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Wavelet representations of general signals

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1. Introduction

Wavelet analysis consists of a collection of tools for the analysis and manipulation of signals. Wavelet representations are multidimensional transformations that indicate the joint time–frequency content of a signal. Representations such as wavelet transform, Gabor transform, and the short-time Fourier transform have proven to be powerful tools for signal analysis; however, current techniques are not without their drawbacks. Contribution and applications on continuous wavelet transforms have been carried out in various areas [1–3, 7–9]. This paper presents a new approach to time-frequency analysis which provides a unified framework to analyze both continuous and discrete time signals in the same way as the distribution function in probability allows a unified treatment of discrete and continuous random variables. In the heart of our approach lies the idea of analyzing and processing an integrated signal. This provides a natural way to encompass all possible (countable) discrete-time scales and continuous-time scales in a single model. In consequence, the analysis and treatment of signals supported by various time scales can be achieved in a fashion that permits one (at least theoretically) to process information carried by all of the time scales corresponding to the underlying signal. This is particularly important when applying transformation techniques to manipulation of signals subjected to random shock whose time scales cannot be predetermined for the processing purposes.

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An important issue in analysis and processing of signals is the issue of maintaining all relevant information contained in a signal when transforming it. In mathematical terms this issue leads to investigation of kernels of linear transforms in various normed spaces. In this paper we demonstrate that the integral transforms we consider are injective, which allows for recovery of all essential informational content of a signal from its transform.

The signals considered in this paper are essentially tempered distributions. The wavelet transformation of tempered distribution is also treated in [10, 12]. In [12] the wavelet series characterization of various classes of tempered distribution is presented. However, the classes of tempered studied in [12] are typically consisting of derivatives of L_p, $p \geq 1$, functions. In this paper we allow for more generality in the sense that the signals considered here may arise as distributional derivatives of functions which are not integrable. Also, our approach is a more explicit one: we describe signals directly in terms of the continuous- and discrete-time scales, which may have more appeal to the signal processing community. The distributional point of view will be used in the future for deriving a multiresolution analysis of the spaces of signals considered here.

This paper is organized as follows. In Section 2 we introduce Wavelet–Stieltjes transforms and prove uniqueness theorem for WST. In Section 3 we show an inversion formula for WST. In the fourth section we give some properties of the WST and compare them with properties of the "classical" continuous wavelet transform.

Notation. $g(x+)$, $g(x-)$ stand for the right and left limits, respectively, of a function g of x.

2. Wavelet–Stieltjes transform (WST)

2.1. Definition of WST

In this section, we will be considering real-valued functions only. Let us begin with the following definition.

Definition 2.1.

$$BV(1,2) = \left\{ F : R \to R \mid F(\bullet) = \int_{-\infty}^{\bullet} f(t)\,\mathrm{d}t + \sum_{s \leq \bullet} \rho(s), \right.$$

$$\left. f \in L_1(R) \cap L_2(R),\ \rho \in \tilde{l}_1 \cap \tilde{l}_2 \right\}, \tag{1}$$

where

$$\tilde{l}_i = \left\{ \rho : R \to R \mid \rho(x) = 0 \text{ except at countable many points and} \right.$$

$$\left. \sum_{x\in R} |\rho(x)|^i < \infty \right\} \quad i = 1, 2.$$

Remark 2.1. *Note that any function F in $BV(1,2)$ is of bounded first variation, where the integral part is the absolutely continuous component of F and the sum part is the jump component of F. Of course, the singular component is absent from any function F in $BV(1,2)$. Also, note that we have chosen the functions in $BV(1,2)$ to be right continuous. We shall write $F \sim (f, \rho)$ to associate F with its absolutely continuous and jump components. (We will recall absolute continuity, singular and jump functions in Definitions 2.3–2.5, respectively.)*

Remark 2.2. *The representative (f, ρ) of F may not be unique. Let J_F denote the support of jumps of F. We have*

$$f(t) = F'(t) \quad \text{for } t \notin J_F,$$
$$\rho(t) = F(t) - F(t-) \quad \text{for } t \in J_F.$$

Observe also that

$$F(x) = \int_{-\infty}^{x} f_0(s)\,\mathrm{d}s + \sum_{s\le x} \rho(s),$$

where

$$f_0(t) = \begin{cases} f(t) & \text{for } t \notin J_F, \\ 0 & \text{for } t \in J_F, \end{cases}$$

i.e., $F \sim (f_0, \rho)$. Hence, without loss of generality, we shall be assuming that

$$(\text{interior}(\operatorname{supp} f)) \cap J_F = \emptyset, \tag{2}$$

where $F \sim (f, \rho)$ and J_F is the support of jump component of F, that is $J_F = \operatorname{supp} \rho$. Consequently, we have $f(t)\rho(t) = 0,\ t \in R$.

