this is indeed the maximum. Since $e^c - 1$ is convex and $e^0 - 1 = 0$, we have that $c \le e^c - 1$ for all c, so that $0 \le 1 - e^c \le -c$ for $c < 0$ and hence

$$\sup_{t\ge 0,\, 0 > c \ge -1} f(t,c)\,(1-e^c) = \sup_{0 > c \ge -1} \frac{e^c - 1}{ce} \le \frac{1}{e} < 1. \quad (5.38)$$

Therefore

$$|xy(e^c - 1)e^{cx+cy}| = |xe^{cx} y e^{cy} (e^c - 1)| \quad (5.39)$$

$$\le xe^{cx} < x. \quad (5.40)$$

Since by assumption $\mathbb{E}x < \infty$ we can invoke the dominated convergence theorem to conclude that

$$\lim_{c\uparrow 0} \mathbb{E}xy(e^c - 1)e^{cx+cy} = \mathbb{E}\lim_{c\uparrow 0} xy(e^c - 1)e^{cx+cy} = 0. \quad (5.41)$$

■

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Finite-Dimensional Filters with Nonlinear Drift X: Explicit Solution of DMZ Equation

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***Abstract*—In this note, we consider the explicit solution of Duncan–Mortensen–Zakai (DMZ) equation for the finite-dimensional filtering system. We show that Yau filtering system $((\partial f_j/\partial x_i) - (\partial f_i/\partial x_j) = c_{ij} =$ constant for all (i, j) can be solved explicitly with an arbitrary initial condition by solving a system of ordinary differential equations and a Kolmogorov-type equation. Let n be the dimension of state space. We show that we need only n sufficient statistics in order to solve the DMZ equation.**

***Index Terms*—Duncan–Mortensen–Zakai (DMZ) equation, finite-dimensional filter, Kolmogorov equation, nonlinear drift.**

I. INTRODUCTION

In 1961, Kalman and Bucy [11] published a historically important mathematics paper on filtering that is highly influential to modern industry. Since then, the Kalman–Bucy filter has proved useful in many areas such as navigational and guidance systems, radar tracking, solar mapping, and satellite orbit determination. Despite its usefulness, however, the Kalman–Bucy filter is not perfect. One of its weaknesses is that it needs the Gaussian assumption for the initial data. Another weakness is that it is restricted to linear dynamical system. In the 1960s, Duncan [9], Mortensen [14], and Zakai [25] independently derived the so-called DMZ equation for the nonlinear filtering problem. Unfortunately, since the DMZ equation is a stochastic differential equation, there is no easy way to derive a recursive algorithm for solving this equation.

The idea of using estimation algebras to construct finite-dimensional nonlinear filters was first proposed by Brockett and Clark [4], Brockett [3], and Mitter [13]. The motivation came from the Wei–Norman approach [18] of using Lie algebraic ideas to solve a linear time-varying differential equation. The advantage of this approach is that as long as the estimation algebra is finite dimensional, we will get a finite-dimensional recursive filter; there is no need to make any assumption in the initial data. Moreover, the approach applies well to nonlinear dynamical systems. This approach has been worked out in detail in [17] and especially in the so-called Yau filtering system described in [5]. In [19], [20], it was shown that only n sufficient statistics are needed to solve the DMZ equation explicitly. Therefore, the procedure in [19] is much less complex than those given in [12], [10], in which n^2 sufficient statistics are needed. For a linear filtering system, it is quite easy to see that the corresponding estimation algebra is finite dimensional. So we can apply the Wei–Norman approach to construct a finite-dimensional recursive filter with arbitrary initial data. However, in the Wei–Norman approach, one has to know explicitly the basis as vector space of the estimation algebra in order to reduce the DMZ equation to a finite system of ordinary differential equations, Kolmogorov equation, and several first-order linear partial differential equations. Classically, one knows the explicit basis for the estimation algebra only in the case that it has maximal rank. Typically people assume that the linear system is controllable and observable.

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Recently, a new direct method was introduced to study the Kalman–Bucy filtering and Beněs filtering systems with arbitrary initial condition for which f, g, and h in (2.1) below are independent of time (cf. [22], [23], [21]). Our approach offers several advantages. It is easy, and the derivation no longer needs controllability and observability. Thus, the algorithm is universal for any linear filtering system. Furthermore, it eliminates the necessity of integrating n first-order linear partial differential equations, as was the case in the Lie algebra method. Moreover, it does not need to solve an algebraic Riccati equation, in contrast with the result of [23]. Finally, the number of sufficient statistics required to compute the conditional probability density of the state in our present method is n; which is substantially smaller than n^2 required by the classical methods of [12] and [10]. The Boeing company is interested in eliminating noise that penetrates the cockpit or cabin of an airplane. A nonlinear filter is needed to estimate the coefficients used in constructing the antinoise. The dimension of the state vector is in the thousands which makes the classical methods impractical.

In this paper, we apply the new direct method to the Yau filtering system which includes both Kalman–Bucy filters and Beněs filters as special cases. Thus, we have a more general result about the explicit solution of the DMZ equation.

