac{d\sigma_2}{ds}\right\}$$
$$\geq\frac{2\varepsilon_1^2}{2\varepsilon_1^3+1}\min_{\sigma\in\mathcal{P}}\left\{\frac{1}{4(t-t_0+\varepsilon)}\int_0^1|\dot{\sigma}_1(s)|^2-\frac{1}{2}\int_0^1c(1+|\sigma_1(s)|)|\frac{d\sigma_1}{ds}|\right.$$

$$-(t-t_0)\int_0^1 c(\frac{c}{4}+\frac{2\varepsilon_1^3+1}{2\varepsilon_1^2})(1+\sigma_1(s)|)^2$$

$$\geq \frac{2\varepsilon_1^2}{2\varepsilon_1^3+1}\min_{\sigma\in\mathcal{P}}\Big\{(\frac{1}{4(t-t_0+\varepsilon)}-\frac{c}{4})\int_0^1|\dot\sigma_1(s)|^2-[(t-t_0)c(\frac{c}{4}+\frac{2\varepsilon_1^3+1}{2\varepsilon_1^2})+\frac{c}{2}]$$
$$\int_0^1(1+|\sigma_1(s)|)^2\Big\}$$

$$=\frac{2\varepsilon_1^2}{2\varepsilon_1^3+1}\min_{\sigma\in\mathcal{P}}\Big\{(\frac{1}{4(t-t_0+\varepsilon)}-\frac{c}{4})\int_0^1|\dot\sigma_1(s)|^2-[(t-t_0)c(\frac{c}{4}+\frac{2\varepsilon_1^3+1}{2\varepsilon_1^2})+\frac{c}{2}]$$
$$-[(t-t_0)c(\frac{c}{4}+\frac{2\varepsilon_1^3+1}{2\varepsilon_1^2})+c]\int_0^1|\sigma_1(s)|$$
$$-[[(t-t_0)c(\frac{c}{4}+\frac{2\varepsilon_1^3+1}{2\varepsilon_1^2})+\frac{c}{2}]\int_0^1|\sigma_1(s)|^2$$

$$\geq \frac{2\varepsilon_1^2}{2\varepsilon_1^3+1}\min_{\sigma\in\mathcal{P}}\Big\{(\frac{1}{4(t-t_0+\varepsilon)}-\frac{c}{4})\int_0^1|\dot\sigma_1(s)|^2-[(t-t_0)c(\frac{c}{4}+\frac{2\varepsilon_1^3+1}{2\varepsilon_1^2})+\frac{c}{2}]\Big\}$$
$$-[(t-t_0)c(\frac{3c}{4}+\frac{3(2\varepsilon_1^3+1)}{2\varepsilon_1^2})+\frac{3c}{2}]\int_0^1|\sigma_1(s)|^2\Big\}$$

$$\geq \frac{2\varepsilon_1^2}{2\varepsilon_1^3+1}\min_{\sigma\in\mathcal{P}}\Big\{(\frac{1}{4(t-t_0+\varepsilon)}-\frac{c}{4}-\frac{3cD}{2}((t-t_0)(\frac{c}{2}+\frac{2\varepsilon_1^3+1}{\varepsilon_1^2})+1)]\int_0^1|\dot\sigma_1(s)|^2$$
$$-[(t-t_0)c(\frac{c}{4}+\frac{2\varepsilon_1^3+1}{2\varepsilon_1^2})+\frac{c}{2}]-\frac{3c}{2}[(t-t_0)(\frac{c}{2}+\frac{2\varepsilon_1^3+1}{\varepsilon_1^2})+1]|\int_0^1\sigma_1(s)|^2$$

$$\geq \frac{2\varepsilon_1^2}{2\varepsilon_1^3+1}\min_{\sigma\in\mathcal{P}}\Big\{(\frac{1}{4(t-t_0+\varepsilon)}-\frac{c}{4}-\frac{3cD}{2}((t-t_0)(\frac{c}{2}+\frac{2\varepsilon_1^3+1}{\varepsilon_1^2})+1)]\int_0^1|\dot\sigma_1|^2$$
$$-[(t-t_0)c(\frac{c}{4}+\frac{2\varepsilon_1^3+1}{2\varepsilon_1^2})+\frac{c}{2}]-\frac{3c}{2}[(t-t_0)(\frac{c}{2}+\frac{2\varepsilon_1^3+1}{\varepsilon_1^2})+1]|\int_0^1\sigma_1(s)|^2\Big\}$$

$$=\frac{2\varepsilon_1^2}{2\varepsilon_1^3+1}\Big\{[\frac{1}{4(t-t_0+\varepsilon)}-\frac{c}{4}-\frac{3cD}{2}((t-t_0)(\frac{c}{2}+\frac{2\varepsilon_1^3+1}{\varepsilon_1^2})+1)]|x-x_0|^2$$
$$-[(t-t_0)c(\frac{c}{4}+\frac{2\varepsilon_1^3+1}{2\varepsilon_1^2})+\frac{c}{2}]-\frac{3c}{2}[(t-t_0)(\frac{c}{2}+\frac{2\varepsilon_1^3+1}{\varepsilon_1^2})+1]|\frac{x+x_0}{2}|^2\Big\}$$

$$=\frac{2\varepsilon_1^2}{2\varepsilon_1^3+1}[\frac{1}{4(t-t_0+\varepsilon)}-(t-t_0)c(\frac{c}{16}+\frac{2\varepsilon_1^3+1}{8\varepsilon_1^2})-\frac{3c}{4}-\frac{3c}{8\pi^2}$$
$$-\frac{3c(\pi^2+1)}{8\pi^2}(t-t_0)(\frac{c}{2}+\frac{2\varepsilon_1^3+1}{\varepsilon_1^2})]x^2+\text{ lower order term in } x.$$

□

In view of Lemma 1.1, we have the following Lemma 1.2

LEMMA 1.2. *Assume that (1.44) - (1.47) hold i.e.* $|f(x,t)|\leq c(1+|x|), \nabla f(x,t)\leq c(1+|x|), |V(x,t)|\leq c(1+|x|^2)$ *and* $|\nabla V(x,t)|\leq c(|1+|x|)$, *where* $c\geq 4$ *is a constant. Let* $\tilde c$ *be a constant strictly less than* $\min(\frac{1}{4},\frac{\varepsilon_1^2}{4\varepsilon_1^3+2})$. *Let* $\rho(x,t)=\frac{\tilde c(1+|x|^2)}{t-t_0+\varepsilon}$. *Then the conditions (i), (ii) and (iii) in Theorem 1.1 are satisfied for* $t-t_0+\varepsilon$ *sufficiently small i.e.,*

(i) $\frac{\partial\rho}{\partial t}+\frac{1}{2}|\nabla\rho|^2-\sum_{i=1}^n f_i\rho_i-\sum_{i=1}^n f_{i,i}-2V\leq 0$ *for* $t-t_0+\varepsilon$ *sufficiently small.*

(ii) $\frac{\partial\rho}{\partial t}+|\nabla\rho+f|^2-2V+|\nabla V|^2\le 0$ *for* $t-t_0+\varepsilon$ *sufficiently small*

(iii) $\frac{\partial\rho}{\partial t}+(\varepsilon_1+\frac{1}{2\varepsilon_1^2})|\nabla\rho|^2+\sum_{i=1}^n\rho_i f_i+\sum_{i=1}^n f_{i,i}\le 0$ *for* $t-t_0+\varepsilon$ *sufficiently small.*

Proof.

$$\frac{\partial\rho}{\partial t}=-\frac{\tilde c(1+|x|^2)}{(t-t_0+\varepsilon)^2},\quad \frac{\partial\rho}{\partial x_i}=\frac{2\tilde c x_i}{t-t_o+\varepsilon},\quad |\nabla\rho|^2=\frac{4\tilde c^2|x|^2}{(t-t_0+\varepsilon)^2}$$

(i)
$$\begin{aligned}&\frac{\partial\rho}{\partial t}+\frac{1}{2}|\nabla\rho|^2-\sum_{i=1}^n f_i\rho_i-\sum_{i=1}^n f_{i,i}-2V\\
\le&\frac{-\tilde c(1+|x|^2)}{(t-t_0+\varepsilon)^2}+\frac{2\tilde c^2|x|^2}{(t-t_0+\varepsilon)^2}+c(1+|x|)\frac{2\tilde c|x|}{t-t_0+\varepsilon}+nc(1+|x|)\\
&+2c(1+|x|)^2\\
\le&\frac{(-\tilde c+2\tilde c^2)}{(t-t_0+\varepsilon)^2}(1+|x|^2)+\frac{2\tilde c c}{t-t_0+\varepsilon}(1+|x|)^2+nc(1+|x|)^2+2c(1+|x|)^2\\
\le&\left(\frac{-\tilde c(1-2\tilde c)}{(t-t_0+\varepsilon)^2}+\frac{4\tilde c c}{t-t_0+\varepsilon}+2nc+4c\right)(1+|x|^2)\\
<&0\end{aligned}$$

for $t-t_0+\varepsilon$ sufficiently small because $0<\tilde c<\frac{1}{4}$.

(ii)
$$\begin{aligned}&\frac{\partial\rho}{\partial t}+|\nabla\rho+f|^2-2V+|\nabla V|^2\\
=&\frac{\partial\rho}{\partial t}+|\nabla\rho|^2+2\nabla\rho\cdot f+|f|^2-2V+|\nabla V|^2\\
\le&-\frac{\tilde c(1+|x|^2)}{(t-t_0+\varepsilon)^2}+\frac{4\tilde c^2|x|^2}{(t-t_0+\varepsilon)^2}+\frac{4\tilde c|x|}{t-t_0+\varepsilon}c(1+|x|)\\
&+c^2(1+|x|^2)+2c(1+|x|^2)+c^2(1+|x|)^2\\
\le&\frac{-\tilde c(1-4\tilde c)}{(t-t_0+\varepsilon)^2}(1+|x|^2)+\frac{4c\tilde c}{t-t_0+\varepsilon}(1+|x|)^2+2c^2(1+|x|)^2+2c(1+|x|^2)\\
\le&\left[\frac{-\tilde c(1-4\tilde c)}{(t-t_0+\varepsilon)^2}+\frac{8c\tilde c}{t-t_0+\varepsilon}+4c^2+2c\right](1+|x|^2)\\
<&0\end{aligned}$$

for $t-t_0+\varepsilon$ sufficiently small because $0<\tilde c<\frac{1}{4}$.

(iii)
$$\begin{aligned}&\frac{\partial\rho}{\partial t}+(\varepsilon_1+\frac{1}{2\varepsilon_1^2})|\nabla\rho|^2+\sum_{i=1}^n\rho_i f_i+\sum_{i=1}^n f_{i,i}\\
\le&\frac{-\tilde c(1+|x|^2)}{(t-t_0+\varepsilon)^2}+(\varepsilon_1+\frac{1}{2\varepsilon_1^2})\frac{4\tilde c^2}{(t-t_0+\varepsilon)^2}|x|^2+\frac{2\tilde c|x|}{t-t_0+\varepsilon}c(1+|x|)\\
&+c(1+|x|)\\
\le&\left[\frac{-\tilde c(1-(4\varepsilon_1+\frac{2}{\varepsilon_1^2})\tilde c)}{(t-t_0+\varepsilon)^2}+\frac{4\tilde c c}{t-t_0+\varepsilon}+2c\right](1+|x|^2)\\
<&0\end{aligned}$$

for $t-t_0+\varepsilon$ sufficiently small because $0<\tilde c<\frac{\varepsilon_1^2}{4\varepsilon_1^3+2}$. □

Consider the parabolic differential equation

$$(1.48) \qquad \frac{\partial u}{\partial t} = \Delta u + \sum_{i=1}^{n} f_i \frac{\partial u}{\partial x_i} - Vu \qquad \text{on } \mathbb{R}^n$$
$$u(x,0) = \psi(x)$$

where f grows at most linearly and V grows at most quadratically satisfying (1.44) – (1.47) respectively. Let f_{2R} and V_{2R} be the functions obatined by multiplying f and V respectively by a cut off function σ which is equal to one in the ball of radius $R \geq 1$ and equal to zero outside a ball of radius $2R$. We can choose σ such that

$$(1.49) \qquad |\nabla\sigma(x)| \leq \frac{4}{1+|x|} \qquad \text{and} \qquad |\Delta\sigma(x)| \leq \frac{4}{1+|x|^2}$$

Consider the following equation

$$(1.50) \qquad \frac{\partial u_{2R}}{\partial t} = \Delta u_{2_R} + \sum_{i=1}^{n} (f_{2R})_i \frac{\partial u_{2R}}{\partial x_i} - V_{2R}u_{2R}$$

in the ball B_{2R} of radius $2R$ with the Neumann condition, where $(f_{2R})_i$ denotes the i-th component of f_{2R}. Let $\psi_{2R} = \psi(x)\sigma(x)$. Then the second initial-boundary problem

$$(1.51) \qquad \begin{cases} \dfrac{\partial u_{2R}}{\partial t} = \Delta u_{2R} + \displaystyle\sum_{i=1}^{n} (f_{2R})_i \frac{\partial u_{2R}}{\partial x_i} - V_{2R}u_{2R} & \text{on } B_{2R} \times (0,T] \\ u_{2R}(x,0) = \psi_{2R}(x) & \text{on } \overline{B}_{2R} \\ \dfrac{\partial u_{2R}}{\partial \nu} = 0 & \text{on } \partial B_{2R} \times (0,T] \end{cases}$$

has an unique solution (c.f. [Fr] p.144 Theorem 2) for $t \in [0,\infty)$.

LEMMA 1.3. *Assume that (1.44) - (1.47) hold, i.e.,* $|f(x,t)| \leq c(1+|x|)$, $|\nabla f(x,t) \leq c(1+|x|)$, $|V(x,t)| \leq c(1+|x|^2)$ *and* $|\nabla V(x,t)| \leq c(1+|x|)$ *where* $c \geq 4$ *is a constant. Let* $\tilde{c} < \min(\frac{1}{4}, \frac{\varepsilon_1^2}{4\varepsilon_1^3+2})$ *be a constant. Choose* T *and* ε *suitably small so that conclusions (i), (ii) and (iii) in Lemma 1.2 hold for* $\rho(x,t) = \dfrac{\tilde{c}(1+|x|^2)}{t+\varepsilon}$ *for all* $0 \leq t \leq T$. *Then for* $0 \leq t \leq T$

(i) $$\int_{B_{2R}\times\{t\}} e^{\rho} u_{2R}^2 \leq \int_{B_{2R}\times\{0\}} e^{\rho} u_{2R}^2$$

(ii) $$\int_{B_{2R}\times\{t\}} e^{\rho}|\nabla u_{2R}|^2 \leq \int_{B_{2R}\times\{0\}} e^{\rho}(\nabla u_{2_R})^2 + \int_0^t \int_{B_{2R}} e^{\rho(x,s)}(u_{2R}(x,s))^2$$

(iii) $$\begin{aligned}\int_{B_{2R}\times\{t\}} e^{\rho}(\Delta u_{2R})^2 \leq{}& \int_{B_{2R}\times\{0\}} e^{\rho}(\Delta u_{2R})^2 \\ &+ O\Big(\int_{B_{2R}\times[0,t]} e^{\rho}|\nabla\rho|^2|f_{2R}|^2|\nabla u_{2R}|^2 + \int_{B_{2R}\times[0,t]} e^{\rho}|\nabla f_{2R}|^2|\nabla u_{2R}|^2 \\ &+ \int_{B_{2R}\times[0,t]} e^{\rho}|f_{2R}||\nabla u_{2R}|^2|\Delta f_{2R}| + \int_{B_{2R}\times[0,t]} e^{\rho}|f_{2R}|^4|\nabla u_{2R}|^2 \\ &+ \int_{B_{2R}\times[0,t]} e^{\rho}|\nabla(V_{2R}u_{2R})|^2 + \int_{B_{2R}\times[0,t]} e^{\rho}|\nabla u_{2R}|^2(\sum_{i=1}^{n}(f_{2R})_{i,i})^2\Big)\end{aligned}$$

Proof. It is easy to show that $|f_{2R}|, |\nabla f_{2R}|, |\nabla V_{2R}|$ have linear growth while $|V_{2R}|$ has quadratic growth, so we can apply Lemma 1.2 and Theorem 1.1 to get our estimates (i), (ii) and (iii) above. □

THEOREM 1.3. *Let u_{2R} be the solution of (1.53) where $R \geq 1$ woth inital condition $u_{2R}(x,0) = \psi_{2R}(x)$. Assume that (1.44) - (1.47) hold i.e., $|f(x,t)| \leq c(1+|x|)$, $|\nabla f(x,t)| \leq c(1+|x|)$, $|V(x,t)| \leq c(1+|x|^2)$ and $|\nabla V(x,t)| \leq c(1+|x|)$, where $c \geq 4$ is a constant. Assume further that $|\Delta f(x,t)| \leq c(1+|x|)$. Let δ and $\widetilde{c}$ be positive constant such that $\widetilde{\widetilde{c}} := \widetilde{c} + \delta < \min(\frac{1}{4}, \frac{\varepsilon_1^2}{4\varepsilon_1^3 + 2})$. Choose T and ε suitably small so that $T + \varepsilon < \delta$ and the conclusions (i), (ii) and (iii) in Lemma 1.3 hold for both*

$$\rho(x,t) = \frac{\widetilde{c}(1+|x|^2)}{t+\varepsilon} \qquad \text{and} \qquad \widetilde{\rho}(x,t) = \frac{\widetilde{\widetilde{c}}(1+|x|^2)}{t+\varepsilon}$$

for any $0 \leq t \leq T$. Then for any $0 \leq t \leq T$

$$(i) \int_{B_{2R}\times\{t\}} e^{\rho} u_{2R}^2, \quad (ii) \int_{B_{2R}\times\{t\}} e^{\rho}|\nabla u_{2R}|^2, \quad (iii) \int_{B_{2R}\times\{t\}} e^{\rho}|\Delta u_{2R}|^2$$

are bounded above independent of R and t.

Proof. By Lemma 1.3, we have

$$\text{(i)} \int_{B_{2R}\times\{t\}} e^{\rho} u_{2R}^2 \leq \int_{B_{2R}\times\{0\}} e^{\rho} u_{2R}^2 = \int_{B_{2R}} e^{\rho(x,0)} (\psi_{2R}(x))^2$$
$$\leq \int_{\mathbf{R}^n} e^{\rho(x,0)} (\psi(x))^2$$

$$\text{(ii)} \int_{B_{2R}\times\{t\}} e^{\rho}|\nabla u_{2R}|^2 \leq \int_{B_{2R}\times\{0\}} e^{\rho}|\nabla u_{2R}|^2 + \int_0^t \int_{B_{2R}} e^{\rho(x,s)} (u_{2R}(x,s))^2$$
$$\leq \int_{B_{2R}} e^{\rho(x,0)} |\nabla \psi_{2R}(x)|^2 + t \int_{B_{2R}} e^{\rho(x,0)} (\psi(x))^2$$
$$\leq \int_{B_{2R}} e^{\rho(x,0)} (|\nabla\sigma||\psi| + |\nabla\psi|)^2 + T \int_{B_{2R}} e^{\rho(x,0)} (\psi(x))^2$$
$$\leq 2\int_{B_{2R}} e^{\rho(x,0)} |\nabla\sigma|^2|\psi|^2 + 2\int_{B_{2R}} e^{\rho(x,0)}|\nabla\psi|^2 + T\int_{B_{2R}} e^{\rho(x,0)}(\psi(x))^2$$
$$\leq 8\int_{B_{2R}} e^{\rho(x,0)}|\psi|^2 + 2\int_{B_{2R}} e^{\rho(x,0)}|\nabla\psi|^2 + T\int_{B_{2R}} e^{\rho(x,0)}(\psi(x))^2$$
$$= (8+T)\int_{\mathbf{R}^n} e^{\rho(x,0)}|\psi|^2 + 2\int_{\mathbf{R}^n} e^{\rho(x,0)}|\nabla\psi|^2$$

$$\text{(iii)} \int_{B_{2R}\times\{t\}} e^{\rho}(\Delta u_{2R})^2 \leq \int_{B_{2R}\times\{0\}} e^{\rho}(\Delta u_{2R})^2$$
$$+ O\Big(\int_{B_{2R}\times[0,t]} e^{\rho}|\nabla\rho|^2|f_{2R}|^2|\nabla u_{2R}|^2 + \int_{B_{2R}\times[0,t]} e^{\rho}|\nabla f_{2R}|^2|\nabla u_{2R}|^2$$
$$+ \int_{B_{2R}\times[0,t]} e^{\rho}|f_{2R}||\nabla u_{2R}|^2|\Delta f_{2R}| + \int_{B_{2R}\times[0,t]} e^{\rho}|f_{2R}|^4|\nabla u_{2R}|^2$$
$$+ \int_{B_{2R}\times[0,t]} e^{\rho}|\nabla(V_{2R}u_{2R})|^2 + \int_{B_{2R}\times[0,t]} e^{\rho}|\nabla u_{2R}|^2 (\sum_{i=1}^n (f_{2R})_{i,i})^2\Big)$$

We need to get upper estimate of the right hand side of the above inequality.

$$\begin{aligned}
\int_{B_{2R}\times\{0\}} e^{\rho}(\Delta u_{2R})^2 &= \int_{B_{2R}} e^{\rho(x,0)}|\Delta(\sigma\psi)|^2 \\
&= \int_{B_{2R}} e^{\rho(x,0)}\left((\Delta\psi)\sigma + 2\nabla\psi\cdot\nabla\sigma + \psi\Delta\sigma\right)^2 \\
&= \int_{B_{2R}} e^{\rho(x,0)}\big[(\Delta\psi)^2\sigma^2 + 4(\nabla\psi\cdot\nabla\sigma)^2 + \psi^2(\Delta\sigma)^2 + 4(\Delta\psi)\sigma(\nabla\psi\cdot\nabla\sigma) \\
&\qquad +2(\Delta\psi)\sigma\psi\Delta\sigma + 4(\nabla\psi\cdot\nabla\sigma)\psi\Delta\sigma\big] \\
&\le \int_{B_{2R}} e^{\rho(x,0)}\left[|\Delta\psi|^2\sigma^2 + 4(\nabla\psi\cdot\nabla\sigma)^2 + \psi^2(\Delta\sigma)^2\right] \\
&\quad + \int_{B_{2R}} e^{\rho(x,0)}\big\{\left[|\Delta\psi|^2\sigma^2 + 4(\nabla\psi\cdot\nabla\sigma)^2\right] + \left[|\Delta\psi|^2\sigma^2 + \psi^2(\Delta\sigma)^2\right] \\
&\qquad +\left[4(\nabla\psi\cdot\nabla\sigma)^2 + (\psi^2\Delta\sigma)^2\right]\big\} \\
&= 3\int_{B_{2R}} e^{\rho(x,0)}\left[|\Delta\psi|^2\sigma^2 + 4(\nabla\psi\cdot\nabla\sigma)^2 + \psi^2(\Delta\sigma)^2\right] \\
&\le 3\int_{B_{2R}} e^{\rho(x,0)}\left[(\Delta\psi|^2 + 4|\nabla\psi|^2|\nabla\sigma|^2 + \psi^2(\Delta\sigma)^2\right] \\
&\le 3\int_{B_{2R}} e^{\rho(x,0)}|\Delta\psi|^2 + 48\int_{B_{2R}} e^{\rho(x,0)}|\nabla\psi|^2 + 12\int_{B_{2R}} e^{\rho(x,0)}\psi^2 \\
&\le 3\int_{\mathbb{R}^n} e^{\rho(x,0)}|\Delta\psi|^2 + 48\int_{\mathbb{R}^n} e^{\rho(x,0)}|\nabla\psi|^2 + 12\int_{\mathbb{R}^n} e^{\rho(x,0)}\psi^2
\end{aligned}$$

$$\begin{aligned}
\int_{B_{2R}\times[0,t]} e^{\rho}|\nabla\rho|^2 f_{2R}|^2|\nabla u_{2R}|^2 &= \int_0^t \int_{B_{2R}\times\{s\}} e^{\rho}\frac{4\tilde{c}|x|^2}{(s+\varepsilon)^2}|\sigma f|^2|\nabla u_{2R}|^2 \\
&\le \int_0^t \frac{8\tilde{c}^2c^2}{(s+\varepsilon)^2}\int_{B_{2R}\times\{s\}} e^{\rho}|x|^2(1+|x|)^2|\nabla u_{2R}|^2|)
\end{aligned}$$