Definition 2.2. For $F \in BV(1,2)$, define

$$\|F\|_1 := \int_{-\infty}^{\infty} |f(t)|\,\mathrm{d}t + \sum_{-\infty<s<\infty} |\rho(s)|, \tag{3}$$

$$\|F\|_2 := \left[\int_{-\infty}^{\infty} |f(t)|^2\,\mathrm{d}t + \sum_{-\infty<s<\infty} |\rho(s)|^2 \right]^{1/2}. \tag{4}$$

In order to see that $\|\cdot\|_2$ is a norm, we need to verify the triangle inequality. Towards this end, let $F, G \in BV(1,2)$ and $F \sim (f,\rho)$, $G \sim (g,\gamma)$, we have

$$\|F+G\|_2 = \left[\int_{-\infty}^{\infty} (f(t)+g(t))^2 \, \mathrm{d}t + \sum_{s \in J_F \cup J_G} (\rho(s)+\gamma(s))^2 \right]^{1/2}.$$

Therefore, applying Minkowski inequality twice, we obtain from the above that

$$\|F+G\|_2 \le \left\{ \left[\left(\int_{-\infty}^{+\infty} f^2(t)\,\mathrm{d}t \right)^{1/2} + \left(\int_{-\infty}^{+\infty} g^2(t)\,\mathrm{d}t \right)^{1/2} \right]^2 \right\}^{1/2}$$
$$+ \left\{ \left[\left(\sum_{s \in J_F} \rho^2(s) \right)^{1/2} + \left(\sum_{s \in J_G} \gamma^2(s) \right)^{1/2} \right]^2 \right\}^{1/2}.$$

Hence,

$$\|F+G\|_2 \le \|F\|_2 + \|G\|_2.$$

It is elementary to check that $\|\cdot\|_1$ is a norm on $BV(1,2)$. We recall Lebesgue decomposition.

Definition 2.3. A function f defined on an interval $[a,b]$ is said to be absolutely continuous on $[a,b]$ if, given any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$$

for every finite system of pairwise disjoint subintervals $(a_k, b_k) \subset [a,b]$, $k = 1, \dots, n$ of total length

$$\sum_{k=1}^{n} (b_k - a_k)$$

less than δ.

Definition 2.4. A continuous function of bounded variation is said to be singular if its derivative vanishes almost everywhere.

Definition 2.5. A jump function is a function of the form

$$\sum_{x_n < x} h_n,$$

where the numbers $h_1, h_2, \ldots, h_n, \ldots$ corresponding to the discontinuity points $x_1, x_2, \ldots, x_n, \ldots$ satisfy the condition

$$\sum_n |h_n| < \infty.$$

Remark 2.3. *In general, any function f of bounded variation can be represented as a sum*

$$f(x) = f_{\mathrm{ac}} + f_{\mathrm{s}} + f_{\mathrm{j}}$$

of an absolutely continuous function f_{ac}, a singular function f_{s} and a jump function f_{j}. This is known as the Lebesgue decomposition [11].

Proposition 2.1. *Let $F \in BV(1,2)$ and $V^1(F)$ be its first variation. Then*

$$V^1(F) = \|F\|_1. \tag{5}$$

Proof. Let h be any real-valued function of bounded first variation. By Lebesgue decomposition, we have

$$V^1(h) = V^1(h_{\mathrm{ac}}) + V^1(h_{\mathrm{s}}) + V^1(h_{\mathrm{j}}), \tag{6}$$

where $h_{\mathrm{ac}}, h_{\mathrm{s}}, h_{\mathrm{j}}$ are the absolutely continuous, singular and jump components of h, respectively.

It is well known that

$$V^1(h_{\mathrm{ac}}) = \int_{-\infty}^{\infty} |h_{\mathrm{ac}}(t)|\,\mathrm{d}t, \tag{7}$$

$$V^1(h_{\mathrm{j}}) = \sum_{-\infty < s < \infty} |\Delta h_{\mathrm{j}}(s)|, \tag{8}$$

where $\Delta h_{\mathrm{j}}(s) = (h_{\mathrm{j}}(s) - h_{\mathrm{j}}(s-)) + (h_{\mathrm{j}}(s+) - h_{\mathrm{j}}(s))$ is the jump of h_{j} at s. This completes the proof since $F_{\mathrm{s}} \equiv 0$. □

Let $BV(2)$ denote the completion of $BV(1,2)$ with respect to the $\|\cdot\|_2$ norm given in Eq. (4). We endow the Hilbert space $BV(2)$ with an inner product defined in Eq. (9) below:
Let $F, G \in BV(2)$ and $F \sim (f, \rho)$, $G \sim (g, \gamma)$ then

$$\begin{aligned} \langle F, G \rangle &:= \int_{-\infty}^{\infty} f(t) g(t)\,\mathrm{d}t + \frac{1}{2} \sum_{s \in J_F} \rho(s)\gamma(s) + \frac{1}{2} \sum_{s \in J_G} \rho(s)\gamma(s) \\ &= \int_{-\infty}^{+\infty} f(t) g(t)\,\mathrm{d}t + \sum_{s \in J_F \cap J_G} \rho(s)\gamma(s). \end{aligned} \tag{9}$$

