Nonlinear Filtering and Time Varying Schrödinger Equation I

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Based on our previous work [24] we have successfully reduced the nonlinear filtering problem for Yau filtering system to the time-varying Schrödinger equation. In order to solve the nonlinear filtering problem, one needs to solve the time-varying Schrödinger equation with an arbitrary initial condition. We then solve the time-varying Schrödinger equation by constructing the fundamental solution explicitly via a system of nonlinear ODEs in case the potential is quadratic in state variables. This system of nonlinear ODEs is solved explicitly by the power series method.

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I. INTRODUCTION

In 1979, Mitter [13] first observed that there are striking similarities between mathematical problems of nonlinear filtering and quantum physics. Thus the mathematical developments in functional analysis, Lie groups and Lie algebras, group representations, and probabilistic methods of quantum theory can serve as a guide to search for an appropriate theory of stochastic systems. Mitter also observed that "unitarity" which plays an important part in quantum mechanics and quantum field theory is not necessarily relevant to nonlinear filtering theory. The partial differential equations that arise in quantum theory are generally wave equations, whereas the partial differential equations arising in the filtering theory are parabolic equations. He suggested the possibility of passing to a wave equation by appropriate analytic continuation from the parabolic equation. Although we do not know how to carry out Mitter's beautiful idea, we are able to establish an explicit relationship between the important part of nonlinear filtering and the theory of time-varying Schrödinger equation.

The central problem of nonlinear filtering is to solve Duncan-Mortensen-Zakai (DMZ) equation in real time and in a memoryless way. For the past quarter of a century, basically only two methods have been available. The first one is to imitate the Wei-Norman approach of using the Lie algebraic method to solve DMZ equation. This idea is due to Brockett [2], Brockett and Clark [3], and Mitter [13] independently. The details of this approach were worked out in [14] and [15]. In the Wei-Norman approach, one has to know explicitly the basis as vector space of the estimation algebra in order to reduce the DMZ equation to a finite system of ordinary differential equations, Kolmogorov equation, and several first-order linear partial differential equations. Recently Stephen Yau [17] and his colleagues [5–9, 11, 20] have completely classified all finite dimensional estimation algebras of maximal rank. In particular, they have proved that all the observation terms $h_i(x)$, $1 \leq i \leq m$, must be a degree one polynomial.

The second approach to solve the DMZ equation is the direct method introduced by [21, 22] and generalized by [10], [18], and [19]. The advantage of this method is that the algorithm is universal for any Yau filtering system [4] which include Kalman-Bucy [12] and Benés [1] filtering systems. Furthermore, it eliminates the necessity of integrating n first-order linear partial differential equations, as was the case in the Lie algebra method. However, in all these direct methods in [21], [18], [19], and [10], they need to assume that all the observation terms $h_i(x)$, $1 \leq i \leq m$, are degree one polynomials.

In many practical situations, the observation terms may be nonlinear. In view of this, we developed a new method to treat the filtering problem with nonlinear observation terms in [25]. The method in [25] works only when drift term $f(x)$ is a degree one polynomial. Moreover, it involves Gaussian approximation of initial distribution. In the Yau filtering system [4], i.e. $f(x) = Lx + \ell + \nabla\phi$, where $L = (\ell_{ij})$, $1 \le i, j \le n$, $\ell^T = (\ell_1, \ldots, \ell_n)$ and ϕ is a C^∞ function of $\mathbb{R}^n$, the method used in [25] does not work. The novelty of this paper is that we are able to reduce the nonlinear filtering problem for Yau filtering system to time-varying Schrödinger equation (Theorem 2). Therefore, in order to solve the nonlinear filtering problem, we need to solve the time-varying Schrödinger equation with an arbitrary initial condition. For this purpose, it suffices to write down the fundamental solution of the Schrödinger equation. In the work presented here we write down the fundamental solution of the Schrödinger equation in terms of the solution of a system of ODEs (Theorem 4) in case the potential is quadratic in state variables. Since this system of ODEs can be solved by the power series method, we have found the fundamental solution explicitly (Theorem 5). We remark that in order to solve the nonlinear filtering problem for the Yau filtering system, the only condition we put is that (14) is a degree two polynomial. There is no requirement that the observation terms $h_i(x)$, $1 \le i \le m$, have to be linear.

