Finite-Dimensional Filters with Nonlinear Drift

XV: New Direct Method for Construction of Universal Finite-Dimensional Filter

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We consider the explicit solution of Duncan-Mortensen-Zakai (DMZ) equation for the finite-dimensional filtering system. We show that under certain conditions the nonlinear filtering system can be solved explicitly with an arbitrary initial condition by solving a system of ordinary differential equations and a Kolmogorov-type equation. Let n be the dimension of state space. We show that we need only n sufficient statistics in order to solve the DMZ equation.

Note to the Reader: Sections III & V of this series on Finite-Dimensional Filters appeared in *IEEE Transactions on Aerospace and Electronic Systems*, **33**, 4 (Oct. 1997), 1277–1308, together with information on locating Parts I, II, & IV. Sections VI through XIV can be found in references [20]–[28] in this paper.

— *The Editors*

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I. INTRODUCTION

In 1961, Kalman and Bucy [7] published a historically important mathematics paper on filtering that was highly influential to the modern industry. Since then, the Kalman–Bucy filtering has proved useful in many areas such as navigational and guidance systems, radar tracking, solar mapping, and satellite orbit determination. Despite its usefulness, however, the Kalman–Bucy filter is not perfect. One of its weaknesses is that it needs Gaussian assumption for the initial data. Another weakness is that it is restricted only to linear dynamical system. In the 1960s, Duncan [6], Mortensen [10], and Zakai [19] independently derived the so-called DMZ equation for the nonlinear filtering problem. Unfortunately, since the DMZ equation is a stochastic differential equation, there is no easy way to derive a recursive algorithm for solving this equation. The idea of using estimation algebras to construct finite-dimensional nonlinear filters was first proposed by Brockett and Clark [2], Brockett [1], and Mitter [9]. The motivation came from the Wei–Norman approach [13] of using Lie algebraic ideas to solve a linear time-varying differential equation. The advantage of this approach is that as long as the estimation algebra is finite dimensional, we will get a finite-dimensional recursive filter; there is no need to make any assumption in the initial data. Moreover, the approach applies well to nonlinear dynamical systems. This approach has been worked out in detail in [11] and especially in the so-called Yau filtering system described in [3]. In [14], it was shown that only n sufficient statistics are needed to solve the DMZ equation explicitly. Therefore, the procedure in [14] is much less complex than those given in [8]. For a linear filtering system, it is quite easy to see that the corresponding estimation algebra is finite dimensional. Therefore we can apply the Wei–Norman approach to construct a finite-dimensional recursive filter with arbitrary initial data. However, in the Wei–Norman approach, one has to know explicitly the basis as vector space of the estimation algebra in order to reduce the DMZ equation to a finite system of ordinary differential equations, Kolmogorov equation, and several first-order linear partial differential equations. Classically, one knows the explicit basis for the estimation algebra only in the case that it has maximal rank. Typically people assume that the linear system is controllable and observable.

Recently, a new direct method was introduced to study the Kalman–Bucy filtering and Beněs filtering systems with arbitrary initial condition for which f, g, and h in (1) below are independent of time (cf. [18], [15–17]) and g is an orthogonal matrix. We apply the new direct method to the nonlinear filterng system with a much weaker assumption on matrix g in (1) below. More precisely, we assume that $g(x)g^T(x)$ is an invertible matrix and each (i,j)-entry of $g(x)g^T(x)$ is a constant function. Under the conditions (C1—C3) which include both Kalman–Bucy filters and Beneš filters as special cases, we have constructed most general finite-dimensional nonlinear filters with arbitrary initial distribution. This approach offers several advantages. The derivation no longer needs controllability and observability. Thus, the algorithm is universal for such nonlinear filtering systems. Furthermore, we eliminate the necessity of integrating n first-order linear partial differential equations, as was the case in Lie algebra method. Moreover, we do not need to solve Riccati differential equations. Finally, the number of sufficient statistics required to compute conditional probability density of the state in our present method is n, which is the best possible case that one can hope for. We expect that our Theorem 4 and Theorem 5 will make a fundamental impact on industry.