Since $s+\varepsilon<\delta$, we have

$$\begin{aligned}
|x|^2(1+|x|)^2 &\le 2|x|^2 + 2|x|^4 \\
&\le e^{\frac{\delta}{s+\epsilon}}\left(1 + \frac{\delta|x|^2}{s+\varepsilon} + \frac{\delta^2|x|^4}{(s+\varepsilon)^2} + \cdots\right) \\
&= e^{\frac{\delta}{s+\epsilon}}e^{\frac{\delta|x|^2}{s+\epsilon}} = e^{\frac{\delta(1+|x|^2)}{s+\epsilon}}
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{B_{2R}\times[0,t]} e^{\rho}|\nabla\rho|^2|f_{2R}|^2|\nabla u_{2R}|^2 \\
&\le \int_0^t \frac{8\tilde{c}^2c^2}{(s+\varepsilon)^2}\int_{B_{2R}\times\{s\}} e^{\frac{\tilde{c}(1+|x|^2)}{s+\epsilon}}e^{\frac{\delta(1+|x|^2)}{s+\epsilon}}(|\nabla u_{2R}|^2) \\
&= \int_0^t \frac{8\tilde{c}^2c^2}{(s+\varepsilon)}\int_{B_{2R}\times\{s\}} e^{\tilde{\rho}(x,s)}|\nabla u_{2R}|^2 \\
&\le \int_0^t \frac{8\tilde{c}^2c^2}{(s+\varepsilon)}\int_{B_{2R}} e^{\tilde{\rho}(x,0)}|\nabla\psi_{2R}|^2
\end{aligned}$$

$$= 8\tilde{c}^2 c^2 \frac{t}{\varepsilon(T+\varepsilon)} \int_{B_{2R}} e^{\tilde{\rho}(x,0)} |(\nabla\sigma)\psi + \sigma(\nabla\psi)|^2$$
$$\le \frac{16\tilde{c}^2 c^2 t}{\varepsilon(t+\varepsilon)} \int_{B_{2R}} e^{\tilde{\rho}(x,0)} (|\nabla\sigma|^2\psi^2 + \sigma^2|\nabla\psi|^2)$$
$$\le \frac{16\tilde{c}^2 c^2 T}{\varepsilon(T+\varepsilon)} \int_{\mathbf{R}^n} e^{\tilde{\rho}(x,0)} (4\psi^2 + |\nabla\psi|^2)$$

$$\int_{B_{2R}\times[0,t]} e^{\rho}|f_{2R}|^2|\nabla u_{2R}|^2 \le \int_0^t \int_{B_{2R}\times\{s\}} e^{\rho}(\sum_{i=1}^{n} \left|(\nabla\sigma)f_i + \sigma\nabla f_i\right|^2 |\nabla u_{2R}|^2$$
$$\le \int_0^t \int_{B_{2R}\times\{s\}} e^{\rho} \sum_{i=1}^{n} (|\nabla\sigma|^2 f_i^2 + \sigma^2|\nabla f_i|^2)|\nabla u_{2R}|^2$$
$$\le \int_0^t \int_{(B_{2R}-B_R)\times\{s\}} (8e^{\rho}|f|^2 + 2e^{\rho}|\nabla f|^2)|\nabla u_{2R}|^2$$
$$\le \int_0^t \int_{(B_{2R}-B_R)\times\{s\}} (10c^2 e^{\rho}(1+|x|)^2|\nabla u_{2R}|^2$$
$$\le \int_0^t \int_{(B_{2R}-B_R)\times\{s\}} 10c^2 e^{\rho} e^{\frac{\delta(1+|x|^2)}{s+\epsilon}} |\nabla u_{2R}|^2$$
$$= \int_0^t \int_{(B_{2R}-B_R)\times\{s\}} 10c^2 e^{\tilde{\rho}} |\nabla u_{2R}|^2$$
$$\le 10c^2 \int_0^t \int_{B_{2R}} e^{\tilde{\rho}(x,0)} |\nabla \psi_{2R}|^2$$
$$\le 10c^2 T \int_{\mathbf{R}^n} e^{\tilde{\rho}(x,0)} (4\psi^2 + |\nabla\psi|^2)$$

$$\int_{B_{2R}\times[0,t]} e^{\rho}|f_{2R}|^2|\triangle f_{2R}||\nabla u_{2R}|^2 = \int_0^t \int_{B_{2R}\times\{s\}} e^{\rho}|f\sigma| \sqrt{\sum_{i=1}^{n} |\triangle(\sigma f_i)|^2} |\nabla u_{2R}|^2$$
$$\le \int_0^t \int_{(B_{2R}-B_R)\times\{s\}} e^{\rho}|f| \sqrt{\sum_{i=1}^{n} \left[(\triangle f_i)\sigma + 2(\nabla f_i)\cdot(\nabla\sigma) + f_i\triangle\sigma\right]^2} |\nabla u_{2R}|^2$$
$$\le \int_0^t \int_{(B_{2R}-B_R)\times\{s\}} \sqrt{3} e^{\rho}|f| \sqrt{\sum_{i=1}^{n} [|\triangle f_i|^2\sigma^2 + 4|\nabla f_i|^2|\nabla\sigma|^2 + |f_i|^2|\triangle\sigma|^2} |\nabla u_{2R}|^2$$
$$\le \sqrt{3} \int_0^t \int_{(B_{2R}-B_R)\times\{s\}} e^{\rho}|f|(|\sigma||\triangle f| + 4|\nabla\sigma||\nabla f| + |\triangle\sigma||f|)|\nabla u_{2R}|^2$$
$$\le \sqrt{3} \int_0^t \int_{(B_{2R}-B_R)\times\{s\}} e^{\rho} c(1+|x|)(c+16c+4c)(1+|x|)|\nabla u_{2R}|^2$$
$$= \sqrt{3} \int_0^t \int_{(B_{2R}-B_R)\times\{s\}} 21c^2 e^{\rho}(1+|x|)^2|\nabla u_{2R}|^2$$
$$\le 21\sqrt{3}c^2 \int_0^t \int_{(B_{2R}-B_R)\times\{s\}} e^{\rho} e^{\frac{\delta(1+|x|^2)}{s+\epsilon}} |\nabla u_{2R}|^2$$

$$\leq 21\sqrt{3}c^2 \int_0^t \int_{B_{2R}\times\{s\}} e^{\tilde{\rho}} |\nabla u_{2R}|^2$$

$$\leq 21\sqrt{3}c^2 \int_0^t \int_{B_{2R}} e^{\tilde{\rho}(x,0)} |\nabla \psi_{2R}|^2$$

$$\leq 21\sqrt{3}c^2 T \int_{\mathbb{R}^n} e^{\tilde{\rho}(x,0)} (4\psi^2 + |\nabla\psi|^2)$$

$$\int_{B_{2R}\times[0,t]} e^{\rho} |f_{2R}|^4 |\nabla u_{2R}|^2 = \int_0^t \int_{B_{2R}\times\{s\}} e^{\rho} c^4 (1+|x|)^4 |\nabla u_{2R}|^2$$

$$\leq \int_0^t \int_{B_{2R}\times\{s\}} e^{\rho} 8c^4 (1+|x|^4) |\nabla u_{2R}|^2$$

$$\leq \int_0^t \int_{B_{2R}\times\{s\}} 8c^4 e^{\rho} e^{\frac{\delta(1+|x|^2)}{s+\epsilon}} |\nabla u_{2R}|^2$$

$$= 8c^4 \int_0^t \int_{B_{2R}\times\{s\}} e^{\tilde{\rho}} |\nabla u_{2R}|^2$$

$$\leq 8c^4 \int_0^t \int_{B_{2R}} e^{\tilde{\rho}(x,0)} |\nabla \psi_{2R}|^2$$

$$\leq 8c^4 T \int_{\mathbb{R}^n} e^{\tilde{\rho}(x,0)} (4\psi^2 + |\nabla\psi|^2)$$

$$\int_{B_{2R}\times[0,t]} e^{\rho} |\nabla (V_{2R} u_{2R})|^2 = \int_0^t \int_{B_{2R}\times\{s\}} e^{\rho} |\nabla(\sigma V u_{2R})|$$

$$= \int_0^t \int_{B_{2R}\times\{s\}} e^{\rho} |(\nabla\sigma) V u_{2R} + \sigma(\nabla V) u_{2R} + \sigma V (\nabla u_{2R})|^2$$

$$\leq 3 \int_0^t \int_{B_{2R}\times\{s\}} e^{\rho} \left(|\nabla\sigma|^2 |V|^2 |u_{2R}|^2 + |\sigma|^2 |\nabla V|^2 |u_{2R}|^2 + |\sigma|^2 |V|^2 |\nabla u_{2R}|^2\right)$$

$$\leq 3 \int_0^t \int_{(B_{2R}-B_R)\times\{s\}} e^{\rho} \big(4c^2 (1+|x|^2)^2 |u_{2R}|^2 + c^2 (1+|x|)^2 |u_{2R}|^2 + c^2 (1+|x|^2)^2 |\nabla u_{2R}|^2\big)$$

$$\leq 3 \int_0^t \int_{(B_{2R}-B_R)\times\{s\}} e^{\rho} \big(8c^2 (1+|x|^4) ||u_{2R}|^2 + 2c^2 (1+|x|^2) |u_{2R}|^2 + 2c^2 (1+|x|^4) |\nabla u_{2R}|^2\big)$$

$$\leq 3 \int_0^t \int_{(B_{2R}-B_R)\times\{s\}} e^{\rho} \left(10c^2 (1+|x|^4) |u_{2R}|^2 + 2c^2 (1+|x|^4) |\nabla u_{2R}|^2\right)$$

$$\leq 3 \int_0^t \int_{B_{2R}\times\{s\}} e^{\rho} e^{\frac{\delta(1+|x|^2)}{s+\epsilon}} \left(10c^2 |u_{2R}|^2 + 2c^2 |\nabla u_{2R}|^2\right)$$

$$= 3 \int_0^t \int_{B_{2R}\times\{s\}} e^{\tilde{\rho}} \left(10c^2 |u_{2R}|^2 + 2c^2 |\nabla u_{2R}|^2\right)$$

$$\leq 3T \int_{B_{2R}} e^{\tilde{\rho}(x,0)} \left(10c^2 |\psi_{2R}|^2 + 2c^2 |\nabla \psi_{2R}|^2\right)$$

$$\leq 3T \int_{\mathbb{R}^n} e^{\tilde{\rho}(x,0)} (18c^2 |\psi|^2 + 2c^2 |\nabla\psi|^2)$$

$$\int_{B_{2R}\times[0,t]} e^{\rho}|\nabla u_{2R}|^2\Big(\sum_{i=1}^{n}(f_{2R})_{i,i}\Big)^2 \le \int_0^t\int_{B_{2R}\times\{s\}} ne^{\rho}|\nabla u_{2R}|^2\sum_{i=1}^{n}(f_{2R})_{i,i}^2$$
$$\le \int_0^t\int_{B_{2R}\times\{s\}} ne^{\rho}|\nabla f|^2|\nabla u_{2R}|^2$$
$$\le \int_0^t\int_{B_{2R}\times\{s\}} ne^{\rho}c^2(1+|x|)^2|\nabla u_{2R}|^2$$
$$\le \int_0^t\int_{B_{2R}\times\{s\}} 2nc^2e^{\rho}(1+|x|^2)|\nabla u_{2R}|^2$$
$$\le \int_0^t\int_{B_{2R}\times\{s\}} 2nc^2e^{\widetilde{\rho}}|\nabla u_{2R}|^2$$
$$\le \int_0^t\int_{B_{2R}} 2nc^2e^{\widetilde{\rho}(x,0)}|\nabla \psi_{2R}|^2$$
$$\le 2nc^2T\int_{\mathbf{R}^n} e^{\widetilde{\rho}(x,0)}(4\psi^2+|\nabla\psi|^2)$$

□

We see from Theorem 1.3 that for $0\le t\le T$, we have estimates of $\int_{B_{2R}\times\{t\}} e^{\rho}u_{2R}^2$, $\int_{B_{2R}\times\{t\}} e^{\rho}|\nabla u_{2R}|^2$ and $\int_{B_{2R}\times\{t\}} e^{\rho}|\Delta u_{2R}|^2$ which are independent of R. Hence we can take $R\to\infty$ and obtain a global solution u up to time T.

THEOREM 1.4. *Assume that* $|f(x,t)|\le c(1+|x|)$, $|\nabla f(x,t)|\le c(1+|x|)$, $|\Delta f(x,t)| \le c(1+|x|)$, $|V(x,t)|\le c(1+|x|^2)$ *and* $|\nabla V(x,t)|\le c(1+|x|)$ *where* $c\ge 4$ *is a constant. Let* δ *and* $\widetilde{c}$ *be positive constant such that* $\widetilde{\widetilde{c}} := \widetilde{c}+\delta < \min(\frac{1}{4}, \frac{\varepsilon_1^2}{4\varepsilon_1^2+2})$. *Choose* T *and* ε *suitably small so that* $T+\varepsilon<\delta$ *and the conclusions (i), (ii) and (iii) of Lemma 1.3 hold for both*

$$\rho(x,t)=\frac{\widetilde{c}(1+|x|^2)}{t+\varepsilon} \qquad \text{and} \qquad \widetilde{\rho}(x,t)=\frac{\widetilde{\widetilde{c}}(1+|x|^2)}{t+\varepsilon}$$

for any $0\le t\le T$. *Then for any initial data* $\psi(x)$ *with* $\int_{\mathbf{R}^n} e^{\rho}(|\psi|^2+|\nabla\psi|^2+|\Delta\psi|^2)<\infty$, *there exist a solution of the equation* $\frac{\partial u}{\partial t}=\Delta u+\sum_{i=1}^{n} f_i\frac{\partial u}{\partial x_i}-Vu$ *up to time* T *and* $u(x,0)=\psi(x)$.

2. Gradient estimate of the solution. We shall first derive the following maximal principle for the parabolic equation which we are interested in.

THEOREM 2.1. *Let* $u>0$ *be a positive solution of the equation.*

$$\frac{\partial u}{\partial t}(x,t)=\Delta u(x,t)+\sum_{i=1}^{n} f_i(x,t)\frac{\partial u}{\partial x_i}(x,t)-V(x,t)u(x,t) \tag{2.1}$$

defined on a compact domain Ω. *Let* $\varphi(x,t) = -\log u_\Omega(x,t)$ *and*

(2.2)
$$\Psi = \varphi_t + |\nabla\varphi|^2 - \sum_{j=1}^n f_j \frac{\partial\varphi}{\partial x_j} - V - \alpha(t)\sqrt{|\nabla\varphi|^2 + \beta(x,t)} - \frac{n}{2t} - 2\sqrt{n}c(t)$$

where

$$0 < \alpha(t) \le \frac{1}{2}\sqrt{\beta(x,t)}, \tag{2.3}$$
$$|\nabla f(x,t)| \le c(t) \tag{2.4}$$

Assume that there exists $c_1(t)$ *such that*

$$|f(x,t)|^2 \le c(t)^2(1+|x|)^2 \le c_1(t)\beta(x,t) \tag{2.5}$$
$$|\nabla\beta(x,t)| \le c_1(t)\sqrt{\beta(x,t)} \tag{2.6}$$

Suppose $\psi < 0$ *on* $\Omega \times [0, t_0)$ *and the following inequalities hold*

$$\frac{3\beta}{2t} + \frac{\beta_t}{2} - \frac{n}{4t^2} > 0 \tag{2.7}$$
$$\alpha'(t) + \frac{3\alpha c}{2\sqrt{n}} - \alpha\lambda - \sqrt{n}|\Delta f| - \frac{\alpha|\nabla V|}{2\sqrt{\beta}} - \frac{1}{2}(c_1 + c_1^{3/2})\alpha \ge 0 \tag{2.8}$$
$$2\sqrt{n}c'(t) + \frac{3c^2}{8} - \Delta V - \frac{1}{6}\sum_{i,j=1}^n \left(\frac{f_{i,j} + f_{j,i}}{2}\right)^2 + \frac{3\sqrt{n}c}{4t} - \frac{\alpha\Delta\beta}{\sqrt{\beta}} \ge 0 \tag{2.9}$$

where λ *is the absolute value of the greatest eigenvalue of* $\left(\frac{f_{i,j} + f_{i,i}}{2}\right)$. *Then* Ψ *cannot have an interior maximum with* $\Psi = 0$ *at* $t = t_0$.

Proof.

$$\frac{\partial\varphi}{\partial t} = -\frac{1}{u}\log u, \qquad \frac{\partial\varphi}{\partial x_i} = -\frac{1}{u}\frac{\partial u}{\partial x_i},$$
$$\frac{\partial^2\varphi}{\partial x_i^2} = \frac{1}{u^2}\left(\frac{\partial u}{\partial x_i}\right)^2 - \frac{1}{u}\frac{\partial^2 u}{\partial x_i^2}$$

Hence

$$\Delta\varphi - |\nabla\varphi|^2 = \frac{1}{u^2}|\nabla u|^2 - \frac{1}{u}\Delta u - \frac{1}{u^2}|\nabla u|^2 = -\frac{1}{u}\Delta u$$

$$\begin{aligned}\varphi_t &= -\frac{1}{u}u_t = -\frac{1}{u}\left(\Delta u + \sum_{i=1}^n f_i u_i - Vu\right) \\ &= \Delta\varphi - |\nabla\varphi|^2 + \sum_{i=1}^n f_i\varphi_i + V\end{aligned} \tag{2.10}$$

$$\Psi_t = \varphi_{tt} + 2\sum_{j=1}^n (\varphi_j)_t\varphi_j - \sum_{j=1}^n (f_i)_t\varphi_j - \sum_{j=1}^n f_j(\varphi_j)_t - V_t - \alpha_t\sqrt{|\nabla\varphi|^2 + \beta}$$

$$-\frac{\alpha}{2}\Big[\sum_{j=1}^{n} 2\varphi_j(\varphi_j)_t + \beta_t\Big](|\nabla\varphi|^2+\beta)^{-\frac{1}{2}} + \frac{n}{2t^2} - 2\sqrt{n}c_t$$

$$\Psi_i = (\varphi_i)_t + 2\sum_{j=1}^{n}\varphi_{ji}\varphi_j - \sum_{j=1}^{n} f_{j,i}\varphi_j - \sum_{j=1}^{n} f_j\varphi_{ji} - V_i$$
$$-\frac{\alpha}{2}\Big[2\sum_{j=1}^{n}\varphi_j\varphi_{ji} + \beta_i\Big](|\nabla\varphi|^2+\beta)^{-\frac{1}{2}}$$

$$\Psi_{ii} = (\varphi_{ii})_t + 2\sum_{j=1}^{n}\varphi_{ji}^2 + 2\sum_{j=1}^{n}\varphi_{jii}\varphi_j - \sum_{j=1}^{n} f_{j,ii}\varphi_j - 2\sum_{j=1}^{n} f_{j,i}\varphi_{ji}$$
$$-\sum_{j=1}^{n} f_j\varphi_{jii} - V_{ii} - \frac{\alpha}{2}\Big[2\sum_{j=1}^{n}\varphi_{ji}^2 + 2\sum_{j=1}^{n}\varphi_j\varphi_{jii} + \beta_{ii}\Big](|\nabla\varphi|^2+\beta)^{-\frac{1}{2}}$$
$$+\frac{\alpha}{4}\Big(2\sum_{j=1}^{N}\varphi_j\varphi_{ji} + \beta_i\Big)^2(|\nabla\varphi|^2+\beta)^{-3/2}$$

$$\Delta\Psi = (\Delta\varphi)_t + 2\sum_{i,j=1}^{n}\varphi_{ji}^2 + 2\sum_{j=1}^{n}(\Delta\varphi)_j\varphi_j - \sum_{j=1}^{n}(\Delta f_j)\varphi_j - 2\sum_{i,j=1}^{n} f_{j,i}\varphi_{ji}$$
$$-\sum_{j=1}^{n} f_j\Delta\varphi_j - \Delta V - \Big[\alpha\sum_{i,j=1}^{n}\varphi_{ji}^2 + \alpha\sum_{j=1}^{n}\varphi_j\Delta\varphi_j + \frac{\alpha}{2}\Delta\beta\Big](|\nabla|^2+\beta)^{-\frac{1}{2}}$$
$$+\alpha\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}\varphi_j\varphi_{ji} + \frac{\beta_i}{2}\Big)^2(|\nabla\varphi|^2+\beta)^{-3/2}$$

By computation, we have

$$\text{(2.11)}\qquad \Psi_t - \Delta\Psi$$
$$= -\Big(|\nabla\varphi|^2 - \sum_{i=1}^{n} f_i\varphi_i - V\Big)_t - 2\sum_{i,j=1}^{n}\varphi_{ji}^2 - 2\sum_{j=1}^{n}\varphi_j\Big(|\nabla\varphi|^2 - \sum_{i=1}^{n} f_i\varphi_i - V\Big)_j$$
$$+2\sum_{i,j=1}^{n} f_{j,i}\varphi_{ji} + \sum_{j=1}^{n} f_j\Big(|\nabla\varphi|^2 - \sum_{i=1}^{n} f_i\varphi_i - V\Big)_j + \sum_{j=1}^{n}\Big[-(f_j)_t + \Delta f_j\Big]\varphi_j$$
$$+\Delta V - V_t + \frac{\alpha\sum_{j=1}^{n}\varphi_j(|\nabla\varphi|^2 - \sum_{i=1}^{n} f_i\varphi_i - V)_j}{\sqrt{|\nabla\varphi|^2+\beta}} + \frac{\alpha\sum_{i,j=1}^{n}\varphi_{ij}^2}{\sqrt{|\nabla\varphi|^2+\beta}}$$
$$-\frac{\frac{\alpha}{2}(\beta_t - \Delta\beta)}{\sqrt{|\nabla\varphi|^2+\beta}} - \frac{\alpha\sum_{i=1}^{n}(\sum_{j=1}^{n}\varphi_j\varphi_{ji} + \frac{\beta_i}{2})^2}{(|\nabla\varphi|^2+\beta)^{3/2}} - \alpha_t\sqrt{|\nabla\varphi|^2+\beta} + \frac{n}{2t^2} - 2\sqrt{n}c_t$$
$$= -2\sum_{j=1}^{n}(\varphi_t)_j\varphi_j + \sum_{i=1}^{n}(f_i)_t\varphi_i + \sum_{i=1}^{n} f_i(\varphi_t)_i + V_t - 2\sum_{i,j=1}^{n}\varphi_{ji}^2$$
$$-2\sum_{j=1}^{n}\varphi_j\Big(\Psi - \varphi_t + \alpha\sqrt{|\nabla\varphi|^2+\beta} + \frac{n}{2t} + c\Big)_j + 2\sum_{i,j=1}^{n} f_{ji}\varphi_{ji}$$
$$+\sum_{j=1}^{n} f_j\Big(\Psi - \varphi_t + \alpha\sqrt{|\nabla\varphi|^2+\beta} + \frac{n}{2t} + c\Big)_j + \sum_{j=1}^{n}\Big[-(f_j)_t + \Delta f_j\Big]\varphi_j$$

$$+\Delta V - V_t + \frac{\alpha\sum_{j=1}^n \varphi_j(|\nabla\varphi|^2 - \sum_{j=1}^n f_i\varphi_i - V)_j}{\sqrt{|\nabla\varphi|^2+\beta}} + \frac{\alpha\sum_{i,j=1}^n \varphi_{ij}^2}{\sqrt{|\nabla\varphi^2|+\beta}}$$

$$-\frac{\frac{\alpha}{2}(\beta_t-\Delta\beta)}{\sqrt{|\nabla\varphi|^2+\beta}} - \frac{\alpha\sum_{i=1}^n(\sum_{j=1}^n\varphi_j\varphi_{ji}+\frac{\beta_i}{2})^2}{(|\nabla\varphi|^2+\beta)^{3/2}} - (\alpha_t - \Delta\alpha)\sqrt{|\nabla\varphi|^2+\beta}$$

$$+\frac{n}{2t^2} - 2\sqrt{n}c_t$$

$$= -2\sum_{i,j=1}^n \varphi_{ji}^2 - 2\sum_{j=1}^n \varphi_j\Psi_j - \alpha\sum_{j=1}^n\varphi_j(\sum_{i=1}^n 2\varphi_i\varphi_{ij}+\beta_j)(|\nabla\varphi|^2)+\beta)^{-\frac{1}{2}}$$

$$+2\sum_{i,j=1}^n f_{j,i}\varphi_{ji} + \sum_{j=1}^n f_j\Psi_j$$

$$+\frac{\alpha}{2}\sum_{j=1}^n f_j(\sum_{i=1}^n 2\varphi_i\varphi_{ij}+\beta_j)(|\nabla\varphi|^2+\beta)^{-\frac{1}{2}} + \sum_{j=1}^n(\Delta f_j)\varphi_j + \Delta V$$

$$+\frac{\alpha\sum_{j=1}^n\varphi_j(\sum_{i=1}^n 2\varphi_i\varphi_{ij} - \sum_{i=1}^n f_{i,j}\varphi_i - \sum_{i=1}^n f_i\varphi_{ij} - V_j)}{\sqrt{|\nabla\varphi|^2+\beta}} + \frac{\alpha\sum_{i,j=1}^n\varphi_{ij}^2}{\sqrt{|\nabla\varphi|^2+\beta}}$$

$$-\frac{\frac{\alpha}{2}(\beta_t-\Delta\beta)}{\sqrt{|\nabla\varphi|^2+\beta}} - \frac{\alpha\sum_{i=1}^n(\sum_{j=1}^n\varphi_j\varphi_{ji}+\frac{\beta_i}{2})^2}{(|\nabla\varphi|^2+\beta)^{3/2}} - \alpha_t\sqrt{|\nabla\varphi|^2+\beta} + \frac{n}{2t^2} - 2\sqrt{n}C_t$$

$$= -2\sum_{j=1}^n\varphi_j\Psi_j + \sum_{j=1}^n f_j\Psi_j - 2\sum_{i,j=1}^n\varphi_{ji}^2 + 2\sum_{i,j=1}^n f_{j,i}\varphi_{ji} + \sum_{j=1}^n(\Delta f_j)\varphi_j$$

$$+\Delta V - \frac{\alpha\sum_{i,j=1}^n f_{i,j}\varphi_i\varphi_j + \alpha\sum_{j=1}^n\varphi_jV_j}{\sqrt{|\nabla\varphi|^2+\beta}} + \frac{\alpha\sum_{i,j=1}^n\varphi_{ij}^2}{\sqrt{|\nabla\varphi|^2+\beta}} - \frac{\frac{\alpha}{2}(\beta_t-\Delta\beta)}{\sqrt{|\nabla\varphi|^2+\beta}}$$

$$-\frac{\alpha\sum_{i=1}^n(\sum_{j=1}^n\varphi_i\varphi_{ji}+\frac{\beta_i}{2})^2}{(|\nabla\varphi|^2+\beta)^{3/2}} - \alpha_t\sqrt{|\nabla\varphi|^2+\beta} + \frac{n}{2t^2} - 2\sqrt{n}c_t$$

$$-\frac{\alpha\sum_{j=1}^n\varphi_j\beta_j}{\sqrt{|\nabla\varphi|^2+\beta}} + \frac{\frac{\alpha}{2}\sum_{j=1}^n f_j\beta_j}{\sqrt{|\nabla\varphi|^2+\beta}}$$

$$= -2\sum_{j=1}^n\varphi_j\Psi_j + \sum_{j=1}^n f_j\Psi_j - 2\sum_{i,j=1}^n\varphi_{ji}^2 + 2\sum_{i,j=1}^n f_{j,i}\varphi_{ji} + \sum_{j=1}^n(\Delta f_j)\varphi_j + \Delta V$$

$$-\frac{\alpha\sum_{i,j=1}^n f_{i,j}\varphi_i\varphi_j + \alpha\sum_{j=1}^n\varphi_jV_j}{\sqrt{|\nabla\varphi|^2+\beta}} + \frac{\alpha\sum_{i,j=1}^n\varphi_{ij}^2}{\sqrt{|\nabla\varphi|^2+\beta}} - \frac{\frac{\alpha}{2}(\beta_t-\Delta\beta)}{\sqrt{|\nabla\varphi|^2+\beta}}$$

$$-\frac{\alpha\sum_{i=1}^n(\sum_{j=1}^n\varphi_j\varphi_{ji}+\frac{\beta_i}{2})^2}{(|\nabla\varphi|^2+\beta)^{3/2}} - \alpha_t\sqrt{|\nabla\varphi|^2+\beta} + \frac{n}{2t^2} - 2\sqrt{n}c_t$$

$$-\frac{\alpha\sum_{j=1}^n\varphi_j\beta_j}{\sqrt{|\nabla\varphi|^2+\beta}} + \frac{\frac{\alpha}{2}\sum_{j=1}^n f_j\beta_j}{\sqrt{|\nabla\varphi|^2+\beta}}$$

Recall that $\Psi < 0$ for $0 \le t < t_0$. If $\Psi = 0$ at some interior point when $t = t_0$, then at such a point, we have

(2.12) $\quad \Psi_t(x_0,t_0) \ge 0, \qquad \nabla\Psi(x_0,t_0) = 0, \qquad \Delta\Psi(x_0,t_0) \le 0$

(2.13) $\quad \Delta\varphi(x_0,t_0) = \alpha(x_0,t_0)\sqrt{|\nabla\varphi(x_0,t_0)|^2+\beta(x_0,t_0)} + \dfrac{n}{2t_0} + 2\sqrt{n}c(t_0)$

(2.13) follows from (2.2) and (2.10).

