where b_L is a quantity similar to b_3 in (63). Finally, we may repeat the same procedure for $iL, i = 1, 2, \cdots$ to obtain an expression for $\|x_{\bar{k}-2+iL}\|$

$$\left\|x_{\bar{k}-2+iL}\right\| \leq \varepsilon b_L \left\|x_{\bar{k}-2+(i-1)L}\right\| + \zeta. \tag{65}$$

Expressions (64) and (65) can be combined to obtain

$$\begin{aligned}\left\|x_{\bar{k}-2+iL}\right\| \leq (\varepsilon b_L)^i \left\|x_{\bar{k}-2}\right\| &+ [(\varepsilon b_L)^{i-1}\zeta \\ &+ (\varepsilon b_L)^{i-2}\zeta + \cdots + \varepsilon b_L \zeta + \zeta].\end{aligned} \tag{66}$$

Factor $[(\varepsilon b_L)^{i-1}\zeta + (\varepsilon b_L)^{i-2}\zeta + \cdots + \varepsilon b_L\zeta + \zeta]$ will be a sum of an infinite number of terms. As $\varepsilon b_L \ll 1$, however, this sum will be bounded. Hence, we will have a bounded decreasing state $\|x_{\bar{k}-2+iL}\|$ as follows:

$$\left\|x_{\bar{k}-2+iL}\right\| \leq (\varepsilon b_L)^i \left\|x_{\bar{k}-2}\right\| + \zeta, \qquad \text{for all } i = 1, \cdots. \tag{67}$$

From (67), we conclude the state $x_{\bar{k}-2+iL}$ for $i = 1, \cdots, \infty$ is bounded in the limit by a term that depends on the noise. The states at other time instants inside the intervals $[k-2+(i-1)L, k-2+iL]$ can be proven to be bounded too, by using the procedure described in (57) up to (63). This process clearly contradicts our assumption that ϕ diverges. Therefore, all signals remain bounded. Furthermore, in view of the previous analysis, the smaller the noise upper bound, the smaller the state becomes in the limit.

V. Conclusions

This paper has presented an indirect adaptive periodic control scheme based on the lifted representation of the plant proposed in [3]. New arguments have been presented, so this technique appears in a better position as a solution to the long-standing problem of singularities in adaptive control of nonminimum phase plants. Contrary to other techniques, the only *a priori* knowledge required on the plant is its order besides the standard controllability/observability assumption. As compared with previous studies in the subject [3], [13], the proposed controller does not require the frequent introduction of periodic n-length sequences of zero inputs. Furthermore, simulations have shown that, in practice, no need of introducing sequences of inputs equal to zero exists. Therefore, the new controller is such that the system always operates in closed loop, which should lead to better performance characteristics. Simulation results have also shown the use of the proposed estimate modification can significantly reduce signal peaks during the transient period.

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Classification of Four-Dimensional Estimation Algebras

Stephen S. T. Yau and Amid Rasoulian

***Abstract*— Of central importance in nonlinear filtering theory is the classification of finite-dimensional estimation algebras, through which we can construct recursive filters. In this paper, we classify all t-dimensional estimation algebras, $t \leq 4$, for arbitrary state-space dimension.**

***Index Terms*— Estimation algebra, nonlinear filter, Wei–Norman approach, Yau filter.**

I. Introduction

Although the Kalman–Bucy filter opened the way to a new era of filtering theory, it has limited applicability because of linearity assumptions and Gaussian assumptions of initial value. Brockett and Mitter [2], [9] assign an estimation algebra to a DMZ equation which plays a central role in filtering theory. The structure of this Lie algebra is of great importance in solving DMZ equation.

In 1983, Brockett [2] in his famous lecture at the International Congress of Mathematicians (Warsaw, Poland) proposed the idea of classifying finite-dimensional estimation algebras. In 1990, Dong *et al.* [8] and Tam *et al.* [12] classified exact finite-dimensional estimation algebras. In [13], Wong introduced the $\Omega = (\omega_{ij})$ matrix related to the following equation (1). During 1991–1995, Yau *et al.*, in a series of papers [3]–[7], [11], [14], [15], introduced new concepts and classified new finite-dimensional estimation algebras. Of particular note is the introduction of a new filtering system that

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The authors are with the Control and Information Laboratory, MSCS, UIC, Chicago, IL 60607-7045 USA (e-mail: yau@uic.edu).

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contains both Kalman–Bucy and Benes filtering systems [14]. Also, in this series, Yau *et al.* introduced the maximal rank estimation algebras and classified these estimation algebras up to $n \leq 4$, n being the state-space dimension.

In this paper, we classify all four-dimensional (4-D) estimation algebras for arbitrary state-space dimension. One-(FD), two-(2-D), and three-dimensional (3-D) estimation algebras are easy to classify.

II. Filtering Equation and Basic Theorems

The filtering problem considered here is based on the following signal observation model:

$$\begin{cases} dx(t) = f(x(t))\,dt + g(x(t))\,dv(t) & x(0) = x_0 \\ dy(t) = h(x(t))\,dt + dw(t) & y(0) = 0 \end{cases} \tag{1}$$

in which x, v, y, and w are, respectively, R^n-, , R^p-, R^m-, and R^m-valued processes, and v and w have components that are independent, standard Brownian processes. We further assume $n = p, f, h$ are C^∞ smooth, and g is an orthogonal matrix.

Let $\rho(t,x)$ denote the conditional probability density of the state given the observation $\{y(s) : 0 \leq s \leq t\}$. It is well known that $\rho(t,x)$ is given by normalizing a function, $\sigma(t,x)$, that satisfies the following Duncan–Mortensen–Zakai equation:

$$\begin{aligned} d\sigma(t,x) &= L_0\sigma(t,x)d(t) + \sum_{i=1}^{m} L_i\sigma(t,x)\,dy_i(t), \\ \sigma(t,x)d(t) &\to \sigma(t,x)\,dt \end{aligned} \tag{2}$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i(x)\frac{\partial}{\partial x_i} - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2$$

and for $i = 1, \cdots, m$, L_i is the 0 degree differential operator of multiplication by h_i. Here, σ_0 is the probability density of the initial point, x_0.

Definition 1: The estimation algebra E of a filtering system (1) is defined as the Lie algebra generated by $\{L_0, L_1, \cdots, L_m\}$ or $E = \langle L_0, L_1, \cdots, L_m\rangle_{\text{L.A.}}$. Also, if for every $1 \leq i \leq n$, $x_i \in E$, we say E is of maximal rank.