II. BASIC FILTERING PROBLEM

The filtering problem considered here is based on the following signal observation model:

$$\begin{cases} dx(t) = f(x(t))dt + g(x(t))dv(t) & x(0) = x_0 \\ dy(t) = h(x(t))dt + dw(t) & y(0) = 0 \end{cases} \tag{1}$$

in which x, v, y, and w are, respectively, R^n, R^p, R^m, and R^m valued processes and v and w have components that are independent, standard Brownian processes. We further assume that $n = p$; f, g and h are vector-valued, matrix-valued and vector-valued C^∞ smooth functions. We refer to $x(t)$ as the state of this system at time t and to $y(t)$ as the observation at time t.

Let $\rho(t,x)$ denote the conditional probability density of the state given the observation $\{y(s) : 0 \leq s \leq t\}$. It is well known (see [5], for example) that $\rho(t,x)$ is given by normalizing a function $\sigma(t,x)$ that satisfies the following DMZ equation:

$$\begin{cases} d\sigma(t,x) = L_0\sigma(t,x)dt + \sum_{i=1}^{m} L_i\sigma(t,x)\,dy_i(t) \\ \sigma(0,x) = \sigma_0(x) \end{cases} \tag{2}$$

where

$$L_0 = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}[g(x)g^T(x)]_{ij} - \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2$$

for $i = 1,\ldots,m, L_i$ is the zero-degree differential operator of multiplication by h_i, and σ_0 is the probability density of the initial point x_0.

We assume here that $[g(x)g^T(x)]_{ij} = G_{ij}$ are constants for all $1 \leq i,j \leq n$ instead of the traditional orthogonal matrix assumption ([14–18]). (Note that $G_{ij} = G_{ji}$.) Then

$$L_0 = \frac{1}{2}\sum_{i,j=1}^{n} G_{ij}\frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2.$$

Equation (2) is a stochastic partial differential equation in the sense of Stratonovich. In real applications, we are interested in constructing robust state estimators from observed sample paths with some property of robustness. Davis in [4] studied this problem and proposed some robust algorithms. In our case, his basic idea reduces to define a new unnormalized density

$$u(t,x) = \exp\left(-\sum_{i=1}^{m} h_i(x)y_i(t)\right)\sigma(t,x).$$

Then we can reduce (2) to the following time varying partial differential equation

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = L_0 u(t,x) + \sum_{i=1}^{m} y_i(t)[L_0,L_i]u(t,x) \\ \qquad + \frac{1}{2}\sum_{i,j=1}^{m} y_i(t)y_j(t)[[L_0,L_i],L_j]u(t,x) \\ u(0,x) = \sigma_0(x) \end{cases} \tag{3}$$

where $[\cdot,\cdot]$ is the Lie bracket defined as follows.

DEFINITION If X and Y are differential operators, the Lie brackett of X and Y, $[X,Y]$ is defined by

$$[X,Y]\xi = X(Y\xi) - Y(X\xi)$$

for any C^∞ function ξ.

LEMMA 1 *Equation* (3) *is equivalent to the following equation:*

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\sum_{i,j=1}^{n} G_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}(t,x) \\ \qquad + \sum_{i=1}^{n}\left\{\sum_{k=1}^{m} y_k(t)\sum_{j=1}^{n} G_{ij}\frac{\partial h_k}{\partial x_j}(x) - f_i(x)\right\} \\ \qquad \times \frac{\partial u}{\partial x_i}(t,x) \\ \qquad + \left\{\frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{m} G_{ij}y_k(t)\frac{\partial^2 h_k}{\partial x_i \partial x_j}(x)\right. \\ \qquad + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{k,l=1}^{m} G_{ij}y_k(t)y_l(t)\frac{\partial h_k}{\partial x_i}(x)\frac{\partial h_l}{\partial x_j}(x) \\ \qquad - \sum_{i=1}^{n}\sum_{k=1}^{m} y_k(t)f_i(x)\frac{\partial h_k}{\partial x_i}(x) \\ \qquad \left. - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) - \frac{1}{2}\sum_{k=1}^{m} h_k^2(x)\right\}u(t,x) \\ u(0,x) = \sigma_0(x) \end{cases} \tag{4}$$

