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# FINITE-DIMENSIONAL FILTERS WITH NONLINEAR DRIFT XIII: CLASSIFICATION OF FINITE-DIMENSIONAL ESTIMATION ALGEBRAS OF MAXIMAL RANK WITH STATE SPACE DIMENSION FIVE*

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**Abstract.** The idea of using estimation algebras to construct finite dimensional nonlinear filters was first proposed by Brockett and Mitter independently. For this approach, one needs to know explicitly the structure of these estimation algebras in order to construct finite dimensional nonlinear filters. Therefore Brockett proposed to classify all finite dimensional estimation algebras. Chiou and Yau [ChYa1] classify all finite dimensional estimation algebras of maximal rank with dimension of the state space less than or equal to two. The purpose of this paper is to give a new result on classification of all finite dimensional estimation algebras of maximal rank with state space dimension less than or equal to five.

**1. Introduction.** The idea of using estimation algebras to construct finite dimensional nonlinear filters was first proposed by Brockett [Br] and Mitter [Mi] independently. The advantage of this finite-dimensional nonlinear filter is at least the same as Kalman-Bucy filter. Moreover, it avoids the disadvantages of Kalman-Bucy filter such as Gaussian initial condition as well as linearity assumption of the drift term. For more detail, we refer the readers to [TWY] and [Ya], in which the links between finite dimensional estimation algebras and finite dimensional filters were discussed. It is clear from the works of [TWY] and [Ya] that one needs to know explicitly the structure of these estimation algebras in order to construct finite dimensional nonlinear filters. In 1983, Brockett proposed to classify all finite dimensional estimation algebras in his talk at the International Congress of Mathematics. If the drift term of the nonlinear filtering system has a potential function (i.e. drift term is a gradient vector filed), then the corresponding estimation algebra is called exact. In [TWY], Tam, Wong and Yau have classified all finite dimensional exact estimation algebras of maximal rank with arbitrary state space dimension. In [ChYa1], Chiou and Yau are able to classify all finite dimensional estimation algebras of maximal rank with state space dimension less than or equal to two. The novelty of their theorem is that there is no assumption on the drift term of the nonlinear filtering system. In [CYL1], Chen, Leung and Yau classify all finite dimensional estimation algebras of maximal rank with state space dimension equal to 3 (without any assumption on the drift term). They introduced a new matrix equation in [CYL2] and showed that this matrix equation has only trivial solution if the state space dimension is at most four. They reduced the classification problem of finite dimensional estimation algebras with maximal rank to the problem of nonexistence of nontrivial solution of this new matrix equation by using the fact that $\eta_4$, homogeneous degree four part of $\eta$, depends only on $x_{k+1},...,x_n$ where $k$ is the quadratic rank of the estimation algebra. Recently Wu, Yau and Hu [WYH] have a direct proof of nonexistence of nontrivial solution of this new matrix equation. In this paper, we develop a completely different technique than [CYL2]. Moreover we

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can classify all finite dimensional estimation algebras with maximal rank with state space dimension $n = 5$ also. The following is our main theorem.

Main Theorem. *Suppose that the state space of the filtering system (2.1) is of dimension* $n \leq 5$. *If* $E$ *is the finite-dimensional estimation algebra of maximal rank, then the drift term* $f$ *must be a linear vector field (i.e. each component is a polynomial of degree one) plus a gradient vector field and* $E$ *is a real vector space of dimension* $2n+2$ *with basis given by* $1, x_1, ..., x_n, D_1, ..., D_n$ *and* $L_0$. *Moreover* $\eta$ *is a degree two polynomial.*

Let $\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$, which was first introduced by Wong [Wo]. Our strategy is to prove that $\omega_{ij}$'s are constants for all $i, j$. Then we can apply the result of [Ya] to finish the proof. This involves two steps. The first step is to prove that $\omega_{ij}$'s are degree one polynomials. This step was completed by Chen and Yau [ChYa2] for arbitrary $n$. The second step is to prove that $\omega_{ij}$'s are actually constants. This is the hard part in the problem of classification of finite dimensional estimation algebras of maximal rank. The purpose of this paper is to deal with the hard part of the problem by proving that $\omega_{ij}$'s are constants for $n \leq 5$. We observed that Chen and Yau [ChYa2] have already proved that $\omega_{ij}$, $1 \leq i,\ j \leq k$, where $k$ is the quadratic rank of the estimation algebra, are constants. In [ChYa3], Chen-Yau introduced many new ideas and claimed to prove that $\omega_{ij}$ 's are constants for either $1 \leq i \leq k$, $1 \leq j \leq n$ or $1 \leq i \leq n$, $1 \leq j \leq k$,and $\omega_{ij}$'s are degree one polynomials in $x_{k+1}$,...,$x_n$ for $k+1 \leq i$, $j \leq n$. Unfortunately, the proof turns out to be incomplete.

Recently, Hu and Yau [HuYa] developed a new method and they proved that $\omega_{ij}$'s, $1 \leq i,\ j \leq k$ or $k+1 \leq i,\ j \leq n$, are constants, and $\omega_{ij}$'s are degree one polynomials in $x_1, ..., x_k$ for $1 \leq i \leq k$, $k+1 \leq j \leq n$ or $k+1 \leq i \leq n$, $1 \leq j \leq k$. By using this result and new technique developed in this paper together with basic theory developed in [ChYa2], not only we can obtain entirely new results for $n = 5$, but also we have simple uniform proof for $n \leq 4$.

This paper is in essence a continuation of [Ya], [ChYa1], [ChYa2] and [HuYa] and we strongly recommend that readers familiarize themselves with the results in these papers. However, every effort will be made to make this paper as self-contained as possible, with minimal duplication of the previous papers.

## 2. Basic concepts.

In this section, we shall recall some basic concepts and results from [Ya], [ChYa2] and [HuYa]. Consider a filtering problem based on the following signal observation model:

$$\left\{ \begin{array}{lcll} dx(t) & = & f(x(t)dt + g(x(t))dv(t), & x(0) = x_0 \\ dy(t) & = & h(x(t)dt + dw(t), & y(0) = 0 \end{array} \right. \tag{2.1}$$

in which $x, v, y$,and $w$ are, respectively, $\mathbf{R}^n, \mathbf{R}^p, \mathbf{R}^m$ and $\mathbf{R}^m$ valued processes and $v$ and $w$ have components which are independent, standard Brownian processes. We further assume that $n = p$, $f$, $h$ are $C^\infty$ smooth and that $g$ is an orthogonal matrix. We shall refer to $x(t)$ as the state of the system at time $t$ and to $y(t)$ as the observation at time $t$.

Let $\rho(t, x)$ denote the conditional probability density of the state given the observation $\{y(s) : 0 \leq s \leq t\}$. It is well-known (see [DaMa]) that $\rho(t, x)$ is given by normalizing a function $\sigma(t, x)$ that satisfies the following Duncan-Mortensen-Zakai equation:

$$d\sigma(t,x) = L_0\sigma(t,x)dt + \sum_{i=1}^{m} L_i\sigma(t,x)dy_i(t), \quad \sigma(0,x) = \sigma_0, \tag{2.2}$$

where $L_0 = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i} - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2$ and for $i = 1, ..., m, L_i$ is the zero-degree differential operator of multiplication by $h_i, \sigma_0$ is the probability density of the initial point $x_0$.

Equation (2.2) is a stochastic partial differential equation. Davis [Da] proposed some robust algorithms. In our case, his basic idea reduces to defining a new unnormalized density

$$u(t,x) = \exp\left(-\sum_{i=1}^{m} h_i(x)y_i(t)\right)\sigma(t,x)$$

Davis reduced (2.2) to the following time-varying partial differential equation, which is called the robust DMZ equation.

$$\left\{\begin{array}{lcl} \frac{\partial u}{\partial t}(t,x) & = & L_o u(t,x) + \sum_{i=1}^{m} y_i(t)[L_o, L_i]u(t,x) \\ & & +\frac{1}{2}\sum_{i,j=1}^{m} y_i(t)y_j(t)[[L_o, L_i], L_j]u(t,x) \\ u(0,x) & = & \sigma_0(x) \end{array}\right. \tag{2.3}$$

Here we have used the following notation.

Definition 2.1. *If $X$ and $Y$ are differential operators, then the Lie bracket of $X$ and $Y$, $[X,Y]$, is defined by $[X,Y]\phi = X(Y\phi) - Y(X\phi)$ for any $C^\infty$ function $\phi$.*

Definition 2.2. *The estimation algebra $E$ of a filtering problem (2.1) is defined to be the Lie algebra generated by $\{L_0, L_1, ..., L_m\}$. $E$ is said to be an estimation algebra of maximal rank if for any $1 \leq i \leq n$ there exists a constant $c_i$ such that $x_i + c_i$ is in $E$.*

Most of the known finit-dimensional estimation algebras are maximal. For example, if (2.1) is linear, i.e., $f(x) = Ax, g(x) = Bx$, and $h(x) = Cx$,and if $(A, B, C)$ also is minimal, then the corresponding estimation algebra is of maximal rank [Ha]. We need the following basic result for later discussion.

Theorem 2.1 (Ocone). *Let $E$ be a finite dimensional estimation algebra. If a function $\xi$ is in $E$, then $\xi$ is a polynomial of degree at most two.*

In [Wo], Wong introduced $\Omega$ matrix whose $(i,j)-$ element $\omega_{ij}$ is $\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$. Define

$$D_i = \frac{\partial}{\partial x_i} - f_i \text{ and } \eta = \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 - \sum_{i=1}^{m} h_i^2.$$

Then $L_0 = \frac{1}{2}(\sum_{i=1}^{n} D_i^2 - \eta)$.

The following theorem proved in [Ya] plays a fundamental role in the classification of finite-dimensional estimation algebras.

Theorem 2.2 (Yau). *Let $E$ be a finite-dimensional estimation algebra of (2.1) such that $\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$ are constant functions. If $E$ is of maximal rank, then $E$ is a real vector space of dimension $2n+2$ with basis given by $1, x_1, x_2, ..., x_n, D_1, D_2, ..., D_n$, and $L_0$.*

Recently Chen and Yau [ChYa2] have made important progress in the program of classification of finite-dimensional estimation algebras of maximal rank. They study the quadratic forms in $E$ and show that the $\Omega$-matrix is linear in the sense that all $\omega_{ij}$ are degree one polynomials.

Definition 2.3. *Let $Q$ be the space of quadratic forms in $n$ variables, namely, realy vector space spanned by $x_i x_j, 1 \leq i \leq j \leq n$. Let $X = (x_1, ..., x_n)^T$. For any quadratic form $p \in Q$, there exists a symmetric matrix $A$ such that $p(x) = X^T AX$. The rank of the quadratic form $p$ is denoted by $r(p)$ and is defined to be the rank of the matrix $A$. A fundamental quadratic form of the estimation algebra $E$ is an element $p_0 \in E \cap Q$ with the greatest positive rank, that is, $r(p_0) \geq r(p)$ for any $p \in E \cap Q$ .The maximal rank of quadratic forms in the estimation algebra $E$ is defined to be $k = r(p_0)$ and is called the quadratic rank of $E$.*

After an orthogonal transformation, $p_0$ can be written as

$$p_0(x) = c_1 x^2 + c_2 x_2^2 + ... + c_k x_k^2, \quad c_i \neq 0, \quad 0 \leq k \leq n$$

From $p_0(x)$, we can construct a sequence of quadratic forms in $E \cap Q$ as follows:

$$\begin{aligned} q_0(x) &= p_0(x) \\ q_j(x) &= [[L_0, q_{j-1}], q_0] = \sum_{i=1}^{k} 4^j c_i^{j+1} x_i^2 \end{aligned}$$

In view of the invertibility of the Vandermonde matrix, we can assume that

$$p_0(x) = x^2 + x_2^2 + ... + \ x_k^2 \in E \tag{2.4}$$

Lemma 2.1 (Chen and Yau) [ChYa2]. *If $p$ is a quadratic form in the estimation algebra $E$ of (2.1), then $p$ is independent of $x_j$ for $j \geq k$, where $k = r(p_0)$ is the quadratic rank of $E$. In other words, $\frac{\partial p}{\partial x_j} = 0$ for $k + 1 \leq j \leq n$.*

Let $p_1 \in E \cap Q$ be an element with least positive rank, that is, $o < r(p_1) \leq r(q)$ for any nonzero $q \in E \cap Q$. After an orthogonal transform that fixes $x_{k+1}, ..., x_n$ varables (i.e. an orthogonal transform on $x_1, x_2, ..., x_k$), and the Vandermonde matrix procedure as above, we can assume

$$p_1 = \sum_{i=1}^{k_1} x_i^2 \in E, \quad 1 \leq k_1 \leq k \tag{2.5}$$

Notice that the orthogonal transform on $x_1, ..., x_k$ leaves $p_0$ invariant. In summary, we deduce that $p_0 = \sum_{i=1}^{k} x_i^2$ has the greatest postive rank and $p_1 = \sum_{i=1}^{k_1} x_i^2$ has the least positive rank. Define

$$S_1 = \{1, 2, ..., k_1\} \subseteq S = \{1, 2, ..., k\} \tag{2.6}$$

and $Q_1 =$ real vector space spanned by $\{x_i x_j : k_1 + 1 \leq i \leq j \leq k\} \subseteq Q$. If $k_1 < k$, then $Q_1 \cap E$ is a nontrivial space, since $p - p_0 \in E \cap Q$. In a similar procedure as above, there exist $k_2 > k_1$ and

$$p_2 = \sum_{i=k_1+1}^{k_2} x_i^2 \in E \cap Q \tag{2.7}$$

with the least positive rank in $E \cap Q$. By induction, we can construct a series of $S_i$, $Q_i$ and $p_i$ such that

$$S_i = \{k_{i-1}+1, ..., k_i\}, \quad k_0 = 0 < k_1 < ... < k_i < ... \leq k \tag{2.8}$$

$$Q_i = \text{real vctor space spanned by} \quad \{x_l x_j : k_i + 1 \leq l \leq j \leq k\} \tag{2.9}$$

$$p_i = \sum_{j=k_{i-1}+1}^{k_i} x_j^2 = \sum_{j \in S_i} x_j^2, \quad i > 0 \tag{2.10}$$

and $p_i$ has the least positive rank in $E \cap Q_{i-1}$, for $i > 0$.

Lemma 2.2 (Chen and Yau) [ChYa2]. *If $p \in E \cap Q$ , then there exists a constant $\lambda$ such that*

$$p(0, ..., 0, x_{k_{i-1}+1}, ..., x_{k_i}, 0, ..., 0) = \lambda p_i, \quad \textit{for } i > 0$$

Lemma 2.3 (Chen and Yau) [ChYa2]. *If $p \in E \cap Q$, then*

$$p(x_1, ..., x_{k_{i-1}+1}, 0, ..., 0, x_{k_{i+1}}, ..., x_n) \in E \quad \textit{for } i > 0$$

The following theroem is the main result of Chen and Yau in [ChYa2].