In view of the arithmetic-geometric inequality $\sum_{i=1}^{n} a_i^2 \geq \frac{1}{n}(\sum_{i=1}^{n} a_i)^2$, we have at the point (x_0, t_0).

$$(2.14)\qquad (2-\frac{\alpha}{\sqrt{|\nabla\varphi|^2+\beta}})\sum_{i,j=1}^{n}\left[\varphi_{ij}-\frac{1}{2}(2-\frac{\alpha}{\sqrt{|\nabla\varphi|^2+\beta}})^{-1}(f_{i,j}+f_{j,i})\right]^2$$

$$\geq \frac{1}{n}(2-\frac{\alpha}{\sqrt{|\nabla\varphi|^2+\beta}})\left[\Delta\varphi-(2-\frac{\alpha}{\sqrt{|\nabla\varphi|^2+\beta}})^{-1}\sum_{i=1}^{n} f_{i,i}\right]^2$$

$$= \frac{1}{n}(2-\frac{\alpha}{\sqrt{|\nabla\varphi|^2+\beta}})\left[\alpha\sqrt{|\nabla\varphi|^2+\beta}+\frac{n}{2t}+2\sqrt{n}c-(2-\frac{\alpha}{\sqrt{|\nabla\varphi|^2+\beta}})^{-1}\sum_{i=1}^{n} f_{i,i}\right]^2$$

$$= \frac{1}{n}(2-\frac{\alpha}{\sqrt{|\nabla\varphi|^2+\beta}})\left[\alpha^2(|\nabla\varphi|^2+\beta)+\frac{n}{t}\alpha\sqrt{|\nabla\varphi|^2+\beta}+\frac{n^2}{4t^2}\right.$$

$$+(2\sqrt{n}c-(2-\frac{\alpha}{\sqrt{|\nabla\varphi|^2+\beta}})^{-1}\sum_{i=1}^{n} f_{i,i})^2+(2\alpha\sqrt{|\nabla\varphi|^2+\beta}+\frac{n}{t})$$

$$\left.(2\sqrt{n}c-(2-\frac{\alpha}{\sqrt{|\nabla\varphi|^2+\beta}})^{-1}\sum_{i=1}^{n} f_{i,i})\right]^2$$

On the other hand from (2.11) and the fact that $\psi_t-\Delta\psi\geq 0$ at the point (x_0, t_0), we have

$$(2.15)\qquad (2-\frac{\alpha}{\sqrt{|\nabla\varphi|^2+\beta}})\sum_{i,j=1}^{n}\left[\varphi_{ij}-\frac{1}{2}(2-\frac{\alpha}{\sqrt{|\nabla\varphi|^2+\beta}})^{-1}(f_{i,j}+f_{j,i})\right]^2$$

$$\leq \sum_{j=1}^{n}(\Delta f_j)\varphi_j+\Delta V+\frac{1}{4}(2-\frac{\alpha}{\sqrt{|\nabla\varphi|^2+\beta}})^{-1}\sum_{i,j=1}^{n}(f_{i,j}+f_{j,i})^2$$

$$-\frac{\alpha\sum_{i,j=1}^{n} f_{i,j}\varphi_i\varphi_j+\alpha\sum_{j=1}^{n}\varphi_j V_j}{\sqrt{|\nabla\varphi|^2+\beta}}-\frac{\frac{\alpha}{2}(\beta_t-\Delta\beta)}{\sqrt{|\nabla\varphi|^2+\beta}}$$

$$-\frac{\alpha\sum_{i=1}^{n}(\sum_{j=1}^{n}\varphi_j\varphi_{ji}+\frac{\beta_i}{2})^2}{(|\nabla\varphi|^2+\beta)^{3/2}}-\alpha_t\sqrt{|\nabla\varphi|^2+\beta}+\frac{n}{2t^2}-2\sqrt{n}c_t$$

$$-\frac{\alpha\sum_{j=1}^{n}\varphi_j\beta_j}{\sqrt{|\nabla\varphi|^2+\beta}}+\frac{\frac{\alpha}{2}\sum_{j=1}^{n} f_j\beta_j}{\sqrt{|\nabla\varphi|^2+\beta}}$$

$$\leq \sum_{j=1}^{n}(\Delta f_j)\varphi_j+\Delta V+\frac{1}{4}(2-\frac{\alpha}{\sqrt{|\nabla\varphi|^2+\beta}})^{-1}\sum_{i,j=1}^{n}(f_{i,j}+f_{j,i})^2$$

$$-\frac{\alpha\sum_{i,j=1}^{n} f_{i,j}\varphi_i\varphi_j+\alpha\sum_{j=1}^{n}\varphi_j V_j}{\sqrt{|\nabla\varphi|^2+\beta}}-\frac{\frac{\alpha}{2}(\beta_t-\Delta\beta)}{\sqrt{|\nabla\varphi|^2+\beta}}-\alpha_t\sqrt{|\nabla\varphi|^2+\beta}$$

$$+\frac{n}{2t^2}-2\sqrt{n}c_t-\frac{\alpha\sum_{j=1}^{n}\varphi_j\beta_j}{\sqrt{|\nabla\varphi|^2+\beta}}+\frac{\frac{\alpha}{2}\sum_{j=1}^{n} f_j\beta_j}{\sqrt{|\nabla\varphi|^2+\beta}}$$

since $\alpha>0$ by assumption. Equation (2.14) and (2.15) imply

$$(2.16)\qquad \frac{1}{n}(2-\frac{\alpha}{\sqrt{|\nabla\varphi|^2+\beta}})\Big[\alpha^2(|\nabla\varphi|^2+\beta)+\frac{n\alpha}{t}\sqrt{|\nabla\varphi|^2+\beta}+\frac{n^2}{4t^2}$$

$$+\ (2\sqrt{n}c-(2-\frac{\alpha}{\sqrt{|\nabla\varphi|^2+\beta}})^{-1}\sum_{i=1}^{n}f_{i,i})^2+(2\alpha\sqrt{|\nabla\varphi|^2+\beta}+\frac{n}{t})(2\sqrt{n}c$$

$$-(2-\frac{\alpha}{\sqrt{|\nabla\varphi|^2+\beta}})^{-1}\sum_{i=1}^{n}f_{i,i})\Big]$$

$$\leq\sum_{j=1}^{n}(\Delta f_j)\varphi_j+\Delta V+\frac{1}{4}(2-\frac{\alpha}{\sqrt{|\nabla\varphi|^2+\beta}})^{-1}\sum_{i,j=1}^{n}(f_{i,j}+f_{j,i})^2$$

$$-\frac{\alpha\sum_{i,j=1}^{n}f_{i,j}\varphi_i\varphi_j+\alpha\sum_{j=1}^{n}\varphi_jV_j}{\sqrt{|\nabla\varphi|^2+\beta}}-\frac{\alpha(\beta_t-\Delta\beta)}{2\sqrt{|\nabla\varphi|^2+\beta}}-\alpha_t\sqrt{|\nabla\varphi|^2+\beta}$$

$$+\frac{n}{2t^2}-2\sqrt{n}c_t-\frac{\alpha\sum_{j=1}^{n}\varphi_j\beta_j}{\sqrt{|\nabla\varphi|^2+\beta}}+\frac{\alpha\sum_{j=1}^{n}f_j\beta_j}{2\sqrt{|\nabla\varphi|^2+\beta}}$$

Let $z=\sqrt{|\nabla\varphi|^2+\beta}$. In view of (2.3), we have

$$2-\frac{\alpha}{z}\geq\frac{3}{2}$$

Observe that

$$|\nabla f|=\sqrt{\sum_{i=1}^{n}|\nabla f_i|^2}\geq\sqrt{\sum_{i=1}^{n}(\frac{\partial f_i}{\partial x_i})^2}\geq\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}$$

Hence

$$2\sqrt{n}c-(2-\frac{\alpha}{z})^{-1}\sum_{i=1}^{n}f_{i,i}\geq2\sqrt{n}c-\frac{3}{2}\sqrt{n}|\nabla f|$$

$$\geq2\sqrt{n}c-\frac{3}{2}\sqrt{n}c=\frac{\sqrt{n}}{2}c$$

The left hand side of (2.16) can be estimated from below as follow

$$\frac{1}{n}(2-\frac{\alpha}{z})\big[\alpha z^2+\frac{n}{t}\alpha z+\frac{n^2}{4t^2}+(2\sqrt{n}c-(2-\frac{\alpha}{z})^{-1}\sum_{i=1}^{n}f_{i,i})^2$$

$$+(2\alpha z+\frac{n}{t})(2\sqrt{n}c-(2-\frac{\alpha}{z})^{-1}\sum_{i=1}^{n}f_{i,i})]$$

$$\geq\frac{3}{2n}\big[\alpha^2z^2+\frac{n\alpha z}{t}+\frac{n}{4}c^2+(2\alpha z+\frac{n}{t})\frac{\sqrt{n}}{2}c\big]+\frac{1}{n}(2-\frac{\alpha}{z})\frac{n^2}{4t^2}$$

$$=\frac{3\alpha^2z^2}{2n}+\frac{3\alpha z}{2t}+\frac{3}{8}c^2+\frac{3}{2\sqrt{n}}\alpha zc+\frac{3\sqrt{n}\,c}{4\ \ t}+\frac{n}{2t^2}-\frac{n\alpha}{4t^2z}$$

(2.6) implies

$$2|\nabla\varphi||\Delta\beta|\leq2c_1|\nabla\varphi|\sqrt{\beta}\leq c_1(|\nabla\varphi|^2+\beta)$$

which is equivalent to

$$\frac{|\nabla\varphi||\nabla\beta|}{|\nabla\varphi|^2+\beta}\leq\frac{c_1}{2}$$

In view of (2.5), (2.6) and the above inequality, the right hand side of (2.16) can be estimated from above as follow.

$$
\begin{aligned}
&\sum_{j=1}^{n}(\Delta f_j)\varphi_j+\Delta V+\frac{1}{4}(2-\frac{\alpha}{z})^{-1}\sum_{i,j=1}^{n}(f_{i,j}+f_{j,i})^2\\
&-\frac{1}{z}(\alpha\sum_{i,j=1}^{n} f_{i,j}\varphi_i\varphi_j+\alpha\sum_{j=1}^{n}\varphi_jV_j)-\frac{\alpha}{2z}(\beta_z-\Delta\beta)-\alpha_t z\\
&+\frac{n}{2t^2}-2\sqrt{n}c_t-\frac{\alpha}{z}\sum_{j=1}^{n}\varphi_j\beta_j+\frac{\alpha}{2z}\sum_{j=1}^{n}f_j\beta_j\\
\le&\sqrt{n}[\sum_{j=1}^{n}(\Delta f_j)^2]^{\frac{1}{2}}z+\Delta V+\frac{1}{4}(2-\frac{\alpha}{z})^{-1}\sum_{i,j=1}^{n}(f_{i,j}+f_{j,i})^2+\frac{\lambda\alpha|\nabla\varphi|^2}{z}\\
&+\frac{\alpha}{z}|\nabla\varphi||\nabla V|-\frac{\alpha(\beta_t-\Delta\beta)}{2z}-\alpha_t z+\frac{n}{2t^2}-2\sqrt{n}c_t\\
&+\frac{\alpha}{z}|\nabla\varphi||\nabla\beta|+\frac{\alpha}{2z}|f||\nabla\beta|\\
\le&\sqrt{n}[\sum_{j=1}^{n}(\Delta f_j)^2]^{\frac{1}{2}}z+\Delta V+\frac{1}{6}\sum_{i,j=1}^{n}(f_{i,j}+f_{j,i})^2+\lambda\alpha z+\frac{\alpha|\nabla\varphi|}{2\sqrt{\beta}}\\
&-\frac{\alpha(\beta_t-\Delta\beta)}{2z}-\alpha'(t)z+\frac{n}{2t^2}-2\sqrt{n}c'(t)+\frac{c_1(t)}{2}\alpha z\\
&+\frac{\alpha z}{2}\frac{|f||\nabla\beta|}{|\nabla\varphi|^2+\beta}\\
\le&\sqrt{n}|\Delta f|z+\Delta V+\frac{1}{6}\sum_{i,j=1}^{n}(f_{i,i}+f_{j,i})^2+\lambda\alpha z+\frac{\alpha|\nabla V|}{2\sqrt{\beta}}\\
&-\frac{\alpha}{2z}(\beta_t-\Delta\beta)-\alpha'(t)z+\frac{n}{2t^2}-2\sqrt{n}c'(t)+\frac{c_1(t)}{2}\alpha z\\
&+\frac{c_1(t)^{3/2}}{2}\alpha z
\end{aligned}
$$

Putting the upper estimate of R.H.S. of (2.16) and lower estimate of L.H.S. of (2.16) together, we have

$$
\begin{aligned}
(2.17)\qquad &\frac{3\alpha^2z^2}{2n}+\frac{3\alpha z}{2t}-\frac{n\alpha}{4t^2z}+\frac{3}{8}c^2+\frac{3}{2\sqrt{n}}\alpha zc+\frac{3\sqrt{n}}{4}\frac{c}{t}\\
&\le\sqrt{n}|\Delta f|z+\Delta V+\frac{1}{6}\sum_{i,j=1}^{n}(f_{i,j}+f_{j,i})^2+\lambda\alpha z+\frac{\alpha|\nabla V|}{2\sqrt{\beta}}\\
&-\frac{\alpha}{2z}(\beta_t-\Delta\beta)-\alpha'(t)z+\frac{c_1+c_1^{3/2}}{2}\alpha z-2\sqrt{n}c'(t)
\end{aligned}
$$

(2.17) can be rewritten in the following form

$$
\begin{aligned}
&\frac{3\alpha^2}{2n}z^2+\big(\alpha'(t)z+\frac{3\alpha zc}{2\sqrt{n}}-\lambda\alpha z-\sqrt{n}|\Delta f|z-\frac{\alpha|\nabla V|}{2\sqrt{\beta}}-\frac{c_1+c_1^{3/2}}{2}\alpha z)\\
&+(2\sqrt{n}c'(t)+\frac{3}{8}c^2-\Delta V-\frac{1}{6}\sum_{i,j=1}^{n}(f_{i,j}+f_{j,i})^2+\frac{3\sqrt{n}}{4}\frac{c}{t}-\frac{\alpha\Delta\beta}{2z})
\end{aligned}
$$

$$+\left(\frac{3\alpha z}{2t}+\frac{\alpha\beta_t}{2z}-\frac{n\alpha}{4t^2 z}\right)\le 0$$

The above inequality implies

$$\begin{aligned}(2.18)\qquad &\frac{3\alpha^2}{2n}z^2+z\big(\alpha'(t)+\frac{3\alpha c}{2\sqrt{n}}-\lambda\alpha-\sqrt{n}|\Delta f|-\frac{\alpha|\nabla V|}{2\sqrt{\beta}}-\frac{c_1+c_1^{3/2}}{2}\alpha\big)\\ &+\big(2\sqrt{n}c'(t)\big)+\frac{3}{8}c^2-\Delta V-\frac{1}{6}\sum_{i,j=1}^{n}(f_{i,j}+f_{j,i})^2+\frac{3\sqrt{n}}{4}\frac{c}{t}-\frac{\alpha\Delta\beta}{\sqrt{\beta}}\\ &+\frac{1}{z}\left(\frac{3\alpha\beta}{2t}+\frac{\alpha\beta_t}{2}-\frac{n\alpha}{4t^2}\right)\le 0\end{aligned}$$

However (2.18) contradicts with (2.3), (2.7), (2.8) and (2.9). Therefore we conclude that Ψ cannot have an interior maximum with $\Psi=0$ at $t=t_0$. □

LEMMA 2.1. *Let $u_\Omega>0$ be a positive solution of the equation (2.1) defined on a compact domain $\Omega=\{x\in\mathbb{R}^n:\theta(x)\le 0\}$ with the Neumann condition*

$$\frac{\partial u_\Omega}{\partial\nu}=0\quad on\quad\partial\Omega$$

where $\nu=\nabla\theta$ is an outward normal of $\partial\Omega$. Let $\varphi(x,t)=-\log u_\Omega(x,t)$ and $\Psi(x,t)$ be defined in (2.2). If $\frac{\partial\Psi}{\partial\nu}(x_0,t_0)\ge 0$ for $x_0\in\partial\Omega$, then at (x_0,t_0)

$$\begin{aligned}(2.19)\qquad \frac{\partial V}{\partial\nu}\le&-2\Big(1-\frac{\alpha(t)}{2\sqrt{|\nabla\varphi|^2+\beta}}\Big)\sum_{i,j=1}^{n}\theta_{ij}\varphi_i\varphi_j-\sum_{i,j=1}^{n}\theta_i\frac{\partial f_j}{\partial x_i}\varphi_j\\ &+\sum_{i,j=1}^{n}\theta_{ij}f_j\varphi_i-\frac{\alpha(t)\sum_{i=1}^{n}\theta_i\beta_i}{2\sqrt{|\nabla\varphi|^2+\beta}}\end{aligned}$$

Proof. Since

$$\frac{\partial\varphi}{\partial\nu}=-\frac{\partial}{\partial\nu}(\log u)=-\frac{1}{u}\frac{\partial u}{\partial\nu}=0\qquad\text{on}\qquad\partial\Omega\,,$$

we have $\frac{\partial\varphi_t}{\partial\nu}=0$ on $\partial\Omega$. On the other hand,

$$\begin{aligned}&\frac{\partial\varphi}{\partial\nu}=0\quad\text{on}\quad\partial\Omega\\ \Longleftrightarrow&\sum_{i=1}^{n}\frac{\partial\theta}{\partial x_i}\frac{\partial\varphi}{\partial x_i}=0\quad\text{on}\quad\partial\Omega\\ \Longrightarrow&\sum_{i=1}^{n}\theta_{ij}\varphi_i+\sum_{i=1}^{n}\theta_i\varphi_{ij}=0\quad\text{on}\quad\partial\Omega\quad\text{for all}\quad 1\le j\le n.\end{aligned}$$

Observe that

$$\begin{aligned}\frac{\partial}{\partial\nu}\Big(\sum_{i=1}^{n}\varphi_i^2\Big)&=\sum_{j=1}^{n}\frac{\partial\theta}{\partial x_j}\frac{\partial}{\partial x_j}\Big(\sum_{i=1}^{n}\varphi_i^2\Big)=2\sum_{i=1}^{n}\sum_{j=1}^{n}\theta_j\varphi_i\varphi_{ij}\\ &=-2\sum_{i=1}^{n}\sum_{j=1}^{n}\theta_{ij}\varphi_i\varphi_j\end{aligned}$$

Therefore, at $(x_0,t_0), x_0 \in \partial\Omega$, we have

$$0\le \frac{\partial}{\partial\nu}\Big[\varphi_t+|\nabla\varphi|^2-\sum_{j=1}^{2} f_j\varphi_j - V - \alpha(t)\sqrt{|\nabla\varphi|^2+\beta}-\frac{n}{2t}-2\sqrt{n}\,c(t)\Big]$$

$$= -2\sum_{i,j=1}^{n}\theta_{ij}\varphi_i\varphi_j - \sum_{i,j=1}^{n}\theta_i\frac{\partial f_j}{\partial x_i}\varphi_j - \sum_{i,j=1}^{n}\theta_i f_j\varphi_{ji} - \frac{\partial V}{\partial\nu}$$

$$-\frac{\alpha(t)\Big[-2\sum_{i,j=1}^{n}\theta_{ij}\varphi_i\varphi_j+\sum_{i=1}^{n}\theta_i\beta_i\Big]}{2\sqrt{|\nabla\varphi|^2+\beta}}$$

$$= -2\Big(1-\frac{\alpha(t)}{2\sqrt{|\nabla\varphi|^2+\beta}}\Big)\sum_{i,j=1}^{n}\theta_{ij}\varphi_i\varphi_j - \sum_{i,j=1}^{n}\theta_i\frac{\partial f_j}{\partial x_i}\varphi_j + \sum_{i,j=1}^{n}\theta_{ij}f_j\varphi_i$$

$$-\frac{\partial V}{\partial\nu} - \frac{\alpha(t)\sum_{i=1}^{n}\theta_i\beta_i}{2\sqrt{|\nabla\varphi|^2+\beta}}$$

(2.19) follows immediately from the above inequality. □

LEMMA 2.2. *Let $u_R > 0$ be a positive solution of the equation (2.1) defined on the closed ball $B_R = \{x\in\mathbb{R}^n : \theta(x) = |x| - R \le 0\}$ with the Neumann condition*

$$\frac{\partial u_R}{\partial\nu} = 0 \quad \text{on} \quad \partial B_R \quad \text{where } \nu = \nabla\theta$$

Suppose $|f| \le c(t)(1+|x|)$, $|\nabla f| = \sqrt{\sum_{i=1}^{n}|\nabla f_i|^2} \le c(t)$, $0<\alpha(t)\le \frac{1}{2}\sqrt{\beta(x,t)}$ and $|\nabla\beta| \le c_1(t)\sqrt{\beta(x,t)}$. Let $\varphi(x,t) = -\log u_R(x,t)$ and $\Psi(x,t)$ be defined in (2.2). If $\frac{\partial\Psi}{\partial\nu}\ge 0$ at (x_0,t_0) with $x_0\in\partial B_R$, then

$$\frac{\partial V}{\partial\nu} \le \frac{R}{6}\Big(1+\frac{\sqrt{n-1}}{R}\Big)^2 + \frac{\alpha(t)}{2}c_1(t) \text{ at } (x_0,t_0)\ . \tag{2.20}$$

Proof. Since $\theta = |x| - R$, we have

$$\theta_i = \frac{x_i}{|x|}\ ,\ \nu = (\theta_1,\cdots,\theta_n) \text{ and } \theta_{ij} = \frac{\delta_{ij}}{|x|} - \frac{x_ix_j}{|x|^3}$$

The last equality implies

$$\sum_{i,j=1}^{n}\theta_{ij}\varphi_i\varphi_j = \frac{1}{R}|\nabla\varphi|^2 - \frac{\big(\sum_{i-1}^{n}x_i\varphi_i\big)^2}{R^3} \quad \text{on} \quad \partial B_R$$

$$= \frac{1}{R}|\nabla\varphi|^2 - \frac{1}{R}\Big|\frac{\partial\varphi}{\partial\nu}\Big|^2 \quad \text{on} \quad \partial B_R$$

$$= \frac{1}{R}|\nabla\varphi| \quad \text{on} \quad \partial B_R$$

Observe that

$$\begin{aligned}
\Big|\sum_{j=1}^{n}\big(\sum_{i=1}^{n}\theta_i\frac{\partial f_j}{\partial x_i}\big)\varphi_j\Big| &\le \sqrt{\sum_{j=1}^{n}\big(\sum_{i=1}^{n}\theta_i\frac{\partial f_j}{\partial x_i}\big)^2}\ |\nabla\varphi| \\
&\le \sqrt{\sum_{j=1}^{n}\big(\sum_{i=1}^{n}\theta_i^2\big)\big(\sum_{i=1}^{n}\big(\frac{\partial f_j}{\partial x_i}\big)^2\big)}\ |\nabla\varphi| \\
&= \sqrt{\sum_{j=1}^{n}|\nabla f_j|^2}\ |\nabla\varphi| = |\nabla f|\ |\nabla\varphi| \\
&\le c(t)|\nabla\varphi|
\end{aligned}$$

$$\begin{aligned}
\Big|\sum_{i,j=1}^{n}\theta_{ij}f_j\varphi_i\Big| &\le \sqrt{\sum_{i=1}^{n}\big(\sum_{j=1}^{n}\theta_{ij}f_j\big)^2}\ |\nabla\varphi| \\
&\le \sqrt{\sum_{i=1}^{n}\big(\sum_{j=1}^{n}\theta_{ij}^2\big)\big(\sum_{j=1}^{n}f_j^2\big)}\ |\nabla\varphi| \\
&= \sqrt{\sum_{i,j=1}^{n}\theta_{i,j}^2}\ |\nabla f|\ |\nabla\varphi| \\
&= \Big[\sum_{i,j=1}^{n}\big(\frac{\delta_{ij}^2}{|x|^2} + \sum_{i,j=1}^{n}\frac{x_i^2x_j^2}{|x|^6} - 2\sum_{i,j=1}^{n}\frac{\delta_{ij}x_ix_j}{|x|^4}\big)\Big]^{\frac{1}{2}}\ |\nabla f|\ |\nabla\varphi| \\
&= \frac{\sqrt{n-1}}{|x|}\ |\nabla f|\ |\nabla\varphi|\ \le \frac{c\sqrt{n-1}}{R}\ |\nabla\varphi| \quad \text{on } \partial B_R
\end{aligned}$$

$$\frac{\alpha}{2\sqrt{|\nabla\varphi|+\beta}} \le \frac{\alpha}{2\sqrt{\beta}} \le \frac{1}{4}$$

$$-2\big(1-\frac{\alpha}{2\sqrt{|\nabla\varphi|+\beta}}\big) \le -\frac{3}{2}$$

$$\begin{aligned}
\frac{-\alpha\sum_{i=1}^{n}\theta_i\beta_i}{2\sqrt{|\nabla\varphi|^2+\beta}} &\le \frac{\alpha}{2}\frac{|\nabla\theta||\nabla\beta|}{\sqrt{|\nabla\varphi|^2+\beta}} = \frac{\alpha}{2}\frac{|\nabla\beta|}{\sqrt{|\nabla\varphi|^2+\beta}} \\
&\le \frac{\alpha}{2}\frac{|\nabla\beta|}{\sqrt{\beta}} \le \frac{\alpha}{2}c_1(t)
\end{aligned}$$