PROOF

$$[L_0,h_i] = \left[\frac{1}{2}\sum_{k,l=1}^{n} G_{kl}\frac{\partial^2}{\partial x_k \partial x_l} - \sum_{k=1}^{n} f_k \frac{\partial}{\partial x_k} - \sum_{k=1}^{n}\frac{\partial f_k}{\partial x_k} - \frac{1}{2}\sum_{k=1}^{m} h_k^2, h_i\right]$$

$$= \frac{1}{2}\sum_{k,l=1}^{n} G_{kl}\left[\frac{\partial^2}{\partial x_k\partial x_l}, h_i\right] - \left[\sum_{k=1}^{n} f_k\frac{\partial}{\partial x_k}, h_i\right]$$

$$= \frac{1}{2}\sum_{k,l=1}^{n} G_{kl}\left(\frac{\partial h_i}{\partial x_k}\frac{\partial}{\partial x_l} + \frac{\partial h_i}{\partial x_l}\frac{\partial}{\partial x_k} + \frac{\partial^2 h_i}{\partial x_k \partial x_l}\right) - \sum_{k=1}^{n} f_k \frac{\partial h_i}{\partial x_k}$$

$$= \frac{1}{2}\sum_{k,l=1}^{n} G_{kl}\left(\frac{\partial h_i}{\partial x_k}\frac{\partial}{\partial x_l} + \frac{\partial h_i}{\partial x_l}\frac{\partial}{\partial x_k}\right) + \frac{1}{2}\sum_{k,l=1}^{n} G_{kl}\frac{\partial^2 h_i}{\partial x_k\partial x_l} - \sum_{k=1}^{n} f_k\frac{\partial h_i}{\partial x_k}$$

$$= \sum_{k,l=1}^{n} G_{kl}\frac{\partial h_i}{\partial x_k}\frac{\partial}{\partial x_l} + \frac{1}{2}\sum_{k,l=1}^{n} G_{kl}\frac{\partial^2 h_i}{\partial x_k \partial x_l} - \sum_{k=1}^{n} f_k \frac{\partial h_i}{\partial x_k}$$

$$[[L_0,h_i],h_j] = \sum_{k,l=1}^{n} G_{kl}\frac{\partial h_i}{\partial x_k}\frac{\partial h_j}{\partial x_l}L_0 + \sum_{i=1}^{m} y_i(t)[L_0,L_i] + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} y_i(t)y_j(t)[[L_0,L_i],L_j]$$

$$= \frac{1}{2}\sum_{i,j=1}^{n} G_{ij}\frac{\partial^2}{\partial x_i\partial x_j} - \sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i} - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2 + \sum_{i=1}^{m}\sum_{k,l=1}^{n} G_{kl}y_i(t)\frac{\partial h_i}{\partial x_k}\frac{\partial}{\partial x_l} + \frac{1}{2}\sum_{i=1}^{m}\sum_{k,l=1}^{n} G_{kl}y_i(t)\frac{\partial^2 h_i}{\partial x_k\partial x_l} - \sum_{i=1}^{m}\sum_{k=1}^{n} y_i(t) f_k\frac{\partial h_i}{\partial x_k} + \frac{1}{2}\sum_{i,j=1}^{m}\sum_{k,l=1}^{n} y_i(t)y_j(t)G_{kl}\frac{\partial h_i}{\partial x_k}\frac{\partial h_j}{\partial x_l}$$

$$= \frac{1}{2}\sum_{i,j=1}^{n} G_{ij}\frac{\partial^2}{\partial x_i\partial x_j} + \sum_{l=1}^{m}\sum_{k,i=1}^{n} G_{ki}y_l(t)\frac{\partial h_l}{\partial x_k}\frac{\partial}{\partial x_i} - \sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2 - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{m}\sum_{k,l=1}^{n} G_{kl}y_i(t)\frac{\partial^2 h_i}{\partial x_k\partial x_l} - \sum_{i=1}^{m}\sum_{k=1}^{n} y_i(t) f_k\frac{\partial h_i}{\partial x_k} + \frac{1}{2}\sum_{i,j=1}^{m}\sum_{k,l=1}^{n} y_i(t)y_j(t)G_{kl}\frac{\partial h_i}{\partial x_k}\frac{\partial h_j}{\partial x_l}$$

$$= \frac{1}{2}\sum_{i,j=1}^{n} G_{ij}\frac{\partial^2}{\partial x_i\partial x_j} + \sum_{i=1}^{n}\left[\sum_{k=1}^{m} y_k(t)\sum_{j=1}^{n} G_{ij}\frac{\partial h_k}{\partial x_j} - f_i(x)\right]\frac{\partial}{\partial x_i} + \left[\frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{m} G_{ij}y_k(t)\frac{\partial^2 h_k}{\partial x_i\partial x_j} + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{k,l=1}^{m} G_{ij}y_k(t)y_l(t)\frac{\partial h_k}{\partial x_i}\frac{\partial h_l}{\partial x_j} - \sum_{i=1}^{n}\sum_{k=1}^{m} y_k(t) f_i(x)\frac{\partial h_k}{\partial x_i} - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{k=1}^{m} h_k^2(x)\right]. \qquad \text{Q.E.D.}$$