Lemma 2.4 (Chen and Yau)[ChYa2]. *Let $p = \sum_{i \in S_{l_1}} \sum_{j \in S_{l_2}} 2a_{ij} x_i x_j \in E$ , where $a_{ij} \in \mathbf{R}$ and $l_1 < l_2$. Then $| S_{l_1} | = | S_{l_2} |$ and $A = (a_{ij}) = bT$ where $b$ is a constant and $T$ is an orthogonal matrix.*

Theorem 2.3 (Chen-Yau). *If $E$ is a finite-dimensional estimation algebra of maximal rank, then all the entries $\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$ of $\Omega$ are degree one polynomials. Let $k$ be the quadratic rank of $E$. Then there exists an orthogonal change of coordinates such that $\omega_{ij}$ are constants for $1 \leq i, j \leq k$, $\omega_{ij}$ are degree one polynomials in $x_1, ..., x_k$ for $1 \leq i \leq k$ or $1 \leq j \leq k$; and $\omega_{ij}$ are degree one polynomials in $x_{k+1}, ..., x_n$ for $k+1 \leq i, j \leq n$.*

For the convenience of the readers, we also list the following elementary lemma which was proven in [Ya] and [ChYa1].

Lemma 2.5.

*(i)* $[X, Y, Z] = X[Y, Z] + [X, Z]Y$, *where $X, Y$ and $Z$ are differential operators;*

*(ii)* $[aD_i, b] = a\frac{\partial b}{\partial x_i}$ , *where $D_i = \frac{\partial}{\partial x_i} - f_i$, $a$ and $b$ are functions defined on $\mathbf{R}^n$;*

*(iii)* $[aD_i, bD_j] = -ab\omega_{ij} + a\frac{\partial b}{\partial x_i} D_j - b\frac{\partial a}{\partial x_j} D_i$, *where $\omega_{ji} = [D_i, D_j] = \frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i}$;*

*(iv)* $[aD_i^2, b] = 2a\frac{\partial b}{\partial x_i} D_i + a\frac{\partial^2 b}{\partial x_i^2}$

*(v)* $[D_i^2, bD_j] = 2\frac{\partial b}{\partial x_i} D_i D_j - 2b\omega_{ij} D_i + \frac{\partial^2 b}{\partial x_i^2} D_j - b\frac{\partial \omega_{ij}}{\partial x_i} h$

*(vi)* $[D_i^2, D_j^2] = 4\omega_{ji} D_j D_i + 2\frac{\partial \omega_{ji}}{\partial x_j} D_i + 2\frac{\partial \omega_{ji}}{\partial x_i} D_j + \frac{\partial^2 \omega_{ji}}{\partial x_i \partial x_j} + 2\omega_{ji}^2$

*(vii)* $[D_k^2, bD_iD_j] = 2\frac{\partial b}{\partial x_k} D_k D_i D_j + 2b\omega_{jk} D_i D_k + 2b\omega_{ik} D_k D_j + \frac{\partial^2 b}{\partial x_k^2} D_i D_j + 2b\frac{\partial \omega_{jk}}{\partial x_i} D_k$
$+ b\frac{\partial \omega_{jk}}{\partial x_k} D_i + b\frac{\partial \omega_{ik}}{\partial x_k} D_j + h\frac{\partial^2 \omega_{jk}}{\partial x_i \partial x_k}$;

*(viii)* $[aD_iD_j, bD_k] = a\frac{\partial b}{\partial x_j}D_iD_k + a\frac{\partial b}{\partial x_i}D_jD_k + ab\omega_{kj}D_i + ab\omega_{ki}D_j + a\frac{\partial^2 b}{\partial x_i\partial x_j}D_k + ab\frac{\partial \omega_{kj}}{\partial x_i} - b\frac{\partial a}{\partial x_k}D_iD_j$

The following lemma was observed in [Ya]

LEMMA 2.6. *Let $E$ be a finite-dimensional estimation algebra with maximal rank. Then $< 1, x_1, ..., x_n, D_1, ..., D_n,\ L_0 >\subseteq E$.*

We need the following important theorem from our previous paper [HuYa] which is frequently used in the proof of our Main Theorem of this paper.

THEOREM 2.4 (Hu and Yau). *Let $E$ be a finite-dimensional estimation algebra of maximal rank and $k$ be the quadratic rank of $E$. Then $\omega_{ij}$ are constants for $1 \leq i, j \leq k$ or $k+1 \leq i, j \leq n$; $\omega_{ij}$ are degree one polynomials in $x_1, ..., x_k$ for $1 \leq i \leq k$ or $1 \leq j \leq k$. Moreover $\alpha_j = \sum_{l=1}^k x_l\omega_{jl}$, for $k+1 \leq j \leq n$, are in $E$ .*

**3. Some useful lemmas.** Suppose that the estimation algebra $E$ of (2.1) is finite-dimensional. Let $k$ be the quadratic rank of $E$. Let $U_i$ be the space of differential operators with order at most $i$. The following lemmas facilitate our proof of the main theorem in section 4.

LEMMA 3.1. *If $x_j^2 \in E$ , $j \leq k$ , then $\frac{\partial \omega_{jl}}{\partial x_j} = 0$ for all $k+1 \leq j \leq n$.*

*Proof.* We shall construct a sequence of elements in $E$ in the following manner:

$$\begin{aligned}
Z_1 &= \frac{1}{2}[L_0, x_j^2] = \frac{1}{4}\sum_{i=1}^n [D_i^2, x_j^2] = x_jD_j + \frac{1}{2} \\
Z_2 &= [L_0, Z_1] = \frac{1}{2}\sum_{i=1}^n [D_i^2, x_jD_j] \\
&= \frac{1}{2}\sum_{i=1}^n \left(2\frac{\partial x_j}{\partial x_i}D_iD_j - 2x_j\omega_{ij}D_i\right) \mod U_0 \\
&= D_j^2 + \sum_{i=1}^n x_j\omega_{ji}D_i \mod U_0 \\
Z_3 &= [L_0, Z_2] = \frac{1}{2}\sum_{i=1}^n \left[D_i^2, D_j^2 + \sum_{l=1}^n x_j\omega_{jl}D_l\right] \mod U_1 \\
&= 2\sum_{i=1}^n \omega_{ji}D_jD_i + \sum_{i=1}^n\sum_{l=1}^n \frac{\partial(x_j\omega_{jl})}{\partial x_i}D_iD_l \\
&= 2\sum_{i=1}^n \omega_{ji}D_jD_i + \sum_{i=1}^n\sum_{l=1}^n \delta_{ji}\omega_{jl}D_iD_l + \sum_{i=1}^n\sum_{l=1}^n x_j\frac{\partial \omega_{jl}}{\partial x_i}D_iD_l \mod U_1 \\
&= 3\sum_{i=1}^n \omega_{ji}D_jD_i + \sum_{i=1}^k\sum_{l=k+1}^n x_j\frac{\partial \omega_{jl}}{\partial x_i}D_iD_l \mod U_1
\end{aligned}$$

$$
\begin{aligned}
[Z_3, Z_1] &= 3\textstyle\sum_{i=1}^{n}[\omega_{ji}D_jD_i, x_jD_j] \\
&\quad + \textstyle\sum_{i=1}^{k}\sum_{l=k+1}^{n}[x_j\frac{\partial\omega_{jl}}{\partial x_i}D_iD_l, x_jD_j] \quad \bmod U_1 \\
&= 3\textstyle\sum_{i=1}^{n}\left(\omega_{ji}\frac{\partial x_j}{\partial x_j}D_iD_j + \omega_{ji}\frac{\partial x_j}{\partial x_i}D_j^2 - x_j\frac{\partial\omega_{jl}}{\partial x_j}D_jD_i\right) \\
&\quad + \textstyle\sum_{i=1}^{k}\sum_{l=k+1}^{n}\left(x_j\frac{\partial\omega_{jl}}{\partial x_i}\frac{\partial x_j}{\partial x_i}D_lD_j + x_j\frac{\partial\omega_{jl}}{\partial x_i}\frac{\partial x_j}{\partial x_l}D_iD_j\right. \\
&\quad \left. -x_j\frac{\partial w_{jl}}{\partial x_i}D_iD_l\right) \quad \bmod U_1 \\
&= 3\textstyle\sum_{i=1}^{n}\omega_{ji}D_jD_i - 3\sum_{i=k+1}^{n}x_j\frac{\partial\omega_{ji}}{\partial x_j}D_jD_i + \sum_{l=k+1}^{n}x_j\frac{\partial\omega_{jl}}{\partial x_j}D_jD_l \\
&\quad - \textstyle\sum_{i=1}^{k}\sum_{l=k+1}^{n}x_j\frac{\partial\omega_{jl}}{\partial x_i}D_iD_l \quad \bmod U_1 \\
&= 3\textstyle\sum_{i=1}^{n}\omega_{ji}D_jD_i - 2\sum_{i=k+1}^{n}x_j\frac{\partial\omega_{ji}}{\partial x_j}D_jD_i \\
&\quad - \textstyle\sum_{i=1}^{k}\sum_{l=k+1}^{n}x_j\frac{\partial\omega_{jl}}{\partial x_i}D_iD_l \quad \bmod U_1
\end{aligned} \tag{3.1}
$$

$$
Z_4 = \frac{1}{2}\left([Z_3, Z_2] + Z_3\right) = 3\sum_{i=1}^{n}\omega_{ji}D_jD_i - \sum_{i=k+1}^{n}x_j\frac{\partial\omega_{ji}}{\partial x_j}D_jD_i \quad \bmod U_1 \tag{3.2}
$$

$$
\begin{aligned}
[Z_4, Z_1] &= 3\sum_{i=1}^{n}[\omega_{ji}D_jD_i, x_jD_j] - \sum_{i=k+1}^{n}[x_j\frac{\partial\omega_{ji}}{\partial x_j}D_jD_i, x_jD_j] \\
&= 3\sum_{i=1}^{n}[\omega_{ji}D_jD_i, x_jD_j] - \sum_{i=k+1}^{n}[x_j\frac{\partial\omega_{ji}}{\partial x_j}D_jD_i, x_jD_j] \quad \bmod U_1 \\
&= 3\sum_{i=1}^{n}\omega_{ji}D_jD_i - 3\sum_{i=k+1}^{n}x_j\frac{\partial\omega_{ji}}{\partial x_j}D_jD_i \\
&\quad - \sum_{i=k+1}^{n}\left(x_j\frac{\partial\omega_{ji}}{\partial x_j}\frac{\partial x_j}{\partial x_j}D_jD_i + x_j\frac{\partial\omega_{ji}}{\partial x_j}\frac{\partial x_j}{\partial x_i}D_j^2 - x_j\frac{\partial w_{ji}}{\partial x_j}D_jD_i\right) \quad \bmod U_1 \\
&= 3\sum_{i=1}^{n}\omega_{ji}D_jD_i - 3\sum_{i=k+1}^{n}x_j\frac{\partial\omega_{ji}}{\partial x_j}D_jD_i \quad \bmod U_1
\end{aligned}
$$

$$
Z_5 = \frac{1}{2}\left(Z_4 - [Z_4, Z_1]\right) = \sum_{i=k+1}^{n}x_j\frac{\partial\omega_{ji}}{\partial x_j}D_jD_i \quad \bmod U_1 \tag{3.3}
$$

$$
\begin{aligned}
A^{(1)} = [L_0, Z_5] &= \frac{1}{2}\sum_{i=1}^{n}\sum_{l=k+1}^{n}[D_i^2, x_j\frac{\partial\omega_{jl}}{\partial x_j}D_jD_l] \quad \bmod U_2 \\
&= \sum_{i=1}^{n}\sum_{l=k+1}^{n}\frac{\partial\left(x_j\frac{\partial\omega_{il}}{\partial x_j}\right)}{\partial x_i}D_iD_jD_l \quad \bmod U_2 \\
&= \left(\sum_{l=k+1}^{n}\frac{\partial\omega_{jl}}{\partial x_j}D_l\right)D_j^2 \quad \bmod U_2
\end{aligned}
$$

$$
A^{(2)} = [A^{(1)}\ , Z_5] = \sum_{l=k+1}^{n}\sum_{i=k+1}^{n}\left(\ \frac{\partial\omega_{jl}}{\partial x_j}D_lD_j^2, x_j\frac{\partial\omega_{ji}}{\partial x_j}D_jD_i\right) \quad \bmod U_3
$$

$$= \sum_{l=k+1}^{n} \sum_{i=k+1}^{n} \left( 2\, \frac{\partial \omega_{jl}}{\partial x_j} \frac{\partial \left( x_j \frac{\partial \omega_{ji}}{\partial x_j} \right)}{\partial x_j} D_l D_j D_j D_i \right) \mod U_3$$

$$= 2 \sum_{l=k+1}^{n} \sum_{i=k+1}^{n} \frac{\partial \omega_{jl}}{\partial x_j} \frac{\partial \omega_{ji}}{\partial x_j}\, D_j^2 D_i D_l \mod U_3$$

$$= 2 \left( \sum_{l=k+1}^{n} \frac{\partial \omega_{jl}}{\partial x_j} D_l \right)^2 D_j^2 \mod U_3$$

Now, we shall prove by induction on $s$ that

$$A^{(s)} := [A^{(s-1)}, Z_5] = 2^{s-1} \left( \sum_{l=k+1}^{n} \frac{\partial \omega_{jl}}{\partial x_j} D_l \right)^s D_j^2 \mod U_{s+1}$$

Suppose that this is true for $s$. We are going to prove that the same formula is true for $s+1$

$$\begin{aligned} A^{(s+1)} &:= [A^{(s)}, Z_5] \\ &= 2^{s-1} \sum_{l_1,\ldots,l_s=k+1}^{n} \sum_{i=k+1}^{n} [\frac{\partial \omega_{jl_1}}{\partial x_j} \ldots \frac{\partial \omega_{jl_s}}{\partial x_j} \frac{\partial \left( x_j \frac{\partial \omega_{ji}}{\partial x_j} \right)}{\partial x_j} D_{l_1} \ldots . D_{l_s} D_j^2, \\ &\quad x_j \frac{\partial \omega_{ji}}{\partial x_j} D_j D_i] \mod U_{s+2} \\ &= 2^{s-1} \sum_{l_1,\ldots,l_s=k+1}^{n} \sum_{i=k+1}^{n} 2 \frac{\partial \omega_{jl_1}}{\partial x_j} \ldots \frac{\partial \omega_{jl_s}}{\partial x_j} \frac{\partial \omega_{ji}}{\partial x_j} D_{l_1} \ldots . D_{l_s} D_j^2 D_i \mod U_{s+2} \\ &= 2^s \left( \sum_{l=k+1}^{n} \frac{\partial \omega_{jl}}{\partial x_j} D_l \right)^{s+1} D_j^2 \mod U_{s+2} \end{aligned}$$