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II. Basic Filtering Problem

The filtering problem considered here is based on the following signal observation model:

$$\begin{cases} dx(t) = f(x(t))\,dt + g(x(t))\,dv(t) & x(0) = x_0 \\ dy(t) = h(x(t))\,dt + dw(t) & y(0) = 0 \end{cases} \tag{2.1}$$

in which x, v, y, and w are, respectively, R^n, R^p, R^m, and R^m valued processes and v and w independent, standard Brownian processes. We further assume that $n = p$; f, g, and h are vector-valued, orthogonal matrix-valued and vector-valued C^∞ smooth functions. We shall refer to $y(t)$ as the observation at time t.

Let $\rho(t, x)$ denote the conditional probability density of the state given the observation $\{y(s): 0 \le s \le t\}$. It is well known (see [7], for example) that $\rho(t, x)$ is given by normalizing a function $\sigma(t, x)$ that satisfies the following DMZ equation:

$$\begin{cases} d\sigma(t,x) = L_0\sigma(t,x)\,dt + \sum_{i=1}^{m} L_i\sigma(t,x)\,dy_i(t) \\ \sigma(0,x) = \sigma_0(x) \end{cases} \tag{2.2}$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i} - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2$$

for $i = 1, \dots, m$, L_i is the zero-degree differential operator of multiplication by h_i, and σ_0 is the probability density of the initial point x_0.

In [6], Davis introduced a new unnormalized density

$$u(t,x) = \exp\left(-\sum_{i=1}^{m} h_i(x)y_i(t)\right)\sigma(t,x).$$

He reduced (2.2) to the following time-varying partial differential equation which is called robust DMZ-equation:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = L_0u(t,x) + \sum_{i=1}^{m} y_i(t)[L_0, L_i]u(t,x) \\ \qquad\qquad + \frac{1}{2}\sum_{i,j=1}^{m} y_i(t)y_j(t)[[L_0, L_i], L_j]u(t,x) \\ u(0,x) = \sigma_0(x) \end{cases} \tag{2.3}$$

where $[\cdot,\cdot]$ is the Lie bracket as described in [20]. It is easy to show [22] that (2.3) is equivalent to the following time-varying partial differential equation:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}(t,x) + \sum_{i=1}^{n}\left(-f_i(x) + \sum_{j=1}^{m} y_j(t)\frac{\partial h_j}{\partial x_i}(x)\right)\frac{\partial u}{\partial x_i}(t,x) \\ \qquad - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x)u(t,x) - \frac{1}{2}\sum_{i=1}^{m} h_i^2(x)u(t,x) \\ \qquad + \frac{1}{2}\sum_{i=1}^{m} y_i(t)\Delta h_i(x)u(t,x) \\ \qquad - \sum_{i=1}^{m}\sum_{j=1}^{n} y_i(t)f_j(x)\frac{\partial h_i}{\partial x_j}(x)u(t,x) \\ \qquad + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{n} y_i(t)y_j(t)\frac{\partial h_i}{\partial x_k}(x) \cdot \frac{\partial h_j}{\partial x_k}(x)u(t,x) \\ u(0,x) = \sigma_0(x) \end{cases} \tag{2.4}$$

In 1990, Yau [19] (cf. [20] for detail version) first studied the filtering system (2.1) with the following conditions:

$$C_1')\quad \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j} = \text{constant (depending on } i, j), \quad \text{for all } 1 \le i, j \le n.$$

This was called Yau filtering system in [5]. Yau's filtering system include Kalman–Bucy filtering systems and Beněs' filtering systems as special cases and finite dimensional filters were constructed explicitly by using Lie algebra [17], [19], [20]. Define

$$\eta(x) = \sum_{i=1}^{n} f_i^2(x) + \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) + \sum_{i=1}^{m} h_i^2(x). \tag{2.5}$$

The following theorems are proved in [20].

Theorem 2.1: $C_1')$ holds if and only if

$$(f_1, \dots, f_n) = (l_1, \dots, l_n) + \left(\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n}\right)$$

where $l_1, \dots, l_n$ are polynomials of degree one and F is a C^∞ function.

Theorem 2.2: Let E be a finite-dimensional estimation algebra of (2.1) satisfying $C_1')$. Then $h_1, \dots, h_m$ are polynomials of degree at most one.

From Theorem 2.1, we know that $C_1')$ is equivalent to the following condition:

$$C_1)\quad f_i(x) = l_i(x) + \frac{\partial F}{\partial x_i}(x), \qquad 1 \le i \le n \tag{2.6}$$

where $l_i(x) = \sum_{j=1}^n d_{ij}x_j + d_i$ for $1 \le i \le n$ and F is a C^∞ function.

Theorem 2.2 tells us that $h_1, \ldots, h_m$ are polynomials of degree at most one if Yau filtering system is a finite dimensional estimation algebra. So we list the following condition:

$$C_2) \quad h_i(x) = \sum_{j=1}^n c_{ij}x_j + c_i, \qquad 1 \le i \le m \tag{2.7}$$

where c_{ij} and c_i are constants.

Moreover, we know that $\eta(x)$ is a polynomial of degree at most two (quadratic form in x) for most interesting filtering systems [19], [17]. Hence we assume the following condition:

$$C_3) \quad \eta(x) = \sum_{i,j=1}^n \eta_{ij}x_ix_j + \sum_{i=1}^n \eta_i x_i + \eta_0 \tag{2.8}$$

where η_{ij}, η_i, and η_0 are constants. We remark that in [1], Beněs also requires this condition.