III. EXPLICIT SOLUTION OF DMZ EQUATION

Define

$$\eta(x) = \sum_{i,j=1}^{n} G^{ij} f_i(x) f_j(x) + \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) + \sum_{i=1}^{m} h_i^2(x) \tag{5}$$

where $G^{-1} = [G^{ij}], GG^{-1} = I$, i.e., $\sum_{k=1}^{n} G_{ik}G^{kj} = \delta_{ij}$. Here $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$.

THEOREM 1 *Equation* (4) *is equivalent to the following equation:*

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{n} G_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}(t,x) + \displaystyle\sum_{i=1}^{n}\theta_i(t,x)\frac{\partial u}{\partial x_i}(t,x) + \theta(t,x)u(t,x) \\ u(0,x) = \sigma_0(x) \end{cases} \tag{6}$$

where

$$\theta_i(t,x) = \sum_{k=1}^{m} y_k(t)\sum_{j=1}^{n} G_{ij}\frac{\partial h_k}{\partial x_j}(x) - f_i(x)$$

$$\overline{\theta}_i(t,x) = \sum_{k=1}^{m} y_k(t)\frac{\partial h_k}{\partial x_i}(x) - \sum_{j=1}^{n} G^{ij} f_j(x)$$

and

$$\theta(t,x) = \frac{1}{2}\left(\sum_{i=1}^{n}\theta_i(t,x)\overline{\theta}_i(t,x) + \sum_{i=1}^{n}\frac{\partial\theta_i}{\partial x_i}(t,x) - \eta(x)\right).$$

PROOF We need only to show that $\theta(t,x)$ equals the coefficient of $u(t,x)$ term in the equation of (4). We leave it as an easy exercise to the readers. Q.E.D.

THEOREM 3.2 *Suppose $u(t,x)$ is a solution of* (6) *and*

$$\tilde{u}(t,x) = e^{\Lambda(t,x)}u(t,x+b(t)). \tag{7}$$

Then $\tilde{u}(t,x)$ is the solution of the following Kolmogorov equation

$$\begin{cases} \dfrac{\partial\tilde{u}}{\partial t}(t,x) = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{n} G_{ij}\frac{\partial^2\tilde{u}}{\partial x_i\partial x_j}(t,x) \\ \qquad\qquad - \displaystyle\sum_{i=1}^{n} H_i(x)\frac{\partial\tilde{u}}{\partial x_i}(t,x) - P(x)\tilde{u}(t,x) \\ \tilde{u}(0,x) = e^{\Lambda(0,x)}u(0,x+b(0)) \end{cases} \tag{8}$$

if and only if we can choose $H_i(x)$ and $P(x)$ such that $\Lambda(t,x)$ satisfy the following systems:

$$b_i'(t) \equiv \sum_{j=1}^{n} G_{ij}\frac{\partial\Lambda}{\partial x_j}(t,x) - H_i(x) - \theta_i(t,x+b(t)) \tag{9}$$

for $1 \le i \le n$ and

$$\begin{aligned} \frac{\partial\Lambda}{\partial t}(t,x) &= \frac{1}{2}\sum_{i,j=1}^{n} G^{ij}b_i'(t)b_j'(t) \\ &\quad + \sum_{i=1}^{n}\overline{\theta}_i(t,x+b(t))b_i'(t) + \frac{1}{2}\eta(x+b(t)) \\ &\quad - \frac{1}{2}\sum_{i,j=1}^{n} G^{ij}H_i(x)H_j(x) + \frac{1}{2}\sum_{i=1}^{n}\frac{\partial H_i}{\partial x_i}(x) - P(x). \end{aligned} \tag{10}$$

PROOF We need only to write down $\partial\tilde{u}/\partial t$, $\partial\tilde{u}/\partial x_i$, $\partial^2\tilde{u}/\partial x_i\partial x_j$ and put them into (8). By observing the coefficient $\partial u/\partial x_i(t,x+b(t))$ and $u(t,x+b(t))$ and using the facts that $\theta_i(t,x) = \sum_{j=1}^{n} G_{ij}\overline{\theta}_j(t,x)$ and $\overline{\theta}_i(t,x) = \sum_{j=1}^{n} G^{ij}\theta_j(t,x)$, we get (9) and (10). Q.E.D.

