Explicit Solution of DMZ Equation in Nonlinear Filtering via Solution of ODEs

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***Abstract*—In this note, we develop a real-time and accurate solution for nonlinear filtering problems based on the Gaussian distribution. Specifically, we present an explicit solution of the Duncan–Mortensen–Zakai equation of the Yau filtering system, which includes the linear filtering system and the exact filtering system. The solution is given in terms of a solution of a system of ordinary differential equations. In particular, our method can be implemented in hardware. The complexity of our algorithms is the same as those of Kalman–Bucy filters in the case of linear filtering systems.**

***Index Terms*—Duncan–Mortensen–Zakai (DMZ) equation, Gaussian distribution, nonlinear filter.**

I. Introduction

The nonlinear filtering problem involves the estimation of a stochastic process $x = \{x_t\}$ (called the signal or state process) that cannot be observed directly. Information containing x is obtained from observations of a related process $y = \{y_t\}$ (the observation process). The goal of nonlinear filtering is to determine the conditional density $\rho(t, x)$ of x_t given the observation history $\{y_s : 0 \leq s \leq t\}$. In 1961, Kalman and Bucy [15] published a historically important paper on filtering that is highly influential in modern industry. Since then, nonlinear filtering has proved useful in science and engineering, for example in navigational and guidance systems, radar tracking, sonar ranging, and satellite and airplane orbit determination [13], [14]. Despite its usefulness, however, the Kalman–Bucy filter is not perfect. The main weakness is that it is restricted only to linear dynamical systems. In the 1960s, Duncan [11], Mortensen [18], and Zakai [28] independently derived the so-called Duncan–Mortensen–Zakai (DMZ) equation for the nonlinear filtering problem. Unfortunately, since the DMZ equation is a stochastic differential equation, there is no easy way to derive a recursive algorithm for solving this equation.

The idea of using estimation algebras to construct finite dimensional nonlinear filters was first proposed in [5], [4], and [17]. The advantage of this approach is that as long as the estimation algebra is finite dimensional, we will get a finite-dimensional recursive filter. The approach applies well to nonlinear dynamical systems and has been worked out in detail in [21], especially for the so-called Yau filtering system described in [6]. For a linear filtering system, it is quite easy to see that the corresponding estimation algebra is finite dimensional. So one can apply the Wei–Norman approach to construct a finite-dimensional recursive filter. However, in the Wei–Norman approach, one has to know explicitly a basis of the estimation algebra as a vector space in order to reduce the DMZ equation to a finite system of ordinary differential equations, a Kolmogorov equation, and several first-order linear partial differential equations. Classically, one knows an explicit basis for the estimation algebra only in the case that it has maximal rank. Typically people assume that the linear system is controllable and observable.

Recently, a new direct method has been introduced to study the Kalman–Bucy and the Benés filtering systems with arbitrary initial condition for which f, g and h in (2.1) are independent of time (cf. [26], [27], and [22]). This approach offers several advantages. It is easy, and the derivation no longer needs controllability and observability. Furthermore, the necessity of integrating n first-order linear partial differential equations in the Lie algebra method is eliminated. More recently, Yau and Hu [23] applied the new direct method successfully to the Yau filtering systems, which include both the Kalman–Bucy filters and the Benés filters as special cases.

The purpose of this note is to solve the robust DMZ equation explicitly in terms of a system of ordinary differential equations (ODEs). More specifically, the solution of the robust DMZ equation is reduced to the online solution of a linear system of ODEs and the offline solution of a nonlinear system of ODEs. Our result is built on the previous result of Yau and Hu [22], which states that the solution of the robust DMZ equation can be reduced to the online solution of a linear system of ODEs and the offline solution of a Kolmogorov type PDE. In [16], Liang, Yau, and Yau have found a closed form solution to the Kolmogorov equation arising from linear filtering. Although the computation is offline, it is not easy to obtain a numerical solution especially when the state dimension is large. This is because the analytic solution in [16] involves a convolution operation which requires integration over $\mathbb{R}^n$. In particular, it is difficult to implement their analytic solution in hardware (cf. [12]). The major advantage of our note is that we can use a simple system of nonlinear ODEs to compute the solution of the Kolmogorov equation which makes the computation feasible even if the state dimension is fairly large. Thus our method can be implemented in hardware.