III. Explicit Solution of DMZ Equation

Lemma: Equation (2.4) is equivalent to the following equation:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + \sum_{i=1}^n \theta_i(t,x) \\ \qquad \cdot \frac{\partial u}{\partial x_i}(t,x) + \theta(t,x)u(t,x) \\ u(0,x) = \sigma_0(x) \end{cases} \tag{3.1}$$

where

$$\theta_i(t,x) = \sum_{j=1}^m y_j(t)\frac{\partial h_j}{\partial x_i}(x) - f_i(x)$$

and

$$\theta(t,x) = \frac{1}{2}\left(\sum_{i=1}^n \theta_i^2(t,x) + \sum_{i=1}^n \frac{\partial \theta_i}{\partial x_i}(t,x) - \eta(x)\right).$$

Proof: We need only to show that the right-hand side of (3.1) equals to the right-hand side of (2.4). Since

$$\begin{aligned} \theta_i^2(t,x) &= \sum_{j=1}^m\sum_{k=1}^m y_j(t)y_k(t)\frac{\partial h_j}{\partial x_i}(x)\frac{\partial h_k}{\partial x_i}(x) \\ &\quad - 2f_i(x)\sum_{j=1}^m y_j(t)\frac{\partial h_j}{\partial x_i}(x) + f_i^2(x) \\ \frac{\partial \theta_i}{\partial x_i}(t,x) &= \sum_{j=1}^m y_j(t)\frac{\partial^2 h_j}{\partial x_i^2}(x) - \frac{\partial f_i}{\partial x_i}(x) \end{aligned}$$

by (2.5)

$$\begin{aligned} \theta(t,x) = \frac{1}{2}&\sum_{i=1}^n\sum_{j=1}^m\sum_{k=1}^m y_j(t)y_k(t)\frac{\partial h_j}{\partial x_i}(x)\frac{\partial h_k}{\partial x_i}(x) \\ &- \sum_{i=1}^n f_i(x)\sum_{j=1}^m y_j(t)\frac{\partial h_j}{\partial x_i}(x) \\ &+ \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^m y_j(t)\frac{\partial^2 h_j}{\partial x_i^2}(x) - \sum_i^n \frac{\partial f_i}{\partial x_i}(x) \\ &- \frac{1}{2}\sum_{i=1}^m h_i^2(x). \end{aligned}$$

Theorem 3.1: Suppose $u(t,x)$ is a solution of (3.1) and

$$\tilde{u}(t,x) = e^{\Lambda(t,x)}u(t, x+b(t)). \tag{3.2}$$

Then $\tilde{u}(t,x)$ is the solution of the following Kolmogorov equation:

$$\begin{cases} \frac{\partial \tilde{u}}{\partial t}(t,x) = \frac{1}{2}\Delta\tilde{u}(t,x) - \sum_{i=1}^n H_i(x)\frac{\partial \tilde{u}}{\partial x_i}(t,x) \\ \qquad - P(x)\tilde{u}(t,x) \\ \tilde{u}(0,x) = e^{\Lambda(0,x)}u(0, x+b(0)) \end{cases} \tag{3.3}$$

if we can choose $H_i(x)$ and $P(x)$ such that $\Lambda(t,x)$ satisfy the following systems:

$$\begin{aligned} &b_i'(t) - \frac{\partial \Lambda}{\partial x_i}(t,x) + H_i(x) \\ &\quad + \theta_i(t, x+b(t)) \equiv 0, \qquad 1 \le i \le n \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\frac{\partial \Lambda}{\partial t}(t,x) - \frac{1}{2}\sum_{i=1}^n (b_i'(t))^2 - \sum_{i=1}^n \theta_i(t, x+b(t))b_i'(t) \\ &\quad - \frac{1}{2}\eta(x+b(t)) + \frac{1}{2}\sum_{i=1}^n H_i^2(x) \\ &\quad - \frac{1}{2}\sum_{i=1}^n \frac{\partial H_i}{\partial x_i}(x) + P(x) \equiv 0. \end{aligned} \tag{3.5}$$

Moreover if u, $(\partial u/\partial x_1), \ldots, (\partial u/\partial x_n)$ are linearly independent, then (3.4) and (3.5) are also necessary conditions for (3.3).

Proof:

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial t}(t,x) &= e^{\Lambda(t,x)}\frac{\partial \Lambda}{\partial t}(t,x)u(t, x+b(t)) \\ &\quad + e^{\Lambda(t,x)}\frac{\partial u}{\partial t}(t, x+b(t)) + e^{\Lambda(t,x)} \\ &\quad \cdot \sum_{i=1}^n b_i'(t)\frac{\partial u}{\partial x_i}(t, x+b(t)) \end{aligned} \tag{3.6}$$