Therefore (2.19) implies

$$\begin{aligned}
\frac{\partial V}{\partial\nu} &\le -\frac{3}{2R}|\nabla\varphi|^2 + c(t)\big(1+\frac{\sqrt{n-1}}{R}\big)|\nabla\varphi| + \frac{\alpha}{2}c_1(t) \\
&= -\frac{3}{2R}\Big[|\nabla\varphi|^2 - \frac{2R}{3}\big(1+\frac{\sqrt{n-1}}{R}\big)|\nabla\varphi| + \frac{R^2}{9}\big(1+\frac{\sqrt{n-1}}{R}\big)^2\Big] \\
&\quad +\frac{R}{6}\big(1+\frac{\sqrt{n-1}}{R}\big)^2 + \frac{\alpha}{2}c_1(t)
\end{aligned}$$

$$= -\frac{3}{2R}\Big[|\nabla\varphi| - \frac{R}{3}(1+\frac{\sqrt{n-1}}{R})\Big]^2 + \frac{R}{6}(1+\frac{\sqrt{n-1}}{R})^2 + \frac{\alpha}{2}c_1(t)$$
$$\leq \frac{R}{6}(1+\frac{\sqrt{n-1}}{R})^2 + \frac{\alpha(t)}{2}c_1(t)$$

□

THEOREM 2.2. *Let $u_R > 0$ be a positive solution of the equation*

$$\frac{\partial u}{\partial t} = \Delta u + \sum_{i=1}^{n} f_i u_i - Vu$$

defined on the closed ball $B_R = \{x \in \mathbb{R}^n : \theta(x) = |x| - R \leq 0\}$, wth the Neumann condition

$$\frac{\partial u_R}{\partial \nu} = 0 \quad on \quad \partial B_R \quad where \quad \nu = \nabla\theta \tag{2.21}$$

Let $\varphi(x,t) = -\log u_R(x,t)$ and

$$\Psi(x,t) = \varphi_t + |\nabla\varphi|^2 - \sum_{j=1}^{n} f_j \frac{\partial\varphi}{\partial x_j} - V - \alpha(t)\sqrt{|\nabla\varphi|^2 + \beta(x,t)} - \frac{n}{2t} - 2\sqrt{n}\ c(t)$$

Suppose

$$|f|^2 \leq c(t)^2(1+|x|)^2 \leq c_1(t)\beta(x,t) \tag{2.22}$$
$$|\nabla f| = \sqrt{\sum_{i=1}^{n} |\nabla f_i|^2} \leq c(t) \tag{2.23}$$
$$0 < \alpha(t) \leq \frac{1}{2}\sqrt{\beta(x,t)} \tag{2.24}$$
$$|\nabla\beta| \leq c_1(t)\sqrt{\beta(x,t)} \tag{2.25}$$
$$\frac{\beta_t}{2} + \frac{3\beta}{2t} - \frac{n}{4t^2} > 0 \tag{2.26}$$
$$\alpha'(t) + \alpha\Big(\frac{3c}{2\sqrt{n}} - \lambda - \frac{|\nabla V|}{2\sqrt{\beta}} - \frac{1}{2}(c_1+c_1)^{3/2}\Big) - \sqrt{n}\ |\nabla f| \geq 0 \tag{2.27}$$
$$c'(t) + \frac{3}{16\sqrt{n}}c^2 + \frac{3}{8t}c - \frac{1}{2\sqrt{n}}\Delta V - \frac{1}{12\sqrt{n}}\sum_{i,j=1}^{n}(\frac{f_{i,j}+f_{j,i}}{2})^2 - \frac{\alpha\Delta\beta}{2\sqrt{n}\sqrt{\beta}} \geq 0 \tag{2.28}$$

where λ is the absolute value of the greatest eigenvalue of $\left(\frac{f_{i,j}+f_{j,i}}{2}\right)$.

$$\frac{\partial V}{\partial \nu} > \frac{R}{6}\Big(1+\frac{\sqrt{n-1}}{R}\Big)^2 + \frac{\alpha(t)}{2}c_1(t) \quad on \quad \partial B_R \quad for \quad t > 0 \tag{2.29}$$

If $\Psi(x,0) < 0$, then $\Psi(x,t) < 0$ for all $t > 0$, i.e.
$$\varphi_t + |\nabla\varphi|^2 - \sum_{j=1}^{n} f_j \frac{\partial\varphi}{\partial x_j} - V < \alpha(t)\sqrt{|\nabla\varphi|^2 + \beta(x,t)} + \frac{n}{2t} + 2\sqrt{n}\ c(t) \ for\ all\ t \geq 0.$$

Proof. If the conclusion is not true, then there exists t_0 such that $\Psi < 0$ for $0 \le t < t_0$ and $\Psi = 0$ at some point x_0 when $t = t_0$. In view of the hypothesis (2.22)-(2.28), we can apply Theorem 2.1 to conclude that $x_0 \in \partial B_R$. It is clear that $\frac{\partial \Psi}{\partial \nu}(x_0, t_0) \ge 0$. In view of Lemma 2.2, we have inequality (2.20) which contradicts to our assumptions (2.29). □

THEOREM 2.3. *Let $u_R > 0$ be a positive solution of the equation*

$$\frac{\partial u}{\partial t} = \Delta u + \sum_{i=1}^{n} f_i u_i - Vu$$

defined on the local ball $B_R = \{x \in \mathbb{R}^n : \theta(x) = |x| - R \le 0\}$ with the Neumann condition

$$\frac{\partial u_R}{\partial \nu} = 0 \quad \text{on} \quad \partial B_R \quad \text{where} \quad \nu = \nabla \theta \tag{2.30}$$

Let $\varphi(x,t) = -\log u_R(x,t)$ and

$$\Psi(x,t) = \varphi_t + |\nabla \varphi|^2 - \sum_{j=1}^{n} f_j \frac{\partial \varphi}{\partial x_j} - V - \alpha(t)\sqrt{|\nabla \varphi|^2 + \beta(x,t)} - \frac{n}{2t} - 2\sqrt{n}\, c(t)$$

Let $\beta(x,t) = c_2(t)|x|^2 + 4\alpha^2(t) + \frac{n}{2t}$ with $c_2(t) > 0$ and $\alpha(t) > 0$. Suppose that there exists $c_1(t) > 0$ such that

$$|f| \le c(t)(1 + |x|) \tag{2.31}$$

$$|\nabla f| = \sqrt{\sum_{i=1}^{n} |\nabla f_i|^2} \le c(t), |\nabla V| \le c(1 + |x|) \tag{2.32}$$

$$|\Delta V| \le c, |\Delta f| \le c \tag{2.33}$$

$$2c^2(t) \le c_1(t) c_2(t) \tag{2.34}$$

$$2c^2(t) \le c_1(t)(4\alpha^2(t) + \frac{n}{2t}) \tag{2.35}$$

$$4c_2(t) \le c_1^2(t) \tag{2.36}$$

$$c_2' + \frac{3}{t} c_2 \ge 0 \tag{2.37}$$

$$2(\alpha^2)' + \frac{6}{t}\alpha^2 + \frac{n}{4t^2} > 0 \tag{2.38}$$

$$\alpha'(t) + \alpha[\frac{3c}{2\sqrt{n}} - \lambda - \frac{1}{2}(\sqrt{c_1} + c_1 + c_1^{3/2}] - \sqrt{n}c \ge 0 \tag{2.39}$$

$$c'(t) + \frac{5}{48\sqrt{n}}c^2 + (\frac{3}{8t} - \frac{1}{2\sqrt{n}})c - \frac{\sqrt{n}}{2}c_2 > 0 \tag{2.40}$$

where λ is the absolute value of the greatest eigenvlaue of $(\frac{f_{i,j} + f_{j,i}}{2})$

$$\frac{\partial V}{\partial \nu} > \frac{R}{6}(1 + \frac{\sqrt{n-1}}{R})^2 + \frac{\alpha(t)}{2} c_1(t) \quad \text{on} \quad \partial B_R \text{ for } \quad t > 0 \tag{2.41}$$

If $\Psi(x,0) < 0$, then $\Psi(x,t) < 0$ for all $t > 0$, i.e.,

$$\varphi_t + |\nabla\varphi|^2 - \sum_{j=1}^{n} f_j \frac{\partial\varphi}{\partial x_j} - V < \alpha(t)\sqrt{|\nabla\varphi|^2 + \beta(x,t)} + \frac{n}{2t} + 2\sqrt{n}\ c(t) \quad \textit{for all} \quad t \geq 0. \tag{2.42}$$

Proof. We only need to prove that (2.31)–(2.40) imply (2.22)–(2.28).
Observe that

$$c(t)^2(1+|x|)^2 \leq 2c(t)^2(1+|x|^2) \leq c_1(t)c_2(t)|x|^2 + (\frac{n}{2t} + 4\alpha^2(t))c_1(t) = c_1(t) = c_1(t)\beta(x,t)$$

by (2.34) and (2.35). Hence (2.22) follows. (2.23) is part of our assumption (2.32) while (2.24) follows from our definition of $\beta(x,t)$ and the hypothesis $c_2(t) > 0$.

(2.25) is equivalent to $2c_2|x| \leq c_1\sqrt{c_2|x|^2 + 4\alpha^2 + \frac{n}{2t}}$, i.e., $4c_2^2|x|^2 \leq c_1^2(c_2|x|^2 + 4\alpha^2 + \frac{n}{2t})$. Hence (2.36) implies (2.25)

$$\begin{aligned}\frac{\beta_t}{2} + \frac{3\beta}{2t} - \frac{n}{4t^2} &= \frac{1}{2}(c_2'|x|^2 + 8\alpha\alpha' - \frac{n}{2t^2}) + \frac{3c_2}{2t}|x|^2 + \frac{6\alpha^2}{t} + \frac{3n}{4t^2} - \frac{n}{4t^2} \\ &= (\frac{1}{2}c_2' + \frac{3}{2t}c_2)|x|^2 + 4\alpha'\alpha + \frac{6}{t}\alpha^2 + \frac{n}{4t^2}\end{aligned}$$

Therefore (2.37) and (2.38) imply (2.26)

$$\begin{aligned}&\alpha'(t) + \alpha[\frac{3c}{2\sqrt{n}} - \lambda - \frac{|\nabla V|}{2\sqrt{\beta}} - \frac{1}{2}(c_1 + c_1^{3/2})] - \sqrt{n}|\Delta f| \\ \geq &\alpha'(t) + \alpha[\frac{3c}{2\sqrt{n}} - \lambda - \frac{c(1+|x|)}{2\sqrt{\beta}} - \frac{1}{2}(c_1 + c_1^{3/2})] - \sqrt{n}\ c \\ \geq &\alpha'(t) + \alpha[\frac{3c}{2\sqrt{n}} - \lambda - \frac{1}{2}(\sqrt{c_1} + c_1 + c_1^{3/2})] - \sqrt{n}\ c\end{aligned}$$

Hence (2.39) implies (2.27).
Observe that

$$\begin{aligned}\sum_{i,j=1}^{n}(\frac{f_{i,j}+f_{j,i}}{2})^2 &= \frac{1}{2}\sum_{i,j=1}^{n} f_{i,j}^2 + \frac{1}{2}\sum_{i,j=1}^{n} f_{i,j}f_{j,i} \\ &\leq \frac{1}{2}\sum_{i,j=1}^{n} f_{i,j}^2 + \frac{1}{4}(\sum_{i,j=1}^{n} f_{i,j}^2 + \sum_{i,j=1}^{n} f_{j,i}^2) \\ &= \sum_{i,j=1}^{n} f_{i,j}^2 = \sum_{i=1}^{n}|\nabla f_i|^2 = |\nabla f|^2 \leq c^2\end{aligned}$$

$$\begin{aligned}&c'(t) + \frac{3}{16\sqrt{n}}c^2 + \frac{3}{8t}c - \frac{1}{2\sqrt{n}}\Delta V - \frac{1}{12\sqrt{n}}\sum_{i,j=1}^{n}(\frac{f_{i,j}+f_{j,i}}{2})^2 - \frac{\alpha\Delta\beta}{2\sqrt{n}\sqrt{\beta}} \\ \geq\ &c'(t) + \frac{3}{16\sqrt{n}}c^2 + \frac{3}{8t}c - \frac{1}{2\sqrt{n}}c(t) - \frac{1}{12\sqrt{n}}c^2 - \frac{2nc_2}{2\sqrt{n}}\cdot\frac{1}{2} \\ =\ &c'(t) + \frac{5}{48\sqrt{n}}c^2 + (\frac{3}{8t} - \frac{1}{2\sqrt{n}})c - \frac{\sqrt{n}}{2}\ c_2\end{aligned}$$

(2.28) follows immediately from (2.40). □

THEOREM 2.4. *Let $u > 0$ be a positive solution of the equation*

$$\frac{\partial u}{\partial t} = \Delta u + \sum_{i=1}^{n} f_i u_i + Vu$$

on $\mathbb{R}^n$. Let $\varphi(x,t) = -\log u(x,t)$ and

$$\Psi(x,t) = \varphi_t + |\nabla\varphi|^2 - \sum_{j=1}^{n} f_j \frac{\partial \varphi}{\partial x_j} - V - \alpha(t)\sqrt{|\nabla\varphi|^2 + \beta(x,t)} - \frac{n}{2t} - 2\sqrt{n}\, c(t)$$

Let $\beta(x,t) = c_2(t)|x|^2 + 4\alpha^2(t) + \frac{n}{2t}$ with $c_2(t) > 0$ and $\alpha(t) > 0$. Suppose that there exists $c_1(t) > 0$ such that

(2.43) $|f| \le c(t)(1 + |x|)$

(2.44) $|\nabla f| = \sqrt{\sum_{i=1}^{n} |\nabla f_i|^2} \le c(t), |\nabla V| \le c(1 + |x|)$

(2.45) $|\Delta V| \le c, |\Delta f| \le c$

(2.46) $2c^2(t) \le c_1(t) c_2(t)$

(2.47) $2c^2(t) \le c_1(t)\big(4\alpha^2(t) + \frac{n}{2t}\big)$

(2.48) $4c_2(t) \le c_1^2(t)$

(2.49) $c_2'(t) + \frac{3}{t} c_2 \ge 0$

(2.50) $2(\alpha^2)' + \frac{6}{t}\alpha^2 + \frac{n}{4t^2} > 0$

(2.51) $\alpha'(t) + \alpha\Big[\frac{3c}{2\sqrt{n}} - \lambda - \frac{1}{2}(\sqrt{c_1} + c_1 + c_1^{3/2})\Big] - \sqrt{n}c \ge 0$

(2.52) $c'(t) + \frac{5}{48\sqrt{n}}c^2 + (\frac{3}{8t} - \frac{1}{2\sqrt{n}})c - \frac{\sqrt{n}}{2}c_2 > 0$

where λ is the absolute value of the greatest eigenvalue of $(\frac{f_{i,j} + f_{j,i}}{2})$. If $\Psi(x,0) < 0$, then $\Psi(x,t) < 0$ for all $t > 0$, i.e.

(2.53)
$$\Delta\varphi = \varphi_t + |\nabla\varphi|^2 - \sum_{j=1}^{n} f_j \frac{\partial\varphi}{\partial x_j} - V$$
$$< \alpha(t)\sqrt{|\nabla\varphi|^2 + \beta(x,t)} + \frac{n}{2t} + 2\sqrt{n}c(t) \text{ for all } t \ge 0\,.$$

Proof. Choose a function $A_R(x,t)$ which is convex in x direction with the following properties.

(2.54) $A_R(x,t) \equiv 0$ for $|x| \le R/2$

(2.55) $|\nabla V_R| \le c(1 + |x|), |\Delta V_R| \le c$

(2.56) $\frac{\partial V_R}{\partial \nu} > \frac{R}{6}(1 + \frac{\sqrt{n-1}}{R})^2 + \frac{\alpha(t)}{2}c_1(t)$ on ∂B_R for $t > 0$

where $V_R = V + A_R$. Let $f_R = f \cdot \sigma$ where σ is a cut off function with the following properties

$$\sigma(x) = 1 \text{ for } |x| \leq \frac{R}{2} \text{ and } \sigma(x) = 0 \text{ for } |x| \geq R \tag{2.57}$$

$$|\nabla f_R| \leq c, |\Delta f| \leq c \tag{2.58}$$

Let u_R be the positive solution of the equation

$$\frac{\partial u_R}{\partial t} = \Delta u_R + \sum_{i=1}^{n} (f_R)_i (u_R)_i - V_R u_R$$

defined on the closed ball $B_R = \{x \in \mathbb{R}^n : \theta(x) = |x| - R \leq 0\}$ with the Neumann condition $\frac{\partial u_R}{\partial \nu} = 0$ on ∂B_R where $\nu = \nabla\theta$. In view of Theorem 2.3, we know that (2.42) holds for u_R. Since u is the limit of u_R as R goes to infinity, we conclude that (2.53) holds. □

Remark : It is easy to choose c, c_1, c_2 and α such that (2.43)-(2.52) are satisfied. For instance, we can take $c_1 = 2c^{2/3}, c_2 = c^{4/3}$ and $\alpha(t) = \frac{1}{T-t}$ where c and T are positive constant and T is sufficiently small.

3. Harnack Inequality. In this section, we shall use the gradient estimate obtained in Theorem 2.3 to deduce Harnack inequality.

PROPOSITION 3.1. *Let $u_R > 0$ be a positive solution of the equation*

$$\frac{\partial u}{\partial t} = \Delta u + \sum_{i=1}^{n} f_i \frac{\partial u}{\partial x_i} - Vu \tag{3.1}$$

defined on the closed ball $B_R = \{x \in \mathbb{R}^n : \theta(x) = |x| - R \leq 0\}$ with the Neumann condition

$$\frac{\partial u_R}{\partial \nu} = 0 \quad on \quad \partial B_R \quad where \quad \nu = \nabla\theta \tag{3.2}$$

Let $\varphi(x,t) = -\log u_R(x,t)$ and $\beta(x,t) = c_2(t)|x|^2 + 4\alpha^2(t) + n/(2t)$ with $c_2(t) > 0$ and $\alpha(t) > 0$. Suppose that

$$\varphi_t + |\nabla\varphi|^2 - \sum_{j=1}^{n} f_j \frac{\partial\varphi}{\partial x_j} - V \leq \alpha(t)\sqrt{|\nabla\varphi|^2 + \beta(x,t)} + \frac{n}{2t} + 2\sqrt{n}\, c(t) \tag{3.3}$$

Let $t > t_0$ and $\mathcal{P} = \{$differentiable path $\sigma = (\sigma_1, \sigma_2) : [0,1] \longrightarrow B_R \times \mathbb{R}$ such that $\sigma(0) = (\sigma_1(0), \sigma_2(0)) = (x_0, t_0), \sigma(1) = (x,t), \sigma_2'(s) > 0\}$. Define

$$d((x_0,t_0),(x,t)) := \inf_{\sigma\in\mathcal{P}} \Big\{ \frac{1}{2}\int_0^1 < \dot{\sigma_1}, f > ds + \frac{1}{4}\int_0^1 \frac{d\sigma_2}{ds} \Big[\frac{|\dot{\sigma_1}|}{\frac{d\sigma_2}{ds}} + \sqrt{2}\alpha\Big]^2 ds \\ + \int_0^1 \frac{d\sigma_2}{ds} \Big(\sqrt{2}\alpha\sqrt{\frac{|f|^2}{4} + \frac{\beta}{2}} + V + \frac{1}{4}\sum_{i=1}^{n} f_i^2\Big) ds \Big\} \tag{3.4}$$

Then

$$\frac{u_R(x,t)}{u_R(x_0,t_0)} \geq \Big(\frac{t}{t_0}\Big)^{-\frac{n}{2}} \exp\Big(-\int_{t_0}^{t} 2\sqrt{n}\, c(\tau) d\tau\Big) \exp\Big[-d((x_0,t_0),(x,t))\Big] \tag{3.5}$$

Proof. We shall rewrite the inequality (3.1) in the following form

$$\varphi_t + |\nabla\varphi - \frac{f}{2}|^2 - \frac{|f|^2}{4} - V = \varphi_t + |\nabla\varphi|^2 - \sum_{i=1}^{n} f_i\varphi_i - V \tag{3.6}$$
$$\leq \alpha\sqrt{|\nabla\varphi|^2 + \beta} + \frac{n}{2t} + 2\sqrt{n}\,c(t)$$
$$\leq \alpha\sqrt{(|\nabla\varphi - \frac{f}{2}| + |\frac{f}{2}|)^2 + \beta} + \frac{n}{2t} + 2\sqrt{n}\,c(t)$$
$$\leq \alpha\sqrt{2|\nabla\varphi - \frac{f}{2}|^2 + \frac{|f|^2}{2} + \beta} + \frac{n}{2t} + 2\sqrt{n}\,c(t)$$
$$= \alpha\sqrt{2}\sqrt{|\nabla\varphi - \frac{f}{2}|^2 + \frac{|f|^2}{4} + \frac{\beta}{2}} + \frac{n}{2t} + 2\sqrt{n}\,c(t)$$
$$\leq \alpha\sqrt{2}(|\nabla\varphi - \frac{f}{2}| + \sqrt{\frac{|f|^2}{4} + \frac{\beta}{2}}) + \frac{n}{2t} + 2\sqrt{n}\,c(t)$$

Let $\sigma = (\sigma_1, \sigma_2)$ be a path in $\mathcal{P}$. Then

$$-\varphi(x,t) + \varphi(x_0,t_0) = -\int_0^1 \frac{d}{ds}\varphi(\sigma_1(s), \sigma_2(s))ds \tag{3.7}$$
$$= -\int_0^1 < \dot{\sigma_1}, \nabla\varphi > - \int_0^1 \frac{d\sigma_2}{ds}\frac{\partial\varphi}{\partial t}$$
$$\geq -\int_0^1 < \dot{\sigma_1}, \nabla\varphi - \frac{f}{2} > - \frac{1}{2}\int_0^1 < \dot{\sigma_1}, f > + \int_0^1 \frac{d\sigma_2}{ds}\Big(|\nabla\varphi - \frac{f}{2}|^2$$
$$-\alpha\sqrt{2}|\nabla\varphi - \frac{f}{2}| - \alpha\sqrt{2}\sqrt{\frac{|f|^2}{4} + \frac{\beta}{2}} - \frac{|f|^2}{4} - V - \frac{n}{2\sigma_2(s)} - 2\sqrt{n}c(\sigma_2(s))\Big)$$
$$\geq -\frac{1}{2}\int_0^1 < \dot{\sigma_1}, f > - \int_0^1 \frac{d\sigma_2}{ds}\Big(\alpha\sqrt{2}\sqrt{\frac{|f|^2}{4} + \frac{\beta}{2}} + \frac{|f|^2}{4} + V + \frac{n}{2\sigma_2(s)} + 2\sqrt{n}c(\sigma_2(s))\Big)$$
$$+ \int_0^1 \frac{d\sigma_2}{ds}\Big[|\nabla\varphi - \frac{f}{2}|^2 - (\alpha\sqrt{2} + \frac{|\dot{\sigma_1}|}{\frac{d\sigma_2}{ds}})|\nabla\varphi - \frac{f}{2}|\Big]$$
$$= -\frac{1}{2}\int_0^1 < \dot{\sigma_1}, f > - \int_0^1 \frac{d\sigma_2}{ds}\Big[\alpha\sqrt{2}\sqrt{\frac{|f|^2}{4} + \frac{\beta}{2}} + \frac{|f|^2}{4} + V + \frac{n}{2\sigma_2(s)} + 2\sqrt{n}c(\sigma_2(s))\Big]$$
$$-\frac{1}{4}\int_0^1 \frac{d\sigma_2}{ds}(\alpha\sqrt{2} + \frac{|\dot{\sigma_1}|}{\frac{d\sigma_2}{ds}})^2$$
$$+ \int_0^1 \frac{d\sigma_2}{ds}\Big[|\nabla\varphi - \frac{f}{2}|^2 - \frac{1}{2}(\alpha\sqrt{2} + \frac{|\dot{\sigma_1}|}{\frac{d\sigma_2}{ds}})\Big]^2$$
$$\geq -\frac{1}{2}\int_0^1 < \dot{\sigma_1}, f > - \int_0^1 \frac{d\sigma_2}{ds}\Big[\alpha\sqrt{2}\sqrt{\frac{|f|^2}{4} + \frac{\beta}{2}} + \frac{|f|^2}{4} + V + \frac{n}{2\sigma_2(s)} + 2\sqrt{n}c(\sigma_2(s))\Big]$$
$$-\frac{1}{4}\int_0^1 \frac{d\sigma_2}{ds}(\alpha\sqrt{2} + \frac{|\dot{\sigma_1}|}{\frac{d\sigma_2}{ds}})^2$$
$$= -\frac{1}{2}\int_0^1 < \dot{\sigma_1}, f > - \int_0^1 \frac{d\sigma_2}{ds}\Big[\alpha\sqrt{2}\sqrt{\frac{|f|^2}{4} + \frac{\beta}{2}} + \frac{|f|^2}{4} + V\Big]$$

$$-\frac{1}{4}\int_0^1 \frac{d\sigma_2}{ds}(\alpha\sqrt{2}+\frac{|\dot{\sigma_1}|}{\frac{d\sigma_2}{ds}})^2 - \int_0^1 \frac{n}{2\sigma_2(s)}d\sigma_2(s) - \int_0^1 2\sqrt{n}c(\sigma_2(s))d\sigma_2(s)$$

$$=-\frac{1}{2}\int_0^1 < \dot{\sigma_1}, f > - \int_0^1 \frac{d\sigma_2}{ds}\Big[\alpha\sqrt{2}\sqrt{\frac{|f|^2}{4}+\frac{\beta}{4}}+\frac{|f|^2}{4}+V\Big]$$

$$-\frac{1}{4}\int_0^1 \frac{d\sigma_2}{ds}\Big(\alpha\sqrt{2}+\frac{|\dot{\sigma_1}|}{\frac{d\sigma_2}{ds}}\Big)^2 - \frac{n}{2}\log(\frac{t}{t_0}) - \int_{t_0}^t 2\sqrt{n}\ c(\tau)d\tau$$