Hence we get a sequence of elements in $E$ of the form

$$A^{(s)} = 2^{s-1} \left( \sum_{l=k+1}^{n} \frac{\partial \omega_{jl}}{\partial x_j} D_l \right)^s D_j^2 \mod U_{s+1}$$

Since $E$ is finite dimensional, we conclude that

$$\frac{\partial \omega_{jl}}{\partial x_j} = 0, \quad \text{for all} \quad k+1 \le l \le n$$

□

LEMMA 3.2. *If $x_i^2 \in E$ and $x_j^2 \in E$ ,$1 \le i,\ j \le k,\ i \ne j$, then $\frac{\partial \omega_{il}}{\partial x_j} = 0 = \frac{\partial \omega_{jl}}{\partial x_i}$ for all $k+1 \le l \le n$.*

*Proof.* From (3.1), (3.2) and (3.3) in the proof of Lemma 3.1, we have the following elements in $E$

$$[Z_3, Z_1] = 3 \sum_{i=1}^{n} \omega_{js} D_j D_s - 2 \sum_{s=k+1}^{n} x_j \frac{\partial \omega_{js}}{\partial x_j} D_j D_s$$

$$-\sum_{s=1}^{k}\sum_{l=k+1}^{n} x_j \frac{\partial \omega_{jl}}{\partial x_s} D_s D_l \mod U_1$$

$$Z_4 = 3\sum_{s=1}^{n} \omega_{is} D_j D_s - \sum_{s=k+1}^{n} x_j \frac{\partial \omega_{js}}{\partial x_j} D_j D_s \mod U_1$$

$$Z_5 = \sum_{s=k+1}^{n} x_j \frac{\partial \omega_{js}}{\partial x_j} D_j D_s \mod U_1$$

$$\overline{Z}_4 = -[Z_3, Z_1] + Z_4 - Z_5 = \sum_{s=1}^{k}\sum_{l=k+1}^{n} x_j \frac{\partial \omega_{jl}}{\partial x_s} D_s D_l \mod U_1$$

$$\overline{Z}_1 = [L_0, x_i^2] = \frac{1}{2}\sum_{s=1}^{n} [D_s^2, x_i^2] = x_i D_i + \frac{1}{2}$$

$$\begin{aligned}\overline{Z}_5 = [\overline{Z}_4, \overline{Z}_1] &= \sum_{s=1}^{k}\sum_{l=k+1}^{n} [x_j \frac{\partial \omega_{jl}}{\partial x_s} D_s D_l, x_i D_i] \mod U_1 \\ &= \sum_{s=1}^{k}\sum_{l=k+1}^{n} x_j \frac{\partial \omega_{jl}}{\partial x_s} \frac{\partial x_i}{\partial x_s} D_l D_i \mod U_1 \\ &= \sum_{l=k+1}^{n} x_j \frac{\partial \omega_{jl}}{\partial x_i} D_l D_i \mod U_1\end{aligned}$$

By interchanging the role of $i$ and $j$, we get the following element in $E$

$$\overline{Z}_6 = \sum_{l=k+1}^{n} x_i \frac{\partial \omega_{il}}{\partial x_j} D_l D_j \mod U_1$$

By the cyclic relation $\frac{\partial \omega_{il}}{\partial x_j} + \frac{\partial \omega_{ji}}{\partial x_l} + \frac{\partial \omega_{lj}}{\partial x_i} = 0$ and Theorem 2.3, we deduce easily that $\frac{\partial \omega_{il}}{\partial x_j} = \frac{\partial \omega_{jl}}{\partial x_i}$ because $\omega_{ij}$ is a constant.

Hence $\overline{Z}_6$ can be rewritten as

$$\overline{Z}_6 = \sum_{l=k+1}^{n} x_i \frac{\partial \omega_{jl}}{\partial x_i} D_l D_j$$

We shall construct from $\overline{Z}_5$ and $\overline{Z}_6$ an infinite sequence of elements in $E$

$$\begin{aligned}\overline{A}^{(1)} = [L_0, \overline{Z}_5] &= \frac{1}{2}\sum_{s=1}^{n}\sum_{l=k+1}^{n} [D_s^2, x_j \frac{\partial \omega_{jl}}{\partial x_j} D_l D_i] \mod U_2 \\ &= \sum_{s=1}^{n}\sum_{l=k+1}^{n} \frac{\partial \left( x_j \frac{\partial \omega_{jl}}{\partial x_i} \right)}{\partial x_s} D_s D_l D_i \mod U_2 \\ &= \left( \sum_{l=k+1}^{n} \frac{\partial \omega_{jl}}{\partial x_i} D_l \right) D_j D_i \mod U_2\end{aligned}$$

$$\overline{A}^{(2)} = \left[\overline{A}^{(1)}, \overline{Z}_5\right] = \sum_{l=k+1}^{n}\sum_{s=k+1}^{n} [\frac{\partial \omega_{jl}}{\partial x_i} D_l D_j D_i, x_j \frac{\partial \omega_{js}}{\partial x_i} D_s D_i] \mod U_3$$

$$
\begin{aligned}
&= \sum_{l=k+1}^{n} \sum_{s=k+1}^{n} \frac{\partial \omega_{jl}}{\partial x_i} \frac{\partial \left( x_j \frac{\partial \omega_{js}}{\partial x_i} \right)}{\partial x_j} D_l D_i D_s D_i \mod U_3 \\
&= \sum_{l=k+1}^{n} \sum_{s=k+1}^{n} \frac{\partial \omega_{jl}}{\partial x_i} \frac{\partial \omega_{js}}{\partial x_i} D_l D_s D_i^2 \mod U_3 \\
&= \left( \sum_{l=k+1}^{n} \frac{\partial \omega_{jl}}{\partial x_i} D_l \right)^2 D_i^2 \mod U_3
\end{aligned}
$$

$$
\begin{aligned}
\overline{A}^{(3)} &= \left( \overline{A}^{(2)}, \overline{Z}_6 \right) = \sum_{l_1,l_2=k+1}^{n} \sum_{l=k+1}^{n} [\frac{\partial \omega_{jl_1}}{\partial x_i} \frac{\partial \omega_{jl_2}}{\partial x_i} D_{l_1} D_{l_2} D_i^2, x_i \frac{\partial \omega_{jl}}{\partial x_i} D_l D_j] \mod U_4 \\
&= \sum_{l_1,l_2=k+1}^{n} \sum_{l=k+1}^{n} 2 \frac{\partial \omega_{jl_1}}{\partial x_i} \frac{\partial \omega_{jl_2}}{\partial x_i} \frac{\partial \left( x_i \frac{\partial \omega_{jl}}{\partial x_i} \right)}{\partial x_i} D_{l_1} D_{l_2} D_i D_l D_j \qquad \mod U_4 \\
&= 2 \sum_{l_1,l_2=k+1}^{n} \sum_{s=k+1}^{n} \frac{\partial \omega_{jl_1}}{\partial x_i} \frac{\partial \omega_{jl_2}}{\partial x_i} \frac{\partial \omega_{jl}}{\partial x_i} D_{l_1} D_{l_2} D_l D_i D_j \mod U_4 \\
&= 2 \left( \sum_{l=k+1}^{n} \frac{\partial \omega_{jl}}{\partial x_i} D_l \right)^3 D_i D_j \mod U_4
\end{aligned}
$$

We shall prove by induction on $s$ that

$$
\overline{A}^{(2s-1)} := \overline{[A}^{(2s-2)}, \overline{Z}_6] = 2^{s-1} \left( \sum_{l=k+1}^{s} \frac{\partial \omega_{jl}}{\partial x_i} D_l \right)^{2s-1} D_i D_j \mod U_{2s+1}
$$

and

$$
\overline{A}^{(2s)} := \overline{[A}^{(2s-1)}, \overline{Z}_5] = 2^{s-1} \left( \sum_{l=k+1}^{n} \frac{\partial \omega_{jl}}{\partial x_i} D_l \right)^{2s} D_i^2 \mod U_{2s+1}
$$

Suppose that this is true for $2s-1$ and $2s$. We are going to prove that the same formulas are true for $2s+1$ and $2s+2$

$$
\begin{aligned}
\overline{A}^{(2s+1)} &= \overline{[A}^{(2s)}, \overline{Z}_6] \\
&= 2^{s-1} \sum_{l_1,\dots,l_{2s}=k+1}^{n} \sum_{l=k+1}^{n} [\frac{\partial \omega_{jl_1}}{\partial x_i} \dots \frac{\partial \omega_{jl_{2s}}}{\partial x_i} D_{l_1} \dots D_{l_{2s}} D_i^2, \\
&\qquad x_i \frac{\partial \omega_{jl}}{\partial x_i} D_l D_j] \mod U_{2s+2} \\
&= 2^{s} \sum_{l_1,\dots,l_{2s}=k+1}^{n} \sum_{l=k+1}^{n} [\frac{\partial \omega_{jl_1}}{\partial x_i} \dots \frac{\partial \omega_{jl_{2s}}}{\partial x_i} \frac{\partial \omega_{jl}}{\partial x_i} D_{l_1} \dots D_{l_{2s}} D_i \\
&\qquad D_l D_j] \mod U_{2s+2} \\
&= 2^{s} \left( \sum_{l=k+1}^{n} \frac{\partial \omega_{jl}}{\partial x_i} D_l \right)^{2s+1} D_i D_j \mod U_{2s+2}
\end{aligned}
$$

$$\begin{aligned}\overline{A}^{(2s+2)} &= [\overline{A}^{(2s+1)}, \overline{Z}_5] = 2^s \sum_{l_1,\ldots,l_{2s}=k+1}^{n} \sum_{l=k+1}^{n} [\frac{\partial \omega_{jl_1}}{\partial x_i} \ldots \frac{\partial \omega_{jl_{2s+1}}}{\partial x_i} D_{l_1} \ldots D_{l_{2s+1}} D_i D_j, \\ &\quad x_j \frac{\partial \omega_{jl}}{\partial x_i} D_l D_i] \mod U_{2s+3} \\ &= 2^s \sum_{l_1,\ldots,l_{2s+1}=k+1}^{n} \sum_{l=k+1}^{n} [\frac{\partial \omega_{jl_1}}{\partial x_i} \ldots \frac{\partial \omega_{jl_{2s+1}}}{\partial x_i} \frac{\partial \omega_{jl}}{\partial x_i} D_{l_1} \ldots D_{l_{2s+1}} D_i \\ &\quad D_l D_i] \mod U_{2s+3} \\ &= 2^s \left( \sum_{l=k+1}^{n} \frac{\partial \omega_{jl}}{\partial x_i} D_l \right)^{2s+2} D_i^2 \mod U_{2s+3}\end{aligned}$$

Since $E$ is finite−dimensional, we conclude that

$$\frac{\partial \omega_{jl}}{\partial x_i} = 0 \quad \text{for all} k+1 \leq l \leq n$$

Similarly, we have $\frac{\partial \omega_{il}}{\partial x_j} = 0$ for all $k+1 \leq l \leq n$. □

Lemma 3.3. *If* $x_{k_1}^2 + \ldots + x_k^2 \in E$ *and* $\frac{\partial \omega_{jl}}{\partial x_i} = 0$ *for all* $k+1 \leq l \leq n$ *and* $k_1 \leq i \neq j \leq k$*, then* $\frac{\partial \omega_{il}}{\partial x_i} = 0$ *for all* $k+1 \leq l \leq n$ *and* $k_1 \leq i \leq k$.

*Proof.* We shall construct a sequence of elements in $E$ in the following manner.

$$Z_1 = \frac{1}{2}\left[L_0, \quad x_{k_1}^2 + \ldots + x_k^2\right] = \sum_{i=k_1}^{k} x_i D_i + \frac{1}{2}(k - k_1 + 1)$$

$$\begin{aligned}Z_2 = [L_0,\ Z_1] &= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=k_1}^{k} [D_i^2, x_j D_j] \mod U_0 \\ &= \sum_{i=1}^{n} \sum_{j=k_1}^{k} \left( \frac{\partial x_j}{\partial x_i} D_i D_j - x_j \omega_{ji} D_i \right) \mod U_0 \\ &= \sum_{j=k_1}^{k} D_j^2 + \sum_{i=1}^{n} \sum_{j=k_1}^{k} x_j \omega_{ji} D_i \mod U_0\end{aligned}$$

$$\begin{aligned}Z_3 = [L_0,\ Z_2] &= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=k_1}^{k} [D_i^2, D_j^2] + \frac{1}{2} \sum_{i=1}^{n} \sum_{l=1}^{n} \sum_{j=k_1}^{k} [D_i^2, x_j \omega_{jl} D_l] \mod U_1 \\ &= 2 \sum_{i=1}^{n} \sum_{j=k_1}^{k} \omega_{ji} D_j D_i + \sum_{i=1}^{n} \sum_{l=1}^{n} \sum_{j=k_1}^{k} \frac{\partial (x_j \omega_{jl})}{\partial x_i} D_i D_l \mod U_1 \\ &= 2 \sum_{i=1}^{n} \sum_{j=k_1}^{k} \omega_{ji} D_j D_i + \sum_{l=1}^{n} \sum_{j=k_1}^{k} \omega_{jl} D_j D_l \\ &\quad + \sum_{i=1}^{k} \sum_{l=k+1}^{n} \sum_{j=k_1}^{k} x_j \frac{\partial \omega_{jl}}{\partial x_i} D_i D_l \mod U_1\end{aligned}$$

$$= 3\sum_{i=1}^{n}\sum_{j=k_1}^{k}\omega_{ji}D_jD_i + \sum_{i=1}^{k}\sum_{l=k+1}^{n}\sum_{j=k_1}^{k}x_j\frac{\partial\omega_{jl}}{\partial x_i}D_iD_l \mod U_1$$