The following conditions (C1–C3) are assumed in the sequel.

(C1) $h_i(x) = \sum_{j=1}^{n} c_{ij}x_j + c_i$, $1 \le i \le m$, where c_{ij} and c_i are constants.

(C2) $f_i(x) = \sum_{j=1}^{n} G_{ij}(\partial F/\partial x_j)(x) + l_i(x)$, $1 \le i \le n$, where $l_i(x) = \sum_{j=1}^{n} d_{ij}x_j + d_i$, d_{ij} and d_i are constants.

(C3) $\eta(x) = \sum_{i,j=1}^{n}\eta_{ij}x_ix_j + \sum_{i=1}^{n}\eta_ix_i + \eta_0$, where η_{ij}, η_i and η_0 are constants.

We choose $\Lambda(t,x) = c(t) + S(x) + \sum_{i=1}^{n} a_i(t)x_i - F(x+b(t))$ and simplify the systems (9) and (10).

THEOREM 3 (Main Theorem) *Consider the filtering system* (1) *with conditions* ($C1$–$C3$). *Then the solution $u(t,x)$ for the DMZ equation* (4) *or* (6) *is reduced to the solution $\tilde{u}(t,x)$ for the Kolmogorov equation*

$$\begin{cases} \dfrac{\partial\tilde{u}}{\partial t}(t,x) = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{n} G_{ij}\frac{\partial^2\tilde{u}}{\partial x_i\partial x_j}(t,x) \\ \qquad\qquad - \displaystyle\sum_{i=1}^{n} H_i(x)\frac{\partial\tilde{u}}{\partial x_i}(t,x) - P(x)\tilde{u}(t,x) \\ \tilde{u}(0,x) = e^{\Lambda(0,x)}u(0,x+b(0)) \end{cases} \tag{11}$$

where

$$\tilde{u}(t,x) = e^{c(t)+S(x)+\sum_{i=1}^{n} a_i(t)x_i - F(x+b(t))}u(t,x+b(t))$$

and $a_i(t), b_i(t)$, and $c(t)$ satisfy the following system of ODEs. For $1 \le i \le n$,

$$\begin{cases} b_i'(t) = \displaystyle\sum_{j=1}^{n} G_{ij}a_j(t) + \sum_{j=1}^{n} d_{ij}b_j(t) \\ \qquad\quad - \displaystyle\sum_{k=1}^{m}\sum_{j=1}^{n} G_{ij}c_{kj}y_k(t) \\ b_i(0) = 0 \end{cases} \tag{12}$$

$$\begin{cases} a_i'(t) = \dfrac{1}{2}\displaystyle\sum_{j=1}^{n}(\eta_{ij} + \eta_{ji})b_j(t) \\ \qquad\quad - \displaystyle\sum_{j=1}^{n}\sum_{l=1}^{n} G^{lj}d_{ji}b_l'(t) \\ a_i(0) = 0 \end{cases} \tag{13}$$

$$\begin{cases} c'(t) = -\dfrac{1}{2}\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} G^{ij}b_i'(t)b_j'(t) \\ \qquad + \displaystyle\sum_{i=1}^{n} a_i(t)b_i'(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} G^{ij}d_jb_i'(t) \\ \qquad + \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{n}\eta_{ij}b_i(t)b_j(t) + \frac{1}{2}\sum_{i=1}^{n}\eta_ib_i(t) \\ c(0) = 0 \end{cases} \tag{14}$$

if we can choose $H(x)$, $S(x)$ and $P(x)$ such that

$$\frac{1}{2}\sum_{i,j=1}^{n} G^{ij}H_i(x)H_j(x) - \frac{1}{2}\sum_{i=1}^{n}\frac{\partial H_i}{\partial x_i}(x) - \frac{1}{2}\eta(x) + P(x) \equiv 0 \tag{15}$$