In Section II, we shall recall the basic formulation of the filtering problem. In Section III, we recall an explicit algorithm for real time computation of DMZ equation by means of the off-time computation of Kolmogorov equation. In Section IV, we show that in order to solve the nonlinear filtering problem for Yau filtering system, it suffices to solve a time-varying Schrödinger equation. In Section V, we solve the time-varying Schrödinger equation by constructing the fundamental solution explicitly in case the potential is quadratic in state variables. The fundamental solution is constructed via a system of nonlinear ODEs (Theorem 4). This system of nonlinear ODEs is solved explicitly by power series method (Theorem 5). Finally we conclude our results in Section VI.

II. BASIC CONCEPTS

The filtering problem considered here is based on the following signal observation model:

$$\begin{cases} dx(t) = f(x(t))dt + g(x(t))dv(t) & x(0) = x_0 \\ dy(t) = h(x(t))dt + dw(t) & y(0) = 0 \end{cases} \tag{1}$$

in which x, v, y, and w, are, respectively, $\mathbb{R}^n$, $\mathbb{R}^p$, $\mathbb{R}^m$, and $\mathbb{R}^m$ valued processes, and v and w have components which are independent, standard Brownian processes. We further assume that $n = p$, f and h are C^∞ smooth, and that g is an orthogonal matrix. We refer to $x(t)$ as the state of the system at time t and $y(t)$ as the observation at time t.

Let $\rho(t,x)$ denote the conditional density of the state given the observation $\{y(s) : 0 \le s \le t\}$. It is well known that $\rho(t,x)$ is given by normalizing a function $\sigma(t,x)$, which satisfies the DMZ equation,

$$\begin{cases} d\sigma(t,x) = L_0\sigma(t,x)dt + \sum_{i=1}^{m} L_i\sigma(t,x)dy_i(t) \\ \sigma(0,x) = \sigma_0 \end{cases} \tag{2}$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i} - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2$$

and for $i = 1, \ldots, m$, L_i is the zero-degree differential operator of multiplication by h_i, σ_0 is the probability density of the initial point x_0. Let

$$u(t,x) = \exp\left(-\sum_{i=1}^{m} h_i(x)y_i(t)\right)\sigma(t,x). \tag{3}$$

It is easy to show that $u(t,x)$ satisfies the following time-varying partial differential equation which is called robust DMZ equation:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = L_0u(t,x) + \sum_{i=1}^{m} y_i(t)[L_0, L_i]u(t,x) \\ \qquad\qquad + \frac{1}{2}\sum_{i,j=1}^{m} y_i(t)y_j(t)[[L_0, L_i], L_j]u(t,x) \\ u(0,x) = \sigma_0 \end{cases} \tag{4}$$

where $[\cdot,\cdot]$ is the Lie bracket. Equation (4) is equivalent to the following time-varying partial differential equation:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}(t,x) \\ \qquad + \sum_{i=1}^{n}\left(-f_i(x) + \sum_{j=1}^{m} y_j(t)\frac{\partial h_j}{\partial x_i}(x)\right)\frac{\partial u}{\partial x_i}(t,x) \\ \qquad - \Bigg[\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) + \frac{1}{2}\sum_{i=1}^{m} h_i^2(x) - \frac{1}{2}\sum_{i=1}^{m} y_i(t)\Delta h_i(x) \\ \qquad + \sum_{i=1}^{m}\sum_{j=1}^{n} y_i(t)f_j(x)\frac{\partial h_i}{\partial x_j}(x) \\ \qquad - \frac{1}{2}\sum_{i,j=1}^{m}\sum_{k=1}^{n} y_i(t)y_j(t)\frac{\partial h_i}{\partial x_k}(x)\frac{\partial h_j}{\partial x_k}(x)\Bigg]u(t,x) \\ u(0,x) = \sigma_0(x). \end{cases} \tag{5}$$

III. REDUCTION TO OFF-LINE COMPUTATION OF KOLMOGOROV EQUATION

In this section, we briefly recall our algorithm [24, 25] which solves the nonlinear filtering problem with arbitrary initial distribution by reducing it to solve the Kolmogorov equation.