Definition 2: We define

$$\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}, \quad \Omega = (\omega_{ij}), \qquad 1 \leq i, j \leq n.$$

Clearly, Ω is skew symmetric and $(\partial\omega_{jk}/\partial x_i) + (\partial\omega_{ki}/\partial x_j) + (\partial\omega_{ij}/\partial x_k) = 0$ for every $1 \leq i, j, k \leq n$. Also, let

$$D_i = \frac{\partial}{\partial x_i} - f_i, \quad \eta = \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{m} h_i^2.$$

Then

$$L_0 = \tfrac{1}{2}\left(\sum_{i=1}^{n} D_i^2 - \eta\right).$$

Theorem 1 (Ocone): Let E be a finite-dimensional estimation algebra. If a function ψ is in E, ψ is a polynomial of degree ≤ 2.

Mitter [9] conjectured that any function (in E) is at most linear. Yau [14] proved the following theorem.

Theorem 2 (Yau): If $\dim E < \infty$ and $\omega_{ij} = (\partial f_j/\partial x_i) - (\partial f_i/\partial x_j)$ being constant for all i and j, h_i is a polynomial of degree at most one for each i, $1 \leq i \leq m$.

The following theorem, proven in [14], is very useful in the classification of estimation algebras.

Theorem 3 (Yau): Let $F(x_1, \cdots, x_n)$ be a polynomial on R^n. Suppose a polynomial path c: $R \to R^n$ exists such that $\lim_{t\to\infty} c(t) = \infty$ and $\lim_{t\to\infty} F(c(t)) = -\infty$. Then, no C^∞ function on R^n satisfies the equation

$$\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = F.$$

Let U_i be the space of differential operators of order i. In [13], Wong proved the following two theorems.

Theorem 4 (Wong): If $Y = \sum_{1\leq i<j\leq n}\gamma_{ij}D_iD_j \mod U_1$ is an element in E and $\dim E < \infty$, γ_{ij} are polynomials in x for all i, j.

Theorem 5 (Wong): If $Y = \sum_{i=1}^{n}\gamma_i D_i \mod U_0$ is an element in E and $\dim E < \infty$, γ_i's are polynomials in x for all i.

Let ζ be a C^∞-function on R^n. By $E_l(\zeta)$, $l \leq n$, we mean

$$E_l(\zeta) = x_1\frac{\partial\zeta}{\partial x_1} + x_2\frac{\partial\zeta}{\partial x_2} + \cdots + x_l\frac{\partial\zeta}{\partial x_l}$$

which is again an element of $C^\infty(R^n)$.

Theorem 6: Let $E_l = x_1(\partial/\partial x_1) + x_2(\partial/\partial x_2) + \cdots + x_l(\partial/\partial x_l)$ be an Euler operator in $x_1, x_2, \cdots, x_l$ variables. Suppose $m \in Z$ is a constant integer and ζ is a C^∞-function on R^n such that $E_l(\zeta) + m\zeta$ is a polynomial of degree k, k a positive integer, in $x_1, x_2, \cdots, x_l$ variables with coefficients in C^∞-functions of $x_{l+1}, \cdots, x_n$ variables. If $k + m \geq 0$, ζ is a polynomial of degree k in $x_1, x_2, \cdots, x_l$ variables with coefficients in C^∞ functions of $x_{l+1}, \cdots, x_n$. If $k + m < 0$, ζ is a polynomial of degree at most $m' = -m$ in $x_1, x_2, \cdots, x_l$ variables with coefficients in C^∞ functions of $x_{l+1}, \cdots, x_n$.

Proof: See Theorem A1 in the Appendix. □

III. One-, Two-, and Three-Dimensional Estimation Algebras

Let $E = \langle L_o, h_1, h_2, \cdots, h_m\rangle$ be the estimation algebra related to (1). By Ocone's theorem, $h_1, h_2, \cdots, h_m$ are polynomials of at most degree two. Let $h_1, h_2, \cdots, h_k$, $k \leq m$ be independent and the other h_i's, $k+1 \leq i \leq m$ be linear combinations of h_i's for $1 \leq i \leq k$.

If $k = 1$ and $h_1 = \text{constant}$, it is obvious E is 1-D if and only if $h_1 = 0$, and E is 2-D if and only if $h_1 = 1$.

If all h_i's, $1 \leq i \leq k$, are linear, the elements $L_0, h_1, \cdots, h_k, Y_1, \cdots, Y_k, 1$ are independent, and hence E is at least $2 + 2k$-dimensional.

If any of h_i's, say, h_1, is quadratic, by an orthogonal transformation followed by a translation, we may suppose

$$h_1 = \sum_{i=1}^{l} c_i x_i^2 + \sum_{i=l+1}^{n} c_i x_i + d. \tag{3}$$

Then, however, by $X_i = [[L_o, X_{i-1}], X_o]$, $X_o = h_1$, $i > 1$, we get $X_j = 4^j\sum_{i=1}^{l} c_i^{j+1}x_i^2$, $j > 1$, and by the invertibility of the Vandermonde matrix, we may suppose $h_1 = \sum_{i=1}^{l} x_i^2$. It is not difficult to see that, when two independent quadratic forms in E exist, then E is at least five-dimensional (5-D), because if

$$\begin{aligned} h_1 &= \sum_{1\leq i\leq j\leq n}\alpha_{ij}x_ix_j + \sum_{1\leq i\leq n}\beta_i x_i + \beta_0 \\ h_2 &= \sum_{1\leq i\leq j\leq n}\lambda_{ij}x_ix_j + \sum_{1\leq i\leq n}\gamma_i x_i + \gamma_0 \end{aligned}$$

are independent, then

$$Y_1 = [L_0, h_1] = \sum_{1\leq i\leq j\leq n}\alpha_{ij}(x_iD_j + x_jD_i) + \sum_{1\leq i\leq n}\beta_i D_i$$

and

$$Y_2 = [L_0, h_2] = \sum_{1\leq i\leq j\leq n}\lambda_{ij}(x_iD_j + x_jD_i) + \sum_{1\leq i\leq n}\gamma_i D_i$$

are also independent, and now clearly the five elements L_0, h_1, h_2, Y_1, Y_2 are independent, and hence E is at least 5-D. So, from now on, if any quadratic function in E exists, we suppose $k=1$ and $h_1 = \sum_{i=1}^{l} x_i^2$, $l \leq n$. Consider

$$Y_1 = [L_o, h_1] = 2\sum_{i=1}^{l} x_i D_i + l \tag{4}$$

$$\begin{aligned} Y_2 &= [L_o, Y_1] \\ &= 2\sum_{i=1}^{l} D_i^2 + 2\sum_{i=1}^{n}\left(\sum_{j=1}^{l} x_j w_{ji}\right) D_i \\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{l} x_j \frac{\partial w_{ji}}{\partial x_i} + E_l(\eta). \end{aligned} \tag{5}$$

Because $l < n$, the four elements L_o, h_1, Y_1, and Y_2 are independent, and so E is at least 4-D.