Since (3.7) is true for all $\sigma \in \mathcal{P}$, we have

$$\text{(3.8)} \qquad \log u(x,t) - \log u(x_0,t_0)$$

$$\geq \sup_{\sigma\in\mathcal{P}}\Big\{-\frac{1}{2}\int_0^1 < \dot{\sigma_1}, f > - \int_0^1 \frac{d\sigma_2}{ds}\Big[\alpha\sqrt{2}\sqrt{\frac{|f|^2}{4}+\frac{\beta}{2}}+\frac{|f|^2}{4}+V\Big]$$

$$-\frac{1}{4}\int_0^1 \frac{d\sigma_2}{ds}(\alpha\sqrt{2}+\frac{|\dot{\sigma_1}|}{\frac{d\sigma_2}{ds}})^2\Big\} - \frac{n}{2}\log(\frac{t}{t_0}) - \int_{t_0}^t 2\sqrt{n}\ c(\tau)d\tau$$

$$=-\inf_{\sigma\in\mathcal{P}}\Big\{\frac{1}{2}\int_0^1 < \dot{\sigma_1}, f > + \int_0^1 \frac{d\sigma_2}{ds}[\alpha\sqrt{2}\sqrt{\frac{|f|^2}{4}+\frac{\beta}{4}}+\frac{|f|^2}{4}+V]$$

$$+\frac{1}{4}\int_0^1 \frac{d\sigma_2}{ds}(\alpha\sqrt{2}+\frac{|\dot{\sigma_1}|}{\frac{d\sigma_2}{ds}}\Big\}^2 - \frac{n}{2}\log(\frac{t}{t_0}) - \int_{t_0}^t 2\sqrt{n}\ c(\tau)d\tau$$

$$=-d\big((x_0,t_0),(x,t)\big) - \frac{n}{2}\log(\frac{t}{t_0}) - \int_{t_0}^t 2\sqrt{n}\ c(\tau)d\tau$$

It follows that

$$\frac{u(x,t)}{u(x_0,t_0)} \geq (\frac{t}{t_0})^{-\frac{n}{2}}\exp\Big(-\int_{t_0}^t 2\sqrt{n}\ c(\tau)d\tau\Big)\exp\big[-d\big((x_0,t_0),(x,t)\big)\big]$$

□

PROPOSITION 3.2. *Let $d((x_0,t_0),(x,t))$ be defined by (3.4) in Proposition 3.1. Then*

$$\text{(3.9)} \quad d\big((x_0,t_0),(x,t)\big) \leq \frac{|x-x_0|^2}{4(t-t_0)} + \frac{\sqrt{2}}{2}|x-x_0|\int_0^1 \alpha((1-s)t_0+st)ds$$

$$+\frac{t-t_0}{2}\int_0^1 \alpha^2((1-s)t_0+st)ds$$

$$+\frac{1}{2}\int_0^1 < x-x_0, f(1-s)x_0+sx,(1-s)t_0+st) > ds$$

$$+(t-t_0)\int_0^1 \{\sqrt{2}\alpha(1-s)t_0+st)[\frac{1}{4}\sum_{i=1}^n f_i^2((1-s)x_0+sx,(1-s)t_0+st)$$

$$+\frac{1}{2}\beta((1-s)x_0+sx,(1-s)t_0+st)]^{\frac{1}{2}}$$

$$+\frac{1}{4}\sum_{i=1}^n f_i^2\big((1-s)x_0+sx,(1-s)t_0+st\big)+V\big((1-s)x_0+sx,(1-s)t_0+st\big)\big\}ds$$

Proof. Let $(\sigma_1,\sigma_2) = \big((1-s)x_0+sx,(1-s)t_0+st\big)$. Then $\dot{\sigma_1} = x - x_0$ and $\frac{d\sigma_2}{ds} = t-t_0$. By the definition of $d\big((x_0,t_0),(x,t)\big)$, we have

$$
\begin{aligned}
d\big((x_0,t_0),(x,t)\big) \le & \frac{1}{2}\int_0^1 < x-x_0, f\big((1-s)x_0+sx,(1-s)t_0+st\big) > ds \\
& + \int_0^1 (t-t_0)\Big[\sqrt{2}\alpha((1-s)t_0+st) \\
& \sqrt{\frac{|f(1-s)x_0+sx,(1-s)t_0+st)|^2}{4}+\frac{\beta((1-s)x_0+sx,(1-s)t_0+st)}{2}} \\
& + \frac{|f((1-s)x_0+sx,(1-s)t_0+st)|^2}{4} + V((1-s)x_0+sx,(1-s)t_0+st)\Big]ds \\
& + \frac{1}{4}\int_0^1 (t-t_0)\Big(\big(\sqrt{2}\alpha(1-s)t_0+st\big)+\frac{|x-x_0|}{t-t_0}\Big)^2 ds \\
\le & \frac{|x-x_0|^2}{4(t-t_0)} + \frac{\sqrt{2}}{2}|x-x_0|\int_0^1 \alpha\big((1-s)t_0+st\big)ds + \frac{t-t_0}{2}\int_0^1 \alpha^2\big((1-s)t_0+st\big)ds \\
& \frac{1}{2}\int_0^1 < x-x_0, f\big((1-s)x_0+sx,(1-s)t_0+st\big) > ds \\
& + (t-t_0)\int_0^1 \Big\{\frac{1}{4}\sum_{i=1}^n f_i^2\big((1-s)x_0+sx,(1-s)t_0+st\big) \\
& + V\big((1-s)x_0+sx,(1-s)t_0+st\big) \\
& + \sqrt{2}\alpha\big((1-s)t_0+st\big)\Big[\frac{1}{4}\sum_{i=1}^n f_i^2\big((1-s)x_0+sx,(1-s)t_0+st\big) \\
& + \frac{1}{2}\beta((1-s)x_0+sx,(1-s)t_0+st)\Big]^{\frac{1}{2}}\Big]ds
\end{aligned}
$$

□

Similarly we can use Theorem 2.4 to deduce Harnack inequality on $\mathbb{R}^n$.

THEOREM 3.3. *Let $u > 0$ be a positive solution of the equation*

$$\frac{\partial u}{\partial t} = \Delta u + \sum_{i=1}^n f_i u_i - Vu$$

on $\mathbb{R}^n$. Let $\varphi(x,t) = -\log u(x,t)$ and

$$\Psi(x,t) = \varphi_t + |\nabla\varphi|^2 - \sum_{j=1}^n f_j\frac{\partial\varphi}{\partial x_j} - V - \alpha(t)\sqrt{|\nabla\varphi|^2+\beta(x,t)} - \frac{n}{2t} - 2\sqrt{n}\ c(t)$$

Let $\beta(x,t) = c_2(t)|x|^2 + 4\alpha^2(t) + \frac{n}{2t}$ with $c_2(t) > 0$ and $\alpha(t) > 0$. Suppose that there exists $c_1(t)$ such that

$$|f| \le c(1+|x|) \tag{3.10}$$

$$|\nabla f| = \sqrt{\sum_{i=1}^n |\nabla f_i|^2} \le c, |\nabla V| \le c(1+|x|) \tag{3.11}$$

(3.12) $|\Delta V| \le c, |\Delta f| \le c$

(3.13) $2c^2(t) \le c_1(t)c_2(t)$

(3.14) $2c^2(t) \le c_1(t)(4\alpha^2(t) + \frac{n}{2t})$

(3.15) $4c_2(t) \le c_1^2(t)$

(3.16) $c_2'(t) + \frac{3}{t}c_2 \ge 0$

(3.17) $2(\alpha^2)' + \frac{6}{t}\alpha^2 + \frac{n}{4t^2} > 0$

(3.18) $\alpha'(t) + \alpha\Big[\frac{3c}{2\sqrt{n}} - \lambda - \frac{1}{2}(\sqrt{c_1} + c_1 + c_1^{3/2})\Big] - \sqrt{n}\, c \ge 0$

(3.19) $c'(t) + \frac{5}{48\sqrt{n}}c^2 + (\frac{3}{8t} - \frac{1}{2\sqrt{n}})c - \frac{n}{2}c_2 > 0$

where λ *is the absolute value of the greatest eigenvalue of* $(\frac{f_{i,j} + f_{j,i}}{2})$.

Let $t > t_0$ *and* $\mathcal{P} = \{$ *differentiable path* $\sigma = (\sigma_1, \sigma_2) : [0,1] \to \mathbb{R}^n \times \mathbb{R}$ *such that* $\sigma(0) = (\sigma_1(0), \sigma_2(0)) = (x_0, t_0), \sigma(1) = (x,t), \sigma_2'(s) > 0\}$. *Define*

$$d((x_0,t_0),(x,t)) := \inf_{\sigma\in\mathcal{P}} \Big\{\frac{1}{2}\int_0^1 < \dot\sigma_1, f > ds + \int_0^1 \frac{d\sigma_2}{ds}\Big(\sqrt{2}\alpha\sqrt{\frac{|f|^2}{4} + \frac{\beta}{2}} + V + \frac{1}{4}\sum_{i=1}^n f_i^2\Big)ds + \frac{1}{4}\int_0^1 \frac{d\sigma_2}{ds}\Big[\frac{|\dot\sigma_1|}{\frac{d\sigma_2}{ds}} + \sqrt{2}\alpha\Big]^2 ds\Big\} \quad (3.20)$$

If $\Psi(x,0) < 0$, *then*

$$\frac{u(x,t)}{u(x_0,t_0)} \ge \Big(\frac{t}{t_0}\Big)^{-\frac{n}{2}} \exp\Big(-\int_{t_0}^t 2\sqrt{n}c(\tau)d\tau\Big)\exp\Big[-d((x_0,t_0),(x,t))\Big] \quad (3.21)$$

Proof. The proof is same as the proof of Proposition 3.1. □

4. L^1 and Pointwise Estimate. Let us first recall some general theory of parabolic equations for the sake of convenience to the readers.

THEOREM 4.1. *(c.f. P.43 of [Fr]) Consider the operator*

$$Lu := \sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u - \frac{\partial u}{\partial t}$$

on $\mathbb{R}^n \times [0,T]$. *Assume that* L *is parabolic in* $\mathbb{R}^n \times [0,T]$, *i.e. for every* $(x,t) \in \mathbb{R}^n \times [0,T]$ *and for every real vector* $\zeta \ne 0, \sum a_{ij}(x,t)\zeta_i\zeta_j > 0$. *Assume also that the coefficients of* L *are continous functions in* $\mathbb{R}^n \times [0,T]$ *with the following growth conditions*

(4.1) $|a_{ij}(x,t)| \le M, |b_i(x,t)| \le M(1+|x|), c(x,t) \le M(1+|x|^2)$
for $i,j, = 1,\cdots,n, (x,t) \in \mathbb{R}^n \times (0,T], M =$ *positive constant* .

Assume further that $Lu \le 0$ *in* $\mathbb{R}^n \times (0,T]$ *and that*

(4.2) $u(x,t) \ge -B\exp[\beta|x|^2] \quad in \quad \mathbb{R}^n \times [0,T]$

for some positive constants B, β. *If* $u(x,0) \geq 0$ *in* $\mathbb{R}^n$, *then* $u(x,t) \geq 0$ *in* $\mathbb{R}^n \times [0,T]$.

THEOREM 4.2. *(c.f. P. 44[Fr]) Consider the parabolic operator*

$$Lu := \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial x}{\partial x_i} + c(x,t)u - \frac{\partial u}{\partial t}$$

with continous coefficients in $\mathbb{R}^n \times (0,T]$. *Let (4.1) be satisfied. then there exists at most one solution to the Cauchy problem*

$$\begin{cases} Lu = f(x,t) & \text{in } \mathbb{R}^n \times (0,T] \\ u(x,t) = \varphi(x) & \text{in } \mathbb{R}^n \end{cases} \tag{4.3}$$

satisfying

$$|u(x,t)| \leq B \exp[\beta |x|^2] \tag{4.4}$$

for some positive constant B, β.

Let $u_R > 0$ be a positive solution of the equation

$$\frac{\partial u}{\partial t} = \Delta u + \sum_{i=1}^{n} f_i \frac{\partial u}{\partial x_i} - Vu \tag{4.5}$$

on the ball $B_R = \{x \in \mathbb{R}^n : \theta(x) = |x| - R \leq 0\}$ with the Neumann condition $\frac{\partial u_R}{\partial \nu} = 0$ on ∂B_R where $\nu = \nabla\theta$. Assume that (2.31)-(2.41) hold. Then for $\varphi = -\log u_R$, we have

$$\varphi_t + |\nabla\varphi|^2 - \sum_{j=1}^{n} f_j \frac{\partial \varphi}{\partial x_j} - V < \alpha\sqrt{|\nabla\varphi| + \beta} + \frac{n}{2t} + 2\sqrt{n}\, c(t)$$

on B_R as long as it holds for $t = 0$.

From now on, we shall study the behaviour of the solution of equation (4.5) in $\mathbb{R}^n$. If the initial data is nonnegative, then in view of Theorem 4.1 the solution is nonnegative. Hence we can assume that the solution of (4.5) is nonnegative.

THEOREM 4.3. *Let* u *be a nonnegative solution of the equatoin*

$$\frac{\partial u}{\partial t} = \Delta u + \sum_{i=1}^{n} f_i \frac{\partial u}{\partial x_i} - Vu$$

on $\mathbb{R}^n$. *Suppose that* $u(x,0)$ *has compact support**,

$$|f(x,t)| \leq c(t)(1+|x|), |\nabla f(x,t)| = \sqrt{\sum_{i=1}^{n} |\nabla f_i|^2} \leq c(t)(1+|x|)$$

and

$$|V(x,t)| \leq c(t)(1+|x|^2),$$

*It is sufficient for $u(x,0)$ to decay like Gaussian.

Let $a_1(t), a_2(t)$ and $a_3(t)$ satisfy the following ordinary differential equations

$$(\log a_1)' + 2na_2 + a_3^2 + 4a_2c + c = 0 \tag{4.6}$$

$$a_2' + 4a_2^2 + 4a_2c + c = 0 \tag{4.7}$$

$$a_3' + 2a_3c - \sqrt{2n}c = 0 \tag{4.8}$$

with initial conditions suitably chosen (e.g. $a_1(0) > 0, a_2(0) > \frac{1}{2}a_3(0) > 0$) so that $a_1(t) > 0, a_2(t) \geq \frac{1}{2}a_3(t) > 0$ for all $0 \leq t \leq T$ for some $T > 0$. Then, for $0 \leq t \leq T$,

$$\begin{aligned} &a_1(t)\int_{\mathbb{R}^n} u(x,t)\exp(a_2(t)|x|^2 - a_3(t)\sqrt{1+|x|^2}) \\ \leq &a_1(0)\int_{\mathbb{R}^n} u(x,0)\exp(a_2(0)|x|^2 - a_3(0)\sqrt{1+|x|^2}) \end{aligned} \tag{4.9}$$

Proof. Let ρ be any smooth function and $B_R = \{x \in \mathbb{R}^n : |x| \leq R\}$. We have

$$\begin{aligned} &\frac{d}{dt}\int_{B_R}\rho u = \int_{B_R}\rho_t u + \int_{B_R}\rho u_t \\ &= \int_{B_R}\rho_t u + \int_{B_R}\rho\Delta u + \int_{B_R}\rho\sum_{i=1}^n f_i\frac{\partial u}{\partial x_i} - \int_{B_R}\rho V u \\ &= \int_{B_R}\rho_t u + \int_{B_R}(\Delta\rho)u - \int_{\partial B_R} u\frac{\partial\rho}{\partial\nu}ds + \int_{\partial B_R}\rho\frac{\partial u}{\partial\nu}ds \\ &- \int_{B_R}(\rho\sum_{i=1}^n f_{i,i} + \sum_{i=1}^n \rho_i f_i)u + \int_{\partial B_R} u\rho f\cdot\nu ds - \int_{B_R}\rho V u \end{aligned}$$

by Divergence Theorem and Green's first identity. Now we can use simple cut off argument to otbain the following

$$\frac{d}{dt}\int_{\mathbb{R}^n}\rho u = \int_{\mathbb{R}^n}(\rho_t + \Delta\rho - \sum_{i=1}^n f_i\rho_i - \sum_{i=1}^n f_{i,i}\rho - V\rho)u \tag{4.10}$$

Observe that

$$\begin{aligned} \left|\sum_{i=1}^n f_{i,i} + V\right| &\leq \sum_{i=1}^n |f_{i,i}| + |V| \\ &\leq \sqrt{n}\sqrt{\sum_{i=1}^n |f_{i,i}|^2} + c(1+|x|^2) \\ &\leq \sqrt{n}\, c(1+|x|) + c(1+|x|^2) \\ &\leq \sqrt{2n}\, c(1+|x|^2)^{\frac{1}{2}} + c(1+|x|^2) \end{aligned}$$

$$\begin{aligned} \left|\sum_{i=1}^n x_i f_i\right| &\leq |x||f| \leq c|x|(1+|x|) \\ &\leq c(1+|x|)^2 \leq 2c(1+|x|^2) \end{aligned}$$

Set

$$\rho = a_1(t)\exp\left[a_2(t)|x|^2 - a_3(t)\sqrt{1+|x|^2}\right]$$

Then

$$\begin{aligned}
\rho_t &= \left(a_t' + a_1(t)a_2'(t)|x|^2 - a_1(t)a_3'(t)\sqrt{1+|x|^2}\right)\exp[a_2(t)|x|^2 - a_3(t)\sqrt{1+|x|^2}] \\
&= \left((\log a_1)' + a_2'|x|^2 - a_3'\sqrt{1+|x|^2}\right)\rho \\
\rho_i &= \left(2x_i a_2(t) - a_3(t)\frac{x_i}{\sqrt{1+|x|^2}}\right)\rho \\
\rho_{ii} &= \left(2a_2(t) - \frac{a_3(t)}{\sqrt{1+|x|^2}} + \frac{a_3(t)x_i^2}{(1+|x|^2)^{3/2}}\right)\rho \\
&\quad + \left(2x_i a_2(t) - a_3(t)\frac{x_i}{\sqrt{1+|x|^2}}\right)^2\rho \\
&= \left[2a_2(t) - \frac{a_3(t)}{\sqrt{1+|x|^2}} + \frac{a_3(t)x_i^2}{(1+|x|^2)^{3/2}} + \left(2a_2(t) - \frac{a_3(t)}{\sqrt{1+|x|^2}}\right)^2 x_i^2\right]\rho \\
\Delta\rho &= \left[2na_2(t) - \frac{na_3(t)}{\sqrt{1+|x|^2}} + \frac{a_3(t)|x|^2}{(1+|x|^2)^{3/2}} + \left(2a_2(t) - \frac{a_3(t)}{\sqrt{1+|x|^2}}\right)^2|x|^2\right]\rho
\end{aligned}$$

It follows that

$$\begin{aligned}
(4.11)\quad & \rho_t + \Delta\rho - \sum_{i=1}^n f_i\rho_i - \left(\sum_{i=1}^n f_{i,i} + V\right)\rho \\
&= \Big[(\log a_1)' + a_2'|x|^2 - a_3'\sqrt{1+|x|^2} + 2na_2 - \frac{((n-1)|x|^2+n)a_3}{(1+|x|^2)^{3/2}} \\
&\quad + \left(2a_2 - \frac{a_3}{\sqrt{1+|x|^2}}\right)^2|x|^2 - \left(2a_2 - \frac{a_3}{\sqrt{1+|x|^2}}\right)\sum_{i=1}^n x_i f_i - \left(\sum_{i=1}^n f_{i,i} + V\right)\Big]\rho
\end{aligned}$$

Recall that

$$2a_2(t) \geq a_3(t) \geq \frac{a_3(t)}{\sqrt{1+|x|^2}} \quad 0 \leq t \leq T$$

Therefore (4.11) has the following upper estimates

$$\begin{aligned}
(4.12)\quad & \rho_t + \Delta\rho - \sum_{i-1}^n f_i\rho_i - \left(\sum_{i=1}^n f_{i,i} + V\right)\rho \\
&\leq \Big[(\log a_1)' + a_2'|x|^2 - a_3'\sqrt{1+|x|^2} + 2na_2 - \frac{((n-1)|x|^2+n)a_3}{(1+|x|^2)^{3/2}} \\
&\quad + \left(4a_2^2 - \frac{4a_2a_3}{\sqrt{1+|x|^2}} + \frac{a_3^2}{1+|x|^2}\right)|x|^2 + \left(2a_2 - \frac{a_3}{\sqrt{1+|x|^2}}\right)2c(1+|x|^2) \\
&\quad + \sqrt{2n}\, c(1+|x|^2)^{\frac{1}{2}} + c(1+|x|^2)\Big]\rho \\
&= \Big[(\log a_1)' + a_2'|x|^2 - a_3'\sqrt{1+|x|^2} + 2na_2 - \frac{(n-1)a_3}{(1+|x|^2)^{1/2}} \\
&\quad - \frac{a_3}{(1+|x|^2)^{3/2}} + 4a_2^2|x|^2 - \frac{4a_2a_3|x|^2}{\sqrt{1+|x|^2}} + a_3^2 - \frac{a_3^2}{1+|x|^2} \\
&\quad + 4a_2c + 4a_2c|x|^2 - 2a_3c\sqrt{1+|x|^2} + \sqrt{2n}\, c(1+|x|^2)^{\frac{1}{2}} \\
&\quad + c + c|x|^2\Big]\rho
\end{aligned}$$

$$
\begin{aligned}
=&\Big[(\log a_1)' + 2na_2 + a_3^2 + 4a_2c + c + (a_2' + 4a_2^2 + 4a_2c + c)|x|^2 \\
&+(-a_3' - 2a_3c + \sqrt{2n}\, c)(1+|x|^2)^{\frac{1}{2}} - \frac{(n-1)a_3}{(1+|x|^2)^{\frac{1}{2}}} \\
&-\frac{a_3}{(1+|x|^2)^{3/2}} - \frac{4a_2a_3|x|^2}{\sqrt{1+|x|^2}} - \frac{a_3^2}{1+|x|^2}\Big]\rho \\
\le&\Big[(\log a_1)' + 2na_2 + a_3^2 + 4a_2c + c + (a_2' + 4a_2^2 + 4a_2c + c)|x|^2 \\
&+(-a_3' - 2a_3c + \sqrt{2nc})(1+|x|^2)^{\frac{1}{2}}\Big]\rho \\
\le&0
\end{aligned}
$$

because of (4.6), (4.7) and (4.8) and the assumption $a_2(t) \ge \frac{1}{2}a_3(t) > 0$ for all $0 \le t \le T$ (4.10) and (4.12) imply

$$\frac{d}{dt}\int_{\mathbb{R}^n} \rho u \le 0$$

(4.9) follows immediately from the above inequality. □

Theorem 4.3 above gives L^1-estimate for u. Combining the Harnack inequality of §3, we shall have the following pointwise estimate of u.

THEOREM 4.4. *Let u be a nonnegative solution of the equation*

$$\frac{\partial u}{\partial t} = \Delta u + \sum_{i=1}^{n} f_i \frac{\partial u}{\partial x_i} - Vu$$

on $\mathbb{R}^n$. Suppose that $u(x,0)$ has compact support† *and $|f(x,t)| \le c(t)(1+|x|)$, $|\nabla f(x,t)| = \sqrt{\sum_{i=1}^{n}|\nabla f_i|^2} \le c(t)(1+|x|), |V(x,t)| \le c(t)(1+|x|^2)$. Let $a_1(t), a_2(t)$ and $a_3(t)$ satisfy the ordinary differential equations (4.6), (4.7) and (4.8) with initial conditions suitably chosen (e.g. $a_1(0) > 0, a_2(0) > \frac{1}{2}a_3(0) > 0$) so that $a_1(t) > 0, a_2(t) \ge \frac{1}{2}a_3(t) > 0$ for all $0 \le t \le 2T$ for some $T > 0$. Assume that the Harnack inequality holds*

$$\frac{u(x,t)}{u(x_0,t_0)} \ge \left(\frac{t}{t_0}\right)^{-\frac{n}{2}}\Big[\exp\Big(-\int_{t_0}^{t} 2\sqrt{n}\, c(\tau)d\tau\Big)\Big]\exp\Big[-d\big((x_0,t_0),(x,t)\big)\Big]$$

where

$$
\begin{aligned}
d((x_0,t_0),(x,t)) := \inf_{\sigma\in\mathcal{P}}\Big\{&\frac{1}{2}\int_0^1 <\dot\sigma_1, f> ds + \int_0^1 \frac{d\sigma_2}{ds}\Big(\sqrt{2}\alpha\sqrt{\frac{|f|^2}{4}+\frac{\beta}{2}} + V \\
&+\frac{1}{4}\sum_{i=1}^{n} f_i^2\Big)ds + \frac{1}{4}\int_0^1 \frac{d\sigma_2}{ds}\Big[\frac{|\dot\sigma_1|}{\frac{d\sigma_2}{ds}} + \sqrt{2}\alpha\Big]^2 ds\Big\}
\end{aligned}
$$

†It is sufficient for $u(x,0)$ to decay like Gaussian

Here $\mathcal{P} = \{$ *differentiable path* $\sigma = (\sigma_1, \sigma_2) : [0,1] \to \mathbb{R}^n \times \mathbb{R}$ *such that* $\sigma(0) = (\sigma_1(0), \sigma_2(0)) = (x_0, t_0), \sigma(1) = (x,t), \sigma_2'(s) > 0\}$ *and* α, β *are defined in Theorem 2.3. Then for* $t \leq T$,

$$u(x,t) \leq 2^{\frac{n}{2}} \frac{a_1(0)}{a_1(2t)} \Big(\exp \int_t^{2t} 2\sqrt{n}c(\tau)d\tau \Big)$$
$$\Big[\int_{x\in\mathbb{R}^n} u(x,0) \exp \big(a_2(0)|x|^2 - a_3(0)\sqrt{1+|x|^2} \big) \Big]$$
$$\Big\{ \int_{y\in\mathbb{R}^n} \big[\exp \big(-d(x,t),(y,2t) \big) \big] \exp \big[a_2(2t)|y|^2 - a_3(2t)\sqrt{1+|y|^2} \big] \Big\}^{-1}$$

Proof. By the L^1-estimate for u and Harnack inequality, we have

$$a_1(0) \int_{x\in\mathbb{R}^n} u(x,0) \exp \big[a_2(0)|x|^2 - a_3(0)\sqrt{1+|x|^2} \big]$$
$$\geq a_1(2t) \int_{y\in\mathbb{R}^n} u(y,2t) \exp \big[a_2(2t)|y|^2 - a_3(2t)\sqrt{1+|y|^2} \big]$$
$$\geq a_1(2t) \int_{y\in\mathbb{R}^n} u(x,t) (\frac{2t}{t})^{-\frac{n}{2}} \Big[\exp \Big(- \int_t^{2t} 2\sqrt{n}c(\tau)d\tau \Big) \Big] \exp \Big[\big(-d((x,t),(y,2t)) \big) \Big]$$
$$\Big[\exp \big(a_2(2t)|y|^2 - a_3(2t)\sqrt{1+|y|^2} \big) \Big]$$
$$= (\frac{1}{2})^{\frac{n}{2}} a_1(2t) u(x,t) \Big[\exp \Big(- \int_t^{2t} 2\sqrt{n}c(\tau)d\tau \Big) \Big] \int_{y\in\mathbb{R}^n} \Big[\exp \big(-d(x,t),(y,2t) \big) \Big]$$
$$\Big[\exp \big(a_2(2t)|y|^2 - a_3(2t)\sqrt{1+|y|^2} \big) \Big]$$