Suppose that $u(t,x)$ is the solution of the robust DMZ equation (5) and we want to compute $u(\tau,x)$. Let $\mathcal{P}_k = \{0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_k = \tau\}$ be a partition of $[0,\tau]$. Let $u_i(t,x)$ be a solution of the following partial differential equation for $\tau_{i-1} \le t \le \tau_i$

$$\begin{cases} \dfrac{\partial u_i}{\partial t}(t,x) = \dfrac{1}{2}\Delta u_i(t,x) \\ \qquad + \displaystyle\sum_{\ell=1}^{n}\left(-f_\ell(x) + \sum_{j=1}^{m} y_j(\tau_{i-1})\frac{\partial h_j}{\partial x_\ell}(x)\right)\frac{\partial u_i}{\partial x_\ell}(t,x) \\ \qquad - \left(\displaystyle\sum_{\ell=1}^{n}\frac{\partial f_\ell}{\partial x_\ell}(x) + \frac{1}{2}\sum_{\ell=1}^{m} h_\ell^2(x) - \frac{1}{2}\sum_{j=1}^{m} y_j(\tau_{i-1})\Delta h_j(x)\right. \\ \qquad + \displaystyle\sum_{j=1}^{m}\sum_{\ell=1}^{n} y_j(\tau_{i-1}) f_\ell(x)\frac{\partial h_j}{\partial x_\ell}(x) \\ \qquad \left. - \dfrac{1}{2}\displaystyle\sum_{p=1}^{n}\sum_{j,\ell=1}^{m} y_j(\tau_{i-1}) y_\ell(\tau_{i-1})\frac{\partial h_j}{\partial x_p}(x)\frac{\partial h_\ell}{\partial x_p}(x)\right) u_i(t,x) \\ u_i(\tau_{i-1},x) = u_{i-1}(\tau_{i-1},x). \end{cases} \tag{6}$$

In [24, 25], we have proved that in both point-wise sense and L^2 sense,

$$u(\tau,x) = \lim_{|\mathcal{P}_k|\to 0} u_k(\tau,x). \tag{7}$$

Therefore it remains to describe an algorithm to compute $u_k(\tau_k,x)$. In [24, 25], we showed that $u_1(\tau_1,x)$ can be computed by $\tilde{u}_1(\tau_1,x)$ where $\tilde{u}_1(t,x)$ for $0 \le t \le \tau_1$ satisfies the following Kolmogorov equation

$$\begin{cases} \dfrac{\partial \tilde{u}_1}{\partial t}(t,x) = \dfrac{1}{2}\Delta \tilde{u}_1(t,x) - \displaystyle\sum_{j=1}^{n} f_j(x)\frac{\partial \tilde{u}_1}{\partial x_j}(t,x) \\ \qquad - \left(\displaystyle\sum_{j=1}^{n}\frac{\partial f_j}{\partial x_j}(x) + \frac{1}{2}\sum_{j=1}^{m} h_j^2(x)\right)\tilde{u}_1(t,x) \\ \tilde{u}_1(0,x) = \sigma_0(x). \end{cases} \tag{8}$$