If $l = n$ (i.e., if $h_1 = \sum_{i=1}^{n} x_i^2$), $Y_1 = [L_o, h_1] = 2\sum_{i=1}^{n} x_i D_i + n$ and

$$\begin{aligned} Y_2 &= 4L_o - [L_o, Y_1] \\ &= 2\sum_{i,j=1}^{n} x_j w_{ij} D_i + \sum_{i,j=1}^{n} x_i \frac{\partial w_{ij}}{\partial x_j} - E_n(\eta) - 2\eta. \end{aligned} \tag{6}$$

Now if E is 3-D, two constants $\alpha, \beta \in R$ should exist such that $Y_2 = \alpha Y_1 + \beta h_1$, which implies

$$\begin{aligned} &2\sum_{i=1}^{n}\sum_{j=1}^{n} x_i w_{ij} D_j + \sum_{i,j=1}^{n} x_i \frac{\partial w_{ij}}{\partial x_j} + E_n(\eta) \\ &\quad = -2\eta + \alpha\left(2\sum_{i=1}^{n} x_i D_i + n\right) + \beta \sum_{i=1}^{n} x_i^2 \end{aligned} \tag{7}$$

or

$$\sum_{i=1}^{n} x_i w_{ij} = \alpha x_j \quad 1 \leq j \leq n \tag{8}$$

$$\sum_{i,j=1}^{n} x_i \frac{\partial w_{ij}}{\partial x_j} + E_n(\eta) = \beta \sum_{i=1}^{n} x_i^2 - 2\eta + \alpha n. \tag{9}$$

Equation (8) implies $\sum_{i=1}^{n} x_i (\partial w_{ij}/\partial x_j) = \alpha$ for every $1 \leq j \leq n$. Putting this function in the last equation, we get

$$\begin{aligned} n\alpha + E_n(\eta) &= \beta \sum_{i=1}^{n} x_i^2 - 2\eta + \alpha n \\ \Rightarrow E_n(\eta) + 2\eta &= \beta \sum_{i=1}^{n} x_i^2 \in E. \end{aligned}$$

By Theorem 6, however, η is a polynomial of degree two. Now, we have

$$\begin{aligned} &\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 + h_1^2 = \eta \\ &\Rightarrow \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = -\left(\sum_{i=1}^{n} x_i^2\right)^2 + \eta. \end{aligned}$$

By Theorem 3, however, no C^∞ solution $f = (f_1, \cdots, f_n)$ exists for this equation, a contradiction.

We have proven the first theorem.

Theorem 7: For an arbitrary state-space dimension, 3-D estimation algebra does not exist.

IV. Four-Dimensional Estimation Algebras

For state-space dimension equal to one, the following theorem was proven in [3], [8], and [10].

Theorem 8: Suppose the state space of the filtering system (1) is of dimension one. Then, the observation function, $h(x)$, is linear and the linear span of $\nabla h_1, \cdots, \nabla h_m$ is 1-D. Assume $h_1(x) = x_1$. Then, the 4-D estimation algebra has a basis given by $1, x, D = (\partial/\partial x) - f(x)$, and $L_0 = \frac{1}{2}(D^2 - \eta)$. Moreover, $[L_0, x] = D, [D, x] = 1$, $[L_0, D] = \frac{1}{2}(\partial\eta/\partial x)$, where $\eta = \alpha x^2 + 2\beta x + \gamma$. Here, α, β, γ are constants.

In particular, f has to satisfy the equation, $f' + f^2 = (\alpha - 1)x^2 + 2\beta x + \gamma$, where $\alpha - 1 \geq 0$ and $\sqrt{\alpha - 1} \geq (\beta^2/\alpha - 1) - \gamma$.

Now, let $n \geq 2$. As we discussed earlier, we need to consider only three possible cases: $h_1 = x_1, h_1 = \frac{1}{2}\sum_{i=1}^{l} x_i^2, l < n$, and $h_1 = \frac{1}{2}\sum_{i=1}^{n} x_i^2$.

First, let $h_1 = x_1$. Then

$$\begin{aligned} &[L_o, h_1] = D_1 \quad [D_1, h_1] = 1 \\ &[L_o, D_1] = \sum_{j=1}^{n}\left(w_{1j} D_j + \frac{1}{2}\frac{\partial w_{1j}}{\partial x_j}\right) + \frac{1}{2}\frac{\partial \eta}{\partial x_1}. \end{aligned}$$

The four elements L_o, x_1, D_1, and 1 are independent, and $[L_o, D_1]$ should be a linear combination of these elements, which implies $w_{1j} = 0$, $j \geq 1$ and $\eta = \alpha x_1^2 + 2\beta x_1 + g(x_2, \cdots, x_n)$ for some $g \in C^\infty(R^{n-1})$. Also, $f = (f_1, \cdots, f_n)$ should satisfy the following partial differential equation:

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = (\alpha - 1)x_1^2 + 2\beta x_1 + g(x_2, \cdots, x_n).$$

A C^∞ function $f = (f_1, \cdots, f_n)$ exists that satisfies this PDE. One example is provided at the end of this section.

Now, suppose $h_1 = \frac{1}{2}\sum_{i=1}^{l} x_i^2$, $l < n$. Then

$$\begin{aligned} Y_1 &= [L_o, h] = \sum_{i=1}^{l} x_i D_i + \frac{l}{2} \\ Y_2 &= [L_o, Y_1] = \sum_{i=1}^{l} D_i^2 - \sum_{i=1}^{n} \alpha_i D_i - \frac{1}{2}\beta_n + \frac{1}{2}E_l(\eta) \end{aligned}$$

where

$$\alpha_i = \sum_{j=1}^{l} x_j w_{ij}, \quad \beta_n = \sum_{i=1}^{n} \frac{\partial \alpha_i}{\partial x_i} = \sum_{i=1}^{n}\sum_{j=1}^{l} x_j \frac{\partial w_{ij}}{\partial x_i}.$$

Also

$$\begin{aligned} Z_{21} &= [Y_2, Y_1] \\ &= 2\sum_{i=1}^{l} D_i^2 - 3\sum_{i=1}^{l} \alpha_i D_i + \sum_{i=1}^{n} \alpha_i D_i \\ &\quad + \sum_{i=1}^{n}\sum_{j,k=1}^{l} x_k x_j \frac{\partial w_{ij}}{\partial x_k} D_i - \sum_{i,j=1}^{l} x_j \frac{\partial w_{ij}}{\partial x_i} \\ &\quad + \sum_{i=1}^{n}\sum_{j,k=1}^{l} x_k x_j w_{ij} w_{ik} + \frac{1}{2}E_l(\beta_n) - \frac{1}{2}E_l(E_l(\eta)) \\ &= 2\sum_{i=1}^{l} D_i^2 + \sum_{i=1}^{n} E_l(\alpha_i) D_i - 3\sum_{i=1}^{l} \alpha_i D_i - \beta_l \\ &\quad + \sum_{i=1}^{n} \alpha_i^2 + \frac{1}{2}E_l(\beta_n) - \frac{1}{2}E_l(E_l(\eta)) \end{aligned}$$

where $\beta_l = \sum_{i=1}^{l}(\partial\alpha_i/\partial x_i) = \sum_{i=1}^{l}\sum_{j=1}^{l} x_j(\partial\omega ij/\partial x_i)$ and $\sum_{i=1}^{n}\alpha_i^2 = \sum_{i=1}^{n}\sum_{j,k=1}^{l} x_k x_j \omega_{ij}\omega_{ik}$

$$\begin{aligned}Z_{21} - 2Y_2 &= \sum_{i=1}^{l}(E_l(\alpha_i) - \alpha_i)D_i + \sum_{i=l+1}^{n}(E_l(\alpha_i) + 2\alpha_i)D_i \\ &\quad + \sum_{i=1}^{n}\alpha_i^2 - \beta_l + \tfrac{1}{2}(E_l(\beta_n) + \beta_n) - \tfrac{1}{2}(E_l(E_l(\eta)) \\ &\quad + 2E_l(\eta)).\end{aligned}$$