$$\begin{aligned}
[Z_3, Z_1] &= 3\sum_{i=1}^{n}\sum_{j=k_1}^{k}\sum_{l=k_1}^{k}[\omega_{ji}D_jD_i, x_lD_l] \\
&\quad + \sum_{i=1}^{k}\sum_{l=k+1}^{n}\sum_{j=k_1}^{k}\sum_{p=k_1}^{k}[x_j\frac{\partial\omega_{jl}}{\partial x_i}D_iD_l, x_pD_p] \mod U_1 \\
&= 3\sum_{i=1}^{n}\sum_{j=k_1}^{k}\sum_{l=k_1}^{n}\left(\omega_{ji}\frac{\partial x_l}{\partial x_i}D_jD_l + \omega_{ji}\frac{\partial x_l}{\partial x_j}D_iD_l - x_l\frac{\partial\omega_{ji}}{\partial x_l}D_jD_i\right) \\
&\quad + \sum_{i=1}^{k}\sum_{l=k+1}^{n}\sum_{j=k_1}^{k}\sum_{p=k_1}^{k}\left(x_j\frac{\partial\omega_{jl}}{\partial x_i}\frac{\partial x_p}{\partial x_i}D_lD_p + x_j\frac{\partial\omega_{jl}}{\partial x_i}\frac{\partial x_p}{\partial x_l}D_iD_p\right. \\
&\quad \left. - x_p\frac{\partial x_j}{\partial x_p}\frac{\partial\omega_{jl}}{\partial x_i}D_iD_l\right) \mod U_1 \\
&= 3\sum_{j=k_1}^{k}\sum_{l=k_1}^{k}\omega_{jl}D_jD_l + 3\sum_{i=1}^{n}\sum_{j=k_1}^{k}\omega_{ji}D_iD_j - 3\sum_{i=k+1}^{n}\sum_{j=k_1}^{k}\sum_{l=k_1}^{k}x_l\frac{\partial\omega_{ji}}{\partial x_l}D_jD_i \\
&\quad + \sum_{i=k_1}^{k}\sum_{l=k+1}^{n}\sum_{j=k_1}^{k}x_j\frac{\partial\omega_{jl}}{\partial x_i}D_lD_i - \sum_{i=1}^{k}\sum_{l=k+1}^{n}\sum_{j=k_1}^{k}x_j\frac{\partial\omega_{jl}}{\partial x_i}D_iD_l \mod U_1 \\
&= 3\sum_{i=1}^{n}\sum_{j=k_1}^{k}\omega_{ji}D_iD_j - 2\sum_{i=k_1}^{k}\sum_{l=k+1}^{n}\sum_{j=k_1}^{k}x_j\frac{\partial\omega_{jl}}{\partial x_i}D_iD_l \\
&\quad - \sum_{i=1}^{k}\sum_{l=k+1}^{n}\sum_{j=k_1}^{k}x_j\frac{\partial\omega_{jl}}{\partial x_i}D_iD_l \mod U_1 \\
Z_4 &= \frac{1}{2}\left([Z_3, Z_1] + Z_3\right) = 3\sum_{i=1}^{n}\sum_{j=k_1}^{k}\omega_{ji}D_iD_j \\
&\quad - \sum_{i=k_1}^{k}\sum_{l=k+1}^{n}\sum_{j=k_1}^{k}x_j\frac{\partial\omega_{jl}}{\partial x_i}D_iD_l \mod U_1 \\
[Z_4, Z_1] &= 3\sum_{i=1}^{n}\sum_{j=k_1}^{k}\sum_{l=k_1}^{k}[\omega_{ji}D_iD_j, x_lD_l] - \sum_{i=k_1}^{k}\sum_{l=k+1}^{n}\sum_{j=k_1}^{k}\sum_{p=k_1}^{k}[x_j\frac{\partial\omega_{jl}}{\partial x_i}D_iD_l, x_pD_p] \\
&= 3\sum_{i=1}^{n}\sum_{j=k_1}^{k}\omega_{ji}D_iD_j - 3\sum_{i=k_1}^{k}\sum_{l=k+1}^{n}\sum_{j=k_1}^{k}x_j\frac{\partial\omega_{jl}}{\partial x_i}D_iD_l \mod U_1 \\
Z_5 &= \frac{1}{2}\left(Z_4 - [Z_4, Z_1]\right) = \sum_{i=k_1}^{k}\sum_{l=k+1}^{n}\sum_{j=k_1}^{k}x_j\frac{\partial\omega_{jl}}{\partial x_i}D_iD_l \mod U_1
\end{aligned}$$

Since $\frac{\partial \omega_{jl}}{\partial x_i} = 0$ for all $k+1 \leq l \leq n$ and $k_1 \leq i \neq j \leq k$ , by hypothesis, we have

$$\begin{aligned} Z_5 &= \sum_{i=k_1}^{k} \sum_{l=k+1}^{n} x_i \frac{\partial \omega_{il}}{\partial x_i} D_i D_l \mod U_1 \\ &= \sum_{i=k_1}^{k} x_i \left( \sum_{l=k+1}^{n} \frac{\partial \omega_{il}}{\partial x_i} D_l \right) D_i \mod U_1 \end{aligned}$$

$$\begin{aligned} A^{(1)} = [\, L_{0,}\ Z_5] &= \frac{1}{2} \sum_{p=1}^{n} \sum_{i=k_1}^{k} \sum_{l=k+1}^{n} [D_p^2, x_i \frac{\partial \omega_{il}}{\partial x_i} D_i D_l] \mod U_2 \\ &= \sum_{p=1}^{n} \sum_{i=k_1}^{k} \sum_{l=k+1}^{n} \frac{\partial \left( x_i \frac{\partial \omega_{il}}{\partial x_i} \right)}{\partial x_p} D_p D_i D_l \mod U_2 \\ &= \sum_{i=k_1}^{k} \sum_{l=k+1}^{n} \frac{\partial \omega_{il}}{\partial x_i} D_i^2 D_l \mod U_2 \\ &= \sum_{i=k_1}^{k} \left( \sum_{l=k+1}^{n} \frac{\partial \omega_{il}}{\partial x_i} D_l \right) D_i^2 \mod U_2 \end{aligned}$$

$$\begin{aligned} A^{(2)} &= [A^{(1)}, Z_5] = \sum_{i=k_1}^{k} \sum_{l=k+1}^{n} \sum_{p=k_1}^{k} \sum_{q=k+1}^{n} [\frac{\partial \omega_{il}}{\partial x_i} D_i^2 D_l, x_p \frac{\partial \omega_{pq}}{\partial x_p} D_p D_q] \mod U_3 \\ &= \sum_{i=k_1}^{k} \sum_{l=k+1}^{n} \sum_{p=k_1}^{k} \sum_{q=k+1}^{n} 2 \frac{\partial \omega_{il}}{\partial x_i} \frac{\partial \left( x_p \frac{\partial \omega_{pq}}{\partial x_p} \right)}{\partial x_i} D_i D_l D_p D_q \mod U_3 \\ &= 2 \sum_{i=k_1}^{k} \sum_{l=k+1}^{n} \sum_{q=k+1}^{n} \frac{\partial \omega_{il}}{\partial x_i} \frac{\partial \omega_{iq}}{\partial x_i} D_l D_q D_i^2 \mod U_3 \\ &= 2 \sum_{i=k_1}^{k} \left( \sum_{l=k+1}^{n} \frac{\partial \omega_{il}}{\partial x_i} D_l \right)^2 D_i^2 \mod U_3 \end{aligned}$$

$$\begin{aligned} A^{(3)} &= [A^{(2)}, Z_5] = 2 \sum_{i=k_1}^{k} \sum_{l_1=k+1}^{n} \sum_{l_2=k+1}^{n} \sum_{p=k_1}^{k} \sum_{q=k+1}^{n} [\frac{\partial \omega_{il_1}}{\partial x_i} \frac{\partial \omega_{il_2}}{\partial x_i} D_{l_1} D_{l_2} D_i^2, \\ &\quad x_p \frac{\partial \omega_{pq}}{\partial x_p} D_p D_q] \mod U_4 \\ &= 2^2 \sum_{i=k_1}^{k} \sum_{l_1=k+1}^{n} \sum_{l_2=k+1}^{n} \sum_{p=k_1}^{k} \sum_{q=k+1}^{n} \frac{\partial \omega_{il_1}}{\partial x_i} \frac{\partial \omega_{il_2}}{\partial x_i} \frac{\partial \left( x_p \frac{\partial \omega_{pq}}{\partial x_p} \right)}{\partial x_i} D_{l_1} D_{l_2} D_i \\ &\quad D_p D_q \mod U_4 \\ &= 2^2 \sum_{i=k_1}^{k} \sum_{l_1=k+1}^{n} \sum_{l_2=k+1}^{n} \sum_{q=k+1}^{n} \frac{\partial \omega_{il_1}}{\partial x_i} \frac{\partial \omega_{il_2}}{\partial x_i} \frac{\partial \omega_{iq}}{\partial x_i} D_{l_1} D_{l_2} D_i^2 D_q \mod U_4 \\ &= 2^2 \sum_{i=k_1}^{k} \left( \sum_{l=k+1}^{n} \frac{\partial \omega_{il}}{\partial x_i} D_l \right)^3 D_i^2 \mod U_4 \end{aligned}$$

By induction, we get an infinite sequence in $E$ of the form

$$A^{(s)} = 2^{s-1} \sum_{i=k_1}^{k} \left( \sum_{l=k+1}^{n} \frac{\partial \omega_{il}}{\partial x_i} D_l \right)^s D_i^2 \quad \text{mod } U_{s-1}$$

Since $E$ is finite dimensional, we conclude that

$$\frac{\partial \omega_{il}}{\partial x_i} = 0 \quad \text{for all} \quad k+1 \le l \le n \quad \text{and} \quad k_1 \le i \le k$$

□

**4. Classificaton of finite-dimensional estimation algebras of maximal rank with state space less than or equal to five.** In order to prove our Main Theorem in section 1, we only need to prove that $\omega_{ij}$, $1 \le i, j \le n$ are constants because of Theorem 2.2. Let $k$ be the quadratic rank of the estimation algebra $E$. In view of Theorem 2.4, we can assume that $0 < k < n$ . In the subsequent discussion, we shall let $k_1, k_2, ...$ be the sequence defined in $(2.6) - (2.10)$.

**4.1. State space dimension $n$=2.**

In this case, we only need to to consider $k = 1$. By Theorem 2.4, the $\Omega$ matrix is of the following form

$$\Omega = (\omega_{ij}) = \begin{pmatrix} 0 & \omega_{12} \\ \omega_{21} & 0 \end{pmatrix}$$

where $\omega_{12} = -\omega_{21} = a_{12}^1 x_1 + c_{12}$. Since $k = 1$, we have $x_1^2 \in E$.

By Lemma 3.1, we have $\frac{\partial \omega_{12}}{\partial x_1} = 0$ which implies $\omega_{12}$ is a constant.

Therefore the following result of Chiou-Yau follows from Theorem 2.2.

Theorem 4.1. *Suppose that the state space of the filtering system (2.1) is of dimension two. If $E$ is the finite-dimensional estimation algebra with maximal rank, then $E$ is a real vector space of dimension 6 with basis given by $1, x_1, x_2, D_1, D_2$ and $L_0$.*

**4.2. State space dimension $n$=3.**

In this case, we only need to consider two subcases: $k = 1$ or $k = 2$. **case(I):** $k = 1$ By Theorem 2.4, the $\Omega$-matrix is of the following form

$$\Omega = (\omega_{ij}) = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ \cdots & \cdots & \cdots \\ \omega_{21} & 0 & c_{23} \\ \omega_{31} & c_{32} & 0 \end{pmatrix}$$

where:

$$\omega_{12} = -\omega_{21} = a_{12}^1 x_1 + c_{12}$$
$$\omega_{13} = -\omega_{31} = a_{13}^1 x_1 + c_{13}$$

Since $k = 1$, we have $x_1^2 \in E$. Lemma 3.1 implies $\frac{\partial \omega_{12}}{\partial x_1} = 0 = \frac{\partial \omega_{13}}{\partial x_1}$

Hence $\omega_{12}$ and $\omega_{13}$ are also constants.

**case(II):** $\boldsymbol{k}$**=2** By Theorem 2.4, the $\Omega$-matrix is of the following form

$$\Omega = (\omega_{ij}) = \begin{pmatrix} 0 & c_{12} & \omega_{13} \\ c_{21} & 0 & \omega_{23} \\ \cdots & \cdots & \cdots \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix}$$

where

$$\begin{aligned} \omega_{13} &= -\omega_{31} = a_{13}^1 x_1 + a_{13}^2 x_2 + c_{13} \\ \omega_{23} &= -\omega_{32} = a_{23}^1 x_1 + a_{23}^2 x_2 + c_{23} \end{aligned}$$

We need to consider two subcases.

**subcase(IIa):** $k_1 = 1 < k_2 = k = 2$. In this case, we have $x_1^2 \in E$ and $x_2^2 \in E$. By Lemma 3.1, we have $\frac{\partial \omega_{13}}{\partial x_1} = 0$ and $\frac{\partial \omega_{23}}{\partial x_2} = 0$.

On the other hand, $\frac{\partial \omega_{13}}{\partial x_2} = 0 = \frac{\partial \omega_{23}}{\partial x_1}$ by Lemma 3.2. Therefore $\omega_{13}$ and $\omega_{23}$ are constants in this case.

**subcase(IIb):** $k_1 = k = 2$. In this case, we have $x_1^2 + x_2^2 \in E$. By cyclic relation $\frac{\partial \omega_{13}}{\partial x_2} + \frac{\partial \omega_{21}}{\partial x_3} + \frac{\partial \omega_{32}}{\partial x_1} = 0$, we have $\frac{\partial \omega_{13}}{\partial x_2} = \frac{\partial \omega_{23}}{\partial x_1}$.

This implies $a_{13}^2 = a_{23}^1$. Since $\alpha_3 = \sum_{l=1}^2 x_l \omega_{3l}$, we have $a_{13}^1 x_1^2 + 2a_{13}^2 x_1 x_2 + a_{23}^2 x_2^2 \in E$.

By Lemma 2.2, any quadratic form in $E$ must be a constant multiple of $x_1^2 + x_2^2 \in E$. Therefore $a_{13}^2 = a_{23}^1 = 0$ and $a_{13}^1 = a_{23}^2$. In view of Lemma 3.3, we have $\frac{\partial \omega_{13}}{\partial x_1} = 0 = \frac{\partial \omega_{23}}{\partial x_2}$ . Hence $\omega_{13}$ and $\omega_{23}$ are constants. Consequently, the following result of [CYL$_1$] follows from Theorem 2.2.

Theorem 4.2. *Suppose that the state space of the filtering system (2.1) is of dimension three. If $E$ is the finite-dimensional estimation algebra with maximal rank, then $E$ is a real vector space of dimension 8 with a basis given by* $1,\ x_1,\ x_2,\ x_3,\ D_1,\ D_2,\ D_3$ *and* $L_0$.