The idea of our note is quite simple. Finite dimensional approximations are obtained by exploiting the fact that a large class of non-Gaussian initial densities can be approximated by a finite sum of Gaussian densities. Under certain conditions, the DMZ equation can be reduced to the linear Kolmogorov equation (cf. Theorem 3.1). By the linearity of the Kolmogorov equation, an approximate solution can be obtained by solving a finite number of Kolmogorov equations with Gaussian initial conditions. This gives rise to a finite-dimensional approximation because the solution of the Kolmogorov equation with a Gaussian initial condition can be written in terms of ODEs (as shown in Theorem 3.2). The technique of approximating the non-Gaussian initial condition by linear combination of Gaussians was first used by Ahmed and Radaideh [1]. They used Galerkin numerical scheme to construct the solution of the DMZ equation. However unlike our method their method is not a theoretically justifiable approximation method for the nonlinear filtering problem. On the other hand, Ocone and Pardoux [20] has shown in the case of linear filtering that a conditional density filter forgets the initial condition asymptotically at an exponential rate. A similar result for Benés filters was obtained in [19]. (Also, stability results for filters based on Lyapunov exponents have been explored in [9] and [2]. Dey and Charalambous [8] investigated the problem of asymptotic forgetting of initial conditions by risk-sensitive filters for linear time-invariant systems).

An outline of the note is as follows. In Section II, we shall recall the basic filtering problem. In Section III, we shall solve the robust DMZ equation in terms of the online solution of a system of linear ODEs and the offline solution of a system of nonlinear ODEs. Finally we conclude our results in Section IV. We thank the referees for their valuable suggestions in revising this note.

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II. Basic Filtering Problem

The filtering problem considered here is based on the following signal observation model:

$$\begin{cases} dx(t) = f(x(t))dt + g(x(t))dv(t) & x(0) = x_0 \\ dy(t) = h(x(t))dt + dw(t) & y(0) = 0 \end{cases} \tag{2.1}$$

in which x, v, y, and w are, respectively, $\mathbb{R}^n$, $\mathbb{R}^p$, $\mathbb{R}^m$ and $\mathbb{R}^m$ valued processes and v and w are independent, standard Brownian processes. We further assume that $n = p$ and that f, g, and h are, respectively, vector-valued, orthogonal matrix-valued and vector-valued C^∞ smooth functions. We shall refer to $x(t)$ as the state of the system at time t and $y(t)$ as the observation at time t.

Let $\rho(t,x)$ denote the conditional probability density of the state given the observation $\{y(s) : 0 \leq s \leq t\}$. It is well known (see [10], for example) that $\rho(t,x)$ is given by normalizing a function $\sigma(t,x)$ that satisfies the following DMZ equation:

$$\begin{cases} d\sigma(t,x) = & L_0\sigma(t,x)dt + \sum_{i=1}^m L_i\sigma(t,x)dy_i(t) \\ \sigma(0,x) = & \sigma_0(t) \end{cases} \tag{2.2}$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^n f_i\frac{\partial}{\partial x_i} - \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^m h_i^2$$

L_i is the zero-degree differential operator given by multiplication by h_i, for $i = 1,\ldots,m$, and σ_0 is the probability density of the initial point x_0. In [7], Davis introduced a new unnormalized density

$$u(t,x) = \exp\left(-\sum_{i=1}^m h_i(x)y_i(t)\right)\sigma(t,x).$$

He reduced (2.2) to the following time-varying partial differential equation which is called the robust DMZ equation:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = & L_0u(t,x) + \sum_{i=1}^m y_i(t)[L_0,L_i]u(t,x) \\ & +\frac{1}{2}\sum_{i,j=1}^m y_i(t)y_j(t)\Big[[L_0,L_i],L_j\Big]u(t,x) \\ u(0,x) = & \sigma_0(x) \end{cases} \tag{2.3}$$

where $[\cdot,\cdot]$ is the Lie bracket as described in [21]. It is easy to show [26] that (2.3) is equivalent to the following time-varying partial differential equation; see (2.4) shown at the bottom of the page.

In 1990, Yau [24] (cf. [25] for a detailed version) first studied the filtering system (2.1) with the following conditions:

$$(C_1')\ \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j} = \text{constant (depending on } i,j\text{), for all } 1 \leq i,j \leq n.$$

This was called the Yau filtering system in [6]. The Yau filtering systems include the Kalman–Bucy filtering systems and the Benés filtering systems as special cases (see Theorem 2.1) and finite dimensional filters were constructed explicitly by using Lie algebra methods [21], [24], [25]. Define

$$\eta(x) = \sum_{i=1}^n f_i^2(x) + \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}(x) + \sum_{i=1}^m h_i^2(x). \tag{2.5}$$

The following theorems are proved in [25].