The conclusion follows immediately. □

COROLLARY 4.1. *Under the assumptions of Theorem 2.4, Theorem 4.4,* $u(x,t)$ *decays like a Gaussian.*

Proof. In view of Theorem 4.4, we need the following lower estimate of

$$\int_{y\in\mathbb{R}^n} \Big[\exp -d\Big(\big((x,t),(y,2t) \big) \Big) \Big] \Big[\exp \big(a_2(2t)|y|^2 - a_3(2t)\sqrt{1+|y|^2} \big) \Big]$$
$$\geq \int_{y\in\mathbb{R}^n} \Big[\exp \Big\{ -\frac{1}{2} \int_0^1 < y-x, f\big((1-s)x+sy, (1-s)t+2st \big) > ds$$
$$+ t \int_0^1 \Big\{ \sqrt{2}\alpha\big((1-s)t+2st \big)$$
$$\Big[\frac{1}{4} \sum_{i=1}^n f_i^2\big((1-s)x+sy, (1-s)t+2st \big) + \frac{1}{2}\beta\big((1-s)x+sy, (1-s)+2st \big) \Big]^{\frac{1}{2}}$$
$$+ \frac{1}{4} \sum_{i=1}^n f_i^2\big((1-s)x+sy, (1-s)t+2st \big) + V\big((1-s)x+sy, (1-s)t+2st \big) \Big\} ds$$
$$+ \frac{|y-x|^2}{4t} + \frac{\sqrt{2}}{2}|y-x| \int_0^1 \alpha\big((1-s)t+2st \big) ds + \frac{t}{2} \int_0^1 \alpha^2\big((1-s)t+2st \big) ds \Big\} \Big]$$
$$\exp \big(a_2(2t)|y|^2 - a_3(2t)\sqrt{1+|y|^2} \big)$$

Observe that in view of the Remark after Theorem 2.4, we have

$$\begin{aligned}
&\Big[\frac{|f((1-s)x+sy,(1+s)t)|^2}{4}+\frac{\beta((1-s)x+sy,(1+s)t)}{2}\Big]^{\frac12}\\
&\le \frac{|f((1-s)x+sy,(1-s)t)|}{2}+\frac{1}{\sqrt2}\sqrt{\beta((1-s)x+sy,(1+s)t)}\\
&\le \frac{c}{2}(1+(1-s)|x|+s|y|)+\frac{1}{\sqrt2}\sqrt{c^{4/3}\big|(1-s)x+sy\big|^2+\frac{4}{(T-t)^2}+\frac{n}{2t}}\\
&\le \frac{c}{2}(1+(1-s)|x|+s|y|)+\frac{c^{2/3}}{\sqrt2}((1-s)|x|+s|y|)+\frac{1}{\sqrt2}\big(\frac{4}{(T-t)^2}+\frac{n}{2t}\big)^{\frac12}\\
&\le \Big(\frac{c}{2}+\frac{c^{2/3}}{\sqrt2}\Big)(1-s)|x|+\big(\frac{c}{2}+\frac{c^{2/3}}{\sqrt2}\big)s|y|+\frac{c}{2}+\frac{2}{T-t}+\frac{\sqrt n}{\sqrt{2t}}
\end{aligned}$$

Hence

$$\begin{aligned}
&\sqrt2\alpha((1-s)t+2st)\Big[\frac{|f((1-s)x+sy,(1+s)t)|^2}{4}\\
&\qquad+\frac{\beta((1-s)x+sy,(1-s)t+2st)}{2}\Big]^{\frac12}\\
&\ge -\frac{\sqrt2}{T-(1+s)t}\Big[\big(\frac{c}{2}+\frac{c^{2/3}}{\sqrt2}\big)(1-s)|x|+\big(\frac{c}{2}+\frac{c^{2/3}}{\sqrt2}\big)s|y|+\frac{c}{2}+\frac{2}{T-t}+\frac{\sqrt n}{\sqrt{2t}}\Big]
\end{aligned}$$

On the other hand, we also have the following estimates

$$\begin{aligned}
V((1-s)x+sy,(1-s)t+2st) &\le c(1+|(1-s)x+sy|^2)\\
&\le c(1+2(1-s)^2|x|^2+2s^2|y|^2)\\
\frac14\big|f((1-s)x+sy,(1-s)t+2st)\big|^2 &\le \frac14 c^2(1+|(1-s)x+sy|)^2\\
&\le \frac{c^2}{2}(1+|(1-s)x+sy|^2)\\
&\le \frac{c^2}{2}(1+2(1-s)^2|x|^2+2s^2|y|^2)\\
< y-x, f((1-s)x+sy,(1-s)t+2st) >&\\
&\le |y-x|\,|f((1-s)x+sy,(1+s)t)|\\
&\le (|x|+|y|)c(1+(1-s)|x|+s|y|)\\
&= c((1-s)|x|^2+|x||y|+|x|+s|y|^2+|y|)
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_{y\in\mathbf{R}^n}\Big\{\exp\big[-d((x,t),(y,2t))\big]\Big\}\exp\big(a_2(2t)|y|^2-a_3(2t)\sqrt{1+|y|^2}\big)\\
&\ge\int_{y\in\mathbf{R}^n}\Big[\exp\Big\{-\frac{c}{2}\int_0^1[(1-s)|x|^2+|x||y|+|x|+s|y|^2+|y|]ds\\
&\quad -t\int_0^1\Big\{\frac{\sqrt2}{T-(1+s)t}\big[\frac{c}{2}+\frac{c^{2/3}}{\sqrt2}\big)((1-s)|x|+s|y|)+\frac{c}{2}+\frac{2}{T-t}+\frac{\sqrt n}{\sqrt{2t}}\Big]\\
&\quad+\frac{c^2}{2}(1+2(1-s)^2|x|^2+2s^2+|y|^2)+c\big(1+2(1-s)^2|x|^2+2s^2|y|^2\big)\Big\}ds
\end{aligned}$$

$$+\frac{|x-y|^2}{4t}+\frac{\sqrt{2}}{2}|x-y|\int_0^1\frac{ds}{T-(1+s)t}+\frac{t}{2}\int_0^1\frac{ds}{[T-(1+s)t]^2}\}]$$
$$\exp\left(a_2(2t)|y|^2-a_3(2t)\sqrt{1+|y|^2}\right)$$
$$=\int_{y\in\mathbb{R}^n}\exp\left\{[-\frac{c}{4}-\frac{t}{3}(c^2+2c)+\frac{1}{4t}]|x|^2+\text{ lower order term in }|x|\right\}$$
$$\exp\left(a_2(2t)|y|^2-a_3(2t)\sqrt{1+|y|^2}\right)$$

Observe that when t is sufficiently small, $\frac{1}{4t}-\frac{t}{2}(c^2+2c)-\frac{c}{4}$ is very positive. Therefore $u(x,t)$ decays like a Gaussian. □

5. The Duncan-Mortensen-Zakai equation. In the nonlinear filtering, we have the following signal observation model:

$$\text{(5.1)}\qquad \begin{cases} dx(t)=f(x(t))dt+g(x(t))dv(t) & , \quad x(0)=x_0\\ dy(t)=h(x(t))dt+dw(t) & , \quad y(0)=0\end{cases}$$

in which x,v,y and w are vector valued processes and v and w have components which are independent, standard Brownian processes.

Let ρ denote the conditional probability density of the state given the observation $\{y(t):0\le s\le t\}$. Then ρ can be obtained by normalizing σ which satisfies the Duncan-Mortensen-Zakai equation.

$$d\sigma=L_0(\sigma)dt+\sum_{i=1}^m L_i(\sigma)dy_i\ ,\ \sigma(0,x)=\sigma_0$$

where $L_0=\frac{1}{2}\Delta-\sum_{i=1}^n f_i\frac{\partial}{\partial x_i}-\sum_{n=1}^n\frac{\partial f_i}{\partial x_i}-\frac{1}{2}\sum_{i=1}^m h_i^2$ and L_i is the multiplication operator by h_i.

While (5.2) is a stochastic differential equation, Davis reduces it to a time varying partial differential equation by introducing a new unnormalized density $u=\exp(\frac{1}{2}\sum_{i=1}^m h_i^2(x)y_i(t))\sigma$, which satisfies the following equation

$$\text{(5.2)}\qquad \frac{\partial u}{\partial t}=L_0u+\sum_{i=1}^m y_i(t)[L_0,L_i]u+\frac{1}{2}\sum_{i,j=1}^n y_i(t)y_j(t)[[L_0,L_i],L_j]u$$

We can rewrite this equation as

(5.3)
$$\frac{\partial u}{\partial t}=\frac{1}{2}\Delta u+\sum_{i=1}^n\left(-f_i(x)+\sum_{j=1}^m y_j\frac{\partial h_j}{\partial x_i}\right)\frac{\partial u}{\partial x_i}-\left(\sum_{i=1}^n f_{i,i}+\frac{1}{2}\sum_{i=1}^M h_i^2\right.$$
$$\left.-\frac{1}{2}\sum_{i=1}^m y_i\Delta h_i+\sum_{i=1}^m\sum_{j=1}^n y_if_j\frac{\partial h_i}{\partial x_j}-\frac{1}{2}\sum_{i=1}^m\sum_{j=1}^m\sum_{k=1}^n y_iy_j\frac{\partial h_i}{\partial x_k}\frac{\partial h_j}{\partial x_k}\right)u$$

By changing variables from x_i to $\sqrt{2}\ x_i$ and by letting

$$\tilde{u}(x,t)\ =\ u(\frac{x}{\sqrt{2}},t)$$

$$\bar{f}_i(x) = f_i(\frac{x}{\sqrt{2}})$$
$$\bar{h}_i(x) = h_i(\frac{x}{\sqrt{2}}) ,$$

we obtain

(5.4)
$$\frac{\partial \bar{u}}{\partial t} = \Delta \bar{u} + \sum_{i=1}^{n} \big(-\sqrt{2}\bar{f}_i + 2\sum_{j=1}^{m} y_j \frac{\partial \bar{h}_j}{\partial x_i}\big) \frac{\partial \bar{u}}{\partial x_i} - \big(\sqrt{2}\sum_{i=1}^{n} \bar{f}_{i,i} + \frac{1}{2}\sum_{i=1}^{m} \bar{h}_i^2$$
$$- \sum_{i=1}^{m} y_i \Delta \bar{h}_i + \sqrt{2} \sum_{i=1}^{m}\sum_{j=1}^{n} y_i \bar{f}_j \frac{\partial \bar{h}_i}{\partial x_j} - \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{n} y_i y_i \frac{\partial \bar{h}_i}{\partial x_k}\frac{\partial \bar{h}_j}{\partial x_k}\big)\bar{u}$$

Hence (5.4) can be rewritten as

$$\bar{u}_t = \Delta \bar{u} + \sum_{i=1}^{n} \tilde{f}_i \bar{u}_i - \tilde{V}\bar{u} \tag{5.5}$$

where

$$\tilde{f}_i = -\sqrt{2}\bar{f}_i + 2\sum_{j=1}^{m} y_j \frac{\partial \bar{h}_j}{\partial x_i} \tag{5.6}$$

$$\tilde{V} = \sqrt{2}\sum_{i=1}^{n} \bar{f}_{i,i} + \frac{1}{2}\sum_{i=1}^{m} \bar{h}_i^2 - \sum_{i=1}^{m} y_i \Delta \bar{h}_i + \sqrt{2}\sum_{i=1}^{m}\sum_{j=1}^{n} y_i \bar{f}_j \frac{\partial \bar{h}_i}{\partial x_j} \tag{5.7}$$
$$- \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{n} y_i y_j \frac{\partial \bar{h}_i}{\partial x_k}\frac{\partial \bar{h}_j}{\partial x_k}$$

THEOREM 5.1. *Let $\bar{u}_R > 0$ equation (5.5) on the closed ball $B_R = \{x \in \mathbb{R}^n : |x| \le R\}$ with the Neumann condition $\frac{\partial \bar{u}_R}{\partial \nu} = 0$ on ∂B_R. Let $\bar{\varphi}(x,t) = -\log \bar{u}_R(x,t)$ and*

$$\bar{\Psi}(x,t) = \bar{\varphi}_t + |\nabla\varphi|^2 - \sum_{j=1}^{n} \tilde{f}_j \frac{\partial \bar{\varphi}}{\partial x_j} - \tilde{V} - \bar{\alpha}(t)\sqrt{|\nabla\varphi|^2 + \bar{\beta}(x,t)} - \frac{n}{2t} - 2\sqrt{n}\,\bar{c}(t)$$

Let $\beta(x,t) = \bar{c}_2(t)|x|^2 + 4\bar{\alpha}^2(t) + \frac{n}{2t}$ with $\bar{c}_2(t) > 0$ and $\bar{\alpha}(t) > 0$. Suppose that there exists $\bar{c}_1(t) > 0$ such that

$$|\tilde{f}| \le \bar{c}(t)(1+|x|) \tag{5.8}$$

$$|\nabla \tilde{f}| = \sqrt{\sum_{i=1}^{n} |\nabla \tilde{f}_i|^2} \le \bar{c}(t), |\nabla \tilde{V}| \le \bar{c}(1+|x|) \tag{5.9}$$

$$|\Delta \tilde{V}| \le \bar{c}, |\Delta \tilde{f}| \le \bar{c} \tag{5.10}$$

$$2\bar{c}^2(t) \le \bar{c}_1(t)\bar{c}_2(t) \tag{5.11}$$

$$2\bar{c}^2(t) \le \bar{c}_1(t)\big(4\bar{\alpha}^2(t) + \frac{n}{2t}\big) \tag{5.12}$$

$$(5.13) \qquad 4\bar{c}_2(t) \le \bar{c}_1^2(t)$$

$$(5.14) \qquad \bar{c}_2' + \frac{3}{t}\bar{c}_2 \ge 0$$

$$(5.15) \qquad 2(\bar{\alpha}^2)' + \frac{6}{t}\bar{\alpha}^2 + \frac{n}{4t^2} > 0$$

$$(5.16) \qquad \bar{\alpha}'(t) + \bar{\alpha}\Big[\frac{3\bar{c}}{2\sqrt{n}} - \bar{\lambda} - \frac{1}{2}(\sqrt{\bar{c}_1} + \bar{c}_1 + \bar{c}_1^{3/2})\Big] - \sqrt{n}\,\bar{c} \ge 0$$

$$(5.17) \qquad \bar{c}'(t) + \frac{5}{48\sqrt{n}}\bar{c}^2 + (\frac{3}{8t} - \frac{1}{2\sqrt{n}})\bar{c} - \frac{\sqrt{n}}{2}\,\bar{c}_2 > 0$$

where $\bar{\lambda}$ *is the absolute value of the greatest eigenvalue of* $(\frac{\tilde{f}_{i,j} + \tilde{f}_{j,i}}{2})$.

$$(5.18) \qquad \frac{\partial \tilde{V}}{\partial \nu} > \frac{R}{6}(1 + \frac{\sqrt{n-1}}{R})^2 + \frac{\bar{\alpha}(t)}{2}\bar{c}_1(t) \text{ on } \partial B_R \text{ for } t > 0 \,.$$

If $\bar{\Psi}(x,0) < 0$, *then* $\bar{\Psi}(x,t) < 0$ *for all* $t > 0$, *i.e.,*

$$(5.19) \qquad \bar{\varphi}_t + |\nabla\bar{\varphi}|^2 - \sum_{j=1}^{n} \tilde{f}_j \frac{\partial \bar{\varphi}}{\partial x_j} - \tilde{V} < \bar{\alpha}(t)\sqrt{|\nabla\bar{\varphi}|^2 + \bar{\beta}(x,t)} + \frac{n}{2t} + 2\sqrt{n}\,\bar{c}(t) \text{ for all } t \ge 0$$

PROPOSITION 5.2. *Assume*

$$|f| \le c(1+|x|), |\nabla f| = \sqrt{\sum_{i=1}^{n} |\nabla f_i|^2} \le c, |H(f)| = \sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{n}(\frac{\partial^2 f_j}{\partial x_k \partial x_i})^2} \le c \sqrt{\sum_{i=1}^{n} |\nabla \Delta f_i|^2} \le c, |\nabla h| = \sqrt{\sum_{j=1}^{m} |\nabla h_j|^2} \le c, |H(h)| = \sqrt{\sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{k=1}^{n}(\frac{\partial^2 h_j}{\partial x_k \partial x_i})^2} \le \frac{c}{1+|x|}$$

$$\sqrt{\sum_{i=1}^{m} |\nabla \Delta h_i|^2} \le \frac{c}{1+|x|} \text{ and } \sqrt{\sum_{i=1}^{m} (\Delta\Delta h_i)^2} \le c$$

Let $\tilde{f}_i$ *and* $\tilde{V}$ *be defined as in (5.6) and (5.7). then*

(i) $|\tilde{f}| \le \sqrt{2}\, c\Big(1 + \sqrt{\sum_{j=1}^{m} y_j^2}\Big)\,(1+|x|)$

(ii) $|\nabla \tilde{f}| \le \sqrt{2}\, c\Big(1 + \sqrt{\sum_{j=1}^{m} y_j^2}\Big)$

(iii) $|\tilde{V}| \le \Big(\frac{\sqrt{2n}\,c}{2} + c^2\Big)\Big(1 + \sqrt{\sum_{j=1}^{m} y_j^2}\Big)^2 (1+|x|^2)$

(iv) $|\nabla \tilde{V}| \leq \left(\frac{\sqrt{n}}{2}c + \frac{nc^2}{2}\right)\left(1 + \sqrt{\sum_{i=1}^{m} y_i^2}\right)^2 (1 + |x|)$

(v) $|\Delta \tilde{f}| \leq \left(1 + \sqrt{\sum_{j=1}^{m} y_j^2}\right) c$

(vi) $|\Delta \tilde{V}| \leq \left(\frac{\sqrt{n}}{2}c + c^2\right)\left(1 + \sqrt{\sum_{j=1}^{m} y_j^2}\right)^2$

Proof.

$$(i)\ |\tilde{f}| = |\sqrt{2}\bar{f} + 2\nabla \sum_{j=1}^{n} y_j \bar{h}_j|$$

$$\leq \sqrt{2}|f(\frac{x}{\sqrt{2}})| + \sqrt{2}|\sum_{j=1}^{n} y_j (\nabla h_j)(\frac{x}{\sqrt{2}})|$$

$$\leq \sqrt{2}c(1 + \frac{|x|}{\sqrt{2}}) + \sqrt{2}\sqrt{\sum_{j=1}^{m} y_j^2}\sqrt{\sum_{j=1}^{n} |\nabla h_j|^2 (\frac{x}{\sqrt{2}})}$$

$$\leq \sqrt{2}c(1 + |x|) + \sqrt{2}\ c\ \sqrt{\sum_{j=1}^{m} y_j^2}$$

$$\leq \sqrt{2}c\left(\ 1 + \sqrt{\sum_{j=1}^{m} y_j^2}\ \right)(1 + |x|)$$

$$(ii)\ |\nabla \tilde{f}| = \sqrt{\sum_{i=1}^{n} |\nabla \tilde{f}_i|^2} = \sqrt{\sum_{i=1}^{n} | - \sqrt{2}\nabla \bar{f}_i + 2\nabla \sum_{j=1}^{m} y_j \frac{\partial \bar{h}_j}{\partial x_i}|^2}$$

$$= \sqrt{\sum_{i=1}^{n} | - (\nabla f_i)(\frac{x}{\sqrt{2}}) + \left[\nabla(\sum_{j=1}^{m} y_i \frac{\partial h_j}{\partial x_i})\right](\frac{x}{\sqrt{2}})|^2}$$

$$\leq \sqrt{2}\sqrt{\sum_{i=1}^{n} |(\nabla f_i)(\frac{x}{\sqrt{2}})|^2 + \sum_{i=1}^{n} |\left[\nabla(\sum_{j=1}^{m} y_i \frac{\partial h_j}{\partial x_i})\right](\frac{x}{\sqrt{2}})|^2}$$

$$\leq \sqrt{2}\sqrt{\sum_{i=1}^{n} |(\nabla f_i)(\frac{x}{\sqrt{2}})|^2} + \sqrt{2}\sqrt{\sum_{i=1}^{n} |\left[\nabla(\sum_{j=1}^{m} y_j \frac{\partial h_j}{\partial x_i})\right](\frac{x}{\sqrt{2}})|^2}$$

$$\leq \sqrt{2}\ c + \sqrt{2}\sqrt{\sum_{i=1}^{n}\sum_{k=1}^{n}(\sum_{j=1}^{m} y_j \frac{\partial^2 h_j}{\partial x_k \partial x_i})^2 (\frac{x}{\sqrt{2}})}$$

$$\leq \sqrt{2}\ c + \sqrt{2}\sqrt{\sum_{i=1}^{n}\sum_{k=1}^{n}(\sum_{j=1}^{m} y_j^2)(\sum_{j=1}^{m}(\frac{\partial^2 h_j}{\partial x_k \partial x_i})^2 (\frac{x}{\sqrt{2}}))}$$

$$\leq \sqrt{2}\ c + \sqrt{2}\sqrt{\sum_{j=1}^{m} y_j^2}\sqrt{\sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{k=1}^{n}(\frac{\partial^2 h_j}{\partial x_k \partial x_i})^2 (\frac{x}{\sqrt{2}})}$$

$$\leq \sqrt{2}\,c + \sqrt{2}\,c\sqrt{\sum_{j=1}^{m} y_j^2} = \sqrt{2}\,c(1 + \sqrt{\sum_{j=1}^{m} y_j^2})$$

$$(iii)\ |\bar{V}| = |\sqrt{2}\sum_{i=1}^{n}\bar{f}_{i,i} + \frac{1}{2}\sum_{i=1}^{m}\bar{h}_i^2 - \sum_{i=1}^{m} y_i\Delta\bar{h}_i + \sqrt{2}\sum_{i=1}^{m}\sum_{j=1}^{n} y_i\bar{f}_j\frac{\partial\bar{h}_i}{\partial x_j} - \sum_{k=1}^{n}(\sum_{i=1}^{m} y_i\frac{\partial\bar{h}_i}{\partial x_k})^2|$$

$$= |\frac{1}{\sqrt{2}}\sum_{i=1}^{n} f_{i,i}(\frac{x}{\sqrt{2}}) + \frac{1}{2}\sum_{i=1}^{m} h_i^2(\frac{x}{\sqrt{2}}) - \frac{1}{2}\sum_{i=1}^{m} y_i(\Delta h_i)(\frac{x}{\sqrt{2}})$$

$$+ \sum_{i=1}^{m}\sum_{j=1}^{n} y_i f_j\frac{\partial h_i}{\partial x_j}(\frac{x}{\sqrt{2}}) - \frac{1}{2}\sum_{k=1}^{n}(\sum_{i=1}^{m} y_i\frac{\partial h_i}{\partial x_k})^2(\frac{x}{\sqrt{2}})|$$

$$\leq \frac{1}{\sqrt{2}}\sum_{i=1}^{n}|\frac{\partial f_i}{\partial x_i}|(\frac{x}{\sqrt{2}}) + \frac{1}{2}\sum_{i=1}^{m} h_i^2(\frac{x}{\sqrt{2}}) + \frac{1}{2}\sqrt{\sum_{i=1}^{m} y_i^2}\sqrt{\sum_{i=1}^{m}(\Delta h_i)^2(\frac{x}{\sqrt{2}})}$$

$$+ \sum_{i=1}^{m}|y_i|\sqrt{\sum_{j=1}^{n}|f_j|^2(\frac{x}{\sqrt{2}})}\sqrt{\sum_{j=1}^{n}(\frac{\partial h_i}{\partial x_j})(\frac{x}{\sqrt{2}})}$$

$$+ \frac{1}{2}\sum_{k=1}^{n}(\sum_{i=1}^{m} y_i^2)(\sum_{i=1}^{m}(\frac{\partial h_i}{\partial x_k})^2(\frac{x}{\sqrt{2}}))$$

$$\leq \sqrt{\frac{n}{2}}\sqrt{\sum_{i=1}^{n}(\frac{\partial f_i}{\partial x_i})^2(\frac{x}{\sqrt{2}})} + \frac{1}{2}\sum_{i=1}^{m} h_i^2(\frac{x}{\sqrt{2}}) + \frac{1}{2}\sqrt{\sum_{i=1}^{m} y_i^2}|\Delta h|(\frac{x}{\sqrt{2}})$$

$$+ \sqrt{\sum_{j=1}^{n} f_j^2(\frac{x}{\sqrt{2}})}\sqrt{\sum_{i=1}^{m} y_i^2}\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}(\frac{\partial h_i}{\partial x_j})^2(\frac{x}{\sqrt{2}})} + \frac{1}{2}(\sum_{i=1}^{m} y_i^2)\sum_{i=1}^{m}|\nabla h_i|^2(\frac{x}{\sqrt{2}})$$

$$\leq \sqrt{\frac{n}{2}}\,c + \frac{c^2}{2}(1+|x|)^2 + \frac{c}{2}\sqrt{\sum_{i=1}^{m} y_i^2} + c^2(1+\frac{|x|}{\sqrt{2}})\sqrt{\sum_{i=1}^{m} y_i^2} + \frac{c^2}{2}(\sum_{i=1}^{m} y_i^2)$$

$$\leq \sqrt{\frac{n}{2}}\,c + \frac{c^2}{2}(1+|x|^2) + \frac{c}{2}\sqrt{\sum_{i=1}^{m} y_i^2} + c^2(1+|x|)\sqrt{\sum_{i=1}^{m} y_i^2} + \frac{c^2}{2}(\sum_{i=1}^{m} y_i^2)$$

$$\leq \Big[\frac{\sqrt{2n}}{2}c + c^2 + (\frac{c}{2}+c^2)\sqrt{\sum_{i=1}^{m} y_i^2} + \frac{c^2}{2}\sum_{i=1}^{m} y_i^2\Big](1+|x|^2)$$

$$\leq (\frac{\sqrt{2n}c}{2} + c^2)(1 + \sqrt{\sum_{i=1}^{m} y_i^2})^2(1+|x|^2)$$