**4.3. State space dimension** $\boldsymbol{n = 4}$**.** In this case, we only need to consider three subcases: $k = 1$, $k = 2$ or $k = 3$

**case(I):** $k = 1$ By theorem 2.4, the $\Omega$-matrix is of the following form

$$\Omega = (\omega_{ij}) = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} & \omega_{14} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_{21} & 0 & c_{23} & c_{24} \\ \omega_{31} & c_{32} & 0 & c_{34} \\ \omega_{41} & c_{42} & c_{43} & 0 \end{pmatrix}$$

where

$$\begin{aligned} \omega_{12} &= -\omega_{21} = a_{12}^1 x_1 + c_{12} \\ \omega_{13} &= -\omega_{31} = a_{13}^1 x_1 + c_{13} \\ \omega_{14} &= -\omega_{41} = a_{14}^1 x_1 + c_{14} \end{aligned}$$

Since k = 1, we have $x_1^2 \in E$. Lemma 3.1 implies $\frac{\partial \omega_{12}}{\partial x_1} = 0 = \frac{\partial \omega_{13}}{\partial x_1} = \frac{\partial \omega_{14}}{\partial x_1}$. Hence $\omega_{12}, \omega_{13}$ and $\omega_{14}$ are constants.

**case(II):** $k = 2$ By theorem 2.4, the $\Omega$-matrix is of the following form

$$\Omega = (\omega_{ij}) = \begin{pmatrix} 0 & c_{12} & \omega_{13} & \omega_{14} \\ c_{21} & 0 & \omega_{23} & \omega_{24} \\ \hdotsfor{4} \\ \omega_{31} & \omega_{32} & 0 & c_{34} \\ \omega_{41} & \omega_{42} & c_{43} & 0 \end{pmatrix}$$

where

$$\begin{aligned} \omega_{13} = -\omega_{31} &= a^1_{13}x_1 + a^2_{13}x_2 + c_{13} \\ \omega_{14} = -\omega_{41} &= a^1_{14}x_1 + a^2_{14}x_2 + c_{14} \\ \omega_{23} = -\omega_{32} &= a^1_{23}x_1 + a^2_{23}x_2 + c_{23} \\ \omega_{24} = -\omega_{42} &= a^1_{24}x_1 + a^2_{24}x_2 + c_{24} \end{aligned}$$

We need to consider two subcases.

**subcase(IIa):** $k_1 = 1 < k_2 = k = 2$. In this case we have $x_1^2 \in E$ and $x_2^2 \in E$. By Lemma 3.1, we have $\frac{\partial \omega_{13}}{\partial x_1} = \frac{\partial \omega_{14}}{\partial x_1} = \frac{\partial \omega_{23}}{\partial x_2} = \frac{\partial \omega_{24}}{\partial x_2} = 0$.

On the other hand, in view of Lemma 3.2, we have $\frac{\partial \omega_{13}}{\partial x_2} = 0 = \frac{\partial \omega_{23}}{\partial x_1}$ and $\frac{\partial \omega_{14}}{\partial x_2} = 0 = \frac{\partial \omega_{24}}{\partial x_1}$. Therefore $\omega_{13}, \omega_{14}, \omega_{23}$ and $\omega_{24}$ are constants.

**subcase(IIb):** $k_1 = \ k = 2$. In this case we have $x_1^2 + x_2^2 \in E$.

By cyclic relations $\frac{\partial \omega_{13}}{\partial x_2} + \frac{\partial \omega_{21}}{\partial x_3} + \frac{\partial \omega_{32}}{\partial x_1} = 0$, $\frac{\partial \omega_{14}}{\partial x_2} + \frac{\partial \omega_{21}}{\partial x_4} + \frac{\partial \omega_{42}}{\partial x_1} = 0$, we have $a^2_{13} = \frac{\partial \omega_{13}}{\partial x_2} = \frac{\partial \omega_{23}}{\partial x_1} = a^1_{23}$ and $a^2_{14} = \frac{\partial \omega_{14}}{\partial x_2} = \frac{\partial \omega_{24}}{\partial x_1} = a^1_{24}$.

By Theorem 2.4, the following elements are in $E$

$$\begin{aligned} -\alpha_3 &= \sum_{l=1}^{2} x_l \omega_{l3} = a^1_{13}x_1^2 + 2a^2_{13}x_1x_2 + a^2_{23}x_2^2 + c_{13}x_1 + c_{23}x_2 \in E \\ -\alpha_4 &= \sum_{l=1}^{2} x_l \omega_{24} = a^1_{14}x_1^2 + 2a^2_{14}x_1x_2 + a^2_{24}x_2^2 + c_{14}x_1 + c_{24}x_2 \in E \end{aligned}$$

As $E$ is of maximal rank, we have $a^1_{13}x_1^2 + 2a^2_{13}x_1x_2 + a^2_{23}x_2^2 \in E$ and $a^1_{14}x_1^2 + 2a^2_{14}x_1x_2 + a^2_{24}x_2^2 \in E$. In view of Lemma 2.3, we have $a^2_{13} = a^1_{23} = 0$ and $a^2_{14} = a^1_{24} = 0$. By Lemma 3.3, we have $\frac{\partial \omega_{13}}{\partial x_1} = 0 = \frac{\partial \omega_{23}}{\partial x_2}$, $\frac{\partial \omega_{14}}{\partial x_1} = 0 = \frac{\partial \omega_{24}}{\partial x_2}$. Therefore $\omega_{13}, \omega_{14}, \omega_{23}$ and $\omega_{24}$ are constants.

**case(III):** $k = 3$ By theorem 2.4, the $\Omega-$ matrix is of the following form

$$\Omega = (\omega_{ij}) = \begin{pmatrix} 0 & c_{12} & c_{13} & \omega_{14} \\ c_{21} & 0 & c_{23} & \omega_{24} \\ c_{31} & c_{32} & 0 & \omega_{34} \\ \hdotsfor{4} \\ \omega_{41} & \omega_{42} & \omega_{43} & 0 \end{pmatrix}$$

where

$$\begin{aligned} \omega_{14} = -\omega_{41} &= a^1_{14}x_1 + a^2_{14}x_2 + a^3_{14}x_3 + c_{14} \\ \omega_{24} = -\omega_{42} &= a^1_{24}x_1 + a^2_{24}x_2 + a^3_{24}x_3 + c_{24} \\ \omega_{34} = -\omega_{43} &= a^1_{34}x_1 + a^2_{34}x_2 + a^3_{34}x_3 + c_{34} \end{aligned}$$

**subcase(IIIa):** $k_1 = 1 < k_2 = 2 < k_3 = k = 3$. In this case we have $x_1^2 \in E$ and $x_2^2 \in E$ and $x_3^2 \in E$. By Lemma 3.1, we have $\frac{\partial \omega_{14}}{\partial x_1} = 0 = \frac{\partial \omega_{24}}{\partial x_2} = \frac{\partial \omega_{34}}{\partial x_3}$. By Lemma 3.2, we have $\frac{\partial \omega_{14}}{\partial x_2} = \frac{\partial \omega_{24}}{\partial x_1} = 0, \frac{\partial \omega_{14}}{\partial x_3} = \frac{\partial \omega_{34}}{\partial x_1} = 0$ and $\frac{\partial \omega_{24}}{\partial x_3} = \frac{\partial \omega_{34}}{\partial x_2} = 0$. Therefore $\omega_{14}$, $\omega_{24}$ and $\omega_{34}$ are constants in this case.

For subcase(IIIb) and subcase(IIIc) below, we shall use the following observations. The cyclic relations

$$\frac{\partial \omega_{14}}{\partial x_2} + \frac{\partial \omega_{21}}{\partial x_4} + \frac{\partial \omega_{42}}{\partial x_1} = 0, \frac{\partial \omega_{14}}{\partial x_3} + \frac{\partial \omega_{31}}{\partial x_4} + \frac{\partial \omega_{43}}{\partial x_1} = 0 \text{ and } \frac{\partial \omega_{24}}{\partial x_3} + \frac{\partial \omega_{32}}{\partial x_4} + \frac{\partial \omega_{43}}{\partial x_2} = 0$$

imply

$$a_{14}^2 = \frac{\partial \omega_{14}}{\partial x_2} = \frac{\partial \omega_{24}}{\partial x_1} = a_{24}^1,\ a_{14}^3 = \frac{\partial \omega_{14}}{\partial x_3} = \frac{\partial \omega_{34}}{\partial x_1} = a_{34}^1,$$
$$\text{and } a_{24}^3 = \frac{\partial \omega_{24}}{\partial x_3} = \frac{\partial \omega_{34}}{\partial x_2} = a_{34}^2.$$

**subcase(IIIb):** $k_1 = 1 < k_2 = 3 = k$. In this case, we have $x_1^2 \in E$, $x_2^2 + x_3^2 \in E$. By Lemma 3.1, we have $\frac{\partial \omega_{14}}{\partial x_1} = 0$ . In view of Theorem 2.4, we have

$$\begin{aligned} -\alpha_4 = \sum_{l=1}^{3} x_l \omega_{l4} = a_{14}^1 x_1^2 + 2a_{24}^2\ x_2^2 + a_{34}^3 x_3^2 + 2a_{14}^2 x_1 x_2 + 2a_{14}^3\ x_1 x_3 \\ +2a_{24}^3 x_2 x_3 + c_{14} x_1 + c_{24} x_2 + c_{34} x_3 \in E \end{aligned}$$

Since $E$ is of maximal rank,

$$\widetilde{\alpha_4} = a_{14}^1 x_1^2 + a_{24}^2\ x_2^2 + a_{34}^3 x_3^2 + 2a_{14}^2 x_1 x_2 + 2a_{14}^3\ x_1 x_3 + 2a_{24}^3 x_2 x_3 \in E \tag{4.1}$$

By Lemma 2.2, $-\alpha_4\,(0, x_2, x_3) = a_{24}^2\ x_2^2 + a_{34}^3 x_3^2 + 2a_{24}^3 x_2 x_3$ is a constant multiple of $x_2^2 + x_3^2$. We have $a_{24}^3 = 0 = a_{34}^2$. As $x_1^2 \in E$, we have $\frac{\partial \omega_{14}}{\partial x_1} = a_{14}^1 = 0$ by Lemma 3.1. On the other hand, by Lemma 3.3, we have $\frac{\partial \omega_{24}}{\partial x_2} = \frac{\partial \omega_{34}}{\partial x_3} = 0$. By (4.1), we have $2a_{14}^2 x_1 x_2 + 2a_{14}^3\ x_1 x_3 \in E$

In view of Lemma 2.4 and the fact that $|\{1\}| < |\{2, 3\}|$ , we conclude that $a_{14}^2 = 0 = a_{14}^3$. Hence $\omega_{13}, \omega_{14}, \omega_{23}$ and $\omega_{24}$ are constants.

**subcase(IIIc):** $k_1 = 3 = k$. In this case, we have $x_1^2 + x_2^2 + x_3^2 \in E$. As in the proof of subcase(IIIb), we have
$\widetilde{\alpha_4} = a_{14}^1 x_1^2 + a_{24}^2\ x_2^2 + a_{34}^3 x_3^2 + 2a_{14}^2 x_1 x_2 + 2a_{14}^3\ x_1 x_3 + 2a_{24}^3 x_2 x_3 \in E$

By Lemma 2.2 this quadratic form is a constant multiple of $x_1^2 + x_2^2 + x_3^2$. Consequantly, we have $a_{14}^2 = a_{14}^3 = a_{24}^3 = 0$. It follows that $\frac{\partial \omega_{14}}{\partial x_1} = 0 = \frac{\partial \omega_{24}}{\partial x_2} = \frac{\partial \omega_{34}}{\partial x_3}$ by Lemma 3.3. Therefore $\omega_{14}, \omega_{24}$ and $\omega_{34}$ are constants.

We have proved the following result which was claimed in [CLY2].

Theorem 4.3. *Suppose that the state space of filtering system(2.1) is of dimension four. If $E$ is the finite-dimensional estimation algebra with maximal rank, then $E$ is a real vector space of dimension 10 with a base given by* $1, x_1, x_2, x_3, x_4, D_1, D_2, D_3, D_4$ *and* $L_0$.

**4.4. State Space Dimemsion $n = 5$.** In this case, we only need to consider four subcases: $k = 1, k = 2, k = 3$ and $k = 4$.

**case(I):** $k = 1$. By Theorem 2.4, the $\Omega$-matrix is of the following form

$$\Omega = (\omega_{ij}) = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} & \omega_{14} & \omega_{15} \\ \hdashline \omega_{21} & 0 & c_{23} & c_{24} & c_{25} \\ \omega_{31} & c_{32} & 0 & c_{34} & c_{35} \\ \omega_{41} & c_{42} & c_{43} & 0 & c_{45} \\ \omega_{51} & c_{52} & c_{53} & c_{54} & 0 \end{pmatrix}$$

where

$$\begin{aligned} \omega_{12} = -\omega_{21} &= a_{12}^1 x_1 + \ c_{12} \\ \omega_{13} = -\omega_{31} &= a_{13}^1 x_1 + \ c_{13} \\ \omega_{14} = -\omega_{41} &= a_{14}^1 x_1 + \ c_{14} \\ \omega_{15} = -\omega_{51} &= a_{15}^1 x_1 + \ c_{15} \end{aligned}$$

By Lemma 3.1, we have $\frac{\partial \omega_{1j}}{\partial x_1} = 0$, for all $2 \le j \le 5$, which means that $\omega_{12}, \omega_{13}$, $\omega_{14}$ and $\omega_{15}$ are constants.

**case(II):** $k = 2$. By Theorem 2.4, the $\Omega$-matrix is of the following form

$$\Omega = (\omega_{ij}) = \begin{pmatrix} 0 & c_{12} & \omega_{13} & \omega_{14} & \omega_{15} \\ c_{21} & 0 & \omega_{23} & \omega_{24} & \omega_{25} \\ \hdashline \omega_{31} & \omega_{32} & 0 & c_{34} & c_{35} \\ \omega_{41} & \omega_{42} & c_{43} & 0 & c_{45} \\ \omega_{51} & \omega_{52} & c_{53} & c_{54} & 0 \end{pmatrix}$$

where

$$\begin{aligned} \omega_{13} = -\omega_{31} &= a_{13}^1 x_1 + a_{13}^2 x_2 + c_{13} \\ \omega_{14} = -\omega_{41} &= a_{14}^1 x_1 + a_{14}^2 x_2 + c_{14} \\ \omega_{15} = -\omega_{51} &= a_{15}^1 x_1 + a_{15}^2 x_2 + c_{15} \\ \omega_{23} = -\omega_{32} &= a_{23}^1 x_1 + a_{23}^2 x_2 + c_{23} \\ \omega_{24} = -\omega_{42} &= a_{24}^1 x_1 + a_{24}^2 x_2 + c_{24} \\ \omega_{25} = -\omega_{52} &= a_{25}^1 x_1 + a_{25}^2 x_2 + c_{25} \end{aligned}$$

**subcase(IIa):** $k_1 = 1 < k_2 = 2 \ = k$. In this case we have $x_1^2 \in E$ and $x_2^2 \in E$. By Lemma 3.1, we have $\frac{\partial \omega_{1j}}{\partial x_1} = 0 \ = \frac{\partial \omega_{2j}}{\partial x_2}$ for all $3 \le j \le 5$. By Lemma 3.2, we have $\frac{\partial \omega_{1j}}{\partial x_2} = \frac{\partial \omega_{2j}}{\partial x_1} = 0$, for all $3 \le j \le 5$. Therefore $\omega_{ij}$, for all $1 \le i \le 2$ and $3 \le j \le 5$ are constants.