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial x_i}(t,x) &= e^{\Lambda(t,x)}\frac{\partial \Lambda}{\partial x_i}(t,x)u(t, x+b(t)) \\ &\quad + e^{\Lambda(t,x)}\frac{\partial u}{\partial x_i}(t, x+b(t)) \end{aligned} \tag{3.7}$$

$$\begin{aligned} \frac{\partial^2 \tilde{u}}{\partial x_i^2}(t,x) &= e^{\Lambda(t,x)}\left[\frac{\partial \Lambda}{\partial x_i}(t,x)\right]^2 u(t, x+b(t)) \\ &\quad + e^{\Lambda(t,x)}\frac{\partial^2 \Lambda}{\partial x_i^2}(t,x)u(t, x+b(t)) \\ &\quad + 2e^{\Lambda(t,x)}\frac{\partial \Lambda}{\partial x_i}(t,x)\frac{\partial u}{\partial x_i}(t, x+b(t)) \\ &\quad + e^{\Lambda(t,x)}\frac{\partial^2 u}{\partial x_i^2}(t, x+b(t)). \end{aligned} \tag{3.8}$$

Putting (3.6)–(3.8) into (3.3), we have

$$\begin{aligned} &\frac{\partial \Lambda}{\partial t}(t,x)u(t, x+b(t)) + \frac{\partial u}{\partial t}(t, x+b(t)) \\ &\quad + \sum_{i=1}^n b_i'(t)\frac{\partial u}{\partial x_i}(t, x+b(t)) \\ &\quad - \frac{1}{2}\sum_{i=1}^n\left[\frac{\partial \Lambda}{\partial x_i}(t,x)\right]^2 u(t, x+b(t)) \\ &\quad - \frac{1}{2}\sum_{i=1}^n \frac{\partial^2 \Lambda}{\partial x_i^2}(t,x)u(t, x+b(t)) \\ &\quad - \sum_{i=1}^n \frac{\partial \Lambda}{\partial x_i}(t,x)\frac{\partial u}{\partial x_i}(t, x+b(t)) \\ &\quad - \frac{1}{2}\Delta u(t, x+b(t)) + \sum_{i=1}^n \frac{\partial \Lambda}{\partial x_i}(t,x) \\ &\quad \cdot H_i(x)u(t, x+b(t)) \\ &\quad + \sum_{i=1}^n H_i(x)\frac{\partial u}{\partial x_i}(t, x+b(t)) \\ &\quad + P(x)u(t, x+b(t)) \equiv 0. \end{aligned} \tag{3.9}$$

From (3.1), we know that

$$\frac{\partial u}{\partial t}(t, x+b(t)) - \frac{1}{2}\Delta u(t, x+b(t)) = \sum_{i=1}^{n} \theta_i(t, x+b(t)) \frac{\partial u}{\partial x_i}(t, x+b(t)) + \theta(t, x+b(t))u(t, x+b(t)).$$

By observing the coefficient $(\partial u/\partial x_i)(t, x+b(t))$ and $u(t, x+b(t))$ of (3.9), we have

$$b_i'(t) - \frac{\partial \Lambda}{\partial x_i}(t,x) + H_i(x) + \theta_i(t, x+b(t)) \equiv 0, \qquad 1 \le i \le n \tag{3.10}$$

$$\frac{\partial \Lambda}{\partial t}(t,x) - \frac{1}{2}\sum_{i=1}^{n}\left[\frac{\partial \Lambda}{\partial x_i}(t,x)\right]^2 - \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 \Lambda}{\partial x_i^2}(t,x) + \sum_{i=1}^{n}\frac{\partial \Lambda}{\partial x_i}(t,x)H_i(x) + \frac{1}{2}\sum_{i=1}^{n}\theta_i^2(t, x+b(t)) + \frac{1}{2}\sum_{i=1}^{n}\frac{\partial \theta_i}{\partial x_i}(t, x+b(t)) - \frac{1}{2}\eta(x+b(t)) + P(x) \equiv 0. \tag{3.11}$$

In view of (3.10), we have

$$\frac{\partial \Lambda}{\partial x_i}(t,x) = b'i(t) + H_i(x) + \theta_i(t, x+b(t)) \tag{3.12}$$

and

$$\frac{\partial^2 \Lambda}{\partial x_i^2}(t,x) = \frac{\partial H_i}{\partial x_i}(x) + \frac{\partial \theta_i}{\partial x_i}(t, x+b(t)). \tag{3.13}$$

Putting (3.12), (3.13) into (3.11), we get (3.5). Q.E.D.

Now if we assume that condition C_3) holds, then

$$\eta(x+b(t)) = \eta(x) + \sum_{i=1}^{n} B_i(t)x_i + B(t) \tag{3.14}$$

where

$$B_i(t) = \sum_{j=1}^{n}(\eta_{ij} + \eta_{ji})b_j(t)$$

and

$$B(t) = \sum_{i,j=1}^{n}\eta_{ij}b_i(t)b_j(t) + \sum_{i=1}^{n}\eta_i b_i(t).$$

The following theorem says that in many situations if the solution of (3.1) can be represented in the form of (3.2) and (3.3), then condition C_1 in (2.6) holds.