Remark 2.4. *In the above definition, we have slightly abused notation by writing $F \sim (f, \rho)$ for $F \in BV(2)$. It is clear that each F in $BV(2)$ is naturally corresponding to some pair $(f, \rho) \in L_2(R) \times \tilde{l}_2$ and vice versa.*

Remark 2.5. *It is clear that the inner product introduced in Eq.* (9) *is compatible with norm* (4). *In order to define the Wavelet–Stieltjes Transform, we need to introduce the following class of functions. For any* $F \sim (f, \rho)$, *write*

$$F_\psi := \int_{-\infty}^{\infty} \psi(t) f(t)\, \mathrm{d}t + \sum_{-\infty < s < \infty} \psi(s)\rho(s). \tag{10}$$

Let

$$\Psi_F := \{\psi \in L_1(\mathbf{R}) \mid F_\psi < \infty\},$$

and

$$\Psi = \bigcap_{F \in BV(1,2)} \Psi_F.$$

Note that for $\psi \in \Psi$, *Eq.* (10) *is well-defined and* $F_\psi = \int_{-\infty}^{\infty} \psi(t)\, \mathrm{d}F(t)$, *the last integral being the Lebesgue–Stieltjes integral.*

For any function $h : R \to R$ *and* $a, b \in R$, *the following notation will be used*:

$$h^{a,b}(t) := \begin{cases} h\left(\frac{t-b}{a}\right), & a \neq 0, \\ 0, & a = 0. \end{cases} \tag{11}$$

Definition 2.6. The integral transform of $F \in BV(1,2)$ given by

$$(WS_\psi F)(a,b) := F_{\psi^{a,b}} \tag{12}$$

for $\psi \in \Psi$, is called the Wavelet–Stieltjes transform (WST) of F with kernel $\psi^{a,b}$.

Remark 2.6. *Recall that the "classical" wavelet transform is defined as*

$$W_\psi f(a,b) = |a|^{-1/2} \int_{-\infty}^{+\infty} f(t)\psi\left(\frac{t-b}{a}\right) \mathrm{d}t. \tag{13}$$

We shall present some properties of both WST and the "classical" wavelet transform in Section 4.

The classical WT extends naturally to tempered distributions in the sense of the Gelfand triplet,

$$\mathscr{S}(\mathbf{R}) \subset L^2(\mathbf{R}) \subset \mathscr{S}'(\mathbf{R}),$$

provided the wavelet is a Schwartz function. Indeed, the WT is just an inner product $\langle \psi^{a,b} | s \rangle$, *which makes sense for* $s \in \mathscr{S}'(\mathbf{R})$, *if* $\psi \in \mathscr{S}(\mathbf{R})$. *In fact, by taking a delta function as signal one can have a simplified reconstruction formula.*

Remark 2.7. *Using Theorem* 3.1 *of Section* 3, *we naturally extend the definition of WST to all functions in* $BV(2)$.

2.2. Uniqueness of the WST

In the following proposition, we prove uniqueness of the WST for functions in $BV(1,2)$. Uniqueness for functions in $BV(2)$ will follow from the results of Section 3.

We need 2 Lemmas which are proved in [6].

Lemma 2.1 (Dudlley [6], Lemma 9.5.2). *Let P be a probability Law on R^k with characteristic function*

$$g(t)=\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}P(x).$$

Then $P^{(\sigma)}=P*\mathrm{N}(0,\sigma^2 I)$ has a density $f^{(\sigma)}$ which satisfies

$$f^{(\sigma)}(x)=(2\pi)^{-k}\int_{-\infty}^{\infty} g(t)\exp\left(\mathrm{i}xt-\frac{\sigma^2|t|^2}{2}\right)\mathrm{d}t$$

Lemma 2.2 (Dudlley [6], Lemma 9.5.3). *For any probability measure $P, P^{(\sigma)}$ converge to P as $\sigma\to 0$.*

Theorem 2.1. *Let $F_1, F_2\in BV(1,2)$ be such that*

$$WS_\psi F_1 \equiv WS_\psi F_2$$

for some $\psi\in\Psi$ such that

$$0<c_\psi:=\int_{-\infty}^{\infty}\frac{|\hat{\psi}(w)|^2}{|w|}\,\mathrm{d}w<\infty.$$

Then

$$F_1\equiv F_2.$$

Proof. Let F_1, F_2 be arbitrary functions in $BV(1,2)$ and let F_i^+, F_i^-, $i=1,2$ be the nonnegative, nondecreasing functions in the Jordan decomposition of F_i, $i=1,2$, that is

$$F_i=F_i^+ - F_i^-,\quad i=1,2.$$

Once we prove the uniqueness of the WST for nonnegative, nondecreasing functions in $BV(1,2)$, we will know that

$$WS_\psi F_1 = WS_\psi F_2,$$

$$WS_\psi(F_1^+ + F_2^-) = WS_\psi(F_2^+ + F_1^-),$$

$$F_1^+ + F_2^- \equiv F_2^+ + F_1^-,$$

$$F_1\equiv F_2.$$

Therefore, without loss of generality, we can assume that $F_i\geq 0$ and $V^1(F_i) = 1$, $i=1,2$.