**subcase(IIb):** $k_1 = \ k = 2$. In this case, we only have $x_1^2 + \ x_2^2 \in E$. By the cyclic relations $\frac{\partial \omega_{13}}{\partial x_2} + \frac{\partial \omega_{21}}{\partial x_3} + \frac{\partial \omega_{32}}{\partial x_1} = 0$, $\frac{\partial \omega_{14}}{\partial x_2} + \frac{\partial \omega_{21}}{\partial x_4} + \frac{\partial \omega_{42}}{\partial x_1} = 0$ and $\frac{\partial \omega_{15}}{\partial x_2} + \frac{\partial \omega_{21}}{\partial x_5} + \frac{\partial \omega_{52}}{\partial x_1} = 0$ imply

$a_{13}^2 = \frac{\partial \omega_{13}}{\partial x_2} = \frac{\partial \omega_{23}}{\partial x_1} = a_{23}^1$ , $a_{14}^2 = \frac{\partial \omega_{14}}{\partial x_2} = \frac{\partial \omega_{24}}{\partial x_1} = a_{24}^1$, and $a_{15}^2 = \frac{\partial \omega_{15}}{\partial x_2} = \frac{\partial \omega_{25}}{\partial x_1} = a_{25}^1$. Since

$$-\alpha_3 = \sum_{l=1}^{2} x_l \omega_{l3,} - \alpha_4 = \sum_{l=1}^{2} x_l \omega_{l4} \text{ and } - \alpha_5 = \sum_{l=1}^{2} x_l \omega_{l5}$$

are in $E$, we deduce easily that the following elements are in $E$ .

$$\begin{aligned} \widetilde{\alpha_3} &= a_{13}^1 x_1^2 + 2a_{13}^2 x_1 x_2 + \ a_{23}^3 x_2^2 \in E \\ \widetilde{\alpha_4} &= a_{14}^1 x_1^2 + 2a_{14}^2 x_1 x_2 + \ a_{24}^3 x_2^2 \in E \\ \widetilde{\alpha_5} &= a_{15}^1 x_1^2 + 2a_{15}^2 x_1 x_2 + \ a_{25}^2 x_2^2 \in E \end{aligned}$$

By Lemma 2.2, $\widetilde{\alpha_3}, \widetilde{\alpha_4}$ and $\widetilde{\alpha_5}$ are constant multiple of $x_1^2 + x_2^2$ . Hence we have $a_{13}^2 = 0 = a_{14}^2 = a_{15}^2$. In view of Lemma 3.3, we have $\frac{\partial \omega_{il}}{\partial x_i} = 0$ for all $1 \le i \le 2$ and $3 \le j \le 5$. Therefore $\omega_{ij}$ , for all $1 \le i \le 2$ and $3 \le j \le 5$ are constants.

**case(III):** $k = 3$. By Theorem 2.4, the $\Omega$-matrix is of the following form

$$\Omega = (\omega_{ij}) = \begin{pmatrix} 0 & c_{12} & c_{13} & \omega_{14} & \omega_{15} \\ c_{21} & 0 & c_{23} & \omega_{24} & \omega_{25} \\ c_{31} & c_{32} & 0 & \omega_{34} & \omega_{35} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \omega_{41} & \omega_{42} & \omega_{43} & 0 & c_{45} \\ \omega_{51} & \omega_{52} & \omega_{53} & c_{54} & 0 \end{pmatrix}$$

where

$$\begin{aligned} \omega_{14} = -\omega_{41} &= a_{14}^1 x_1 + a_{14}^2 x_2 + a_{14}^3 x_3 + c_{14} \\ \omega_{15} = -\omega_{51} &= a_{15}^1 x_1 + a_{15}^2 x_2 + a_{15}^3 x_3 + c_{15} \\ \omega_{24} = -\omega_{42} &= a_{24}^1 x_1 + a_{24}^2 x_2 + a_{24}^3 x_3 + c_{24} \\ \omega_{25} = -\omega_{52} &= a_{25}^1 x_1 + a_{25}^2 x_2 + a_{25}^3 x_3 + c_{25} \\ \omega_{34} = -\omega_{43} &= a_{34}^1 x_1 + a_{34}^2 x_2 + a_{34}^3 x_3 + c_{34} \\ \omega_{35} = -\omega_{53} &= a_{35}^1 x_1 + a_{35}^2 x_2 + a_{35}^3 x_3 + c_{35} \end{aligned}$$

**subcase(IIIa):** $k_1 = 1 < k_2 = 2 < k_3 \ = k$. In this case we have $x_1^2 \in E$ ,$x_2^2 \in E$ and $x_3^2 \in E$. By Lemma 3.1, we have $\frac{\partial \omega_{1j}}{\partial x_1} = 0 \ = \frac{\partial \omega_{2j}}{\partial x_2} \ = \frac{\partial \omega_{3j}}{\partial x_3}$ for all $4 \le j \le 5$.

By Lemma 3.2, we have $\frac{\partial \omega_{1j}}{\partial x_2} = 0 = \frac{\partial \omega_{1j}}{\partial x_3} = \frac{\partial \omega_{2j}}{\partial x_3} = \frac{\partial \omega_{2j}}{\partial x_1} = \frac{\partial \omega_{3j}}{\partial x_1} = \frac{\partial \omega_{3j}}{\partial x_2}$, for all $4 \le j \le 5$. Therefore $\omega_{ij}$, for all $1 \le i \le 3$ and $4 \le j \le 5$, are constants.

For subcase(IIIb) and subcase(IIIc) below, we shall use the following observations. The cyclic relations $\frac{\partial \omega_{1j}}{\partial x_2} + \frac{\partial \omega_{21}}{\partial x_j} + \frac{\partial \omega_{j2}}{\partial x_1} = 0$, $\frac{\partial \omega_{3j}}{\partial x_1} + \frac{\partial \omega_{13}}{\partial x_j} + \frac{\partial \omega_{j1}}{\partial x_3} = 0$, $\frac{\partial \omega_{3j}}{\partial x_2} + \frac{\partial \omega_{23}}{\partial x_j} + \frac{\partial \omega_{j2}}{\partial x_3} = 0$ for $j = 4, 5$ imply $a_{1j}^2 = \frac{\partial \omega_{1j}}{\partial x_2} = \frac{\partial \omega_{2j}}{\partial x_1} = a_{2j}^1$ , $a_{3j}^1 = \frac{\partial \omega_{3j}}{\partial x_1} = \frac{\partial \omega_{1j}}{\partial x_3} = a_{1j}^3$ , $a_{3j}^2 = \frac{\partial \omega_{3j}}{\partial x_2} = \frac{\partial \omega_{2j}}{\partial x_3} = a_{2j}^3$ , for $j = 4, 5$.

**subcase(IIIb):** $k_1 = 1 < k_2 = 3 \ = k$. In this case we have $x_1^2 \in E$ ,$x_2^2 + x_3^2 \ \in E$. By Lemma 3.1, we have $a_{14}^1 = \frac{\partial \omega_{14}}{\partial x_1} = 0$ and $a_{15}^1 = \frac{\partial \omega_{15}}{\partial x_1} = 0$ . Since $-\alpha_4 = \sum_{l=1}^{3} x_l \omega_{l4}$ and $-\alpha_5 = \sum_{l=1}^{3} x_l \omega_{l5}$ are in $E$ by Theorm 2.4, it is easy to see that the following quadratic forms are in $E$.

$$\widetilde{\alpha_4} = 2a_{14}^2 x_1 x_2 + 2a_{14}^3 \ x_1 x_3 + a_{24}^2 \ x_2^2 + a_{34}^3 x_3^2 + 2a_{24}^3 x_2 x_3 \ \in E \tag{4.2}$$

$$\widetilde{\alpha_5} = 2a^2_{15}x_1x_2 + 2a^3_{15}\ x_1x_3 + a^2_{25}\ x_2^2 + a^3_{35}x_3^2 + 2a^3_{25}x_2x_3 \ \in E \tag{4.3}$$

By Lemma 2.2, $\widetilde{\alpha_4}\,(0, x_2, x_3)$ and $\widetilde{\alpha_5}\,(0, x_2, x_3)$ are constant multiple of $x_2^2 + x_3^2$. Hence we have $a^3_{24} = 0 = a^3_{25}$. In view of Lemma 3.3 we have $\frac{\partial\omega_{24}}{\partial x_2} = 0 = \frac{\partial\omega_{34}}{\partial x_3} = \frac{\partial\omega_{25}}{\partial x_2} = \frac{\partial\omega 5}{\partial x_3}$. By (4.2) and (4.3) , we have

$$\widetilde{\alpha_4} = 2a^2_{14}x_1x_2 + 2a^3_{14}\ x_1x_3 \in E \quad \text{and} \quad \widetilde{\alpha_5} = 2a^2_{15}x_1x_2 + 2a^3_{15}\ x_1x_3 \in E.$$

In view of Lemma 2.4 and the fact that $|\{1\}| < |\{2,3\}|$ , we conclude that $a^2_{14} = a^3_{14} = 0 = a^2_{15} = a^3_{15}$. Hence $\omega_{14}, \omega_{15}, \omega_{24}, \omega_{25}, \omega_{34}$ and $\omega_{35}$ are constants.

**Subcase(IIIc):** $k_1 = 3 = k$. In this case, we have $x_1^2 + x_2^2 + x_3^2 \ \in E$.

Since $-\alpha_4 = \sum_{l=1}^3 x_l\omega_{l4}$ and $-\alpha_5 = \sum_{l=1}^3 x_l\omega_{l5}$ are in $E$ by Theorem 2.4, it is easy to see that the following quadratic forms are in $E$.

$$\begin{aligned}\widetilde{\alpha_4} &= a^1_{14}x_1^2 + 2a^2_{14}\ x_1x_2 + 2a^3_{14}x_1x_3 + a^2_{24}x_2^2 + a^3_{34}x_3^2 + 2a^3_{24}x_2x_3 \ \in E\\ \widetilde{\alpha_5} &= a^1_{15}x_1^2 + 2a^2_{15}\ x_1x_2 + 2a^3_{15}\ x_1x_3 + a^2_{25}x_2^2 + a^3_{35}x_3^2 + 2a^3_{25}x_2x_3 \ \in E\end{aligned}$$

By Lemma 2.2, $\widetilde{\alpha_4}$ and $\widetilde{\alpha_5}$ are constant multiple of $x_1^2 + x_2^2 + x_3^2$. Therefore we have $a^2_{14} = a^3_{14} = a^3_{24} = 0 = a^2_{15} = a^3_{15} = a^3_{25}$. In view of Lemma 3.3, we have $\frac{\partial\omega_{il}}{\partial x_i} = 0$ for all $4 \le l \le 5$ and $1 \le i \le 3$. Hence $\omega_{ij}$, for all $1 \le i \le 3$ and $4 \le j \le 5$, are also constants.

**Case(IV):** $k = 4$. By Theorem 2.4, the $\Omega$-matrix is of the following form

$$\Omega = (\omega_{ij}) = \begin{pmatrix} 0 & c_{12} & c_{13} & c_{14} & \omega_{15} \\ c_{21} & 0 & c_{23} & c_{24} & \omega_{25} \\ c_{31} & c_{32} & 0 & c_{34} & \omega_{35} \\ c_{41} & c_{42} & c_{43} & 0 & c_{45} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \omega_{51} & \omega_{52} & \omega_{53} & \omega_{54} & 0 \end{pmatrix}$$

where

$$\begin{aligned}\omega_{15} = -\omega_{51} &= a^1_{15}x_1 + a^2_{15}x_2 + a^3_{15}x_3 + a^4_{15}x_4 + c_{15}\\ \omega_{25} = -\omega_{52} &= a^1_{25}x_1 + a^2_{25}x_2 + a^3_{25}x_3 + a^4_{25}x_4 + c_{25}\\ \omega_{35} = -\omega_{53} &= a^1_{35}x_1 + a^2_{35}x_2 + a^3_{35}x_3 + a^4_{35}x_4 + c_{35}\\ \omega_{45} = -\omega_{54} &= a^1_{45}x_1 + a^2_{45}x_2 + a^3_{45}x_3 + a^4_{45}x_4 + c_{45}\end{aligned}$$

**subcase(IVa):** $k_1 = 1 < k_2 = 2 < k_3 < k_4 = 4 = k$. In this case we have $x_1^2 \in E$ , $x_2^2 \in E$, $x_3^2 \in E$ and $x_4^2 \in E$. By Lemma 3.1, we have $\frac{\partial\omega_{l5}}{\partial x_l} = 0$ for all $1 \le l \le 4$. On the other hand, Lemma 3.2 says that $\frac{\partial\omega_{l5}}{\partial x_l} = 0$ for all $1 \le l, m \le 4, l \ne m$. Therefore $\omega_{15}, \omega_{25}, \omega_{35}$ and $\omega_{45}$ are constants.