$$(iv)\ |\nabla\bar{V}| = |\sqrt{2}\sum_{i=1}^{m}\nabla\bar{f}_{i,i} + \frac{1}{2}\sum_{i=1}^{m}\nabla(\bar{h}_i^2) - \sum_{i=1}^{m} y_i\nabla(\Delta\bar{h}_i) + \sqrt{2}\sum_{i=1}^{m}\sum_{j=1}^{n} y_i\nabla(\bar{f}_j\frac{\partial\bar{h}_i}{\partial x_j})$$

$$\sqrt{2}\sum_{i=1}^{m}\nabla\bar{f}_{i,i} + \frac{1}{2}\sum_{i=1}^{m}\nabla(\bar{h}_i^2)$$

$$= |\frac{1}{\sqrt{2}}\sum_{i=1}^{n}(\nabla\frac{\partial f_i}{\partial x_i})(\frac{x}{\sqrt{2}}) + \frac{1}{\sqrt{2}}\sum_{i=1}^{m} h_i(\frac{x}{\sqrt{2}})(\nabla h_i)(\frac{x}{\sqrt{2}})$$

$$-\frac{1}{2\sqrt{2}}\sum_{i=1}^{m} y_i(\nabla\Delta h_i)(\frac{x}{\sqrt{2}}) + \frac{1}{\sqrt{2}}\sum_{i=1}^{m}\sum_{j=1}^{n}[\nabla(f_j\frac{\partial h_i}{\partial x_j})](\frac{x}{\sqrt{2}})$$

$$-\frac{1}{2\sqrt{2}}\sum_{i=1}^{m}[\sum_{k=1}^{n} y_i\frac{\partial h_i}{\partial x_k}(\frac{x}{\sqrt{2}})]\Big[\sum_{k=1}^{n} y_i(\nabla\frac{\partial h_i}{\partial x_k})(\frac{x}{\sqrt{2}})\Big]$$

$$\leq \frac{1}{\sqrt{2}}\sum_{i=1}^{n}|\nabla\frac{\partial f_i}{\partial x_i}|(\frac{x}{\sqrt{2}}) + \frac{1}{\sqrt{2}}\sum_{i=1}^{m}|h_i(\frac{x}{\sqrt{2}})||\nabla h_i|(\frac{x}{\sqrt{2}})$$

$$+\frac{1}{2\sqrt{2}}\sqrt{\sum_{i=1}^{m} y_i^2}\sqrt{\sum_{i=1}^{m}|\nabla\Delta h_i|^2(\frac{x}{\sqrt{2}})}$$

$$+\frac{1}{\sqrt{2}}\sqrt{\sum_{i=1}^{m} y_i^2}\sqrt{\sum_{i=1}^{m}\Big\{\sum_{i=1}^{n}|\nabla h_j|^2(\frac{x}{\sqrt{2}})\Big\}^2}$$

$$+\frac{1}{2\sqrt{2}}\sum_{i=1}^{m}\Big[\sqrt{n}|y_i|\sqrt{\sum_{k=1}^{n}(\frac{\partial h_i}{\partial x_k})^2(\frac{x}{\sqrt{2}})}\sqrt{n}|y_i|\sqrt{\sum_{k=1}^{n}|\nabla\frac{\partial h_i}{\partial x_k}|^2(\frac{x}{\sqrt{2}})}$$

$$\leq \frac{\sqrt{n}}{2}\sqrt{\sum_{i=1}^{n}|\nabla\frac{\partial f_i}{\partial x_i}|^2(\frac{x}{\sqrt{2}})} + \frac{1}{\sqrt{2}}\sqrt{\sum_{i=1}^{m}|h_i(\frac{x}{\sqrt{2}}|^2}\sqrt{\sum_{i=1}^{m}|\nabla h_i|^2(\frac{x}{\sqrt{2}})}$$

$$+\frac{1}{2\sqrt{2}}\sqrt{\sum_{i=1}^{m} y_i^2}\sqrt{\sum_{i=1}^{m}|\nabla\Delta h_i|^2(\frac{x}{\sqrt{2}})}$$

$$+\frac{1}{\sqrt{2}}\sqrt{\sum_{i=1}^{m} y_i^2}\sqrt{\sum_{i=1}^{m}\Big\{\sum_{j=1}^{n}|(\nabla f_j)\frac{\partial h_i}{\partial x_j} + f_j\nabla\frac{\partial h_i}{\partial x_j}|(\frac{x}{\sqrt{2}})\Big\}^2}$$

$$+\frac{n}{2\sqrt{2}}\sqrt{\sum_{i=1}^{m} y_i^2\sum_{i=1}^{m}\sum_{k=1}^{n}(\frac{\partial h_i}{\partial x_k})^2(\frac{x}{\sqrt{2}})}\sqrt{\sum_{i=1}^{m} y_i^2\sum_{i=1}^{m}\sum_{k=1}^{n}|\nabla\frac{\partial h_i}{\partial x_k}|^2(\frac{x}{\sqrt{2}})}$$

$$\leq \frac{\sqrt{n}}{2}\sqrt{\sum_{i=1}^{n}\sum_{k=1}^{n}(\frac{\partial^2 f_i}{\partial x_k\partial x_i})^2(\frac{x}{\sqrt{2}})} + \frac{1}{\sqrt{2}}|h(\frac{x}{\sqrt{2}})||\nabla h|(\frac{x}{\sqrt{2}})$$

$$+\frac{1}{2\sqrt{2}}\sqrt{\sum_{i=1}^{m} y_i^2}\sqrt{\sum_{i=1}^{m}|\nabla(\Delta h_i)|^2(\frac{x}{\sqrt{2}})}$$

$$+\frac{1}{\sqrt{2}}\sqrt{\sum_{i=1}^{m} y_i^2}\sqrt{\sum_{i=1}^{m} n\sum_{j=1}^{n}[|\nabla f_j||\frac{\partial h_i}{\partial x_j}| + |f_j||\nabla\frac{\partial h_i}{\partial x_j}|]^2(\frac{x}{\sqrt{2}})}$$

$$+\frac{n}{2\sqrt{2}}(\sum_{i=1}^{m} y_i^2)\sqrt{\sum_{i=1}^{m}|\nabla h_i|^2(\frac{x}{\sqrt{2}})}\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n}(\frac{\partial^2 h_i}{\partial x_j\partial x_k})^2(\frac{x}{\sqrt{2}})}$$

$$\leq \frac{\sqrt{n}}{2}c + \frac{1}{\sqrt{2}}c(1 + \frac{|x|}{\sqrt{2}})c + \frac{1}{2\sqrt{2}}\sqrt{\sum_{i=1}^{m} y_i^2}\, c$$

$$+\frac{1}{\sqrt{2}}\sqrt{\sum_{i=1}^{m}y_i^2}\sqrt{\sum_{i=1}^{m}2n\sum_{j=1}^{n}[|\nabla f_j|^2|\frac{\partial h_i}{\partial x_j}|^2+|f_j|^2|\nabla\frac{\partial h_i}{\partial x_j}|^2](\frac{x}{\sqrt{2}})}$$

$$+\frac{n}{2\sqrt{2}}(\sum_{i=1}^{m}y_i^2)c^2$$

$$\leq\frac{\sqrt{n}}{2}c+\frac{c^2}{2}(1+|x|)+\frac{c}{2\sqrt{2}}\sqrt{\sum_{i=1}^{m}y_i^2}$$

$$+\frac{1}{\sqrt{2}}\sqrt{\sum_{i=1}^{m}y_i^2}\sqrt{\sum_{i=1}^{m}2n\sum_{j=1}^{n}|\nabla f_j|^2|\frac{\partial h_i}{\partial x_j}|^2(\frac{x}{\sqrt{2}})}$$

$$+\frac{1}{\sqrt{2}}\sqrt{\sum_{i=1}^{m}y_i^2}\sqrt{\sum_{i=1}^{m}2n\sum_{j=1}^{n}|f_j|^2|\nabla\frac{\partial h_i}{\partial x_j}|^2(\frac{x}{\sqrt{2}})}+\frac{n}{2\sqrt{2}}(\sum_{i=1}^{m}y_i^2)c^2$$

$$=\frac{\sqrt{n}}{2}c+\frac{c^2}{2}(1+|x|)+\frac{c}{2\sqrt{2}}\sqrt{\sum_{i=1}^{m}y_i^2}+\frac{1}{\sqrt{2}}\sqrt{\sum_{i=1}^{m}y_i^2}\sqrt{2nc^2\sum_{i=1}^{m}|\nabla h_i|^2(\frac{x}{\sqrt{2}})}$$

$$+\frac{1}{\sqrt{2}}\sqrt{\sum_{i=1}^{m}y_i^2}\sqrt{2nc^2\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n}(\frac{\partial^2 h_i}{\partial x_k\partial x_j})^2(\frac{x}{\sqrt{2}})}+\frac{n}{2\sqrt{2}}(\sum_{i=1}^{m}y_i^2)c^2$$

$$\leq\frac{\sqrt{n}}{2}c+\frac{c^2}{2}(1+|x|)+\frac{c}{2\sqrt{2}}\sqrt{\sum_{i=1}^{m}y_i^2}+\frac{1}{\sqrt{2}}\sqrt{\sum_{i=1}^{m}y_i^2\sqrt{2n}c^2}$$

$$+\frac{1}{\sqrt{2}}\sqrt{\sum_{i=1}^{m}y_i^2\sqrt{2n}c^2}+\frac{n}{2\sqrt{2}}(\sum_{i=1}^{m}y_i^2)c^2$$

$$\leq\Big[\frac{\sqrt{n}}{2}c+\frac{c^2}{2}+(\frac{c}{2\sqrt{2}}+2\sqrt{n}c^2)\sqrt{\sum_{i=1}^{m}y_i^2}+\frac{nc^2}{2\sqrt{2}}(\sum_{i=1}^{m}y_i^2)\Big](1+|x|)$$

$$\leq(\frac{\sqrt{n}}{2}c+\frac{nc^2}{2})(1+\sqrt{\sum_{i=1}^{m}y_i^2})^2(1+|x|)$$

$$(v)\ |\Delta\tilde{f}|=\sqrt{\sum_{i=1}^{n}(\Delta\tilde{f}_i)^2}=\sqrt{\sum_{i=1}^{n}(-\sqrt{2}\Delta\tilde{f}_i+2\sum_{j=1}^{m}y_j\Delta\frac{\partial\tilde{h}_j}{\partial x_i})^2}$$

$$=\sqrt{\sum_{i=1}^{n}[-\frac{1}{\sqrt{2}}(\Delta f_i)(\frac{x}{\sqrt{2}})+\frac{1}{\sqrt{2}}\sum_{j=1}^{m}y_j(\Delta\frac{\partial h_j}{\partial x_i})(\frac{x}{\sqrt{2}})]^2}$$

$$\leq\sqrt{2}\sqrt{\sum_{i=1}^{n}\frac{1}{2}(\Delta f_i)^2(\frac{x}{\sqrt{2}})+\sum_{i=1}^{n}\frac{1}{2}[\sum_{j=1}^{m}y_j(\Delta\frac{\partial h_j}{\partial x_i})(\frac{x}{\sqrt{2}})]^2}$$

$$\le \sqrt{\sum_{i=1}^{n}(\triangle f_i)^2(\frac{x}{\sqrt{2}})} + \sqrt{\sum_{i=1}^{n}[\sum_{j=1}^{m} y_j(\triangle \frac{\partial h_j}{\partial x_i})(\frac{x}{\sqrt{2}})]^2}$$

$$\le c + \sqrt{\sum_{i=1}^{n}(\sum_{j=1}^{m} y_j^2)(\sum_{j=1}^{m}(\triangle \frac{\partial h_j}{\partial x_i})^2(\frac{x}{\sqrt{2}}))}$$

$$= c + \sqrt{\sum_{j=1}^{m} y_j^2}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}(\frac{\partial \triangle h_j}{\partial x_i})^2(\frac{x}{\sqrt{2}})}$$

$$= c + \sqrt{\sum_{j=1}^{m} y_j^2}\sqrt{\sum_{j=1}^{m}|\nabla(\triangle h_j)(\frac{x}{\sqrt{2}})}$$

$$\le \Big(1 + \sqrt{\sum_{j=1}^{m} y_j^2}\Big)c$$

(vi) $$|\triangle \bar{V}| = \Big|\sqrt{2}\sum_{i=1}^{n}\triangle \bar{f}_{i,i} + \frac{1}{2}\sum_{i=1}^{m}\triangle \bar{h}_i^2 - \sum_{i=1}^{m} y_i \triangle\triangle \bar{h}_i + \sqrt{2}\sum_{i=1}^{m}\sum_{j=1}^{n} y_i \triangle(\bar{f}_j \frac{\partial \bar{h}_i}{\partial x_j})$$
$$-\sum_{k=1}^{n}\triangle\Big[(\sum_{i=1}^{m} y_i \frac{\partial \bar{h}_i}{\partial x_k})^2\Big]\Big|$$

$$= \Big|\sqrt{2}\sum_{i=1}^{n}\triangle \bar{f}_{i,i} + \sum_{i=1}^{m}\bar{h}_i\triangle \bar{h}_i + \sum_{i=1}^{m}|\nabla \bar{h}_i|^2 - \sum_{i=1}^{m} y_i\triangle\triangle \bar{h}_i + \sqrt{2}\sum_{i=1}^{m}\sum_{j=1}^{n} y_i \bar{f}_j \triangle \frac{\partial \bar{h}_i}{\partial x_j}$$
$$+\sqrt{2}\sum_{i=1}^{m}\sum_{j=1}^{n} y_i(\triangle \bar{f}_j)\frac{\partial \bar{h}_i}{\partial x_j} + 2\sqrt{2}\sum_{i=1}^{m}\sum_{j=1}^{n} y_i \nabla \bar{f}_j \cdot \nabla \frac{\partial \bar{h}_i}{\partial x_j}$$
$$-2\sum_{k=1}^{n}(\sum_{i=1}^{m} y_i \frac{\partial \bar{h}_i}{\partial x_k})\triangle(\sum_{i=1}^{m} y_i \frac{\partial \bar{h}_i}{\partial x_k}) - 2\sum_{k=1}^{n}|\sum_{i=1}^{m} y_i \nabla(\frac{\partial \bar{h}_i}{\partial x_k})|^2\Big|$$

$$= \Big|\frac{1}{2}\sum_{i=1}^{n}(\triangle f_{i,i})(\frac{x}{\sqrt{2}}) + \frac{1}{2}(h_i\triangle h_i)(\frac{x}{\sqrt{2}}) + \frac{1}{2}\sum_{i=1}^{m}|\nabla h_i|^2(\frac{x}{\sqrt{2}})$$
$$-\frac{1}{4}\sum_{i=1}^{m} y_i\triangle\triangle h_i + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} y_i(f_j\triangle \frac{\partial h_i}{\partial x_j})(\frac{x}{\sqrt{2}})$$
$$+\sum_{i=1}^{m}\sum_{j=1}^{n} y_i(\nabla f_j \cdot \nabla \frac{\partial h_i}{\partial x_j})(\frac{x}{\sqrt{2}}) + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} y_i(\triangle f_j)\frac{\partial h_i}{\partial x_j}(\frac{x}{\sqrt{2}})$$
$$-\frac{1}{2}\sum_{k=1}^{n}(\sum_{i=1}^{m} y_i \frac{\partial h_i}{\partial x_k})(\sum_{i=1}^{m} y_i\triangle \frac{\partial h_i}{\partial x_k})(\frac{x}{\sqrt{2}}) - \frac{1}{2}\sum_{k=1}^{n}|\sum_{i=1}^{m} y_i\nabla(\frac{\partial h_i}{\partial x_k})|^2(\frac{x}{\sqrt{2}})\Big|$$

$$\le \frac{1}{2}\sum_{i=1}^{n}|\frac{\partial \triangle f_i}{\partial x_i}|(\frac{x}{\sqrt{2}}) + \frac{1}{2}\sqrt{\sum_{i=1}^{m} h_i^2(\frac{x}{\sqrt{2}})}\sqrt{\sum_{i=1}^{m}(\triangle h_i)^2(\frac{x}{\sqrt{2}})} + \frac{1}{2}\sum_{i=1}^{m}|\nabla h_i|^2(\frac{x}{\sqrt{2}})$$
$$+\frac{1}{4}\sqrt{\sum_{i=1}^{m} y_i^2}\sqrt{\sum_{i=1}^{m}(\triangle\triangle h_i)^2(\frac{x}{\sqrt{2}})} + \frac{1}{2}\sum_{i=1}^{m}|y_i|\sqrt{\sum_{j=1}^{n} f_j^2(\frac{x}{2})}\sqrt{\sum_{j=1}^{n}[\triangle \frac{\partial h_i}{\partial x_j}]^2(\frac{x}{\sqrt{2}})}$$

$$+\sum_{i=1}^{m}|y_i|\sqrt{\sum_{j=1}^{n}|\nabla f_j|^2(\frac{x}{\sqrt{2}})}\sqrt{\sum_{j=1}^{n}|\nabla\frac{\partial h_i}{\partial x_j}|^2(\frac{x}{\sqrt{2}})}$$

$$+\frac{1}{2}\sum_{i=1}^{m}|y_i|\sqrt{\sum_{j=1}^{n}(\Delta f_j)^2(\frac{x}{\sqrt{2}})}\sqrt{\sum_{j=1}^{n}(\frac{\partial h_i}{\partial x_j})^2(\frac{x}{\sqrt{2}})}$$

$$+\frac{1}{2}\sum_{k=1}^{n}\sqrt{\sum_{i=1}^{m}y_i^2}\sqrt{\sum_{i=1}^{m}(\frac{\partial h_i}{\partial x_k})^2(\frac{x}{\sqrt{2}})}\sqrt{\sum_{i=1}^{m}y_i^2}\sqrt{\sum_{i=1}^{m}(\Delta\frac{\partial h_i}{\partial x_k})^2(\frac{x}{\sqrt{2}})}$$

$$+\frac{1}{2}\sum_{k=1}^{n}(\sum_{i=1}^{m}y_i^2)(\sum_{i=1}^{m}|\nabla\frac{\partial h_i}{\partial x_k}|^2(\frac{x}{\sqrt{2}})$$

$$\leq\frac{\sqrt{n}}{2}\sqrt{\sum_{i=1}^{n}(\frac{\partial\Delta f_i}{\partial x_i})^2}+\frac{1}{2}c(1+\frac{|x|}{2})\frac{c}{1+\frac{|x|}{2}}+\frac{c^2}{2}$$

$$+\frac{1}{4}\sqrt{\sum_{i=1}^{m}y_i^2c}+\frac{1}{2}c(1+\frac{|x|}{2})\sqrt{\sum_{i=1}^{m}y_i^2}\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}(\frac{\partial}{\partial x_j}\Delta h_i)^2(\frac{x}{\sqrt{2}})}$$

$$+c\sqrt{\sum_{i=1}^{m}y_i^2}\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|\nabla\frac{\partial h_i}{\partial x_j}|^2(\frac{x}{\sqrt{2}})}+\frac{c}{2}\sqrt{\sum_{i=1}^{m}y_i^2}\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}(\frac{\partial h_i}{\partial x_j})^2(\frac{x}{\sqrt{2}})}$$

$$+\frac{1}{2}(\sum_{i=1}^{m}y_i^2)\sqrt{\sum_{k=1}^{n}\sum_{i=1}^{m}(\frac{\partial h_i}{\partial x_k})^2(\frac{x}{\sqrt{2}})}\sqrt{\sum_{k=1}^{n}\sum_{i=1}^{m}(\Delta\frac{\partial h_i}{\partial x_k})^2(\frac{x}{\sqrt{2}})}$$

$$+\frac{1}{2}\sum_{i=1}^{m}y_i^2\sum_{k=1}^{n}\sum_{i=1}^{m}|\nabla\frac{\partial h_i}{\partial x_k}|^2(\frac{x}{\sqrt{2}})$$

$$\leq\frac{\sqrt{n}}{2}\sqrt{\sum_{i=1}^{n}|\nabla\Delta f_i|^2}+\frac{c^2}{2}+\frac{c^2}{2}+\frac{c}{4}\sqrt{\sum_{i=1}^{m}y_i^2}$$

$$+\frac{c}{2}(1+\frac{|x|}{2})\sqrt{\sum_{i=1}^{m}y_i^2}\sqrt{\sum_{i=1}^{m}|\nabla\Delta h_i|^2(\frac{x}{\sqrt{2}})}$$

$$+c\sqrt{\sum_{i=1}^{m}y_i^2}\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n}(\frac{\partial^2 h_i}{\partial x_j\partial x_k})^2(\frac{x}{\sqrt{2}})}$$

$$+\frac{c^2}{2}\sqrt{\sum_{i=1}^{m}y_i^2}+\frac{1}{2}(\sum_{i=1}^{m}y_i^2)\sqrt{\sum_{i=1}^{m}|\nabla h_i|^2(\frac{x}{\sqrt{2}})}\sqrt{\sum_{i=1}^{m}|\nabla\Delta h_i|^2(\frac{x}{\sqrt{2}})}$$

$$+\frac{1}{2}\sum_{i=1}^{m}y_i^2\sum_{k=1}^{n}\sum_{i=1}^{m}\sum_{j=1}^{n}(\frac{\partial^2 h_i}{\partial x_j\partial x_k})^2(\frac{x}{\sqrt{2}})$$

$$\leq\frac{\sqrt{n}}{2}c+\frac{c^2}{2}+\frac{c^2}{2}+\frac{c}{4}\sqrt{\sum_{i=1}^{m}y_i^2}+\frac{c^2}{2}\sqrt{\sum_{i=1}^{m}y_i^2}+c^2\sqrt{\sum_{i=1}^{m}y_i^2}$$

$$+\frac{c^2}{2}\sqrt{\sum_{i=1}^m y_i^2}+\frac{c^2}{2}(\sum_{i=1}^m y_i^2)+\frac{c^2}{2}\sum_{i=1}^m y_i^2$$

$$=\frac{\sqrt{n}}{2}c+c^2+\frac{c}{4}\sqrt{\sum_{i=1}^m y_i^2}+2c^2\sqrt{\sum_{i=1}^m y_i^2}+c^2(\sum_{i=1}^m y_i^2)$$

$$=(\frac{\sqrt{n}}{2}+c)c+(\frac{c}{4}+2c^2)\sqrt{\sum_{i=1}^m y_i^2}+c^2(\sum_{i=1}^m y_i^2)$$

$$\leq(\frac{\sqrt{n}}{2}c+c^2)(1+\sqrt{\sum_{i=1}^m y_i^2})^2$$

Remark 5.1 Under the assumptions of Proposition 5.2, we can take

$$\bar{c}(t)=\Big[\frac{\sqrt{n}}{2}c+\frac{(n+1)c^2}{2}\Big](1+\sqrt{\sum_{i=1}^m y_i^2})^2$$

so that (5.8), (5.9), (5.10) are satisfied. □

In view of Proposition 3.1, we have the following Harnack inequality.

THEOREM 5.3. *Let $\bar{u}_R>0$ be a positive solution of the equation (5.5) on the closed ball $B_R=\{x\in\mathbb{R}^n:|x|\leq R\}$ with the Neumann condition $\frac{\partial\bar{u}_R}{\partial\nu}=0$ on ∂B_R. Let $\bar{\varphi}(x,t)=-\log\bar{u}_R(x,t)$ and $\bar{\beta}(x,t)=\bar{c}_2(t)|x|^2+4\bar{\alpha}^2(t)+\frac{n}{2t}$ with $\bar{c}_2(t)>0$ and $\bar{\alpha}(t)>0$. Suppose that (5.8)-(5.18) hold so that*

$$\bar{\varphi}_t+|\nabla\bar{\varphi}|^2-\sum_{j=1}^n\bar{f}_j\frac{\partial\bar{\varphi}}{\partial x_j}-\bar{V}<\bar{\alpha}(t)\sqrt{|\nabla\bar{\varphi}|^2+\bar{\beta}(x,t)}+\frac{n}{2t}+2\sqrt{n}\bar{c}(t)$$

for all $t>0$. Let $t>t_0$ and $\mathcal{P}=\{$ differentiable path $\sigma=(\sigma_1,\sigma_2):[0,1]\to B_R\times\mathbb{R}$ such that $\sigma(0)=(\sigma_1(0),\sigma_2(0))=(x_0,t_0),\sigma(1)=(x,t),\sigma_2'(s)>0\}$. Define

(5.20)

$$d((x_0,t_0),(x,t))=\inf_{\sigma\in\mathcal{P}}\Big\{\frac{1}{2}\int_0^1<\dot{\sigma}_1,\bar{f}>ds+\int_0^1\frac{d\sigma_2}{ds}\Big(\sqrt{2}\bar{\alpha}\sqrt{\frac{|\bar{f}|^2}{4}+\frac{\bar{\beta}}{2}}+\bar{V}+\frac{1}{4}\sum_{i=1}^n\bar{f}_i^2\Big)ds+\frac{1}{4}\int_0^1\frac{d\sigma_2}{ds}\Big[\frac{|\dot{\sigma}_1|}{\frac{d\sigma_2}{ds}}+\sqrt{2}\bar{\alpha}\Big]^2ds\Big\}$$

Then

(5.21) $$\bar{u}_R(x,t)\geq\bar{u}_R(x_0,t_0)(\frac{t}{t_0})^{-\frac{n}{2}}\exp\Big(-\int_{t_0}^t2\sqrt{n}\bar{c}(\tau)d\tau\Big)\exp\Big[-d((x_0,t_0),(x,t))\Big]$$

Similarly Theorem 3.3 gives us the following Harnack inequality.