Let g_σ be the density of a Normal distribution $N(0,\sigma^2)$ with $\sigma>0$. Let G_σ be the corresponding distribution function. Here we have

$$|a|^{-1/2} WS_\psi G_\sigma \equiv W_\psi g_\sigma, \tag{14}$$

where W_ψ is the classical wavelet transform (13) (see Remark 2.6). From the inversion formula (1.3.3) in [5], we have, for all $s\in R$,

$$g_\sigma(s)=\frac{1}{c_\psi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(W_\psi g_\sigma)(a,b)|a|^{-1/2}\psi^{a,b}(s)\frac{\mathrm{d}a\,\mathrm{d}b}{a^2}. \tag{15}$$

Let f_σ be the density of $G_\sigma * F_1$. We have, for $s\in R$, (by Eq. (15) and Fubini's theorem)

$$\begin{aligned} f_\sigma(s) &= \int_{-\infty}^{\infty} g_\sigma(t-s)\,\mathrm{d}F_1(t)\\ &= \frac{1}{c_\psi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}(W_\psi g_\sigma)(a,b)|a|^{-1/2}\psi^{a,b}(t-s)\frac{\mathrm{d}a\,\mathrm{d}b}{a^2}\right\}\mathrm{d}F_1(t)\\ &= \frac{1}{c_\psi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(WS_\psi F_1)(a,b-s)|a|^{-1/2}(W_\psi g_\sigma)(a,b)\frac{\mathrm{d}a\,\mathrm{d}b}{a^2}. \end{aligned} \tag{16}$$

In view of our assumption we also obtain

$$f_\sigma(s)=\int_{-\infty}^{\infty} g_\sigma(t-s)\,\mathrm{d}F_2(t) \tag{17}$$

for all $s\in R$, and consequently

$$G_\sigma * F_1 \equiv G_\sigma * F_2. \tag{18}$$

Taking limit on both sides of Eq. (18) as σ goes to zero, and by Lemmas 2.1 and 2.2, we get $F_1\equiv F_2$. □

2.3. Example 2.1

Let us consider the following example, similar to the one given on p. 4 of [5]. Let $F\sim(f,\rho)$. Where for some $T>0$ and $t_1,t_2\in[-T,T]$,

$$f(t)=\begin{cases}\sin(2\pi v_1 t)+\sin(2\pi v_2 t), & \text{when } t\neq t_1,t_2,\ t\in[-T,T],\\ 0, & \text{when } t=t_1,t_2 \text{ or } t\notin[-T,T],\end{cases} \tag{19}$$

$$\rho(t_1)=\rho(t_2)=\gamma+\sin(2\pi v_1 t_1)+\sin(2\pi v_2 t_2). \tag{20}$$

Take any $\psi\in\Psi$. Note that the "classical" wavelet transform does not distinguish between the two signals f and $f+\rho$ in the sense that

$$W_\psi f\equiv W_\psi(f+\rho). \tag{21}$$

However, consider $F_1\sim(f,0)$ and $F_2\sim(f,\rho)$ with f and ρ as above. The Wavelet–Stieltjes transform does discriminate between F_1 and F_2 which is due to the fact that

$$WS_\psi F_1\not\equiv WS_\psi F_2. \tag{22}$$

3. Inversion formula

In this section we will develop an inversion formula for WST. We first show Parseval Identity for WST. This is a generalization of classical Parseval Identity for Wavelet transform. Then we present the inversion formula.

We begin with the following lemma whose proof is in the appendix. As usually "$\wedge$" and "$-$" will stand for Fourier transform and complex conjugate, and "$\sim$" denotes the inverse Fourier transform.

Lemma 3.1. *Let $G,F\in BV(1,2)$ and $G\sim(g,\gamma)$, $F\sim(f,\rho)$. Then*

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\hat{\gamma}(w)\bar{\hat{\rho}}(w)\,\mathrm{d}w=\sum_{s\in J_G\cap J_F}\gamma(s)\rho(s),$$

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\hat{f}(w)\bar{\hat{\gamma}}(w)\,\mathrm{d}w=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\hat{g}(w)\bar{\hat{\rho}}(w)\,\mathrm{d}w=0.$$

Proof. See Appendix.