For subcases(IVb), (IVc), (IVd) and (IVe) below. We shall use the following observations. The cyclic relations $\frac{\partial\omega_{25}}{\partial x_1} + \frac{\partial\omega_{12}}{\partial x_5} + \frac{\partial\omega_{51}}{\partial x_2} = 0$, $\frac{\partial\omega_{35}}{\partial x_1} + \frac{\partial\omega_{13}}{\partial x_5} + \frac{\partial\omega_{51}}{\partial x_3} = 0$, $\frac{\partial\omega_{45}}{\partial x_1} + \frac{\partial\omega_{14}}{\partial x_5} + \frac{\partial\omega_{51}}{\partial x_4} = 0$, $\frac{\partial\omega_{35}}{\partial x_2} + \frac{\partial\omega_{23}}{\partial x_5} + \frac{\partial\omega_{52}}{\partial x_3} = 0$, $\frac{\partial\omega_{45}}{\partial x_2} + \frac{\partial\omega_{24}}{\partial x_5} + \frac{\partial\omega_{52}}{\partial x_4} = 0$ and $\frac{\partial\omega_{45}}{\partial x_3} + \frac{\partial\omega_{34}}{\partial x_5} + \frac{\partial\omega_{53}}{\partial x_4} = 0$ imply $a^1_{25} = \frac{\partial\omega_{25}}{\partial x_1} = \frac{\partial\omega_{15}}{\partial x_2} = a^2_{15}$, $a^1_{35} = \frac{\partial\omega_{35}}{\partial x_1} = \frac{\partial\omega_{15}}{\partial x_3} = a^3_{15}$, $a^1_{45} = \frac{\partial\omega_{45}}{\partial x_1} = \frac{\partial\omega_{15}}{\partial x_4} = a^4_{15}$, $a^3_{25} = \frac{\partial\omega_{25}}{\partial x_3} = \frac{\partial\omega_{35}}{\partial x_2} = a^2_{35}$, $a^2_{45} = \frac{\partial\omega_{45}}{\partial x_2} = \frac{\partial\omega_{25}}{\partial x_4} = a^4_{25}$ and $a^3_{45} = \frac{\partial\omega_{45}}{\partial x_3} = \frac{\partial\omega_{35}}{\partial x_4} = a^4_{35}$

**subcase(IVb):** $k_1 = 1 < k_2 = 2 < k_3 = 4 = k$. In this case we have $x_1^2 \in E$ , $x_2^2 \in E$, $x_3^2 + x_4^2 \in E$. By Lemma 3.1, we have $a_{15}^1 = \frac{\partial \omega_{15}}{\partial x_1} = 0$ and $a_{25}^2 = \frac{\partial \omega_{25}}{\partial x_2} = 0$ . On the other hand, Lemma 3.2 says that $a_{15}^2 = \frac{\partial \omega_{15}}{\partial x_2} = 0 = \frac{\partial \omega_{25}}{\partial x_1} = a_{25}^1$. Since $-\alpha_5 = \sum_{l=1}^4 x_l \omega_{l5}$ is in $E$ by Theorem 2.4, it is easy to see that the following quadratic form is in $E$.

$$\begin{aligned}\widetilde{\alpha_5} &= 2a_{15}^3\ x_1x_3 + 2a_{15}^4\ x_1x_4 + 2a_{25}^3x_2x_3 + 2a_{25}^4x_2x_4 + a_{35}^3x_3^2 \\ &\quad +2a_{35}^4\ x_3x_4 + a_{45}^4x_4^2\ \in E\end{aligned}$$

As $\widetilde{\alpha_5}\,(0,0,x_3,x_4)$is a constant multiple of $x_3^2 + x_4^2$ , we have $a_{35}^4 = 0$. In view of Lemma 3.3, we have $\frac{\partial \omega_{35}}{\partial x_3} = 0 = \frac{\partial \omega_{45}}{\partial x_4}$.

Hence the following quadratic form is in $E$.

$$\beta_5 = 2a_{15}^3\ x_1x_3 + 2a_{15}^4x_1x_4 + 2a_{25}^3x_2x_3 + 2a_{25}^4x_2x_4\ \in E$$

By Lemma 2.3, we have $\beta_5\,(x_1,0,x_3,x_4) = a_{15}^2x_1x_3 + 2a_{15}^4\ x_1x_4$ is in $E$. In view of Lemma 2.4 and the fact that $|\{1\}| < |\{3,4\}|$ , we conclude that $a_{15}^2 = 0 = a_{15}^4$.

Similarly, because of $\beta_5\,(0,x_2,x_3,x_4) = 2a_{25}^3x_2x_3 + 2a_{25}^4\ x_2x_4$ is in $E$ , we have $a_{25}^3 = 0 = a_{25}^4$ by Lemma 2.4 . Hence $\omega_{15}, \omega_{25}, \omega_{35}$ and $\omega_{45}$ are constants.

**subcase(IVc):** $k_1 = 1 < k_2 = 4 = k$. In this case, we have $x_1^2 \in E$ , $x_2^2+x_3^2+x_4^2 \in E$. By Lemma 3.1, we have $a_{15}^1 = \frac{\partial \omega_{15}}{\partial x_1} = 0$ .

Since $-\alpha_5 = \sum_{l=1}^4 x_l \omega_{l5}$ is in $E$ by Theorem 2.4, it is easy to see that the following quadratic form is in $E$.

$\widetilde{\alpha_5} = 2a_{15}^2x_1x_2 + a_{25}^2x_2^2 + 2a_{15}^3x_1x_3 + 2a_{15}^4x_1x_4 + 2a_{25}^3x_2x_3 + 2a_{25}^4x_2x_4 + a_{35}^3x_3^2 + 2a_{35}^4x_3x_4 + a_{45}^4x_4^2 \quad \in E$

As $\widetilde{\alpha_5}\,(0,x_2,x_3,x_4)$ is a constant multiple of $x_2^2+x_3^2+x_4^2$ by Lemma 2.2 , we have $a_{25}^3 = a_{25}^4 = a_{35}^4 = 0$. By Lemma 3.3, we have $a_{25}^2 = \frac{\partial \omega_{25}}{\partial x_2} = 0$ , $a_{35}^3 = \frac{\partial \omega_{35}}{\partial x_3} = 0$ and $a_{45}^4 = \frac{\partial \omega_{45}}{\partial x_4} = 0$. It follows that $\widetilde{\alpha_5} = 2a_{15}^2x_1x_2 + 2a_{15}^3x_1x_3 + 2a_{15}^4x_1x_4$ is in $E$ . In view of Lemma 2.4 and the fact that $|\{1\}| < |\{2,3,4\}|$ , we conclude $a_{15}^2 = 0 = a_{15}^3 = a_{15}^4$. Therefore $\omega_{15}, \omega_{25}, \omega_{35}$ and $\omega_{45}$ are constants.

**subcase(IVd):** $k_1 = 2 < k_2 = 4 = k$ . In this case, we have $x_1^2 + x_2^2 \in E$ and $x_3^2 + x_4^2 \in E$. Since $-\alpha_5 = \sum_{l=1}^4 x_l \omega_{l5}$ is in by Theorem 2.4, it is easy to see that the following quadratic form is in $E$.

$$\begin{aligned}\widetilde{\alpha_5} &= a_{15}^1x_1^2 + 2a_{15}^2x_1x_2 + 2a_{15}^3x_1x_3 + 2a_{15}^4x_1x_4 + 2a_{25}^2x_2^2 + 2a_{25}^3x_2x_3 \\ &\quad +2a_{25}^4x_2x_4 + a_{35}^2x_3^2 + 2a_{35}^4x_3x_4 + a_{45}^4x_4^2\end{aligned}$$

As $\widetilde{\alpha_5}\,(x_1,x_2,0,0)$ is a constant multiple of $x_1^2 + x_2^2$ and $\widetilde{\alpha_5}\,(0,0,x_3,x_4)$ is a constant multiple of $x_3^2 + x_4^2$ by Lemma 2.2 , we have $a_{15}^2 = 0$ and $a_{35}^4 = 0$. In view of Lemma 3.3, we have

$$a_{15}^1 = \frac{\partial \omega_{15}}{\partial x_1} = 0, \quad a_{25}^2 = \frac{\partial \omega_{25}}{\partial x_2} = 0, \quad a_{35}^3 = \frac{\partial \omega_{35}}{\partial x_3} = 0 \text{ and } a_{45}^4 = \frac{\partial \omega_{45}}{\partial x_4} = 0.$$

Hence $\widetilde{\alpha_5} = 2a_{15}^3x_1x_3 + 2a_{15}^4x_1x_4 + 2a_{15}^3x_2x_3 + 2a_{25}^4x_2x_4 \in E$.

By Lemma 2.4 we know that the matrix $\begin{pmatrix} a_{15}^3 & a_{15}^4 \\ a_{25}^3 & a_{25}^4 \end{pmatrix}$ is a constant multiple of an orthogonal matrix. In particular, we have

$$a_{15}^3a_{25}^3 + a_{15}^4a_{25}^4 = 0 = a_{15}^3a_{15}^4 + a_{25}^3a_{25}^4 \tag{4.4}$$

$$a^1_{35}a^2_{35} + a^1_{45}a^2_{45} = 0 = a^1_{35}a^1_{45} + a^2_{35}a^2_{45} \tag{4.5}$$

We shall construct a sequence of elements in E

$$Z_1 = \frac{1}{2}\left[L_0, x_3^2 + x_4^2\right] = \sum_{i=3}^{4} x_i D_i + 1$$

$$Z_2 = \frac{1}{2}\left[L_0, Z_1\right] = \frac{1}{2}\sum_{i=1}^{5}\sum_{j=3}^{4}\left[D_i^2, x_j D_j\right]$$

$$= \frac{1}{2}\sum_{i=1}^{5}\sum_{j=3}^{4}(2\frac{\partial x_j}{\partial x_i}D_i D_j - 2x_j\omega_{ij}D_i) \mod U_0$$

$$= \sum_{j=3}^{4} D_j^2 + \sum_{i=1}^{5}\sum_{j=3}^{4} x_j\omega_{ji}D_i \mod U_0$$

$$Z_3 = \frac{1}{2}\left[L_0, Z_2\right] = \frac{1}{2}\sum_{i=1}^{5}\sum_{j=3}^{4}\left[D_i^2, D_j^2\right] + \frac{1}{2}\sum_{i=1}^{5}\sum_{l=1}^{5}\sum_{j=3}^{4}\left[D_i^2, x_j\omega_{jl}D_l\right] \mod U_1$$

$$= 2\sum_{i=1}^{5}\sum_{j=3}^{4}\omega_{ji}D_jD_i + \sum_{i=1}^{5}\sum_{l=1}^{5}\sum_{j=3}^{4}\frac{\partial(x_j\omega_{jl})}{\partial x_i}D_iD_l \mod U_1$$

$$= 2\sum_{i=1}^{5}\sum_{j=3}^{4}\omega_{ji}D_jD_i + \sum_{l=1}^{5}\sum_{j=3}^{4}\omega_{ij}D_jD_l + \sum_{i=1}^{4}\sum_{j=3}^{4}x_j\frac{\partial\omega_{j5}}{\partial x_i}D_iD_5 \mod U_1$$

$$= 3\sum_{i=1}^{5}\sum_{j=3}^{4}\omega_{ji}D_jD_i + \sum_{i=1}^{4}\sum_{j=3}^{4}x_j\frac{\partial\omega_{j5}}{\partial x_i}D_iD_5 \mod U_1$$

$$\begin{aligned}
[Z_3, Z_1] = {} & 3\sum_{i=1}^{5}\sum_{j=3}^{4}\sum_{l=3}^{4}\left[\omega_{ij}D_jD_i, x_lD_l\right] \\
& + \sum_{i=1}^{4}\sum_{j=3}^{4}\sum_{p=3}^{4}\left[x_j\frac{\partial\omega_{j5}}{\partial x_i}D_iD_5, x_pD_p\right] \mod U_1 \\
= {} & 3\sum_{i=1}^{5}\sum_{j=3}^{4}\sum_{l=3}^{4}(\omega_{ji}\frac{\partial x_l}{\partial x_i}D_jD_l + \omega_{ji}\frac{\partial x_l}{\partial x_j}D_jD_l - x_l\frac{\partial\omega_{ji}}{\partial x_l}D_jD_l) \\
& + \sum_{i=1}^{4}\sum_{j=3}^{4}\sum_{p=3}^{4}(x_j\frac{\partial\omega_{j5}}{\partial x_i}\frac{\partial x_p}{\partial x_i}D_5D_p + x_j\frac{\partial\omega_{j5}}{\partial x_i}\frac{\partial x_p}{\partial x_5}D_iD_p \\
& - x_p\frac{\partial(x_j\frac{\partial\omega_{j5}}{\partial x_i})}{\partial x_p}D_iD_5) \mod U_1 \\
= {} & 3\sum_{j=3}^{4}\sum_{l=3}^{4}\omega_{jl}D_jD_l + 3\sum_{i=1}^{5}\sum_{j=3}^{4}\omega_{ji}D_iD_j - 3\sum_{j=3}^{4}\sum_{l=3}^{4}\omega_l\frac{\partial\omega_{j5}}{\partial x_l}D_jD_5 \\
& + \sum_{i=3}^{4}\sum_{j=3}^{4}x_j\frac{\partial\omega_{j5}}{\partial x_i}D_5D_i - \sum_{i=1}^{4}\sum_{j=3}^{4}x_j\frac{\partial\omega_{j5}}{\partial x_i}D_iD_5 \mod U_1
\end{aligned}$$

$$
\begin{aligned}
&= 3\sum_{i=1}^{5}\sum_{j=3}^{4}\omega_{ji}D_iD_j - 2\sum_{i=3}^{4}\sum_{j=3}^{4}x_j\frac{\partial\omega_{j5}}{\partial x_i}D_iD_5 \\
&\quad -\sum_{i=1}^{4}\sum_{j=3}^{4}x_j\frac{\partial\omega_{j5}}{\partial x_i}D_iD_5 \mod U_1
\end{aligned}
$$

$$
Z_4 = \frac{1}{2}([Z_3, Z_1] + Z_3) = 3\sum_{i=1}^{5}\sum_{j=3}^{4}\omega_{ji}D_iD_j - \sum_{i=3}^{4}\sum_{j=3}^{4}x_j\frac{\partial\omega_{j5}}{\partial x_i}D_iD_5 \mod U_1
$$

$$
\begin{aligned}
[Z_4, Z_1] &= 3\sum_{i=1}^{5}\sum_{j=3}^{4}\sum_{l=3}^{4}[\omega_{ji}D_iD_j, x_lD_l] \\
&\quad +\sum_{i=1}^{4}\sum_{j=3}^{4}\sum_{p=3}^{4}\left[x_j\frac{\partial\omega_{j5}}{\partial x_i}D_iD_l, x_pD_p\right] \mod U_1 \\
&= 3\sum_{i=1}^{5}\sum_{j=3}^{4}\omega_{ji}D_iD_j - 3\sum_{i=3}^{4}\sum_{j=3}^{4}x_j\frac{\partial\omega_{j5}}{\partial x_i}D_iD_5 \mod U_1
\end{aligned}
$$

$$
\begin{aligned}
Z_5 &= \frac{1}{2}(Z_4 - [Z_4, Z_1]) = \sum_{i=3}^{4}\sum_{j=3}^{4}x_j\frac{\partial\omega_{j5}}{\partial x_i}D_iD_5 \mod U_1 \\
\overline{Z}_4 &= -[Z_3, Z_1] + Z_4 - Z_5 = \sum_{i=1}^{4}\sum_{j=3}^{4}x_j\frac{\partial\omega_{j5}}{\partial x_i}D_iD_5 \mod U_1 \\
\overline{Z}_1 &= \frac{1}{2}\left[L_0, x_1^2 + x_2^2\right] = \sum_{i=1}^{2}x_iD_i + 1 \\
\overline{Z}_5 &= \left[\overline{Z}_4, \overline{Z}_1\right]) = \sum_{i=1}^{4}\sum_{j=3}^{4}\sum_{p=1}^{2}\left[x_j\frac{\partial\omega_{j5}}{\partial x_i}D_iD_5, x_pD_p\right] \mod U_1 \\
&= \sum_{i=1}^{4}\sum_{j=3}^{4}\sum_{p=1}^{2}x_j\frac{\partial\omega_{j5}}{\partial x_i}\frac{\partial x_p}{\partial x_i}D_5D_p \mod U_1 \\
&= \sum_{i=1}^{2}\sum_{j=3}^{4}x_j\frac{\partial\omega_{j5}}{\partial x_i}D_iD_5 \mod U_1 \\
&= \sum_{j=3}^{4}x_j\left[\sum_{i=1}^{2}(\frac{\partial\omega_{j5}}{\partial x_i}D_5)D_i\right] \mod U_1
\end{aligned}
$$