where

$$H_i(x) = \sum_{j=1}^{n} G_{ij}\frac{\partial S}{\partial x_j}(x) + l_i(x).$$

PROOF We use Theorem 2. First by (C1) and the fact that $\Lambda(t,x)$ is in the form of $c(t)+S(x)+\sum_{i=1}^{n} a_i(t)x_i - F(x+b(t))$, we observe that:

$$\theta_i(t,x+b(t)) = \sum_{k=1}^{m} y_k(t)\sum_{j=1}^{n} G_{ij}\frac{\partial h_k}{\partial x_j}(x+b(t)) - f_i(x+b(t))$$

$$= \sum_{k=1}^{m} y_k(t)\sum_{j=1}^{n} G_{ij}c_{kj} - f_i(x+b(t)) \qquad (16)$$

$$\frac{\partial \Lambda}{\partial t}(t,x) = c'(t) + \sum_{i=1}^{n} a_i'(t)x_i - \sum_{j=1}^{n}\frac{\partial F}{\partial x_j}(x+b(t))b_j'(t) \qquad (17)$$

$$\frac{\partial \Lambda}{\partial x_j}(t,x) = \frac{\partial S}{\partial x_j}(x) + a_j(t) - \frac{\partial F}{\partial x_j}(x+b(t)). \qquad (18)$$

By (C2) and (C3), we have

$$f_i(x+b(t)) = \sum_{j=1}^{n} G_{ij}\frac{\partial F}{\partial x_j}(x+b(t)) + l_i(x) + \sum_{j=1}^{n} d_{ij}b_j(t) \qquad (19)$$

and

$$\eta(x+b(t)) = \eta(x) + \sum_{i=1}^{n} B_i(t)x_i + B(t) \qquad (20)$$

where

$$B_i(t) = \sum_{j=1}^{n}(\eta_{ij}+\eta_{ji})b_j(t)$$

and

$$B(t) = \sum_{i,j=1}^{n}\eta_{ij}b_i(t)b_j(t) + \sum_{i=1}^{n}\eta_i b_i(t).$$

Putting (16) and (18) into (9), we have

$$b_i'(t) = \sum_{j=1}^{n} G_{ij}a_j(t) - \sum_{k=1}^{m}\sum_{j=1}^{n} G_{ij}c_{kj}y_k(t) - \sum_{j=1}^{n} G_{ij}\frac{\partial F}{\partial x_j}(x+b(t)) + f_i(x+b(t)) - H_i(x) + \sum_{j=1}^{n} G_{ij}\frac{\partial S}{\partial x_j}(x). \qquad (21)$$

If we choose

$$H_i(x) = \sum_{j=1}^{n} G_{ij}\frac{\partial S}{\partial x_j}(x) + l_i(x), \qquad 1 \le i \le n$$

and combine (19), then (21) gives us (12). Putting (17), (19), and (20) into (10), we have

$$c'(t) = \sum_{i=1}^{n}\frac{\partial F}{\partial x_i}(x+b(t))b_i'(t) - \sum_{j=1}^{n} a_j'(t)x_j + \frac{1}{2}\sum_{i=1}^{n} B_i(t)x_i + \frac{1}{2}B(t) + \frac{1}{2}\eta(x) - \frac{1}{2}\sum_{i,j=1}^{n} G^{ij}H_i(x)H_j(x) + \frac{1}{2}\sum_{i=1}^{n}\frac{\partial H_i}{\partial x_i}(x) - P(x) + \frac{1}{2}\sum_{i,j=1}^{n} G^{ij}b_i'(t)b_j'(t) + \sum_{i=1}^{n}\sum_{k=1}^{m} y_k(t)c_{ki}b_i'(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} G^{ij}b_i'(t)f_j(x+b(t)). \qquad (22)$$

If we choose

$$\frac{1}{2}\sum_{i,j=1}^{n} G^{ij}H_i(x)H_j(x) - \frac{1}{2}\sum_{i=1}^{n}\frac{\partial H_i}{\partial x_i}(x) - \tfrac{1}{2}\eta(x) + P(x) \equiv 0$$

then (22) reduces to

$$c'(t) = \frac{1}{2}\sum_{i,j=1}^{n} G^{ij}b_i'(t)b_j'(t) + \sum_{i=1}^{n}\sum_{k=1}^{m} c_{ki}b_i'(t)y_k(t) - \sum_{i=1}^{n} a_i'(t)x_i + \frac{1}{2}\sum_{i=1}^{n} B_i(t)x_i + \frac{1}{2}B(t) - \sum_{i=1}^{n}\left[\sum_{j=1}^{n} G^{ij}f_j(x+b(t)) - \frac{\partial F}{\partial x_i}(x+b(t))\right]b_i'(t). \qquad (23)$$