We will need the following definition,

Definition 3.1. For $G,F\in BV(1,2)$ and $G\sim(g,\gamma)$, $F\sim(f,\rho)$ and $\psi\in\Psi$ we define

$$\begin{aligned}R_T(G,F,\psi):=&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}I_1(a,b)I_2(a,b)\frac{\mathrm{d}a\,\mathrm{d}b}{|a|^3}\\&+\frac{1}{T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[I_1(a,b)I_2^{T_1a}(b)+I_2(a,b)I_1^{T_1a}(b)\\&+I_1^{T_1a}(b)I_2^{T_1a}(b)]\frac{\mathrm{d}a\,\mathrm{d}b}{|a|^3},\end{aligned}$$

where

$$I_1(a,b)=\int_{-\infty}^{\infty}\psi^{a,b}(s)f(s)\,\mathrm{d}s,$$

$$I_2(a,b)=\int_{-\infty}^{\infty}\psi^{a,b}(s)g(s)\,\mathrm{d}s,$$

$$I_3(a,b)=\sum_{s\in J_G}\psi^{a,b}(s)\gamma(s),$$

$$I_4(a,b)=\sum_{s\in J_F}\psi^{a,b}(s)\rho(s),$$

$$I_1^{T_1a}(b)=\int_{-\infty}^{\infty}I_3(a,b-w)\tilde{\chi}_T(w)\,\mathrm{d}w,$$

$$I_2^{T_1 a}(b) = \int_{-\infty}^{\infty} I_4(a, b-w)\tilde{\chi}_T(w)\,\mathrm{d}w,$$

$$\chi_T(x) = \begin{cases} 1, & -T < x < T, \\ 0 & \text{otherwise.} \end{cases}$$

We now present a "Parseval Identity" for WST, and then we demonstrate the inversion formula.

Theorem 3.1 (Parseval Identity). *For $G, F \in BV(1,2), G \sim (g,\gamma), F \sim (f,\rho)$ and $\psi \in \Psi$. Then*

$$\lim_{T\to\infty} R_T(G,F,\psi) = c_\psi \langle F, G\rangle.$$

Proof.

Step 1: By the classical wavelet Parseval Identity, we get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_1 I_2 \frac{\mathrm{d}a\,\mathrm{d}b}{|a|^3} = c_\psi \int_{-\infty}^{\infty} f(s)g(s)\,\mathrm{d}s.$$

Step 2: We have

$$\hat{I}_4(a,w) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}bw}\left(\sum_{s\in J_F}\psi\left(\frac{b-s}{a}\right)\rho(s)\right)\mathrm{d}b$$
$$= \sum_{s\in J_F}\left(\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}bw}\psi\left(\frac{b-s}{a}\right)\mathrm{d}b\right)\rho(s) = a\hat{\psi}(aw)\hat{\rho}(w),$$

and similarly,

$$\hat{I}_3(a,w) = a\hat{\psi}(aw)\hat{\gamma}(w),$$
$$\hat{I}_1(a,w) = a\hat{\psi}(aw)\hat{f}(w),$$
$$\hat{I}_2(a,w) = a\hat{\psi}(aw)\hat{g}(w),$$

Step 3: Now,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_1^{T_1 a}(b) I_2^{T_1 a}(b)\frac{\mathrm{d}a\,\mathrm{d}b}{|a|^3}$$
$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\hat{I}_3(a,w)\bar{\hat{I}}_u(a,w)x_T(w)\,\mathrm{d}w\right]\frac{\mathrm{d}a}{|a|^3}$$
$$= \int_{-\infty}^{\infty}\int_{-T}^{T} a^2|\hat{\psi}(aw)|^2\hat{\rho}\bar{\hat{\gamma}}(w)\,\mathrm{d}w\frac{\mathrm{d}a}{|a|^3}$$
$$= c_\psi\int_{-T}^{T}\hat{\rho}(w)\bar{\hat{\gamma}}(w)\,\mathrm{d}w.$$

Similarly, we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_1(a,b)I_2^{T_1 a}(b)\frac{\mathrm{d}a\,\mathrm{d}b}{|a|^3} = c_\psi \int_{-T}^{T} \hat{g}(w)\bar{\hat{\rho}}(w)\,\mathrm{d}w,$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_2(a,b)I_1^{T_1 a}(b)\frac{\mathrm{d}a\,\mathrm{d}b}{|a|^3} = c_\psi \int_{-T}^{T} \hat{f}(w)\bar{\hat{\gamma}}(w)\,\mathrm{d}w.$$

Step 4: Use Lemma 3 and Steps 1–3 to conclude

$$\lim_{T\to\infty} R_T(G,F,\psi) = c_\psi\langle G,F\rangle. \qquad \square$$

Example 3.1. Let $f\equiv g\equiv 0$ and

$$\gamma(s)=\rho(s)=\begin{cases}1, & s=0,\\ 0, & s\neq 0.\end{cases}$$

Then

$$\lim_{T\to\infty} R_T(G,F,\psi) = c_\psi \lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T}\hat{\gamma}(w)\bar{\hat{\rho}}(w)\,\mathrm{d}w$$
$$= c_\psi = c_\psi \sum_{s\in J_F\cap J_G}\gamma(s)\rho(s) = c_\psi\langle F,G\rangle.$$