Theorem 2.1: (C_1') holds if and only if

$$(f_1,\ldots,f_n) = (\ell_1,\ldots,\ell_n) + \left(\frac{\partial F}{\partial x_1},\ldots,\frac{\partial F}{\partial x_n}\right)$$

where $\ell_1,\ldots,\ell_n$ are polynomials of degree one and F is a C^∞ function.

Theorem 2.2: Let E be a finite-dimensional estimation algebra of (2.1) satisfying (C_1'). Then, $h_1,\ldots,h_m$ are polynomials of degree at most one.

From Theorem 2.1, we know that (C_1') is equivalent to the following condition:

$$(C_1) \qquad f_i(x) = \ell_i(x) + \frac{\partial F}{\partial x_i}(x) \qquad 1 \leq i \leq n \tag{2.6}$$

where $\ell_i(x) = \sum_{j=1}^n d_{ij}x_j + d_i$, for $1 \leq i \leq n$ and F is a C^∞ function.

Theorem 2.2 tells us that $h_1,\ldots,h_m$ are polynomials of degree at most one if the Yau filtering system has a finite dimensional estimation algebra. So, we list the following condition:

$$(C_2) \qquad h_i(x) = \sum_{j=1}^n c_{ij}x_j + c_i \qquad 1 \leq i \leq m \tag{2.7}$$

where c_{ij} and c_i are constants.

Moreover, we know that $\eta(x)$ is a polynomial of degree at most two in x for most interesting filtering systems [21], [24]. Hence, we assume the following condition:

$$(C_3) \qquad \eta(x) = \sum_{i,j=1}^n \eta_{ij}x_ix_j + \sum_{i=1}^n \eta_i x_i + \eta_0 \tag{2.8}$$

where η_{ij}, η_i, and η_0 are constants. We remark that Kalman–Bucy filtering satisfies (C_3) and Benés [3] also requires this condition.

III. Explicit Solution of DMZ Equation in Terms of Solutions of ODEs

We first begin with the result of Yau–Hu [23].

Theorem 3.1: Consider the filtering system (2.1) with conditions (C_1), (C_2), and (C_3). Then, the solution $u(t,x)$ for the DMZ (2.3) or (2.4) is reduced to the solution $\widetilde{u}(t,x)$ for the Kolmorgorov equation

$$\begin{cases} \frac{\partial \widetilde{u}}{\partial t}(t,x) & = \frac{1}{2}\Delta\widetilde{u}(t,x) - \sum_{i=1}^n \ell_i(x)\frac{\partial \widetilde{u}}{\partial x_i}(t,x) \\ & +\frac{1}{2}\left(\sum_{i=1}^n \ell_i^2(x) - \sum_{i=1}^n \frac{\partial \ell_i}{\partial x_i}(x) - \eta(x)\right)\widetilde{u}(t,x) \\ \widetilde{u}(0,x) & = e^{-F(x)}\sigma_0(x) \end{cases} \tag{3.1}$$

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) & = \frac{1}{2}\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}(t,x) - \sum_{i=1}^n \left(-f_i(x) + \sum_{j=1}^m y_j(t)\frac{\partial h_j}{\partial x_i}(x)\right)\frac{\partial u}{\partial x_i}(t,x) \\ & - \left(\sum_{i=1}^n \frac{\partial f_i}{\partial x_i}(x) + \frac{1}{2}\sum_{i=1}^m h_i^2(x) - \frac{1}{2}\sum_{i=1}^m y_i(t)\Delta h_i(x) + \sum_{i=1}^m\sum_{j=1}^n y_i(t)f_j(x)\frac{\partial h_i}{\partial x_j}(x)\right. \\ & \left. -\frac{1}{2}\sum_{i,j=1}^m\sum_{k=1}^n y_i(t)y_j(t)\frac{\partial h_i}{\partial x_k}(x)\frac{\partial h_j}{\partial x_k}(x)\right)u(t,x) \\ u(0,x) & = \sigma_0(x). \end{cases} \tag{2.4}$$