L_o, h_1, Y_1, and Y_2 are clearly independent; and if E has to be 4-D, $\lambda, \gamma \in R$ exists such that

$$Z_{21} - 2Y_2 = \lambda Y_1 + \gamma h = \lambda\left(\sum_{i=1}^{l} x_i D_i + \frac{l}{2}\right) + \gamma\left(\frac{1}{2}\sum_{i=1}^{l} x_i^2\right)$$

but this implies $E_l(\alpha_i) + 2\alpha_i = 0$ for $l+1 \le i \le n$ and $E_l(\alpha_i) - \alpha_i = 0$ for $1 \le i \le l$. Now, for example, for $i = 1$, we have

$$\begin{aligned}&E_l(\alpha_1) - \alpha_1 = \lambda x_1 \\ &\Rightarrow x_1\frac{\partial\alpha_1}{\partial x_1} + x_2\frac{\partial\alpha_1}{\partial x_2} + \cdots + x_l\frac{\partial\alpha_1}{\partial x_l} - \alpha_1 = \lambda x_1 \\ &\Rightarrow x_1\frac{\partial\left(\sum_{j=1}^{l} x_j w_{1j}\right)}{\partial x_1} + x_2\frac{\partial\left(\sum_{j=1}^{l} x_j w_{1j}\right)}{\partial x_2} \\ &\quad + \cdots + x_l\frac{\partial\left(\sum_{j=1}^{l} x_j w_{1j}\right)}{\partial x_l} - \sum_{j=1}^{l} x_j w_{1j} = \lambda x_1 \\ &\Rightarrow x_1\sum_{j=1}^{l}\left(\frac{\partial x_j}{\partial x_1}w_{1j} + x_j\frac{\partial w_{1j}}{\partial x_1}\right) \\ &\quad + x_2\sum_{j=1}^{l}\left(\frac{\partial x_j}{\partial x_2}w_{1j} + x_j\frac{\partial w_{1j}}{\partial x_2}\right) \\ &\quad + \cdots + x_l\sum_{j=1}^{l}\left(\frac{\partial x_j}{\partial x_l}w_{1j} + x_j\frac{\partial w_{1j}}{\partial x_l}\right) - \sum_{j=1}^{l} x_j w_{1j} = \lambda x_1 \\ &\Rightarrow x_1 w_{11} + x_1\sum_{j=1}^{l} x_j\frac{\partial w_{1j}}{\partial x_1} + x_2 w_{12} + x_2\sum_{j=1}^{l} x_j\frac{\partial w_{1j}}{\partial x_2} \\ &\quad + \cdots + x_l w_{1l} + x_l\sum_{j=1}^{l} x_j\frac{\partial w_{1j}}{\partial x_l} - \sum_{j=1}^{l} x_j w_{1j} = \lambda x_1 \\ &\Rightarrow \sum_{j=2}^{l} x_j E_l(w_{1j}) = \lambda x_1.\end{aligned}$$

This result is impossible [because $E_l(w_{1j})$ is a C^∞ function for $2 \le j \le n$; simply put $x_2 = x_3 = \cdots = x_n = 0$] unless $\lambda = 0$, which implies $E_l(\alpha_i) - \alpha_i = 0$, $1 \le i \le l$.

Now, by Theorem 6, α_i, $1 \le i \le l$, is in the form $\alpha_i = \sum_{i=1}^{l} x_i a_i(x_{l+1}, \cdots, x_n)$, $a_i \in C^\infty(R^{n-l})$ and $\alpha_i = 0, l+1 \le i \le n$. Then, however, $\beta_n = \sum_{i=1}^{n}(\partial\alpha_i/\partial x_i) = \sum_{i=1}^{l}(\partial\alpha_i/\partial x_i) = \beta_l$ and

$$\begin{aligned}E_l(\beta_n) = E_l(\beta_l) &= \sum_{k=1}^{l} x_k\frac{\partial\beta_l}{\partial x_k} = \sum_{k=1}^{l} x_k\frac{\partial\left(\sum_{i=1}^{l}\frac{\partial\alpha_i}{\partial x_i}\right)}{\partial x_k} \\ &= \sum_{k=1}^{l}\sum_{i=1}^{l} x_k\frac{\partial^2\alpha_i}{\partial x_i\partial x_k} = 0\end{aligned}$$

because α_i is linear in $x_1, \cdots, x_l$. Also, $\sum_{i=1}^{n}\alpha_i^2$ is a polynomial of degree two in $x_1, \cdots, x_l$ with coefficients being C^∞ functions in $x_{l+1}, \cdots, x_n$.

Now, go back to $Z_{21} - 2Y_2$, which is $Z_{21} - 2Y_2 = \sum_{i=1}^{n}\alpha_i^2 - \frac{1}{2}(E_l(E_l(\eta) + 2\eta))$. E is 4-D if and only if $Z_{21} - 2Y_2 = \gamma h_1$ for some constant $\gamma \in R$. By Theorem 6, η is a polynomial of degree two in $x_1, \cdots, x_l$ with coefficients being C^∞ functions in $x_{l+1}, \cdots, x_n$. By Theorem 3, no C^∞ solution exists for $\sum_{i=1}^{n}(\partial f_i/\partial x_i) + \sum_{i=1}^{n} f_i^2 = -h_1^2 + \eta$, a contradiction. So no 4-D estimation exists algebra in this case.