Example 3.2. Let

$$f(t)=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-t^2/2},\quad t\in R,$$

$$g\equiv 0,$$

$$\gamma(s)=\begin{cases}1, & s=0,\\ 0, & s\neq 0,\end{cases}$$

$$\rho\equiv 0.$$

Then

$$\lim_{T\to\infty} R_T(G,F,\Psi) = c_\psi \lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T}\hat{f}(w)\bar{\hat{\gamma}}(w)\,\mathrm{d}w = 0 = \langle G,F\rangle.$$

Theorem 3.2 (Inversion Formula)**.** *Let*

$$F\in BV(2),\qquad F\sim(f,\rho),$$

$$\tilde{F}(x)=\frac{F(x+)+F(x-)}{2},$$

and

$$\mu_{\tilde{F}}(x_1,x_2] = \tilde{F}(x_2^+) - \tilde{F}(x_1^-).$$

Then

$$\mu_{\tilde{F}}(x_1,x_2] = \frac{1}{c_\psi}\left\{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[WS_\psi(F_1)(a,b)\int_{x_1}^{x_2}\psi^{a,b}(x)\,\mathrm{d}x\right]\frac{\mathrm{d}a\,\mathrm{d}b}{|a|^3}\right.$$

$$+\lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[(WS_\phi(F_1)(a,\cdot)*\tilde{\chi}_T(b))\right.$$

$$\left.\left.\times\left(\left(\sum_{x_1<s<x_2}\rho(s)\psi^{a,\cdot}(s)\right)*\tilde{\chi}_T(b)\right)\right]\frac{\mathrm{d}a\,\mathrm{d}b}{|a|^3}\right\},$$

where $F_1\sim(f,0)$.

Proof. Let $G\sim(\chi_{(x_1,x_2]},h_{(x_1,x_2]})$, where

$$\chi_{(x_1,x_2]}(x)=\begin{cases}1, & x_1<x\le x_2,\\ 0 & \text{otherwise,}\end{cases}$$

$$h_{(x_1,x_2]}(s)=\begin{cases}1, & \rho(s)\neq 0 \text{ and } x_1<s\le x_2,\\ 0 & \text{otherwise}\end{cases}$$

and apply Theorem 3.1. □

4. Properties of wavelet transforms

In this section we present some basic properties of both "classical" and Stieltjes wavelet transforms.

Theorem 4.1. *Let* $f(x),\ g(x)\in L^2(R)$

(i) $\forall\alpha,\beta\in R$

$$W_\psi(\alpha f+\beta g)(a,b)=\alpha W_\psi f(a,b)+\beta W_\psi g(a,b).$$

(ii) *Let* $g(x)=f(-x)$. *Then*

$$W_\psi g(a,b)=W_\psi f(-a,-b).$$

(iii) *Let* $g(x)=f(x-x_0)$. *Then*

$$W_\psi g(a,b)=W_\psi f(a,b-x_0).$$

(iv) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|W_\psi f(a,b)|^2\frac{\mathrm{d}a\,\mathrm{d}b}{a^2}=c_\psi\|f\|^2$
where $c_\psi=\int\frac{|\hat{\psi}(\xi)|^2}{|\xi|}\,\mathrm{d}\xi$
and $\hat{\psi}$ *denotes Fourier transform of* ψ.

(v) *Let* $g(x)=f(\frac{x}{\lambda})$ $\lambda>0$.

Then $W_\psi g(a,b) = \sqrt{\lambda} W_\psi f(a\lambda^{-1}, b\lambda^{-1})$.

(vi) *Let* $g(x) = f'(x)$.

Then $\frac{\partial W_\psi f}{\partial b}(a,b) = W_\psi g(a,b)$.

(vii) *Let* $g(x) = f * h(x) = \int_{-\infty}^{\infty} f(u)h(x-u)\,\mathrm{d}u$.

Then $W_\psi g(a,b) = f * W_\psi h(a,\cdot)(b) = h * W_\psi f(a,\cdot)(b)$.

Proof. See [4].

Remark 4.1. *Equating both sides of the Parseval Identity* (24), *in general, one can choose* α, β *in the following settings*:

$$W_\psi f(a,b) = |a|^\alpha \int_{-\infty}^{\infty} f(t)\psi\left(\frac{t-b}{a}\right)\,\mathrm{d}t \tag{23}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [(W_\psi f)(b,a)\overline{(W_\psi g)(b,a)}]a^\beta\,\mathrm{d}a\,\mathrm{d}b = c_\psi\langle f, g\rangle, \tag{24}$$

where α, β *can be shown to satisfy* $2\alpha + \beta + 3 = 0$.

For simplicity, we usually choose $\alpha = -\frac{1}{2}$, $\beta = -2$.