Since,

$$f_i(x) - l_i(x) = \sum_{j=1}^{n} G_{ij}\frac{\partial F}{\partial x_j}(x)$$

which implies that

$$\frac{\partial F}{\partial x_i}(x) = \sum_{j=1}^{n} G^{ij}(f_j(x) - l_j(x)).$$

Therefore,

$$\frac{\partial F}{\partial x_i}(x+b(t)) = \sum_{j=1}^{n} G^{ij}f_j(x+b(t)) - \sum_{j=1}^{n} G^{ij}l_j(x+b(t))$$

where

$$\sum_{j=1}^{n} G^{ij}l_j(x+b(t)) = \sum_{j=1}^{n} G^{ij}\left(\sum_{l=1}^{n} d_{jl}x_l + d_j + \sum_{l=1}^{n} d_{jl}b_l(t)\right).$$

Finally, (23) reduces to

$$a_i'(t) = \tfrac{1}{2}B_i(t) - \sum_{j=1}^{n}\sum_{l=1}^{n} G^{lj}d_{ji}b_l'(t) \qquad (24)$$

and

$$c'(t) = \frac{1}{2}\sum_{i,j=1}^{n} G^{ij} b_i'(t) b_j'(t) + \sum_{i=1}^{n}\sum_{k=1}^{m} c_{ki} b_i'(t) y_k(t)$$
$$- \sum_{i=1}^{n}\sum_{j=1}^{n} G^{ij} \sum_{l=1}^{n} d_{jl} b_l(t) b_i'(t)$$
$$+ \tfrac{1}{2}B(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} G^{ij} d_j b_i'(t). \qquad (25)$$

Equation (25) is equivalent to (14) by the following computation

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n} G^{ij} d_{jl} b_l(t) b_i'(t)$$
$$= \sum_{l=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n} G^{li} d_{ij} b_j(t) b_l'(t)$$
$$= \sum_{l=1}^{n}\sum_{i=1}^{n} G^{li} b_l'(t) \left(\sum_{j=1}^{n} d_{ij} b_j(t)\right)$$
$$= \sum_{i=1}^{n}\sum_{l=1}^{n} G^{li} b_i'(t) b_l'(t)$$
$$- \sum_{i=1}^{n}\sum_{l=1}^{n}\sum_{j=1}^{n} G^{li} G_{ij} b_l'(t) a_j(t)$$
$$+ \sum_{i=1}^{n}\sum_{k=1}^{m}\sum_{j=1}^{n} G^{li} G_{ij} c_{kj} b_l'(t) y_k(t)$$
$$= \sum_{i=1}^{n}\sum_{l=1}^{n} G^{li} b_i'(t) b_l'(t)$$
$$- \sum_{j=1}^{n} a_j(t) b_j'(t) + \sum_{k=1}^{m}\sum_{j=1}^{n} c_{kj} b_j'(t) y_k(t).$$

Q.E.D.

We have several choices of $H(x)$, $S(x)$ and $P(x)$. We list some of them as follows. The following two theorems are extremely important for real applications.

THEOREM 4 *Consider the filtering system* (1) *with conditions* (C1–C3). *Then the solution $u(t,x)$ for the DMZ equation* (4) *or* (6) *is reduced to the solution $\tilde{u}(t,x)$ for the Kolmogorov equation*

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t}(t,x) = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{n} G_{ij} \frac{\partial^2 \tilde{u}}{\partial x_i \partial x_j}(t,x) - \sum_{i=1}^{n} l_i(x) \frac{\partial \tilde{u}}{\partial x_i}(t,x) \\ \qquad + \dfrac{1}{2}\left(\displaystyle\sum_{i,j=1}^{n} G^{ij} l_i(x) l_j(x) - \sum_{i=1}^{n} \frac{\partial l_i}{\partial x_i}(x) - \eta(x)\right)\tilde{u}(t,x) \\ \tilde{u}(0,x) = e^{-F(x)}\sigma_0(x) \end{cases}$$

where

$$\tilde{u}(t,x) = e^{c(t)+\sum_{i=1}^{n} a_i(t)x_i - F(x+b(t))} u(t,x+b(t))$$

and $a_i(t)$, $b_i(t)$ and $c(t)$ satisfy ODEs (12)–(14).