THEOREM 5.4. *Let $\bar{u} > 0$ be a positive solution of the equation*

$$\bar{u}_t = \Delta \bar{u} + \sum_{i=1}^{n} \bar{f}_i \bar{u}_i - \bar{V}\bar{u}$$

on $\mathbb{R}^n$. Let $\bar{\varphi}(x,t) = -\log \bar{u}(x,t)$ and $\bar{\beta}(x,t) = \bar{c}_2(t)|x|^2 + 4\bar{\alpha}^2(t) + \frac{n}{2t}$ with $\bar{c}_2(t) > 0$ and $\bar{\alpha}(t) > 0$. Suppose that

$$(5.22)\qquad |\bar{f}| \le \bar{c}(t)(1+|x|)$$

$$(5.23)\qquad |\nabla \bar{f}| = \sqrt{\sum_{i=1}^{n} |\nabla \bar{f}_i|^2} \le \bar{c}(t), |\nabla \bar{V}| \le \bar{c}(1+|x|)$$

$$(5.24)\qquad |\Delta \bar{V}| \le \bar{c}, |\Delta \bar{f}| \le \bar{c}$$

$$(5.25)\qquad 2\bar{c}^2(t) \le \bar{c}_1(t)\bar{c}_2(t)$$

$$(5.26)\qquad 2\bar{c}^2(t) \le \bar{c}_1(t)(4\bar{\alpha}^2(t) + \frac{n}{2t})$$

$$(5.27)\qquad 4\bar{c}_2(t) \le \bar{c}_1^2(t)$$

$$(5.28)\qquad \bar{c}_2' + \frac{3}{t}\bar{c}_2 \ge 0$$

$$(5.29)\qquad 2(\bar{\alpha}^2)' + \frac{6}{t}\bar{\alpha}^2 + \frac{n}{4t^2} > 0$$

$$(5.30)\qquad \bar{\alpha}'(t) + \bar{\alpha}\Big[\frac{3\bar{c}}{2\sqrt{n}} - \bar{\lambda} - \frac{1}{2}(\sqrt{\bar{c}_1} + \bar{c}_1 + \bar{c}_1^{3/2})\Big] - \sqrt{n}\bar{c} \ge 0$$

$$(5.31)\qquad \bar{c}'(t) + \frac{5}{48\sqrt{n}}\bar{c}^+\Big(\frac{3}{8t} - \frac{1}{2\sqrt{n}}\Big)\bar{c} - \frac{\sqrt{n}}{2}\bar{c}_2 > 0$$

where $\bar{\lambda}$ is the absolute value of the greatest eigenvalue of $(\frac{\bar{f}_{i,j} + \bar{f}_{j,i}}{2})$. Let

$$\begin{aligned}\bar{\Psi}(x,t) &= \bar{\varphi}_t + |\nabla\varphi|^2 - \sum_{j=1}^{n} \bar{f}_j \frac{\partial \bar{\varphi}}{\partial x_j} - \bar{V} - \bar{\alpha}(t)\sqrt{|\nabla\varphi|^2 + \bar{\beta}(x,t)} - \frac{n}{2t} - 2\sqrt{n}\bar{c}(t)\\ &= \Delta\varphi - \bar{\alpha}(t)\sqrt{|\nabla\varphi|^2 + \bar{\beta}(x,t)} - \frac{n}{2t} - 2\sqrt{n}\bar{c}(t)\end{aligned}$$

Let $t > t_0$ and $\mathcal{P} = \{$ diferentiable path $\sigma = (\sigma_1, \sigma_2) : [0,1] \to \mathbb{R}^n \times \mathbb{R}$ such that $\sigma(0) = (\sigma_1(0), \sigma_2(0)) = (x_0, t_0), \sigma(1) = (x,t), \sigma_2'(s) > 0\}$. Define

$$(5.32)\quad \begin{aligned} d\big((x_0,t_0),(x,t)\big) = \inf_{\sigma\in\mathcal{P}} \Big\{\frac{1}{2}\int_0^1 < \dot{\sigma}_1, \bar{f} > ds + \int_0^1 \frac{d\sigma_2}{ds}\Big(\sqrt{2}\bar{\alpha}\sqrt{\frac{|\bar{f}|^2}{4} + \frac{\bar{\beta}}{2}} \\ + \bar{V} + \frac{1}{4}\sum_{i=1}^{n} \bar{f}_i^2\Big)ds + \frac{1}{4}\int_0^1 \frac{d\sigma_2}{ds}\Big[\frac{|\dot{\sigma}_1|}{\frac{d\sigma_2}{ds}} + \sqrt{2}\bar{\alpha}\Big]^2 ds\Big\}\end{aligned}$$

If $\Psi(x,0) < 0$, then

$$(5.33)\quad \bar{u}(x,t) \ge \bar{u}(x_0,t_0)\Big(\frac{t}{t_0}\Big)^{-\frac{n}{2}} \exp\Big(-\int_{t_0}^{t} 2\sqrt{n}\bar{c}(\tau)d\tau\Big) \exp\Big[-d\big((x_0,t_0),(x,t)\big)\Big]$$

It is a very interesting question to find path σ that maximize the right hand side of (5.33). It is likely that such a path carries most information of how probability density propagates in time.

In order to apply (5.33) for proving decay of the solution $\bar{u}$, we need some integral estimate of $\bar{u}$.

THEOREM 5.5. *Let $\bar{u}$ be a nonnegatie solution of the equation*

$$\frac{\partial \bar{u}}{\partial t} = \Delta \bar{u} + \sum_{i=1}^{n} \tilde{f}_i \frac{\partial \bar{u}}{\partial x_i} - \tilde{V}\bar{u}$$

on $\mathbb{R}^n$, where $\tilde{f}$ and $\tilde{V}$ are given by (5.6) and (5.7). Suppose that

$$(5.34) \qquad \begin{aligned} &|f| \le c(t)(1+|x|), |\nabla f| = \sqrt{\sum_{i=1}^{n} |\nabla f_i|^2} \le c(t)(1+|x|) \\ &|h| \le c(t)(1+|x|^2), |\nabla h| = \sqrt{\sum_{i=1}^{m} |\nabla h_i|^2} \le c(t) \end{aligned}$$

Let ε_2 and T be small enough so that for $0 \le t \le T$,

$$(5.35) \qquad \begin{aligned} \frac{|x||\bar{f}|\sqrt{2}}{4(t+\varepsilon_2)} \le & \frac{3n}{4(t+\varepsilon_2)} + \frac{|x|^2}{64(t+\varepsilon_2)^2} - \frac{3}{2}\Big(\sum_{j=1}^{m} |\nabla \bar{h}_j|^2\Big)^2 \\ & - \frac{1}{2}\sum_{j=1}^{m} (\Delta \bar{h}_j)^2 + \frac{1}{2}\sum_{i=1}^{m} \bar{h}_i^2 - \frac{\sqrt{2}}{2}\sum_{i=1}^{m}\Big(\sum_{j=1}^{n} \bar{f}_j \frac{\partial \bar{h}_i}{\partial x_j}\Big) \end{aligned}$$

Then

$$\begin{aligned} \int_{\mathbb{R}^n} \rho(x,t)\bar{u}(x,t) \le & \Big\{ \exp\Big[\frac{3}{2}\int_0^T \Big(\sum_{j=1}^{m} y_j^2\Big)^2 + \frac{1+\sqrt{2}}{2}\int_0^T \sum_{j=0}^{m} y_j^2\Big]\Big\} \\ & \int_{\mathbb{R}^n} \rho(x,0)\bar{u}(x,0) \end{aligned}$$

where $\rho(x,t) = (t+\varepsilon_2)^{-n} \exp\left(\dfrac{|x|^2}{8(t+\varepsilon_2)}\right)$

Proof. Let ρ be any smooth function on $\mathbb{R}^n$. We have

$$\begin{aligned} \frac{d}{dt}\int_{B_R} \rho\bar{u} &= \int_{B_R} \rho_t \bar{u} + \int_{B_R} \rho \bar{u}_t \\ &= \int_{B_R} \rho_t \bar{u} + \int_{B_R} \rho \Delta \bar{u} + \int_{B_R} \rho \sum_{i=1}^{n} \tilde{f}_i \frac{\partial \bar{u}}{\partial x_i} - \int_{B_R} \rho \tilde{V}\bar{u} \\ &= \int_{B_R} \rho_t \bar{u} + \int_{B_R} (\Delta\rho)\bar{u} - \int_{\partial B_R} \bar{u}\frac{\partial \rho}{\partial \nu} ds + \int_{\partial B_R} \rho \frac{\partial \bar{u}}{\partial \nu} ds \\ &\quad - \int_{B_R} \Big(\rho \sum_{i=1}^{n} \tilde{f}_{i,i}, + \sum_{i=1}^{n} \rho_i \tilde{f}_i\Big)\bar{u} + \int_{\partial B_R} \bar{u}\rho\tilde{f}\cdot\nu ds - \int_{B_R} \rho\tilde{V}\bar{u} \end{aligned}$$

by Divergence Theorem and Green's first identity. Now we can use simple cut off argument to obtain the following

(5.36)

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^n}\rho\bar{u} &= \int_{\mathbb{R}^n}\Big(\rho_t+\triangle\rho-\sum_{i=1}^n\tilde{f}_i\rho_i-\sum_{i=1}^n\tilde{f}_{i,i}\rho-\tilde{V}\rho\Big)\bar{u}\\
\rho &= (t+\varepsilon_2)^{-n}\exp\Big[\frac{|x|^2}{8(t+\varepsilon_2)}\Big]\\
\rho_i &= \frac{1}{4(t+\varepsilon_2)}(t+\varepsilon_2)^{-n}x_i\exp\Big[\frac{|x|^2}{8(t+\varepsilon_2)}\Big]=\frac{x_i\rho}{4(t+\varepsilon_2)}\\
\rho_{ii} &= \frac{\rho}{4(t+\varepsilon_2)}+\frac{x_i^2\rho}{16(t+\varepsilon_2)^2}\\
\triangle\rho &= \Big[\frac{n}{4(t+\varepsilon_2)}+\frac{|x|^2}{16(t+\varepsilon_2)^2}\Big]\rho\\
\rho_t+\triangle\rho &= -n(t+\varepsilon_2)^{-n-1}\exp\Big[\frac{|x|^2}{8(t+\varepsilon_2)}\Big]-\frac{|x|^2}{8(t+\varepsilon_2)}(t+\varepsilon_2)^{-n}\exp\Big[\frac{|x|^2}{8(t+\varepsilon_2)}\Big]\\
&\quad+\Big[\frac{n}{4(t+\varepsilon_2)}+\frac{|x|^2}{16(t+\varepsilon_2)^2}\Big]\rho\\
&= \Big[\frac{-n}{t+\varepsilon_2}-\frac{|x|^2}{8(t+\varepsilon_2)^2}+\frac{n}{4(t+\varepsilon_2)}+\frac{|x|^2}{16(t+\varepsilon_2)^2}\Big]\rho\\
&= -\Big[\frac{3n}{4(t+\varepsilon_2)}+\frac{|x|^2}{16(t+\varepsilon_2)^2}\Big]\rho
\end{aligned}
$$

Observe that

$$
\begin{aligned}
-\sum_{i=1}^n\tilde{f}_i\rho_i &\le \Big|\sum_{i=1}^n\tilde{f}_i\rho_i\Big|\\
&= \Big|\sum_{i=1}^n\sqrt{2}\bar{f}_ix_i-2\sum_{i=1}^n\sum_{j=1}^m x_iy_i\frac{\partial\bar{h}_j}{\partial x_i}\Big|\frac{\rho}{4(t+\varepsilon_2)}\\
&\le \frac{\sqrt{2}\rho}{4(t+\varepsilon_2)}|x||\bar{f}|+\frac{\rho}{2(t+\varepsilon_2)}\Big[\frac{|x|^2}{16(t+\varepsilon_2)}+4(t+\varepsilon_2)\sum_{i=1}^n\Big(\sum_{j=1}^m y_j\frac{\partial\bar{h}_j}{\partial x_i}\Big)^2\Big]\\
&= \Big[\frac{|x||\bar{f}|\sqrt{2}}{4(t+\varepsilon_2)}+\frac{|x|^2}{32(t+\varepsilon_2)^2}+2\sum_{i=1}^n\Big(\sum_{j=1}^m y_j\frac{\partial\bar{h}_j}{\partial x_i}\Big)^2\Big]\rho
\end{aligned}
$$

$$
\begin{aligned}
-\Big(\sum_{i=1}^n\tilde{f}_{i,i}+\tilde{V}\Big)\rho &= -\Big(\sum_{i=1}^n\tilde{f}_{i,i}+\tilde{V}\Big)\rho\\
&= -\Big[\sum_{i=1}^n\Big(-\sqrt{2}\bar{f}_i+2\sum_{j=1}^m y_j\frac{\partial\bar{h}_j}{\partial x_i}\Big)_{,i}+\sqrt{2}\sum_{i=1}^n\bar{f}_{i,i}+\frac{1}{2}\sum_{i=1}^m\bar{h}_i^2\\
&\quad-\sum_{i=1}^m y_i\triangle\bar{h}_i+\sqrt{2}\sum_{i=1}^m\sum_{j=1}^n y_i\bar{f}_j\frac{\partial\bar{h}_i}{\partial x_j}
\end{aligned}
$$

$$-\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{n} y_i y_j \frac{\partial \bar{h}_i}{\partial x_k}\frac{\partial \bar{h}_j}{\partial x_k}\Big]\rho$$

$$= -\Big[-\sqrt{2}\sum_{i=1}^{n}\bar{f}_{i,i} + 2\sum_{j=1}^{m} y_j \Delta\bar{h}_j + \sqrt{2}\sum_{i=1}^{n}\bar{f}_{i,i} + \frac{1}{2}\sum_{i=1}^{m}\bar{h}_i^2$$

$$-\sum_{i=1}^{m} y_i\Delta\bar{h}_i + \sqrt{2}\sum_{i=1}^{m}\sum_{j=1}^{n} y_i\bar{f}_j\frac{\partial \bar{h}_i}{\partial x_j} - \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{n} y_i y_j \frac{\partial \bar{h}_i}{\partial x_k}\frac{\partial \bar{h}_j}{\partial x_k}\Big]\rho$$

$$= -\Big[2\sum_{j=1}^{m} y_j\Delta\bar{h}_j + \frac{1}{2}\sum_{i=1}^{m}\bar{h}_i^2 + \sqrt{2}\sum_{i=1}^{m}\sum_{j=1}^{n} y_i\bar{f}_j\frac{\partial \bar{h}_i}{\partial x_j}$$

$$-\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{n} y_i y_j \frac{\partial \bar{h}_i}{\partial x_k}\frac{\partial \bar{h}_j}{\partial x_k}\Big]\rho$$

Hence

$$\frac{d}{dt}\int_{\mathbb{R}^n}\rho\bar{u} \le \int_{\mathbb{R}^n}\rho\Big[\frac{-3n}{4(t+\varepsilon_2)} - \frac{|x|^2}{16(t+\varepsilon_2)^2} + \frac{|x||\bar{f}|\sqrt{2}}{4(t+\varepsilon_2)}$$

$$+\frac{|x|^2}{32(t+\varepsilon_2)^2} + 2\sum_{i=1}^{n}\Big(\sum_{j=1}^{m} y_j\frac{\partial \bar{h}_j}{\partial x_i}\Big)^2 - \sum_{j=1}^{m} y_j\Delta\bar{h}_j - \frac{1}{2}\sum_{i=1}^{m}\bar{h}_i^2$$

$$-\sqrt{2}\sum_{i=1}^{m}\sum_{j=1}^{n} y_i\bar{f}_j\frac{\partial \bar{h}_i}{\partial x_j} + \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{n} y_i y_j\frac{\partial \bar{h}_i}{\partial x_k}\frac{\partial \bar{h}_j}{\partial x_k}\Big]\bar{u}$$

$$= \int_{\mathbb{R}^n}\rho\Big[\frac{-3n}{4(t+\varepsilon_2)} - \frac{|x|^2}{32(t+\varepsilon_2)^2} + \frac{|x||\bar{f}|\sqrt{2}}{4(t+\varepsilon_2)}$$

$$+3\sum_{i=1}^{n}\Big(\sum_{j=1}^{m} y_j\frac{\partial \bar{h}_j}{\partial x_i}\Big)^2 - \sum_{j=1}^{m} y_j\Delta\bar{h}_j - \frac{1}{2}\sum_{i=1}^{m}\bar{h}_i^2$$

$$-\sqrt{2}\sum_{i=1}^{m}\sum_{j=1}^{n} y_i\bar{f}_j\frac{\partial \bar{h}_i}{\partial x_j}\Big]\bar{u}$$

By Schwartz inequality, we have

$$\sum_{i=1}^{n}\Big(\sum_{j=1}^{m} y_j\frac{\partial \bar{h}_j}{\partial x_i}\Big)^2 \le \sum_{i=1}^{n}\Big(\sum_{j=1}^{m} y_j^2\Big)\Big(\sum_{j=1}^{m}\frac{\partial \bar{h}_j}{\partial x_i}\Big)^2 = \Big(\sum_{j=1}^{m} y_j^2\Big)\Big(\sum_{j=1}^{m}|\nabla\bar{h}_j|^2\Big)$$

$$\le \frac{1}{2}\Big(\sum_{j=1}^{m} y_j^2\Big)^2 + \frac{1}{2}\Big(\sum_{j=1}^{m}|\nabla\bar{h}_j|^2\Big)^2$$

$$\sum_{j=1}^{m} y_j\Delta\bar{h}_j \le \Big|\sum_{j=1}^{m} y_j\Delta\bar{h}_j\Big| \le \frac{1}{2}\sum_{j=1}^{m} y_j^2 + \frac{1}{2}\sum_{j=1}^{m}(\Delta\bar{h}_j)^2$$

$$-\sqrt{2}\sum_{i=1}^{m}\sum_{j=1}^{n} y_i\bar{f}_j\frac{\partial \bar{h}_i}{\partial x_j} \le \sqrt{2}\Big|\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(\bar{f}_j\frac{\partial \bar{h}_i}{\partial x_j}\Big)y_i\Big|$$

$$\le \frac{\sqrt{2}}{2}\sum_{i=1}^{m}\Big(\sum_{j=1}^{n}\bar{f}_j\frac{\partial \bar{h}_i}{\partial x_j}\Big)^2 + \frac{\sqrt{2}}{2}\sum_{i=1}^{m} y_i^2$$

Hence

$$\frac{d}{dt}\int_{\mathbb{R}^n}\rho\bar{u}\leq\int_{\mathbb{R}^n}\rho\Big[\frac{-3n}{4(t+\varepsilon_2)}-\frac{|x|^2}{32(t+\varepsilon_2)^2}+\frac{\sqrt{2}|x||\bar{f}|}{4(t+\varepsilon_2)}$$
$$+\frac{3}{2}\big(\sum_{j=1}^m y_j^2\big)^2+\frac{3}{2}\big(\sum_{j=1}^m|\nabla\bar{h}_j|^2\big)^2+\frac{1}{2}\sum_{j=1}^m y_j^2+\frac{1}{2}\sum_{j=1}^m(\Delta\bar{h}_j)^2$$
$$-\frac{1}{2}\sum_{i=1}^m\bar{h}_i^2+\frac{\sqrt{2}}{2}\sum_{i=1}^m\big(\sum_{j=1}^m\bar{f}_j\frac{\partial\bar{h}_i}{\partial x_j}\big)^2+\frac{\sqrt{2}}{2}\sum_{i=1}^m y_i^2\Big]\bar{u}$$
$$=\int_{\mathbb{R}^n}\rho\Big[\frac{-3n}{4(t+\varepsilon)}-\frac{|x|^2}{32(t+\varepsilon_2)^2}+\frac{\sqrt{2}|x||f|}{4(t+\varepsilon_2)}+\frac{3}{2}\big(\sum_{i=1}^m y_j^2\big)^2$$
$$+\frac{3}{2}\big(\sum_{j=1}^m|\nabla\bar{h}_j|^2\big)^2+\frac{1+\sqrt{2}}{2}\sum_{j=1}^m y_j^2+\frac{1}{2}\sum_{j=1}^m(\Delta\bar{h}_j)^2-\frac{1}{2}\sum_{i=1}^m\bar{h}_i^2$$
$$+\frac{\sqrt{2}}{2}\sum_{i=1}^m\big(\sum_{j=1}^n\bar{f}_j\frac{\partial\bar{h}_i}{\partial x_j}\big)^2\Big]\bar{u}$$

Choose ε_2 and T small enough so that for $0\leq t\leq T$, we have

$$\frac{|x||f|\sqrt{2}}{4(t+\varepsilon_2)}\leq\frac{3n}{4(t+\varepsilon_2)}+\frac{|x|^2}{64(t+\varepsilon_2)^2}-\frac{3}{2}\big(\sum_{j=1}^m|\nabla\bar{h}_j|^2\big)^2-\frac{1}{2}\sum_{j=1}^m(\Delta\bar{h}_j)^2$$
$$+\frac{1}{2}\sum_{i=1}^m\bar{h}_i^2-\frac{\sqrt{2}}{2}\sum_{i=1}^m\big(\sum_{j=1}^n\bar{f}_j\frac{\partial\bar{h}_i}{\partial x_j}\big)^2$$

Then we have

$$\frac{d}{dt}\int_{\mathbb{R}^n}\rho\bar{u}\leq\Big[\frac{3}{2}\big(\sum_{j=1}^m y_j^2\big)^2+\frac{1+\sqrt{2}}{2}\sum_{j=1}^m y_j^2\Big]\int_{\mathbb{R}^n}\rho\bar{u}$$

This implies

$$\int_{\mathbb{R}^n\times\{t\}}\rho\bar{u}\leq\Big\{\exp\Big[\frac{3}{2}\int_0^T\big(\sum_{j=1}^m y_j^2\big)^2+\frac{1+\sqrt{2}}{2}\int_0^T\sum_{j=1}^m y_j^2\Big]\Big\}\int_{\mathbb{R}^n\times\{0\}}\rho\bar{u}$$

□

Now we are ready to do the pointwise estimate.

THEOREM 5.6. *Let $\bar{u}$ be a nonnegative solution of the equation*

$$\frac{\partial\bar{u}}{\partial t}=\Delta\bar{u}+\sum_{i=1}^n\bar{f}_i\frac{\partial\bar{u}}{\partial x_i}-\bar{V}\bar{u}$$

Suppose that the assumptions of Proposition 5.2 hold. Suppose further that (5.22)-(5.31) hold. Let ε_2 and $2T$ be chosen small enough so that for $0\leq t\leq 2T$, (5.35) holds. Then for $x\in B_{R/2}$.

$$(5.37)\quad \bar{u}(x,t) \le \frac{1}{\omega_n}\Big[\frac{R}{2(2t+\varepsilon_2)}\Big]^{-n}\Big[\exp\Big(\frac{-(\frac{R}{2}-|x|)^2}{8(2t+\varepsilon_2)}\Big)\Big]$$
$$(2)^{\frac{n}{2}}\Big[\exp\int_t^{2t} 2\sqrt{n}\bar{c}(s)ds\Big]\max_{|z-x|\le R/2}\exp\Big[d\big((x,t),(z,2t)\big)\Big]$$
$$\Big\{\exp\Big[\frac{3}{2}\int_0^{2t}\Big(\sum_{j=1}^m y_j^2\Big)^2+\frac{1+\sqrt{2}}{2}\int_0^{2t}\sum_{j=1}^m y_j^2\Big]\Big\}\int_{\mathbb{R}^n}\rho(x,0)\bar{u}(x,0)$$

where $\rho(x,t)=(t+\varepsilon_2)^{-n}\exp\big(\frac{|x|^2}{8(t+\varepsilon_2)}\big)$ *and* ω_n*=volume of unit ball.*

Proof. For $x\in B_{R/2}$ and $z\in B_R$. Theorem 5.4 implies

$$\bar{u}(z,2t)\ge \bar{u}(x,t)\Big(\frac{t}{2t}\Big)^{n/2}\Big[\exp\Big(-\int_t^{2t}2\sqrt{n}\bar{c}(s)ds\Big)\Big]\exp\Big[-d\big((x,t),(z,2t)\big)\Big]$$

which is equivalent to

$$(5.38)\qquad \bar{u}(x,t)\le \bar{u}(z,2t)2^{n/2}\Big[\exp\int_t^{2t}2n\bar{c}(s)ds\Big]\exp\Big[d\big((x,t),(z,2t)\big)\Big]$$

Multiplying both sides of (5.38) by $\rho(z,2t)$ and integrating over a closed ball with center x and radius $R/2$, we get

$$(5.39)\quad \bar{u}(x,t)\int_{|z-x|\le R/2}\rho(z,2t)dz\le 2^{n/2}\Big[\exp\int_t^{2t}2\sqrt{n}\bar{c}(s)ds\Big]$$
$$\int_{|z-x|\le R/2}\bar{u}(z,2t)\rho(z,2t)\exp\Big[d\big((x,t),(z,2t)\big)\Big]dz$$
$$\le 2^{n/2}\Big[\exp\int_t^{2t}2\sqrt{n}\bar{c}(s)ds\Big]\max_{|z-x|\le R/2}d\big((x,t),(z,2t)\big)\int_{\mathbb{R}^n\times\{2t\}}\bar{u}\rho$$
$$\le 2^{n/2}\Big[\exp\int_t^{2t}2\sqrt{n}\bar{c}(s)ds\Big]\max_{|z-x|\le R/2}d\big((x,t),(z,2t)\big)$$
$$\Big\{\exp\Big[\frac{3}{2}\int_t^{2t}\Big(\sum_{j=1}^m y_j^2\Big)^2+\frac{1+\sqrt{2}}{2}\int_0^{2t}\sum_{j=1}^m y_j^2\Big]\Big\}\int_{\mathbb{R}^n}\rho(x,0)\bar{u}(x,0)$$

by Theorem 5.5. Let $z=x+\dfrac{R}{2}z_1$. Then

$$(5.40)\qquad \int_{|z-x|\le R/2}\rho(z,2t)dz=\int_{|z-x|\le R/2}\frac{1}{(2t+\varepsilon_2)^n}\exp\Big[\frac{|z|^2}{8(2t+\varepsilon_2)}\Big]dz$$
$$=\int_{|z_1|\le 1}\frac{(\frac{R}{2})^n}{(2t+\varepsilon_2)^n}\exp\Big[\frac{|x|+\frac{R}{2}z_1|^2}{8(2t+\varepsilon_2)}\Big]dz_1$$
$$\ge \omega_n\Big[\frac{R}{2(2t+\varepsilon_2)}\Big]^n\exp\Big[\frac{(|x|-\frac{R}{2})^2}{8(2t+\varepsilon_2)}\Big]$$

Combining (5.39) and (5.40), we get

$$\begin{aligned}(5.41)\qquad &\bar{u}(x,t)\omega_n\Big[\frac{R}{2(2t+\varepsilon_2)}\Big]^n \exp\Big(\frac{(|x|-\frac{R}{2})^2}{8(2t+\varepsilon_2)}\Big)\\ \leq &\bar{u}(x,t)\int_{|z-x|\leq R/2}\rho(z,2t)dz\\ \leq &2^{\frac{n}{2}}\Big[\exp\int_t^{2t} 2\sqrt{n}\bar{c}(s)ds\Big]\Big\{\max_{|z-x|\leq R/2}\exp\big[d\big((x,t),(z,2t)\big)\big]\Big\}\\ &\Big\{\exp\big[\frac{3}{2}\int_0^{2t}(\sum_{j=1}^m y_j^2)^2+\frac{1+\sqrt{2}}{2}\int_0^{2t}\sum_{j=1}^m y_j^2\big]\Big\}\int_{\mathbb{R}^n}\rho(x,0)\bar{u}(x,0)\end{aligned}$$

(5.37) follows immediately from (5.41). □

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