In fact

$$u_1(t,x) = \tilde{u}_1(t,x), \qquad 0 \le t \le \tau_1. \tag{9}$$

In general, $u_i(\tau_i,x)$ can be computed by $\tilde{u}_i(\tau_i,x)$ where $\tilde{u}_i(t,x)$ for $\tau_{i-1} \le t \le \tau_i$ satisfies the following Kolmogorov equation

$$\begin{cases} \dfrac{\partial \tilde{u}_i}{\partial t}(t,x) = \dfrac{1}{2}\Delta \tilde{u}_i(t,x) - \displaystyle\sum_{j=1}^{n} f_j(x)\frac{\partial \tilde{u}_i}{\partial x_j}(t,x) \\ \qquad - \left(\displaystyle\sum_{j=1}^{n}\frac{\partial f_j}{\partial x_j}(x) + \frac{1}{2}\sum_{j=1}^{m} h_j^2(x)\right)\tilde{u}_i(t,x) \\ \tilde{u}_i(\tau_{i-1},x) = \exp\left[\displaystyle\sum_{j=1}^{m}(y_j(\tau_{i-1}) - y_j(\tau_{i-2}))h_j(x)\right]\tilde{u}_{i-1}(\tau_{i-1},x). \end{cases} \tag{10}$$

In fact

$$u_i(\tau_i,x) = \exp\left[-\sum_{j=1}^{m} y_j(\tau_{i-1})h_j(x)\right]\tilde{u}_i(\tau_i,x).$$

THEOREM 1 [25]. *The unnormalized density σ can be computed via solution $\tilde{u}_i$ of Kolmogorov equation* (10). *More specifically,*

$$\sigma(\tau,x) = \lim_{|\mathcal{P}_k|\to 0} \tilde{u}_k(\tau_k,x). \tag{11}$$

IV. NONLINEAR FILTERING AND TIME-VARYING SCHRÖDINGER EQUATION

Consider the filtering system (1) with

$$f(x) = Lx + \ell + \nabla\phi \tag{12}$$

where $L = (\ell_{ij})$, $1 \le i,\ j \le n$, $\ell^T = (\ell_1,\ldots,\ell_n)$ and ϕ is a C^∞ function on $\mathbb{R}^n$. Such a filtering system is called the Yau filtering system in [4]. Recall that L can be uniquely decomposed as $L = L_1 + L_2$, where $L_1^T = L_1$ and $L_2^T = -L_2$. Observe that $L_1 x = \nabla\phi_1$ for some quadratic polynomial ϕ_1 by Poincare lemma. It follows that $f(x) = L_2 x + \ell + \nabla\tilde{\phi}$ where $\tilde{\phi} = \phi + \phi_1$. Hence, without loss of generality, we assume that in (12)

$$L^T = -L. \tag{13}$$

Let

$$q(x) := \Delta\phi(x) + |\nabla\phi|^2(x) + 2(Lx+\ell)\cdot\nabla\phi(x) + \sum_{i=1}^{m} h_i^2(x) + \mathrm{tr}L. \tag{14}$$

In view of Theorem 1, in order to solve the nonlinear filtering problem with nonlinear observation as in (12), (13), and (14), it suffices to solve the following Kolmogorov equation in real time. For $\tau_{i-1} \le t \le \tau_i$

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t}(t,x) = \dfrac{1}{2}\Delta \tilde{u}(t,x) - \displaystyle\sum_{j=1}^{n} f_j(x)\frac{\partial \tilde{u}}{\partial x_j}(t,x) \\ \qquad - \left(\displaystyle\sum_{j=1}^{n}\frac{\partial f_j}{\partial x_j}(x) + \frac{1}{2}\sum_{j=1}^{m} h_j^2(x)\right)\tilde{u}(t,x) \\ \tilde{u}(\tau_{i-1},x) = \sigma_i(x). \end{cases} \tag{15}$$