PROOF Choose $S(x) \equiv 0$, $P(x) = \frac{1}{2}\eta(x) - \frac{1}{2}\sum_{i,j=1}^{n} G^{ij} l_i(x) l_j(x) + \frac{1}{2}\sum_{i=1}^{n} (\partial l_i/\partial x_i)(x)$ and $H_i(x) = l_i(x)$, $1 \le i \le n$. Then one can show that (15) is satisfied. The result follows from Theorem 3. Q.E.D.

THEOREM 5 *Consider the filtering system* (1) *with conditions* (C1–C3). *Then the solution $u(t,x)$ for the DMZ equation* (4) *or* (6) *is reduced to the solution $\tilde{u}(t,x)$ for the Kolmogorov equation*

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t}(t,x) = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{n} G_{ij} \frac{\partial^2 \tilde{u}}{\partial x_i \partial x_j}(t,x) - \sum_{i=1}^{n} f_i(x) \frac{\partial \tilde{u}}{\partial x_i}(t,x) \\ \qquad + \dfrac{1}{2}\left(\displaystyle\sum_{i,j=1}^{n} G^{ij} f_i(x) f_j(x) - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(x) - \eta(x)\right)\tilde{u}(t,x) \\ \tilde{u}(0,x) = \sigma_0(x) \end{cases}$$

where

$$\tilde{u}(t,x) = e^{c(t)+\sum_{i=1}^{n} a_i(t)x_i + F(x) - F(x+b(t))} u(t,x+b(t))$$

and $a_i(t)$, $b_i(t)$ and $c(t)$ satisfy ODEs (12)–(14).

PROOF Choose $S(x) = F(x)$, $P(x) = \frac{1}{2}\eta(x) - \frac{1}{2}\sum_{i,j=1}^{n} G^{ij} f_i(x) f_j(x) + \frac{1}{2}\sum_{i=1}^{n} (\partial f_i/\partial x_i)(x)$ and $H_i(x) = f_i(x)$, $1 \le i \le n$. Then one can show that (15) is satisfied. The result follows from Theorem 3. Q.E.D.

THEOREM 6 *Consider the filtering system* (1) *with conditions* (C1–C3). *Suppose further that $\sum_{k=1}^{n} G^{ik} d_{kj} = \sum_{k=1}^{n} G^{jk} d_{ki}$ for $1 \le i,j \le n$. Then the solution $u(t,x)$ for the DMZ equation* (4) *or* (6) *is reduced to the solution $\tilde{u}(t,x)$ for the Kolmogorov equation*

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t}(t,x) = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{n} G_{ij} \frac{\partial^2 \tilde{u}}{\partial x_i \partial x_j}(t,x) - \frac{1}{2}\eta(x)\tilde{u}(t,x) \\ \tilde{u}(0,x) = e^{S(x)-F(x)}\sigma_0(x) \end{cases}$$

where

$$\tilde{u}(t,x) = e^{c(t)+S(x)+\sum_{i=1}^{n} a_i(t)x_i - F(x+b(t))} u(t,x+b(t))$$

and $a_i(t)$, $b_i(t)$ and $c(t)$ satisfy ODEs (12)–(14).

PROOF Choose a C^∞ function $S(x)$ such that $\sum_{j=1}^{n} G_{ij}(\partial S/\partial x_j)(x) = -l_i(x)$. Such $S(x)$ exists because of the Poincare lemma and our assumption $\sum_{k=1}^{n} G^{ik} d_{kj} = \sum_{k=1}^{n} G^{jk} d_{ki}$ for $1 \le i,j \le n$. Let $P(x) = \frac{1}{2}\eta(x)$ and $H_i(x) \equiv 0$, $1 \le i \le n$. Then one can show that (15) is satisfied. The result follows from Theorem 3. Q.E.D.

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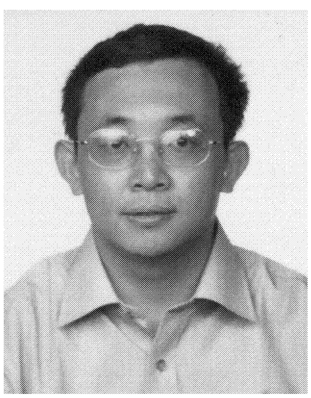

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