where

$$\widetilde{u}(t,x) = \exp\left[c(t) + \sum_{i=1}^{n} a_i(t)x_i - F(x+b(t))\right] \cdot u(t, x+b(t)) \quad (3.2)$$

and $a_i(t)$, $b_i(t)$, and $c(t)$ satisfy ODEs (3.3)–(3.5)

$$\begin{cases} b_i'(t) - & a_i(t) - \sum_{j=1}^{n} d_{ij}b_j(t) + \sum_{j=1}^{m} c_{ji}y_j(t) = 0 \\ b_i(0) = & 0 \qquad 1 \le i \le n \end{cases} \quad (3.3)$$

$$\begin{cases} a_i'(t) - & \frac{1}{2}\sum_{j=1}^{n}(\eta_{ij}+\eta_{ji})b_j(t) + \sum_{j=1}^{n} d_{ji}b_j'(t) = 0 \\ a_i(0) = & 0 \qquad 1 \le i \le n \end{cases} \quad (3.4)$$

$$\begin{cases} c'(t) = & -\frac{1}{2}\sum_{i=1}^{n}(b_i'(t))^2 + \sum_{i=1}^{n} a_i(t)b_i'(t) - \sum_{i=1}^{n} d_ib_i'(t) \\ & +\frac{1}{2}\sum_{i,j=1}^{n}\eta_{ij}b_i(t)b_j(t) + \frac{1}{2}\sum_{i=1}^{n}\eta_ib_i(t) \\ c(0) = & 0. \end{cases} \quad (3.5)$$

In view of Theorem 3.1, in order to give an explicit solution of the DMZ equation in terms of ODEs, it is sufficient to solve (3.1) in terms of ODEs. It is well known that any distribution is well approximated by a finite linear combination of Gaussians of the form $\alpha_1 G_1 + \cdots + \alpha_k G_k$, where α_i's are real numbers and G_i's are Gaussian distributions. Let $\widetilde{u}_i$ be the solution of (3.1) with initial distribution G_i. Since (3.1) is a linear partial differential equation, it follows that the solution of (3.1) is of the form $\alpha_1\widetilde{u}_1 + \cdots + \alpha_k\widetilde{u}_k$. Therefore it remains to solve (3.1) with a Gaussian initial distribution. Theorem 3.2 gives an explicit solution of (3.1) with a Gaussian initial distribution in terms of a solution of ODEs.

Theorem 3.2: Consider the filtering system (2.1) with conditions (C_1), (C_2), and (C_3) and a Kolmogorov equation with Gaussian initial distribution

$$\begin{cases} \frac{\partial\widetilde{u}}{\partial t}(t,x) = & \frac{1}{2}\Delta\widetilde{u}(t,x) - \sum_{i=1}^{n}\ell_i(x)\frac{\partial\widetilde{u}}{\partial x_i}(t,x) \\ & +\frac{1}{2}\left(\sum_{i=1}^{n}\ell_i^2(x) - \sum_{i=1}^{n}\frac{\partial\ell_i}{\partial x_i}(x) - \eta(x)\right)\widetilde{u}(t,x) \\ \widetilde{u}(0,x) = & e^{x^TA(0)x+B^T(0)x+C(0)} \end{cases} \quad (3.6)$$

where $A(0) = \left(A_{ij}(0)\right)$ *is a* $n \times n$ symmetric matrix, $B^T(0) = \left(B_1(0), \ldots B_n(0)\right)$, $x^T = (x_1, \ldots, x_n)$ are $1 \times n$ matrices and $C(0)$ is a scalar. Let

$$\begin{aligned} q(x) =& \frac{1}{2}\left(\sum_{i=1}^{n}\ell_i^2(x) - \sum_{i=1}^{n}\frac{\partial\ell_i}{\partial x_i}(x) - \eta(x)\right) \\ =& x^TQx + p^Tx + r \end{aligned}$$

where $\ell_i(x) = \sum_{j=1}^{n} d_{ij}x_j + d_i$, $Q = (q_{ij})$ a $n \times n$ symmetric matrix, $p^T = (p_1, \ldots, p_n)$ a $1 \times n$ matrix, r a scalar. Then the solution of (3.6) is of the following form