Theorem 3.2: Assume condition C_2) and C_3) hold. Suppose $u(t,x)$ is a solution of (3.1) and

$$\tilde{u}(t,x) = e^{\Lambda(t,x)}u(t, x+b(t))$$

where $b_i'(t)$, $1 \le i \le n$ are linearly independent and $\tilde{u}(t,x)$ is the solution of the following Kolmogorov equation:

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t}(t,x) = \dfrac{1}{2}\Delta \tilde{u}(t,x) - \displaystyle\sum_{i=1}^{n} H_i(x)\frac{\partial \tilde{u}}{\partial x_i}(t,x) - P(x)\tilde{u}(t,x) \\ \tilde{u}(0,x) = e^{\Lambda(0,x)}u(0, x+b(0)). \end{cases}$$

Furthermore, let

$$\Lambda(t,x) = c(t) + G(x) + \sum_{i=1}^{n} a_j(t)x_j - F(x+b(t)).$$

If we can choose $H(x)$, $G(x)$, and $P(x)$ such that

$$\frac{1}{2}\sum_{i=1}^{n}H_i^2(x) - \frac{1}{2}\sum_{i=1}^{n}\frac{\partial H_i}{\partial x_i}(x) - \frac{1}{2}\eta(x) + P(x) \equiv 0 \tag{3.15}$$

then

$$f_i(x) = \frac{\partial F}{\partial x_i}(x) + l_i(x), \qquad 1 \le i \le n$$

where

$$l_i(x) = \sum_{j=1}^{n} d_{ij}x_j + d_i$$

d_{ij} and d_i are some constants, i.e., condition C_1) holds.

Proof: We use Theorem 3.1. First by condition C_2) and the fact that $\Lambda(t,x)$ is in the form of $c(t) + G(x) + \sum_{i=1}^{n} a_j(t)x_j - F(x+b(t))$, we observe that

$$\theta_i(t, x+b(t)) = \sum_{j=1}^{m} y_j(t)\frac{\partial h_j}{\partial x_i}(x+b(t)) - f_i(x+b(t)) = \sum_{j=1}^{m} c_{ji}y_j(t) - f_i(x+b(t)) \tag{3.16}$$

$$\frac{\partial \Lambda}{\partial t}(t,x) = c'(t) + \sum_{j=1}^{n} a_j'(t)x_j - \sum_{j=1}^{n}\frac{\partial F}{\partial x_j}(x+b(t))b_j'(t) \tag{3.17}$$

$$\frac{\partial \Lambda}{\partial x_i}(t,x) = \frac{\partial G}{\partial x_i}(x) + a_i(t) - \frac{\partial F}{\partial x_i}(x+b(t)). \tag{3.18}$$

Putting (3.14)–(3.18) into (3.4) and (3.5), we have

$$b_i'(t) - a_i(t) + \sum_{j=1}^{m} c_{ji}y_j(t) + H_i(x) - \frac{\partial G}{\partial x_i}(x) + \frac{\partial F}{\partial x_i}(x+b(t)) - f_i(x+b(t)) \equiv 0, \qquad 1 \le i \le n \tag{3.19}$$

$$c'(t) - \frac{1}{2}\sum_{i=1}^{n}(b_i'(t))^2 - \frac{1}{2}B(t) - \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ji}y_j(t)b_i'(t) + \sum_{j=1}^{n} a_j'(t)x_j - \frac{1}{2}\sum_{i=1}^{n}B_i(t)x_i + \sum_{i=1}^{n} f_i(x+b(t))b_i'(t) - \sum_{j=1}^{n}\frac{\partial F}{\partial x_j}(x+b(t))b_j'(t) \equiv 0. \tag{3.20}$$

From (3.19), we get

$$f_i(x+b(t)) - \frac{\partial F}{\partial x_i}(x+b(t)) \equiv H_i(x) - \frac{\partial G}{\partial x_i}(x) + b_i'(t) - a_i(t) + \sum_{j=1}^{m} c_{ji}y_j(t). \tag{3.21}$$

Putting (3.21) into (3.20) we get

$$c'(t) - \frac{1}{2}\sum_{i=1}^{n}(b_i'(t))^2 - \frac{1}{2}B(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji}y_j(t)b_i'(t) + \sum_{i=1}^{n}\left(b_i'(t) - a_i(t) + \sum_{j=1}^{m} c_{ji}y_j(t)\right)b_i'(t) + \sum_{j=1}^{n} a_j'(t)x_j - \frac{1}{2}\sum_{i=1}^{n}B_i(t)x_i + \sum_{i=1}^{n}\left(H_i(x) - \frac{\partial G}{\partial x_i}(x)\right)b_i'(t) \equiv 0. \tag{3.22}$$

It follows that

$$\sum_{i=1}^{n}\left(a_i'(t) - \frac{1}{2}B_i(t)\right)x_i + \sum_{i=1}^{n}\left(H_i(x) - \frac{\partial G}{\partial x_i}(x)\right)b_i'(t) \equiv \text{constant}. \tag{3.23}$$

Since $b_i(t)$, $1 \le i \le n$, are linear independent by hypothesis, by considering Taylor expansion of $H_i(x) - (\partial G/\partial x_i)(x)$ for $1 \le i \le n$, we conclude that (3.23) implies

$$H_i(x) - \frac{\partial G}{\partial x_i}(x) = l_i(x) = \sum_{j=1}^{n} d_{ij} x_j + d_i \tag{3.24}$$

for some constants d_{ij} and d_i.