If $l = n$, i.e., $h_1 = \frac{1}{2}\sum_{i=1}^{n} x_i^2$, $Y_1 = [L_o, h_1] = \sum_{i=1}^{n} x_i D_i + (n/2)$ and

$$\begin{aligned}Y_2 = 2L_o - [L_o, Y_1] &= \sum_{i=1}^{n}\alpha_i D_i + \frac{1}{2}\beta_n - \frac{1}{2}E_n(\eta) - \eta \\ &= \sum_{i=1}^{n}\alpha_i D_i + g_0\end{aligned}$$

where $\alpha_i = \sum_{j=1}^{n} x_j w_{ij}$, $1 \le i \le n$, $\beta_n = \sum_{i=1}^{n}(\sum_{j=1}^{l} x_j(\partial\omega_{ij}/\partial x_i)) = \sum_{i=1}^{n}(\partial\alpha_i/\partial x_i)$ and $g_0 = \frac{1}{2}\beta_n - \frac{1}{2}(E_n(\eta) + 2\eta)$.

By Theorem 5, α_i's are polynomials. By the last part of the proof of Theorem 7 (the case $l = n$), these four elements L_0, h_1, Y_1, and Y_2 are independent.

Before we proceed, we check the structure of Y_2. If $Y_2 = \lambda Y_1 + h_2$ for some $\lambda \in R$. Then, $h_2 \ne h_1$ because L_0, h_1, Y_1, Y_2 are independent; but as we showed at the begining of Section III, we have at least five independent elements: $L_0, h_1, h_2, Y_1 = [L_0, h_1], Y_3 = [L_0, h_2]$. So $Y_2 \ne \lambda Y_1 + h_2$ for any $\lambda \in R$ and any h_2 as a function. This process implies $(\alpha_1, \alpha_2, \cdots, \alpha_n) \ne \lambda(x_1, \cdots, x_n)$ for any $\lambda \in R$.

Now, let

$$\begin{aligned}Y_3 &= [L_o, Y_2] \\ &= \sum_{i,j=1}^{n}\left(\frac{\partial\alpha_j}{\partial x_i}D_i D_j - \alpha_j w_{ij} D_i + \frac{1}{2}\frac{\partial^2\alpha_j}{\partial x_i^2}D_j - \frac{1}{2}\alpha_j\frac{\partial w_{ij}}{\partial x_i}\right) \\ &\quad + \sum_{i=1}^{n}\frac{\partial g_0}{\partial x_i}D_i + \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 g_0}{\partial x_i^2} + \frac{1}{2}\sum_{i=1}^{n}\alpha_i\frac{\partial\eta}{\partial x_i}.\end{aligned}$$

If E is 4-D, $\lambda_i \in R$, $1 \le i \le 4$ exist, such that

$$Y_3 = 2\lambda_1 L_0 + \lambda_2 Y_2 + \lambda_3 Y_1 + 2\lambda_4 h_1. \tag{10}$$

This equation implies

$$\begin{aligned}\frac{\partial\alpha_i}{\partial x_i} &= \lambda_1, \qquad 1 \le i \le n \quad \text{and} \\ \frac{\partial\alpha_j}{\partial x_i} + \frac{\partial\alpha_i}{\partial x_j} &= 0, \qquad 1 \le i \ne j \le n.\end{aligned} \tag{11}$$

Then, however, $\beta_n = \sum_{i=1}^{n}(\partial\alpha_i/\partial x_i) = n\lambda_1$, and so $g_0 = \frac{1}{2}n\lambda_1 - \frac{1}{2}(E_n(\eta) + 2\eta)$. We first prove $\lambda_1 = 0$. By (11)

$$\alpha_i = \lambda_1 x_i + g_i(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n), \qquad 1 \le i \le n$$

which contradicts the formula

$$\sum_{i=1}^{n} x_i\alpha_i = 0 \tag{12}$$

if $\lambda_1 \ne 0$ [because $\sum_{i=1}^{n} x_i(\lambda_1 x_i + g_i(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) = \lambda_1\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} x_i g_i(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) = 0$, which is impossible]. Also, by (12), α_i's are constant-free polynomials. So

$$\alpha_i = g_i(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n), \qquad 1 \le i \le n \tag{13}$$

are constant-free polynomials. So, (12) becomes $\sum_{i=1}^{n} x_i g_i = 0$, and we have, for $1 \le j \le n$, $g_j + x_1(\partial g_1/\partial x_j) + x_2(\partial g_2/\partial x_j) + \cdots + x_n(\partial g_n/\partial x_j) = 0$. By using the fact that g_i is x_i-free, and by (11) and (13), we get, for $1 \le j \le n$, $g_j - E_n(g_j) = 0$. Now, by Theorem 6 and using the fact that g_i's are constant free, each g_i is a homogenous polynomial of degree one; i.e., α_i's are homogenous polynomials of degree one.

Now, let

$$\begin{aligned} Y_4 &= [Y_1, Y_2] \\ &= \sum_{i,j=1}^{n} \left(x_j \alpha_i w_{ij} - \alpha_i \delta_{ij} D_j + x_j \frac{\partial \alpha_i}{\partial x_j} D_i \right) + E_n(g_0) \\ &= \sum_{i=1}^{n} (E_n(\alpha_i) - \alpha_i) D_i + \sum_{i=1}^{n} \alpha_i^2 + E_n(g_0). \end{aligned}$$

Because $E_n(\alpha_i) = \alpha_i, 1 \le i \le n$, $Y_4 = \sum_{i=1}^n \alpha_i^2 + E_n(g_0)$ is in E, and we should have $Y_4 = \sum_{i=1}^n \alpha_i^2 + E_n(g_0) = \gamma h_1, \gamma \in R$ (otherwise, we have at least five independent elements). By Theorem A3 in the Appendix, g_0 is a quadratic polynomial, and hence, by Theorem A2 in the Appendix, η is a quadratic polynomial. Now, by Theorem 3, $\sum_{i=1}^n (\partial f_i/\partial x_i) + \sum_{i=1}^n f_i^2 = -(\frac{1}{2}\sum_{i=1}^n x_i^2)^2 + \eta$ has no C^∞ solution, a contradiction.

From all of these notes, we conclude the following.

Theorem 9: Suppose the state space of the filtering system (1) is of a dimension greater than one. Then, the observation function, $h(x)$, is linear and the linear span of $\nabla h_1, \cdots, \nabla h_m$ is 1-D. Assume $h_1(x) = x_1$. Then, the 4-D estimation algebra has a basis given by $1, x_1, D_1 = (\partial/\partial x_1) - f_1(x_1, \cdots, x_n)$ and $L_0 = \frac{1}{2}(\sum_{i=1}^n D_i^2 - \eta)$. Moreover, $\omega_{12} = \omega_{13} = \cdots = \omega_{1n} = 0, [L_0, x_1] = D_1, [D_1, x_1] = 1, [L_0, D_1] = \frac{1}{2}(\partial\eta/\partial x_1) = \alpha x_1 + \beta$, where α, β are constants. Also, $\eta = \alpha x_1^2 + 2\beta x_1 + g(x_2, \cdots, x_n)$, where $g(x_2, \cdots, x_n)$ is in $C^\infty(R^{n-1})$. In particular, $f_1, \cdots, f_n$ have to satisfy the equation

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = (\alpha - 1)x_1^2 + 2\beta x_1 + g(x_2, \cdots, x_n) \qquad (14)$$

where $\alpha \ge 1$.