$$\widetilde{u}(t,x) = e^{x^TAx+B^Tx+C} \quad (3.7)$$

where $A(t) = \left(A_{ij}(t)\right)$ is a $n \times n$ symmetric matrix valued function of t, $B^T(t) = \left(B_1(t), \ldots, B_n(t)\right)$ is a $1 \times n$ matrix valued function of t, and $C(t)$ is a scalar function of t. Moreover, $A(t)$, $B^T(t)$ and $C(t)$ satisfy the following system of nonlinear ODEs:

$$\frac{dA}{dt}(t) = 2A^2(t) - [A(t)D + D^TA(t)] + Q \quad (3.8)$$

$$\frac{dB^T}{dt}(t) = 2B^T(t)A(t) - B^T(t)D - 2d^TA(t) + p^T \quad (3.9)$$

$$\frac{dC}{dt}(t) = trA(t) + \frac{1}{2}B^T(t)B(t) - d^TB(t) + r \quad (3.10)$$

where $D = (d_{ij})$ is a $n \times n$ matrix, $d^T = (d_1, \ldots, d_n)$ is a $1 \times n$ matrix and (3.8) is a Riccati equation.

Proof: Differentiating (3.7) with respect to t and x_k, we get the following equations:

$$\begin{aligned}
\frac{\partial\widetilde{u}}{\partial t} &= \left(x^T\frac{dA}{dt}x + \frac{dB^T}{dt}x + \frac{dC}{dt}\right)\widetilde{u} \\
\frac{\partial\widetilde{u}}{\partial x_k} &= \left[\sum_{i,j=1}^{n} A_{ij}\left(\frac{\partial x_i}{\partial x_k}xj + x_i\frac{\partial x_j}{\partial x_k}\right) + B_k\right]\widetilde{u} \\
&= \left(\sum_{j=1}^{n} A_{kj}x_j + \sum_{i=1}^{n} A_{ik}x_i + B_k\right)\widetilde{u} \\
\nabla\widetilde{u}^T &= \left[(Ax)^T + x^TA + B^T\right]\widetilde{u} \\
&= (x^TA^T + x^TA + B^T)\widetilde{u} \\
\frac{\partial^2\widetilde{u}}{\partial x_k^2} &= \left[2A_{kk} + \left(\sum_{j=1}^{n} A_{kj}x_j + \sum_{i=1}^{n} A_{ik}x_i + B_k\right)^2\right]\widetilde{u} \\
&= \left[2A_{kk} + \sum_{j,\ell=1}^{n} A_{kj}A_{k\ell}x_jx_\ell \right. \\
&\quad + \sum_{i,\ell=1}^{n} A_{ik}A_{\ell k}x_ix_\ell + B_k^2 + 2\sum_{i,j=1}^{n} A_{kj}A_{ik}x_jx_i \\
&\quad \left. + 2B_k\sum_{j=1}^{n} A_{kj}x_j + 2B_k\sum_{i=1}^{n} A_{ik}x_i\right]\widetilde{u} \\
\frac{1}{2}\Delta\widetilde{u} &= \left[trA + \frac{1}{2}(Ax)^T(Ax) + \frac{1}{2}(x^TA)(x^TA)^T \right. \\
&\quad + \frac{1}{2}B^TB + (x^TA)(Ax) \\
&\quad \left. + B^TAx + x^TAB\right]\widetilde{u} \\
&= \left[x^T\left(\frac{1}{2}A^TA + \frac{1}{2}AA^T + A^2\right)x \right. \\
&\quad \left. + B^T(A+A^T)x + trA + \frac{1}{2}B^TB\right]\widetilde{u} \\
\sum_{i=1}^{n}\ell_i(x)\frac{\partial\widetilde{u}}{\partial x_i} &= \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}x_j\frac{\partial u}{\partial x_i} + \sum_{i=1}^{n} d_i\frac{\partial u}{\partial x_i} \\
&= \nabla\widetilde{u}^TDx + d^T\nabla\widetilde{u} \\
&= (x^TA^T + x^TA + B^T)Dx\widetilde{u} \\
&\quad + d^T(Ax + A^Tx + B)\widetilde{u} \\
&= \left[x^T(A^T + A)Dx + (B^TD \right. \\
&\quad \left. + d^TA + d^TA^T)x + d^TB\right]\widetilde{u}.
\end{aligned} \quad (3.11)$$