Remark 4.2. *Convolution theorem only holds for Laplace transform. In general, it is not true for any wavelet transform. This can be shown as follows.*

For simplicity, we let $a = 1$ *in Eq.* (23). *Now, let*

$$g(x) = f * h(x) = \int_{-\infty}^{\infty} f(u)h(x-u)\,\mathrm{d}u \tag{25}$$

suppose

$$W_g \equiv W_f \cdot W_h, \tag{26}$$

where

$$W_g = \int_{-\infty}^{\infty} g(x)\overline{\psi(x-b)}\,\mathrm{d}x, \tag{27}$$

$$W_f = \int_{-\infty}^{\infty} f(x)\overline{\psi(x-b)}\,\mathrm{d}x, \tag{28}$$

$$W_h = \int_{-\infty}^{\infty} h(x)\overline{\psi(x-b)}\,\mathrm{d}x. \tag{29}$$

Substitute Eqs. (27)–(29) *into Eq.* (26). *We have*

$$\int_{-\infty}^{\infty} g(x)\overline{\psi(x-b)}\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\overline{\psi(x-b)}\,\mathrm{d}x \int_{-\infty}^{\infty} h(x)\overline{\psi(x-b)}\,\mathrm{d}x.$$

On the other hand,

$$\int_{-\infty}^{\infty} g(x)\overline{\psi(x-b)}\,\mathrm{d}x = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)h(x-u)\overline{\psi(x-b)}\,\mathrm{d}u\,\mathrm{d}x.$$

By Fubini Theorem,

$$\psi(x+u-b) = \psi(u-b)\psi(x-b).$$

Let $b=0$, *then*

$$\psi(x+u) = \psi(u)\psi(x).$$

Let $u=0$, *then*

$$\psi(0) = 1.$$

Solving the above functional equation, we have, for some constant c,

$$\psi(t) = \mathrm{e}^{ct}.$$

Therefore, W *is Laplace transform.*

In contrast to the above properties, WST has the following properties which generalize some properties of classical wavelet transforms.

Theorem 4.2. *Let* $F \sim (f,\rho)$, $G \sim (g,\sigma)$. *Then*
(i) $\forall \alpha, \beta \in R$

$$\alpha F + \beta G \sim (\alpha f + \beta g, \alpha\rho + \beta\sigma)$$

and hence

$$WS_\psi(\alpha F + \beta G)(a,b) = \alpha WS_\psi[F](a,b) + \beta WS_\psi[G](a,b).$$

(ii) *If* $G(x) = F(-x)$ *then* $f(x) = g(-x)$, $\rho(x) = \sigma(-x)$ *and*

$$WS_\psi[G](a,b) = WS_\psi[F](-a,-b)$$

(iii) *If* $G(x) = F(x - x_0)$
then $g(x) = f(x-x_0), \sigma(x) = \rho(s - x_0)$ *and*

$$WS_\psi[G](a,b) = WS_\psi[F](a, b - x_0).$$

(iv) *Let* $G(x) = F(\frac{x}{\lambda})$, $\lambda > 0$
then $g(x) = f(\frac{x}{\lambda})$, $\sigma(x) = \rho(\frac{x}{\lambda})$

$$WS_\psi[G](a,b) = \lambda \int_{-\infty}^{\infty} f(t)\psi^{\lambda^{-1}a,\lambda^{-1}b}(t)\,\mathrm{d}t + \sum \rho(s)\psi^{\lambda^{-1}a,\lambda^{-1}b}(s)$$

Proof. (i) is obvious.

$$\text{(ii)} \qquad WS_\psi[G](a,b) = \int_{-\infty}^{\infty} g\psi^{a,b}\,\mathrm{d}t + \sum \psi^{a,b}(s)\sigma(s)$$

$$= \int_{-\infty}^{\infty} f(t)\psi^{a,b}(-t)\,\mathrm{d}t + \sum \psi^{a,b}(-s)\rho(s),$$

where

$$\psi^{a,b}(-x) = \psi\left(\frac{-x-b}{a}\right) = \psi\left(\frac{x-(-b)}{a}\right) = \psi^{-a,-b}(x)$$

which implies

$$WS_\psi[G](a,b) = WS_\psi[F](-a,-b)$$

$$\text{(iii)} \qquad WS_\psi[G](a,b) = \int_{-\infty}^{\infty} g(t)\psi^{a,b}(t)\,\mathrm{d}t + \sum \sigma(s)\psi^{a,b}$$

$$= \int_{-\infty}^{\infty} f(t-x_0)\psi^{a,b}(t)\,\mathrm{d}t + \sum \rho(s-x_0)\psi^{a,b}(s)$$

$$= \int_{-\infty}^{\infty} f(t)\psi^{a,b}(t+x_0)\,\mathrm{d}t + \sum \rho(s)\psi^{a,b}(s+x_0)$$

and

$$\psi^{a,b}(t+x_0) = \psi\left(\frac{t+x_0-b}{a}\right) = \psi\left(\frac{t-(b-x_0)}{a}\right) = \psi^{a,b-x_0}$$

which give rise to the proof of (iii).