Example of Theorem 9: If we take

$$\begin{aligned} f_1 &= \sqrt{\alpha - 1}\, x_1 + \frac{\beta}{\sqrt{\alpha - 1}}, \qquad \alpha > 1 \\ f_2 &= f_2(x_2, \cdots, x_n), \cdots, f_n = f_n(x_2, \cdots, x_n) \\ g(x_2, \cdots, x_n) &= \sum_{i=2}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=2}^{n} f_i^2 + \frac{\beta^2}{\alpha - 1} + \sqrt{\alpha - 1} \end{aligned}$$

then $\omega_{12} = \omega_{13} = \cdots = \omega_{1n} = 0$ and (14) is satisfied.

APPENDIX

Let ζ be a C^∞-function on R^n. By $E_l(\zeta)$, $l \le n$, we mean $E_l(\zeta) = x_1(\partial\zeta/\partial x_1) + x_2(\partial\zeta/\partial x_2) + \cdots + x_l(\partial\zeta/\partial x_l)$, which is again an element of $C^\infty(R^n)$.

Theorem A1: Let $E_l = x_1(\partial/\partial x_1) + x_2(\partial/\partial x_2) + \cdots + x_l(\partial/\partial x_l)$ be an Euler operator in $x_1, x_2, \cdots, x_l$ variables. Suppose $m \in Z$ is a constant integer and ζ is a C^∞-function on R^n such that $E_l(\zeta) + m\zeta$ is a polynomial of degree k, k is a positive integer, in $x_1, x_2, \cdots, x_l$ variables with coefficients in C^∞-functions of $x_{l+1}, \cdots, x_n$ variables. If $k + m \ge 0$, ζ is a polynomial of degree k in $x_1, x_2, \cdots, x_l$ variables with coefficients in C^∞ functions of $x_{l+1}, \cdots, x_n$. If $k + m < 0$, ζ is a polynomial of degree at most $m' = -m$ in $x_1, \cdots, x_l$ variables with coefficients in C^∞ functions of $x_{l+1}, \cdots, x_n$.

Proof: First, let $k + m \ge 0$; that is, $k + m + 1 > 0$. Also, let $D = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_l)^{\alpha_l}$, $\alpha_1 + \cdots + \alpha_l = k+1$ be a differential operator of order $k+1$. Because $E_l(\zeta) + m\zeta$ is a polynomial of degree k in $x_1, x_2, \cdots, x_l$ variables with coefficients in C^∞-functions of $x_{l+1}, \cdots, x_n$ variables, we have $D[E_l(\zeta) + m\zeta] = 0$. On the other hand, in view of $(\partial/\partial x_i)E_l = E_l(\partial/\partial x_i) + (\partial/\partial x_i)$ for $1 \le i \le l$, it is easy to see by induction that

$$\begin{aligned} &D[E_l(\zeta) + m\zeta] \\ &\quad = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_{l-1}}\right)^{\alpha_{l-1}} \left(\frac{\partial}{\partial x_l}\right)^{\alpha_l} [E_l(\zeta) + m\zeta] \\ &\quad = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_{l-1}}\right)^{\alpha_{l-1}} \Bigg[E_l(\frac{\partial}{\partial x_l})^{\alpha_l} \zeta \\ &\qquad + (\alpha_l + m)\left(\frac{\partial}{\partial x_l}\right)^{\alpha_l} \zeta \Bigg] \\ &\quad = E_l(D\zeta) + (\alpha_1 + \cdots + \alpha_l + m) D\zeta. \end{aligned}$$

So, we have $E_l(D\zeta) + (k + 1 + m)D\zeta = 0$. Observe

$$\begin{aligned} E_l\left[x_1^{k+1+m} D\zeta\right] &= (k+1+m) x_1^{k+1+m} D\zeta + x_1^{k+1+m} E_l(D\zeta) \\ &= x_1^{k+1+m}[E_l(D\zeta) + (k+1+m)D\zeta] = 0. \end{aligned}$$

Denote $\phi = x_1^{k+1+m} D\zeta$. Because $k + 1 + m > 0$, we have

$$\begin{aligned} &\phi(x_1, \cdots, x_l, \cdots, x_n) - \phi(\epsilon x_1, \cdots, \epsilon x_l, x_{l+1}, \cdots, x_n) \\ &\quad = \int_\epsilon^1 \frac{d\phi}{dt}(tx_1, \cdots, tx_l, x_{l+1}, \cdots, x_n)\, dt \\ &\quad = \int_\epsilon^1 \Bigg(x_1 \frac{d\phi}{dx_1}(tx_1, \cdots, tx_l, x_{l+1}, \cdots, x_n) \\ &\qquad + x_2 \frac{d\phi}{dx_2}(tx_1, tx_2, \cdots, tx_l, x_{l+1}, \cdots, x_n) \\ &\qquad + \cdots + x_l \frac{d\phi}{dx_l}(tx_1, \cdots, tx_l, x_{l+1}, \cdots, x_n) \Bigg) dt \\ &\quad = \int_\epsilon^1 \frac{1}{t}(E_l\phi)(tx_1, \cdots, tx_l, x_{l+1}, \cdots, x_n)\, dt = \int_\epsilon^1 \frac{0}{t}\, dt = 0. \end{aligned}$$

for $\epsilon > 0$. Now, let $\epsilon \to 0$. Then, we get $\phi(x_1, \cdots, x_l, x_{l+1}, \cdots, x_n) = 0$. This result implies

$$D\zeta = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_l}\right)^{\alpha_l} \zeta = 0$$

for all $\alpha_1 + \cdots + \alpha_l = k + 1$ and $\alpha_1 \ge 0, \cdots, \alpha_l \ge 0$. In other words, ζ is a polynomial of degree at most k in $x_1, \cdots, x_l$ variables with coefficients in C^∞-functions of $x_{l+1}, \cdots, x_n$ variables. Now, by two methods, we can prove ζ is a polynomial of degree k. One method is by induction on k and using the same method as above; the other method is by the assumption that $\zeta = \sum_{0 \le i_1 + \cdots + i_l \le s} a_{i_1, \cdots, x_l}(x_{l+1}, \cdots, x_n) x_1^{i_1} \cdots x_l^{i_l}, s \le k$ is a polynomial of degree s, and then using the definition of $E_l(\zeta) + m\zeta$ and the hypothesis that the last one is a degree k polynomial. We provide the proof using the second method. Let ζ be a polynomial of degree s