Therefore, by (3.21), we know that

$$f_i(x) = \frac{\partial F}{\partial x_i}(x) + l_i(x), \qquad 1 \le i \le n. \qquad \text{Q.E.D.}$$

Theorem 3.3: Consider the filtering system (2.1) with conditions $C_1)$, $C_2)$, and $C_3)$. Then the solution $u(t,x)$ for the DMZ equation (2.3), (2.4), or (3.1) is reduced to the solution $\tilde{u}(t,x)$ for the Kolmogorov equation

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t}(t,x) = \dfrac{1}{2}\Delta \tilde{u}(t,x) - \displaystyle\sum_{i=1}^{n} H_i(x)\frac{\partial \tilde{u}}{\partial x_i}(t,x) \\ \qquad - P(x)\tilde{u}(t,x) \\ \tilde{u}(0,x) = e^{G(x)-F(x)}\sigma_0(x) \end{cases}$$

where

$$\tilde{u}(t,x) = \exp\left[c(t) + G(x) + \sum_{i=1}^{n} a_i(t)x_i - F(x+b(t))\right] \cdot u(t, x+b(t))$$

and $a_i(t)$, $b_i(t)$, and $c(t)$ satisfy the following system of ODEs:

$$\begin{cases} b_i'(t) - a_i(t) - \displaystyle\sum_{j=1}^{n} d_{ij} b_j(t) + \sum_{j=1}^{m} c_{ji} y_j(t) = 0 \\ b_i(0) = 0, \end{cases} \quad 1 \le i \le n \tag{3.25}$$

$$\begin{cases} a_i'(t) - \dfrac{1}{2}\displaystyle\sum_{j=1}^{n} (\eta_{ij} + \eta_{ji}) b_j(t) + \sum_{j=1}^{n} d_{ji} b_j'(t) = 0 \\ a_i(0) = 0, \end{cases} \quad 1 \le i \le n \tag{3.26}$$

$$\begin{cases} c'(t) = -\dfrac{1}{2}\displaystyle\sum_{i=1}^{n} (b_i'(t))^2 + \sum_{i=1}^{n} a_i(t) b_i'(t) - \sum_{i=1}^{n} d_i b_i'(t) \\ \qquad + \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{n} \eta_{ij} b_i(t) b_j(t) + \frac{1}{2}\sum_{i=1}^{n} \eta_i b_i(t) \\ c(0) = 0 \end{cases} \tag{3.27}$$

if we can choose $H(x)$, $G(x)$, and $P(x)$ such that

$$\frac{1}{2}\sum_{i=1}^{n} H_i^2(x) - \frac{1}{2}\sum_{i=1}^{n} \frac{\partial H_i}{\partial x_i}(x) - \frac{1}{2}\eta(x) + P(x) \equiv 0 \tag{3.28}$$

where $H_i(x) - (\partial G/\partial x_i)(x) = l_i(x)$.

Proof: Again we use Theorem 3.1 with

$$\Lambda(t,x) = c(t) + G(x) + \sum_{i=1}^{n} a_j(t) x_j - F(x+b(t)).$$

Under the condition of $C_1)$, $C_2)$, and $C_3)$, in view of (3.19) and (3.20), we rewrite (3.4) and (3.5) as follows:

$$b_i'(t) - a_i(t) + l_i(x) + \sum_{j=1}^{m} c_{ji} y_j(t) - l_i(x+b(t)) \equiv 0, \qquad 1 \le i \le n \tag{3.29}$$

and

$$\begin{aligned} c'(t) &- \frac{1}{2}\sum_{i=1}^{n} (b_i'(t))^2 - \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ji} y_j(t) b_i'(t) \\ &- \frac{1}{2}B(t) + \sum_{j=1}^{n} a_j'(t) x_j + \sum_{i=1}^{n} (f_i(x+b(t)) \\ &- \frac{\partial F}{\partial x_i}(x+b(t))) b_i'(t) - \frac{1}{2}\sum_{i=1}^{n} B_i(t) x_i \\ &+ \frac{1}{2}\sum_{i=1}^{n} H_i^2(x) - \frac{1}{2}\sum_{i=1}^{n} \frac{\partial H_i}{\partial x_i}(x) \\ &- \frac{1}{2}\eta(x) + P(x) \equiv 0. \end{aligned} \tag{3.30}$$

Hence, (3.29) gives us (3.25). By (3.28), (3.30) can be further reduced to

$$\begin{aligned} c'(t) &- \frac{1}{2}\sum_{i=1}^{n} (b_i'(t))^2 - \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ji} y_j(t) b_i'(t) - \frac{1}{2}B(t) \\ &+ \sum_{i=1}^{n} a_i'(t) x_i + \sum_{i=1}^{n}\left(\sum_{j=1}^{n} d_{ij}(x_j + b_j(t)) + d_i\right) \\ &\cdot b_i'(t) - \frac{1}{2}\sum_{i=1}^{n} B_i(t) x_i \equiv 0. \end{aligned} \tag{3.31}$$

Therefore

$$\sum_{i=1}^{n}\left(a_i'(t) + \sum_{j=1}^{n} d_{ji} b_j'(t) - \frac{1}{2}B_i(t)\right) x_i \equiv 0 \tag{3.32}$$

and

$$\begin{aligned} c'(t) &- \frac{1}{2}\sum_{i=1}^{n} (b_i'(t))^2 - \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ji} y_j(t) b_i'(t) \\ &- \frac{1}{2}B(t) + \sum_{i=1}^{n} d_i b_i'(t) \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij} b_j(t) b_i'(t) = 0. \end{aligned} \tag{3.33}$$

Equation (3.32) is equivalent to (3.26) and (3.33) is equivalent to (3.27) by combining with (3.25). Q.E.D.