$$\text{(iv)} \qquad WS_\psi[G](a,b) = \int_{-\infty}^{\infty} g(t)\psi^{a,b}(t)\,\mathrm{d}t + \sum \sigma(s)\psi^{a,b}(s)$$

$$= \int_{-\infty}^{\infty} f\left(\frac{t}{\lambda}\right)\psi^{a,b}(t)\,\mathrm{d}t + \sum \rho\left(\frac{s}{\lambda}\right)\psi^{a,b}(s),$$

$$= \lambda\int_{-\infty}^{\infty} f(t)\psi^{a,b}(\lambda t)\,\mathrm{d}t + \sum \rho(s)\psi^{a,b}(s\lambda).$$

On the other hand,

$$\psi^{a,b}(\lambda t) = \psi\left(\frac{\lambda t-b}{a}\right) = \psi\left(\frac{t-\lambda^{-1}b}{\lambda^{-1}a}\right) = \psi\left(\frac{t-\lambda^{-1}b}{\lambda^{-1}a}\right) = \psi^{\lambda^{-1}a,\lambda^{-1}b}(t)$$

Hence (iv) is valid. □

Remark 4.3. *Examples* 2.1 *and* 3.1 *show that WST does distinguish signals with or without jump discontinuities. We have provided a foundation for WST theory. Applications on signal processing and image analysis will be reported in a separate paper.*

Appendix

Proof of Lemma 3.1. It is enough to consider the case where f, g, γ and ρ are probability densities. We will use the following result:

Let μ be a probability measure on R and let

$$\hat{\mu}(w) := \int_{-\infty}^{\infty} e^{-ixw}\, d\mu(x)$$

be its Fourier transform. Then

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \hat{\mu}(w)e^{iwx}\, dw &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\left[\int_{-\infty}^{\infty} e^{-iyw}\, d\mu(y)\right] e^{iwx}\, dw \\ &= \lim_{T\to\infty}\int_{-\infty}^{\infty} \frac{\sin T(y-x)}{T(y-x)}\, d\mu(y) \\ &= \mu(\{x\}). \end{aligned} \tag{A.1}$$

Now, let μ_f, μ_g, μ_ρ and μ_γ be the measures corresponding to f, g, ρ and γ, respectively, that is

$$\mu_f(A) = \int_A f(x)\, dx,$$

$$\mu_g(A) = \int_A g(x)\, dx,$$

$$\mu_\rho(A) = \sum_{x\in A} \rho(x),$$

$$\mu_\gamma(A) = \sum_{x\in A} \gamma(x),$$

for any Borel set $A \subset R$. We have (by letting $\tilde{\mu}(A) = \mu(-A)$)

$$(\mu_f * \tilde{\mu}_\gamma)^\wedge(w) = \hat{f}(w)\overline{\hat{\gamma}(w)}, \tag{A.2}$$

$$(\mu_g * \tilde{\mu}_\rho)^\wedge(w) = \hat{g}(w)\overline{\hat{\rho}(w)} \tag{A.3}$$

and

$$(\mu_\gamma * \tilde{\mu}_\rho)^\wedge(w) = \hat{\gamma}(w)\overline{\hat{\rho}}(w), \tag{A.4}$$

where $\hat{f}$, $\hat{g}$, $\hat{\gamma}$, and $\hat{\rho}$ are the Fourier transforms of f, g, γ and ρ. From Eqs. (A.1)–(A.4) we have, letting $x = 0$,

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \hat{f}(w)\overline{\hat{\gamma}(w)}\, dw &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} (\mu_f * \tilde{\mu}_\gamma)^\wedge(w)\, dw \\ &= (\mu_f * \tilde{\mu}_\gamma)(\{0\}) = 0, \end{aligned} \tag{A.5}$$

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\hat{g}(w)\overline{\hat{\rho}(w)}\,\mathrm{d}w=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}(\mu_g*\mu_\rho)^\wedge(w)\,\mathrm{d}w$$
$$=(\mu_g*\tilde{\mu}_\rho)(\{0\})=0, \tag{A.6}$$

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\hat{\gamma}(w)\overline{\hat{\rho}(w)}\,\mathrm{d}w=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}(\mu_\gamma*\tilde{\mu}_\rho)^\wedge(w)\,\mathrm{d}w$$
$$=(\mu_\gamma*\tilde{\mu}_\rho)(\{0\})=\sum_{s\in T_G\cap T_F}\gamma(s)\rho(s). \tag{A.7}$$

In Eqs. (A.5)–(A.7) we have used the following properties: for any Borel set $A\subset R$:

$$(\mu_f*\tilde{\mu}_\gamma)(A)=\sum_{s\in T_G}\gamma(-s)\int_A f(y-s)\,\mathrm{d}y,$$

$$(\mu_g*\mu_\rho)(A)=\sum_{s\in T_F}\rho(-s)\int_A g(y-s)\,\mathrm{d}y,$$

$$(\mu_\gamma*\tilde{\mu}_\rho)(A)=\sum_{s\in T_F}\rho(-s)\sum_{t\in A\cap T_G}\gamma(t-s). \qquad \square$$

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