$$\begin{aligned} E_l(\zeta) + m\zeta &= E_l\left(\sum_{0 \le |i| \le s} a_i(x_{l+1}, \cdots, x_n) x_1^{i_1} \cdots x_l^{i_l} \right) \\ &\quad + m \sum_{0 \le |i| \le s} a_i(x_{l+1}, \cdots, x_n) x_1^{i_1} \cdots x_l^{i_l} \\ &= \sum_{0 < |i| \le s} |i| a_i(x_{l+1}, \cdots, x_n) x_1^{i_1} \cdots x_l^{i_l} \\ &\quad + m \sum_{0 \le |i| \le s} a_i(x_{l+1}, \cdots, x_n) x_1^{i_1} \cdots x_l^{i_l} \\ &= \sum_{0 < |i| \le s} (|i| + m) a_i(x_{l+1}, \cdots, x_n) x_1^{i_1} \cdots x_l^{i_l} \\ &\quad + m a_0(x_{l+1}, \cdots, x_n) \\ &= \sum_{0 \le |i| \le k} b_i(x_{l+1}, \cdots, x_n) x_1^{i_1}, \cdots, x_l^{i_l} \end{aligned}$$

where $i = (i_1, \cdots, i_l)$ and $|i| = i_1 + \cdots + i_l$ and $b_i(x_{l+1}, \cdots, x_n)$ is C^∞. By looking at the coefficients on both sides, $s = k$ and $(|i| + m)a_i = b_i$ for all $i, 0 < |i| \le k$. That is, ζ is a polynomial of degree k in $x_1, \cdots, x_l$ variables with coefficients being C^∞ functions in $x_{l+1}, \cdots, x_n$.

Now, let $k + m < 0$. In this case, m is a negative integer. Let $m = -m', m' > 0$. Then, $E_l(\zeta) + m\zeta = E_l(\zeta) - m'\zeta = P_k$, where P_k is a polynomial of degree k in $x_1, \cdots, x_l$ variables with coefficients in C^∞-functions of $x_{l+1}, \cdots, x_n$. We have

$$\frac{\partial}{\partial x_{i_1}}(E_l(\zeta) - m'\zeta) = \frac{\partial}{\partial x_{i_1}} P_k = P_{k-1}, \qquad 1 \le i_1 \le l$$
$$\Rightarrow E_l\left(\frac{\partial \zeta}{\partial x_{i_1}}\right) - (m'-1)\frac{\partial \zeta}{\partial x_{i_1}} = P_{k-1}$$

where P_{k-1} is a polynomial of degree $k-1$. Using the same technique, we get

$$\frac{\partial}{\partial x_{i_2}}\left(E_l\left(\frac{\partial \zeta}{\partial x_{i_1}}\right) - (m'-1)\frac{\partial \zeta}{\partial x_{i_1}}\right)$$
$$= \frac{\partial}{\partial x_{i_2}} P_{k-1} = P_{k-2} \qquad 1 \le i_2 \le l$$
$$\Rightarrow E_l\left(\frac{\partial^2 \zeta}{\partial x_{i_1}\partial x_{i_2}}\right) - (m'-2)\frac{\partial^2 \zeta}{\partial x_{i_1}\partial x_{i_2}} = P_{k-2}$$

where P_{k-2} is a polynomial of degree $k-2$. After $m'-1$ times, we have

$$E_l\left(\frac{\partial^{m'-1}\zeta}{\partial x_{i_1}\cdots\partial x_{i_{m'-1}}}\right) - \frac{\partial^{m'-1}\zeta}{\partial x_{i_1}\cdots\partial x_{i_{m'-1}}}$$
$$= P_{k-(m'-1)}, \qquad 1 \le i_{m'-1} \le l$$

where $P_{k-(m'-1)}$ is a polynomial of degree 0 in $x_1, \cdots, x_l$ variables, that is, a C^∞-function in $x_{l+1}, \cdots, x_n$.

Once more

$$\frac{\partial}{\partial x_{i_{m'}}}\left(E_l\left(\frac{\partial^{m'-1}\zeta}{\partial x_{i_1}\cdots\partial x_{i_{m'-1}}}\right) - \frac{\partial^{m'-1}\zeta}{\partial x_{i_1}\cdots\partial x_{i_{m'-1}}}\right) = 0$$
$$1 \le i_{m'} \le l \Rightarrow E_l\left(\frac{\partial^{m'}\zeta}{\partial x_{i_1}\cdots\partial x_{i_{m'}}}\right) = 0.$$

Now let $\epsilon > 0$. By the same technique we have

$$\frac{\partial^{m'}\zeta}{\partial x_{i_1}\cdots\partial x_{i_{m'}}}(x_1, \cdots, x_l, x_{l+1}, \cdots, x_n)$$
$$- \frac{\partial^{m'}\zeta}{\partial x_{i_1}\cdots\partial x_{i_{m'}}}(\epsilon x_1, \cdots, \epsilon x_l, x_{l+1}, \cdots, x_n)$$
$$= \int_\epsilon^1 \frac{d}{dt}\left(\frac{\partial^{m'}\zeta}{\partial x_{i_1}\cdots\partial x_{i_{m'}}}(tx_1, \cdots, tx_l, x_{l+1}, \cdots, x_n)\right) dt$$
$$= \int_\epsilon^1 \frac{1}{t} E_l\left(\frac{\partial^{m'}\zeta}{\partial x_{i_1}\cdots\partial x_{i_{m'}}}(tx_1, \cdots, tx_l, x_{l+1}, \cdots, x_n)\right) dt$$
$$= \int_\epsilon^1 \frac{0}{t}\, dt = 0.$$

Let $\epsilon \to 0$. Then

$$\frac{\partial^{m'}\zeta}{\partial x_{i_1}\cdots\partial x_{i_{m'}}}(x_1, \cdots, x_l, x_{l+1}, \cdots, x_n)$$
$$= \frac{\partial^{m'}\zeta}{\partial x_{i_1}\cdots\partial x_{i_{m'}}}(0, \cdots, 0, x_{l+1}, \cdots, x_n).$$

The right-hand side is a function of $x_{l+1}, \cdots, x_n$, which means $\partial^{m'-1}\zeta/\partial x_{i_1}\cdots\partial x_{i_{m'-1}}$ is a linear function of $x_1, \cdots, x_l$ with coefficients in C^∞-functions of $x_{l+1}, \cdots, x_n$. Now, by induction, we conclude ζ is a polynomial of degree at most m' in $x_1, \cdots, x_l$ variables with coefficients in C^∞-functions of $x_{l+1}, \cdots, x_n$. Q.E.D.

Theorem A2: Let $E_l = x_1(\partial/\partial x_1) + \cdots + x_l(\partial/\partial x_l)$ be an Euler operator in $x_1, \cdots, x_l$ variables. Suppose m is a positive constant and ζ is a C^∞ function on R^n such that $E_l(\zeta) + m\zeta$ is a polynomial of degree k in $x_1, \cdots, x_n$ variables. Then, ζ is a polynomial of degree k in $x_1, \cdots, x_n$ variables.