Therefore, the left-hand side of (3.6) is given by

$$\begin{aligned}
&\frac{1}{2}\Delta\widetilde{u}(t,x) - \sum_{i=1}^{n}\ell_i(x)\frac{\partial\widetilde{u}}{\partial x_i}(t,x) + q(x)\widetilde{u}(t,x) \\
&= \left[x^T\left(\frac{1}{2}A^TA + \frac{1}{2}AA^T + A^2\right)x + B^T(A+A^T)x \right. \\
&\quad \left. + trA + \frac{1}{2}B^TB\right]\widetilde{u} \\
&\quad - \left[x^T(A^T+A)Dx + (B^TD + d^TA + d^TA^T)x + d^TB\right] \\
&\quad \cdot\widetilde{u} + (x^TQx + p^Tx + r)\widetilde{u}.
\end{aligned} \quad (3.12)$$

Equating (3.11) and (3.12) and comparing terms, we get equations (3.8), (3.9), and (3.10). $Q.E.D.$

For the convenience of the reader, we include an example with $n = 1 = m$.

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = & \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x) + (x+1+\frac{dF}{dx} - y(t))\frac{\partial u}{\partial x}(t,x) - [1+\frac{d^2F}{dx^2}+\frac{1}{2}(1+x)^2 \\ & +y(t)(1+x+\frac{dF}{dx}) - \frac{1}{2}y^2(t)]u(t,x) \\ u(0,x) = & \sigma_0(x) \end{cases}. \tag{3.13}$$

Example 3.3: Let $h(x) = x+1$ and $f(x) = x+1+dF/dx$ where $F(x) = \int\{[e^{-(x-1/2)^2} \Big/ \int_{-\infty}^{x} e^{-(x-1/2)^2}dx] - 3/2\}dx$. Then

$$f^2(x) + \frac{df}{dx}(x) + h^2(x) = 2x^2 + x + \frac{9}{4}.$$

The robust DMZ equation is of the form shown in (3.13) at the top of the page. By Theorem 3.1, $u(t,x)$ can be computed via the solution $\widetilde{u}(t,x)$ for the Kolmogorov equation shown in

$$\begin{cases} \frac{\partial \widetilde{u}}{\partial t}(t,x) = & \frac{1}{2}\frac{\partial^2 \widetilde{u}}{\partial x^2}(t,x) - (x+1)\frac{\partial \widetilde{u}}{\partial x}(t,x) + \frac{1}{2}(-x^2+x-\frac{9}{4})\widetilde{u}(t,x) \\ \widetilde{u}(0,x) = & e^{-F(x)}\sigma_0(x) \end{cases} \tag{3.14}$$

where

$$\widetilde{u}(t,x) = \exp[c(t) + a(t)x - F(x+b(t))] \cdot u[t, x+b(t)]$$

and $a(t), b(t)$ and $c(t)$ satisfy the following ODEs:

$$\begin{cases} b'(t) - a(t) - b(t) + y(t) = 0, & b(0) = 0 \\ a'(t) - 2b(t) + b'(t) = 0, & a(0) = 0 \\ c'(t) = b'(t)(-\frac{1}{2}b'(t) + a(t) - 1) + b^2(t) + b(t), & c(0) = 0 \end{cases}.$$

Assume that $\widetilde{u}(0,x) = \alpha_1 G_1 + \cdots + \alpha_n G_n$ where $\alpha_i \in \mathbb{R}$ and $G_i(x) = e^{A_i(0)x^2 + B_i(0)x + C_i(0)}$. Then, $\widetilde{u}(t,x) = \alpha_1 \widetilde{u}_1(t,x) + \cdots + \alpha_n \widetilde{u}_n(t,x)$ where $\widetilde{u}_i(t,x) = e^{A_i(t)x^2 + B_i(t)x + C_i(t)}$ and $A_i(t), B_i(t), C_i(t)$ satisfy the following ODEs:

$$\begin{cases} \frac{dA_i}{dt}(t) = & 2A_i^2(t) - 2A_i(t) - \frac{1}{2} \\ \frac{dB_i}{dt}(t) = & 2B_i(t)A_i(t) - B_i(t) - 2A_i(t) + \frac{1}{2} \\ \frac{dC_i}{dt}(t) = & A_i(t) + \frac{1}{2}B_i^2(t) - B_i(t) - \frac{9}{8} \end{cases}.$$

IV. Conclusion

In this note, we have solved explicitly the robust DMZ equation arising from a Yau filtering system in terms of a system of ODEs. Unlike the closed-form solution of [16], our solution can be implemented in hardware for practical use.

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