We have several choices of $H(x)$, $G(x)$, and $P(x)$. We list some of them as follows.

Corollary 1: Choose a C^∞ function $G(x)$ such that

$$\Delta G(x) + |\nabla G|^2(x) + 2\sum_{i=1}^{n} l_i(x)\frac{\partial G}{\partial x_i}(x) \equiv \eta(x) - \sum_{i=1}^{n} l_i^2(x) - \sum_{i=1}^{n}\frac{\partial l_i}{\partial x_i}(x).$$

(The existence of G can be shown by a similar method as shown in [8] or [16].) Let

$$H_i(x) = \frac{\partial G}{\partial x_i}(x) + l_i(x), \qquad 1 \le i \le n$$

and

$$P(x) = \sum_{i=1}^{n} \frac{\partial H_i}{\partial x_i}(x) = \sum_{i=1}^{n} \left(\frac{\partial^2 G}{\partial x_i^2}(x) + \frac{\partial l_i}{\partial x_i}(x) \right).$$

Then the corresponding Kolmogorov equation is

$$\begin{cases} \frac{\partial \tilde{u}}{\partial t}(t,x) = \frac{1}{2}\Delta \tilde{u}(t,x) - \sum_{i=1}^{n}\left(l_i(x) + \frac{\partial G}{\partial x_i}(x)\right) \frac{\partial \tilde{u}}{\partial x_i}(t,x) \\ \qquad - \sum_{i=1}^{n} \left(\frac{\partial l_i}{\partial x_i}(x) + \frac{\partial^2 G}{\partial x_i^2}(x) \right) \tilde{u}(t,x) \\ \tilde{u}(0,x) = e^{G(x)-F(x)}\sigma_0(x) \end{cases}$$

where

$$\tilde{u}(t,x) = \exp\left[c(t) + G(x) + \sum_{i=1}^{n} a_i(t)x_i - F(x+b(t)) \right] \cdot u(t, x+b(t))$$

and $a_i(t)$, $b_i(t)$, and $c(t)$ satisfy ODEs (3.25)–(3.27).

Corollary 2: Choose

$$G(x) \equiv 0$$
$$P(x) = \frac{1}{2}\eta(x) - \frac{1}{2}\sum_{i=1}^{n} l_i^2(x) - \frac{1}{2}\sum_{i=1}^{n} \frac{\partial l_i}{\partial x_i}(x)$$

and

$$H_i(x) = l_i(x), \qquad 1 \le i \le n.$$

Then the corresponding Kolmogorov equation becomes

$$\begin{cases} \frac{\partial \tilde{u}}{\partial t}(t,x) = \frac{1}{2}\Delta \tilde{u}(t,x) - \sum_{i=1}^{n} l_i(x) \frac{\partial \tilde{u}}{\partial x_i}(t,x) \\ \qquad + \frac{1}{2}\left(\sum_{i=1}^{n} l_i^2(x) - \sum_{i=1}^{n} \frac{\partial l_i}{\partial x_i}(x) - \eta(x) \right) \tilde{u}(t,x) \\ \tilde{u}(0,x) = e^{-F(x)}\sigma_0(x) \end{cases}$$

where

$$\tilde{u}(t,x) = \exp\left[c(t) + \sum_{i=1}^{n} a_i(t)x_i - F(x+b(t)) \right] \cdot u(t, x+b(t))$$

and $a_i(t)$, $b_i(t)$, and $c(t)$ satisfy ODEs (3.25)–(3.27).

Corollary 3: Choose a C^∞ function $G(x)$ such that $(\partial G/\partial x_i)(x) = -l_i(x)$ if $d_{ij} = d_{ji}$ for $1 \le i, j \le n$. Let $P(x) = (1/2)\eta(x)$ and $H_i(x) \equiv 0$, $1 \le i \le n$. Then the corresponding Kolmogorov equation becomes

$$\begin{cases} \frac{\partial \tilde{u}}{\partial t}(t,x) = \frac{1}{2}\Delta \tilde{u}(t,x) - \frac{1}{2}\eta(x)\tilde{u}(t,x) \\ \tilde{u}(0,x) = e^{G(x)-F(x)}\sigma_0(x) \end{cases}$$

where

$$\tilde{u}(t,x) = \exp\left[c(t) + G(x) + \sum_{i=1}^{n} a_i(t)x_i - F(x+b(t)) \right] \cdot u(t, x+b(t))$$

and $a_i(t)$, $b_i(t)$, and $c(t)$ satisfy ODEs (3.25)–(3.27).