Proof: By Theorem A1, $\zeta = \sum_{0 \le |\alpha| \le k} a_\alpha(x_{l+1}, \cdots, x_n)x_1^{\alpha_1}\cdots x_l^{\alpha_l}$, where $\alpha = (\alpha_1, \cdots, \alpha_l)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_l$, and $a_\alpha(x_{l+1}, \cdots, x_n)$ is C^∞.

Hence, we have the following:

$$E_l(\zeta) + m\zeta = \sum_{0<|\alpha|\le k} |\alpha| a_\alpha(x_{l+1}, \cdots, x_n)x_1^{\alpha_1}\cdots x_l^{\alpha_l}$$
$$+ m \sum_{0\le|\alpha|\le k} a_\alpha(x_{l+1}, \cdots, x_n)x_1^{\alpha_1}\cdots x_l^{\alpha_l}$$
$$= \sum_{0<|\alpha|\le k} (|\alpha| + m) a_\alpha(x_{l+1}, \cdots, x_n)x_1^{\alpha_1}\cdots x_l^{\alpha_l}$$
$$+ m a_0(x_{l+1}, \cdots, x_n)$$
$$= \sum_{0\le|\alpha|\le k} p_\alpha(x_{l+1}, \cdots, x_n)x_1^{\alpha_1}\cdots x_l^{\alpha_l}$$

where $p_\alpha(x_{l+1}, \cdots, x_n)$'s are polynomials in $x_{l+1}, \cdots, x_n$ [because $E_l(\zeta) + m\zeta$ is a polynomial in $x_1, \cdots, x_n$, so we may assume it is a polynomial in $x_1, \cdots, x_l$ with coefficients being polynomials in $x_{l+1}, \cdots, x_n$). Now, looking at both sides, we conclude $(|\alpha| + m)a_\alpha = p_\alpha$, for all $\alpha = (\alpha_1, \cdots, \alpha_l), 0 < |\alpha| \le k$; in other words, all $a_\alpha, 0 < |\alpha| \le k$ are polynomials and $a_0 = (1/m)p_0$ is a polynomial, and hence ζ is a polynomial.

Remark: Theorem A2 is false if $m = 0$. It is possible $E_l(\zeta)$ is a polynomial of degree k in $x_1, \cdots, x_n$ variables, but ζ is not a degree k polynomial in $x_1, \cdots, x_n$ variables. For example, we can simply take ζ to be any degree k polynomial in $x_1, \cdots, x_n$ variables plus a transcendental function in $x_{l+1}, \cdots, x_n$ variables.

Theorem A3: Let $E_l = x_1(\partial/\partial x_1) + \cdots + x_l(\partial/\partial x_l)$ be an Euler operator in $x_1, \cdots, x_l$ variables. Suppose ζ is a C^∞ function on R^n such that $E_l(\zeta)$ is a polynomial of degree k in $x_1, \cdots, x_n$ variables. Then, $\zeta = P_k(x_1, \cdots, x_n) + a(x_{l+1}, \cdots, x_n)$, where $P_k(x_1, \cdots, x_n)$ is a polynomial of degree k and $a(x_{l+1}, \cdots, x_n)$ is a C^∞ function in $x_{l+1}, \cdots, x_n$ variables.

Proof: In view of Theorem A1, $\zeta = \sum_{0\le|\alpha|\le k} a_\alpha(x_{l+1}, \cdots, x_n)x_1^{\alpha_1}\cdots x_l^{\alpha_l}$, where $\alpha = (\alpha_1, \cdots, \alpha_l)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_l$, and $a_\alpha(x_{l+1}, \cdots, x_n)$ is C^∞. Then, $E_l(\zeta) = \sum_{0<|\alpha|\le k} |\alpha| a_\alpha(x_{l+1}, \cdots, x_n)x_1^{\alpha_1}\cdots x_l^{\alpha_l}$, which is a polynomial of degree k in $x_1, \cdots, x_n$ variables. Therefore, $a_\alpha(x_{l+1}, \cdots, x_n)$, for $|\alpha| \ge 1$, are polynomials. Theorem A3 follows immediately. Q.E.D.

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Semiglobal Stabilization of Continuous-Time Systems with Bounded Inputs

Rogelio Lozano, Joaquin Collado, and Angel Herrera

***Abstract*—We present a simple linear periodic controller that achieves semi-global stabilization of linear systems with amplitude constrained inputs. The control scheme is applicable to continuous-time null controllable systems, i.e., systems having open-loop poles in the closed left half-plane (LHP).**

***Index Terms*—Bounded inputs, linear time invariant continuous systems, periodic dead-beat controller, semi-global stabilization, stabilization.**

I. Introduction

In the last few years, an increasing interest in obtaining globally or semiglobally stabilizing controllers for linear systems with amplitude constrained inputs has occurred. The interest is because limitations in the control input are omnipresent in practice. For exponentially unstable systems, i.e., systems with poles in the open right half-plane, only locally stabilizing controllers can be obtained. In this paper, we will focus our attention on obtaining a stabilizing controller for linear systems having poles in the closed left half-plane (LHP). This class of systems includes those having multiple poles on the jw-axis that are unstable. One of the first results on that type of systems was obtained by Fuller [1] who showed it is not possible to globally stabilize three or more integrators connected in cascade by simply saturating a stabilizing state feedback control law. Later, Sussman *et al.* [13] showed this constraint can be extended to systems having three or more eigenvalues on the jw-axis.

Teel [14] proposed a control algorithm that achieves global asymptotic stabilization of a chain of n-integrators. The control strategy is based on a series of nested saturation functions. This result was further generalized by Sussman *et al.* [13] to general null controllable linear systems, i.e., controllable systems having all its eigenvalues in the closed LHP. The results in [13] were developed in the continuous-time case and have a discrete-time counterpart, as shown in Yang [15]. Lin and Saberi proposed [5] a linear controller that achieves semiglobal stabilization for null controllable linear systems using state feedback. By semiglobal (exponential) stabilization, we mean the closed-loop system is locally (exponentially) stable and the region of attraction contains any given bounded set. The controller parameters generally depend only on the upper bound of the initial state. Lin and Saberi [6] proposed a semiglobal stabilization scheme by using a linear controller that achieves pole placement. The results in [6] were developed in the continuous-time case and were generalized to the discrete-time case in Yang [15].

Liu *et al.* [4] proposed a global L_p-stable controller based on passivity for neutrally stable null controllable linear systems. Neutrally stable means the eigenvalues on the imaginary axis are associated

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R. Lozano is with the Heudiasyc UMR 6599 CNRS-UTC, 60200 Compiegne, France (e-mail: rlozano@hds.utc.fr).

J. Collado is with the UANL-FIME Apdo, 66451 San Nicolás de los Garza, N.L., Mexico (e-mail: jcollado@ccr.dsi.uanl.mx).

A. Herrera is with the Departamento de Control Automatico, Cinvestav, 07000, Mexico.

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