By switching the roles of $x_1^2 + x_2^2$ and $x_3^2 + x_4^2$, we get the following element in E

$$
\overline{Z}_6 = \sum_{i=1}^{2}\sum_{j=3}^{4}x_i\frac{\partial\omega_{i5}}{\partial x_j}D_jD_5 \mod U_1
$$

The cyclic relation $\frac{\partial\omega_{i5}}{\partial x_j}+\frac{\partial\omega_{ji}}{\partial x_5}+\frac{\partial\omega_{5j}}{\partial x_i}=0$ implies $\frac{\partial\omega_{i5}}{\partial x_j}=\frac{\partial\omega_{j5}}{\partial x_i}$ for $1\le i\le 2$ and $3\le j\le 4$ Therefore

$$\begin{aligned}
\overline{Z}_6 &= \sum_{i=1}^{2}\sum_{j=3}^{4} x_i\frac{\partial\omega_{j5}}{\partial x_i}D_jD_5 \quad \bmod U_1\\
&= \sum_{i=1}^{2} x_i\sum_{j=3}^{4}(\frac{\partial\omega_{j5}}{\partial x_i}D_5)D_j\\
A^{(1)} &= [L_0,\overline{Z}_5] = \frac{1}{2}\sum_{p=1}^{5}\sum_{i=1}^{2}\sum_{j=3}^{4}\left[D_p^2, x_j\frac{\partial\omega_{j5}}{\partial x_i}D_iD_5\right] \quad \bmod U_2\\
&= \sum_{p=1}^{5}\sum_{i=1}^{2}\sum_{j=3}^{4}\frac{\partial(x_j\frac{\partial\omega_{j5}}{\partial x_i})}{\partial x_p}D_pD_iD_5 \quad \bmod U_2\\
&= \sum_{i=1}^{2}\sum_{j=3}^{4}\frac{\partial\omega_{j5}}{\partial x_i}D_jD_iD_5 \quad \bmod U_2\\
&= \sum_{i=1}^{2}\sum_{j=3}^{4}(\frac{\partial\omega_{j5}}{\partial x_i}D_5)D_iD_j \quad \bmod U_2\\
A^{(2)} &= \left[A^{(1)},\overline{Z}_5\right] = \sum_{i=1}^{2}\sum_{j=3}^{4}\sum_{t=3}^{4}\sum_{s=1}^{2}\left[(\frac{\partial\omega_{j5}}{\partial x_i}D_5)D_iD_j, x_t(\frac{\partial\omega_{t5}}{\partial x_s}D_5)D_s\right] \quad \bmod U_3\\
&= \sum_{i=1}^{2}\sum_{j=3}^{4}\sum_{s=1}^{2}(\frac{\partial\omega_{j5}}{\partial x_i}D_5)(\frac{\partial\omega_{j5}}{\partial x_s}D_5)D_iD_s \quad \bmod U_3\\
&= \sum_{i=1}^{2}\left[\sum_{j=3}^{4}(\frac{\partial\omega_{j5}}{\partial x_i}D_5)^2\right]D_i^2 + \sum_{i\neq s;i,s=1}^{2}\sum_{j=3}^{4}\frac{\partial\omega_{j5}}{\partial x_i}\frac{\partial\omega_{j5}}{\partial x_s}D_5^2D_iD_s \quad \bmod U_3
\end{aligned}$$

Observe that for $1\le i\neq s\le 2$

$$\sum_{j=3}^{4}\frac{\partial\omega_{j5}}{\partial x_i}\frac{\partial\omega_{j5}}{\partial x_s}=\sum_{j=3}^{4}a_{j5}^{i}a_{j5}^{s}=0 \quad \text{by (4.5)}$$

For $1\le i\le 2$ and $3\le j\le 4$ , we denote

$$\xi_{ij}=\frac{\partial\omega_{j5}}{\partial x_i}D_5 \quad \text{and} \quad \eta_i=\sum_{j=3}^{4}\xi_{ij}^2,$$

Then

$$\begin{aligned}
A^{(2)} &= \sum_{i=1}^{2}\eta_iD_i^2 \quad \bmod U_3\\
A^{(3)} &= [A^{(2)},\overline{Z}_6] = \sum_{i=1}^{2}\sum_{s=1}^{2}\sum_{j=3}^{4}[\eta_iD_i^2,\xi_{si}x_sD_j] \quad \bmod U4
\end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{i=1}^{2}\sum_{j=3}^{4}\eta_i\xi_{ij}D_iD_j \mod U_4\\
A^{(4)} &= [A^{(3)},\overline{Z}_5] = 2\sum_{i=1}^{2}\sum_{j=3}^{4}\sum_{t=3}^{4}\sum_{s=1}^{2}[\eta_i\xi_{ij}D_iD_j,\xi_{st}x_tD_s] \mod U_5\\
&= 2\sum_{i=1}^{2}\sum_{j=3}^{4}\sum_{s=1}^{2}\eta_i\xi_{ij}\xi_{sj}D_iD_s \mod U_5\\
&= 2\sum_{i=1}^{2}\eta_i^2D_i^2 + 2\sum_{i\neq s;i,s=1}^{2}(\sum_{j=3}^{4}\xi_{ij}\xi_{sj})D_iD_s \mod U_5
\end{aligned}
$$

Observe that for $1 \le i \neq s \le 2$

$$
\sum_{j=3}^{4}\xi_{ij}\xi sj = \sum_{j=3}^{4}\frac{\partial\omega_{j5}}{\partial x_i}D_5\frac{\partial\omega_{j5}}{\partial x_s}D_5 = \sum_{j=3}^{4}a_{j5}^ia_{j5}^sD_5^2 = 0 \quad \text{by (4.5)}
$$

Hence

$$
A^{(4)} = 2\sum_{i=1}^{2}\eta_i^2D_i^2 \mod U_5
$$

By induction, we get an infinite sequence in E of the form

$$
\begin{aligned}
A^{(2s+1)} &= 2^s\sum_{i=1}^{2}\sum_{j=3}^{4}\eta_i^s\xi_{ij}D_iD_j \mod U_{2s+2}\\
A^{(2s+2)} &= 2^s\sum_{i=1}^{2}\eta_i^{s+1}D_i^2 \mod U_{2s+3}
\end{aligned}
$$

Since E is finite dimensional, we conclude that for $1 \le i \le 2$

$$
\eta_i = \sum_{j=3}^{4}\xi_{ij}^2 = \sum_{j=3}^{4}(\frac{\partial\omega_{j5}}{\partial x_i})^2D_5^2 = 0
$$

Therefore $\frac{\partial\omega_{j5}}{\partial x_i} = 0$ for $1 \le i \le 2$ and $3 \le j \le 4$.
We have proved that $\omega_{15}, \omega_{25}, \omega_{35}$ and $\omega_{45}$ are constants.

**Subcase (IVe) :** $k_1 = 4 = k$.

In this case we have $x_1^2 + x_1^2 + x_1^2 + x_1^2 \in E$. Since $-\alpha_5 = \sum_{l=1}^{4} x_l\omega_{l5}$ is in E by Theorem 2.4, it is easy to see that the following quadratic form is in E.

$$
\begin{aligned}
\widetilde{\alpha}_5 = a_{15}^1x_1^2 &+ 2a_{15}^2x_1x_2 + 2a_{15}^3x_1x_3 + 2a_{15}^4x_1x_4 + a_{25}^2x_2^2\\
&+2a_{25}^3x_2x_3 + 2a_{25}^4x_2x_4 + a_{35}^3x_3^2 + 2a_{35}^4x_3x_4 + a_{45}^4x_4^2
\end{aligned}
$$

By Lemma 2.2, $\widetilde{\alpha}_5$ is a constant multiple of $x_1^2 + x_2^2 + x_3^2 + x_4^2$. Therefore we have $\frac{\partial\omega_{j5}}{\partial x_i} = 0$ for $1 \le i \neq j \le 4$. It follows from Lemma 3.3 that $\frac{\partial\omega_{i5}}{\partial x_i} = 0$ for $1 \le i \le 4$. Hence $\omega_{15}, \omega_{25}, \omega_{35}$ and $\omega_{45}$ are constants.

We have proved the following new result.

Theorem 4.4. *Suppose that the state space of the filtering system (2.1) is of dimension five. If E is the finite-dimensional estimation algebra with maximal rank, then E is a real vector space of dimension 12 with a basis given by* $1, x_1, x_2, x_3, x_4, x_5, D_1, D_2, D_3, D_4, D_5$ *and* $L_0$.

Add to the Proof: Recently Yau and Hu [YaHu] have completed the classification of finite dimensional estimation algebra of maximal rank with arbitrary state space dimension.

## REFERENCES

[Br] R. W. Brockett, *Nonlinear systems and nonlinear estimation theory*, in The Mathematics of Filtering and Identification and Applications, M. Hazewinkel and J.C. Willems, eds., Reidel, Dordrecht, 1981.

[BrC1] R. W. Brockett and J. M. C. Clark, *The geometry of the conditional density functions*, in Analysis and Optimization of Stochastic Systems, O. L. R. Jacobs, et. al., eds., Academic Press, New York, 1980, pp. 399–409.

[Ch] J. Chen, *On ubiquity of Yau filters*, in Proceedings of the American Control Conference (Baltimore, Maryland), June 1994, pp. 252–254.

[ChYa1] W. L. Chiou and Stephen S. T. Yau, *Finite-dimensional filters with nonlinear drift II: Brockett's problem on classification of finite-dimensional estimation algebras*, SIAM J. Control and Optimization, 32 (1994), pp. 297–310.

[ChYa2] J. Chen and Stephen S. T. Yau, *Finite-dimensional filters with nonlinear drift VI: Linear structure of* $\Omega$, Mathematics of Control, Signals and Systems, 9 (1996), pp. 370–385.

[ChYa3] J. Chen and Stephen S. T. Yau, *Finite-dimensional filters with nonlinear drift VII: Mitter conjecture and structure of* $\eta$, SIAM J. Control and Optimization, 35:4 (1997), pp. 1116–1131.

[CYL1] J. Chen, Stephen S.T. Yau, and C. W. Leung, *Finite-dimensional filters with nonlinear drift IV: Classification of finite-dimensional estimation algebras of maximal rank with state space dimension 3*, SIAM J. Control and Optimization, 34 (1996), pp. 179–198.

[CYL2] J. Chen, Stephen S. T. Yau, and C. W. Leung, *Finite-dimensional filters with nonlinear drift VIII: Classification of finite-dimensional estimation algebras of maximal rank with state space dimension 4*, SIAM J. Control and Optimization, 35:4 (1997), pp. 1132–1141.

[Da] M. H. A. Davis, *On a multiplicative functional transformation arising in nonlinear filtering theory*, Z. Wahrsch Verw. Gebiete, 54 (1980), pp. 125–139.

[DaMa] M. H. A. Davis and S. I. Marcus, *An introduction to nonlinear filtering*, in The Mathematics of Filtering and Identification and Applications, M. Hazewinkel and J. S. Willems, eds., Reidel, Dordrecht, 1981.

[DTWY] R. T. Dong, L. T. Tam, W. S. Wong, and Stephen S. T. Yau, *Structure and classification theorems of finite-dimensional exact estimation algebras*, SIAM J. Control and Optimization, 29 (1991), pp. 866–877.

[Du] T. E. Duncan, *Probability densities for diffusion processes with applications to nonlinear filtering theory*, Ph.D. thesis, Stanford, 1967.

[Ha] M. Hazewinkel, *Lecture on linear and nonlinear filtering*, in Analysis and Estimation of Stochastic Mechanical Systems, CISM Course and Lectures 303, W. Shiehlen and W. Wedig, eds., Springer, Vienna, 1988.

[Ka] R. E. Kalman, *A new approach to linear filtering and prediction problem*, Trans. ASME., Ser. D.J. Basic Engineering, 82 (1960), pp. 35–45.

[KaBu] R. E. Kalman and R. S. Bucy, *New results in linear filtering and prediction theory*, Trans. ASME Series D.J. Basic Engineering, 83 (1961), pp. 95–108.

[Ma1] A. Makowski, *Filtering formulae for partially observed linear systems with non-Gaussian initial conditions*, Stochastics, 16 (1986), pp. 1–24.

[Ma2] S. Marcus, *Algebraic and geometric methods in nonlinear filtering*, SIAM J. Control and Optimization, 22 (1984), pp. 817–844.

[Mi] S. K. Mitter, *On the analogy between mathematical problems of non-linear filtering and quantum physics*, Ricerche Automat., 10 (1979), pp. 163–216.

[Mo] R. E. Mortensen, *Optimal control of continuous time stochastic systems*, Ph.D. thesis, University of California, Berkeley, 1966.

[Oc] D. Ocone, *Topics in nonlinear filtering theory*, Ph.D. thesis, Massachusetts Institute of Technology, 1980.

[TWY] L. F. Tam, W. S. Wong and S. S. T. Yau, *On a necessary and sufficient condition for finite dimensionality of estimation algebras*, SIAM J. Control and Optimization, 28 (1990), pp. 173–185.

[WeNo] J. Wei and E. Norman, *On the global representation of the solutions of linear differential equations as a product of exponentials*, Proc. Amer. Math. Sci., 15 (1964), pp. 327–334.

[Wo] W. S. Wong, *On a new class of finite-dimensional estimation algebras*, Systems Control Lett., 9 (1987), pp. 79–93.

[WYH] Xi Wu, Stephen S. T. Yau, and G. Q. Hu, *Finite dimensional filters with nonlinear drift XII: Linear and constant structure of* $\Omega$, preprint.

[Ya] S. S. T. Yau, *Finite dimensional filters with nonlinear drift I: A class of filters including both Kalman-Bucy filters and Benes filters*, Journal of Mathematical Systems, Estimation and Control, 4 (1994), pp. 181–203.

[YaHu] S. S.-T. Yau and G. Q. Hu, *Finite-dimensional filters with nonlinear drift XIV: Classification of finite-dimensional estimation algebras of maximal rank with artitrary state space dimension and mitter conjecture*, submitted for publication.

[YaRa] S. S. T. Yau and A. Rasoulian, *Classification of four-dimensional estimation algebras*, to appear in IEEE Transactions on Automation and constrol.

[Za] M. Zakai, *On the optimal filtering of diffusion processes*, Z. Wahrsch. Verw. Geb., 11 (1969), pp. 230–243.