Corollary 4: Choose

$$G(x) = F(x)$$
$$P(x) = \frac{1}{2}\eta(x) - \frac{1}{2}\sum_{i=1}^{n} f_i^2(x) + \frac{1}{2}\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(x)$$

and

$$H_i(x) = f_i(x), \qquad 1 \le i \le n.$$

Then the corresponding Kolmogorov equation becomes

$$\begin{cases} \frac{\partial \tilde{u}}{\partial t}(t,x) = \frac{1}{2}\Delta \tilde{u}(t,x) - \sum_{i=1}^{n} f_i(x) \frac{\partial \tilde{u}}{\partial x_i}(t,x) \\ \qquad + \frac{1}{2}\left(\sum_{i=1}^{n} f_i^2(x) - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(x) - \eta(x) \right) \tilde{u}(t,x) \\ \tilde{u}(0,x) = \sigma_0(x) \end{cases}$$

where

$$\tilde{u}(t,x) = \exp\left[c(t) + \sum_{i=1}^{n} a_i(t)x_i + F(x) - F(x+b(t)) \right] \cdot u(t, x+b(t))$$

and $a_i(t)$, $b_i(t)$, and $c(t)$ satisfy ODEs (3.25)–(3.27).

IV. Explicit Solution of Kolmogorov Equation

Consider the general Kolmogorov equation

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = L(x)u(t,x) \\ u(0,x) = \sigma_0(x) \end{cases} \tag{4.1}$$

where

$$L(x) = \frac{1}{2}\Delta - \sum_{i=1}^{n} H_i(x)\frac{\partial}{\partial x_i} - P(x).$$

Theorem 4.1: Equation (4.1) has a formal asympotic solution on R^n

$$u(t,x) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\sigma_0(\xi)}{\left(\sqrt{2\pi}\right)^n} t^{-(n/2)} \cdot \exp\left[-\frac{1}{2t}\sum_{j=1}^{n}(x_j - \xi_j)^2 \right] b(t,x,\xi)\, d\xi_1 \cdots d\xi_n \tag{4.2}$$

where $b(t,x,\xi) = \sum_{k=0}^{\infty} a_k(x,\xi)t^k$. Here $a_k(x,\xi)$ are described explicitly as follows. Let

$$a(x,\xi) = \int_0^1 \sum_{i=1}^{n} (x_i - \xi_i) H_i[\xi + \tau(x-\xi)]\, d\tau. \tag{4.3}$$

Then

$$a_0(x,\xi) = e^{a(x,\xi)} \tag{4.4}$$

and for $k \ge 1$

$$a_k(x,\xi) = a_0(x,\xi) \int_0^1 \tau^{k-1} e^{-a(\xi+\tau(x-\xi),\xi)} \cdot g_k(\xi + \tau(x-\xi), \xi)\, d\tau \tag{4.5}$$

where $g_k(x,\xi) = L(x)a_{k-1}(x,\xi)$.

Proof: The proof is similar to the proof given in [24]. Q.E.D.

V. Conclusions

We have proposed a new direct strategy for the explicit solution of DMZ equation with arbitrary initial condition. The main result is that: the solution of DMZ equation can be reduced to a solution of Kolmogorov equation (off-line computation), which can be solved explicitly, and an on-line computation which involves a series of ordinary differential equations (n sufficient statistics by eliminating the nonindependent statistics). This approach offers several advantages. It is easy, and the derivation no longer needs controllability and observability.

Thus, the algorithm is universal for any Yau filtering system. Furthermore, they eliminate the necessity of integrating n first-order linear partial differential equations, as was the case in Lie algebra method. Moreover, they do not need to solve an algebraic Riccati equation. Finally, the number of sufficient statistics required to compute conditional probability density of the state in our present method is n; this is the same as in classical Kalman–Bucy filtering, where initial Gaussian distribution is assumed.

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Structure Properties of Min–Max Systems and Existence of Global Cycle Time

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***Abstract*—This paper studies min–max systems which are dynamic systems including three operations (min, max, +) with unknown or stochastic parameters. Some sufficient conditions will be given for the existence of global cycle times. Our results are based on structure properties of min–max systems.**

***Index Terms*—Cycle time, discrete-event systems, synchronization.**

I. INTRODUCTION

Systems in which the operations min, max, and plus appear simultaneously are known as min–max systems, or min–max-plus systems. Such systems are natural extensions of timed discrete-event systems described by max-plus (or min-plus) algebra (e.g., [6] and [1]). It is well known that systems with only maximum timing constraints (or only minimum timing constraints) can be studied by linear methods based on max-algebra. But systems with both maximum and minimum constraints are nonlinear in this sense and only limited results were obtained. We study in this paper the existence conditions of a global cycle time. Global cycle time of a min–max system here refers to the uniform asymptotic growing rate of the system state vector. That is to say, a global cycle time is the real λ satisfying $\lim_{k\to\infty} \mathbf{x}(k)/k = (\lambda, \ldots, \lambda)'$, where $\mathbf{x}(k)$ is the state vector of the system. Since the limit $\lim_{k\to\infty} \mathbf{x}(k)/k$ is usually called the cycle time vector, we can say that the global cycle time exists and is equal to the components of the cycle time vector when it has identical components. For a deterministic system, if a global cycle time exists, it is also the unique (additive, nonlinear) eigenvalue [8]. The associated eigenvectors are also called (generalized) fixed points [10]. For a stochastic system, the global cycle time is known as the Lyapunov exponent [1], [3]. In the early work of [1] and [13], conditions on the existence of eigenvalues of a special class of min–max systems were obtained. Stochastic extensions were made in [12]. Recently some breakthroughs on the study of min–max systems were made. A necessary and sufficient condition on existence of fixed points was obtained in [5]. If a fixed point is known, the global cycle time can be directly determined by evaluation of the

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