Let

$$\tilde{u}(t,x) = e^{\phi(x)}\tilde{v}(t,x). \tag{16}$$

Then

$$\frac{\partial \tilde{u}}{\partial t} = e^{\phi(x)}\frac{\partial \tilde{v}}{\partial t}(t,x) \tag{17}$$

$$\nabla\tilde{u}(t,x) = (\nabla\phi(x))e^{\phi(x)}\tilde{v}(t,x) + e^{\phi(x)}\nabla\tilde{v}(t,x) \tag{18}$$

$$\begin{aligned}\Delta\tilde{u}(t,x) &= (\Delta\phi(x))e^{\phi(x)}\tilde{v}(t,x) + |\nabla\phi(x)|^2 e^{\phi(x)}\tilde{v}(t,x) \\ &\quad + 2e^{\phi(x)}(\nabla\phi(x)\cdot\nabla\tilde{v}(t,x)) + e^{\phi(x)}\Delta\tilde{v}(t,x).\end{aligned} \tag{19}$$

Put (17), (18) and (19) in (15). We get the following equation. For $\tau_{i-1} \le t \le \tau_i$

$$\begin{cases} \dfrac{\partial \tilde{v}}{\partial t}(t,x) = \dfrac{1}{2}\Delta\tilde{v}(t,x) - (Lx+\ell)\cdot\nabla\tilde{v}(t,x) \\ \qquad - \left(\dfrac{1}{2}\Delta\phi(x) + \dfrac{1}{2}|\nabla\phi(x)|^2 + \operatorname{tr}L \right. \\ \qquad \left. + (Lx+\ell)\cdot\nabla\phi(x) + \dfrac{1}{2}\displaystyle\sum_{i=1}^m h_i^2(x)\right)\tilde{v}(t,x) \\ \tilde{v}(\tau_{i-1},x) = \sigma_i(x)e^{-\phi(x)}. \end{cases} \tag{20}$$

Let $B(t) = (b_{ij}(t))$, $1 \le i, j \le n$ and $b^T(t) = (b_1(t),\ldots,b_n(t))$ such that

$$\frac{dB}{dt}(t) = -B(t)L \qquad \text{and} \qquad \frac{db}{dt}(t) = -B(t)\ell. \tag{21}$$

Then

$$B(t) = e^{-Lt} \qquad \text{and} \qquad b(t) = -\int_0^t e^{-Ls}\ell\, ds \tag{22}$$

and $B(t)$ is an orthogonal matrix since $B(t)B^T(t) = e^{-Lt}e^{-L^T t} = e^{-Lt}e^{Lt} = I$. Let

$$\tilde{v}(t,x) = v(t,B(t)x+b(t)). \tag{23}$$

Then

$$\frac{\partial\tilde{v}}{\partial x_i}(t,x) = \sum_{j=1}^n \frac{\partial v}{\partial x_j}(t,B(t)x+b(t))b_{ji}(t) \tag{24}$$

$$\Delta\tilde{v}(t,x) = \Delta v(t,B(t)x+b(t)) \tag{25}$$

$$\begin{aligned} \frac{\partial\tilde{v}}{\partial t}(t,x) &= \frac{\partial v}{\partial t}(t,B(t)x+b(t)) \\ &\quad + \sum_{i,j=1}^n \frac{\partial v}{\partial x_i}(t,B(t)x+b(t))\frac{db_{ij}}{dt}(t)x_j \\ &\quad + \sum_{i=1}^n \frac{\partial v}{\partial x_i}(t,B(t)x+b(t))\frac{db_i}{dt}(t). \end{aligned} \tag{26}$$

Put (23), (24) and (25) in (20). We get the following equation

$$\begin{aligned} \frac{\partial\tilde{v}}{\partial t}(t,x) &= \frac{1}{2}\Delta v(t,B(t)x+b(t)) - \sum_{i=1}^n\left(\sum_{j=1}^n \ell_{ij}x_j\right) \\ &\quad \times\left(\sum_{k=1}^n \frac{\partial v}{\partial x_k}(t,B(t)x+b(t))b_{ki}(t)\right) \\ &\quad - \sum_{i=1}^n \ell_i\left(\sum_{j=1}^n \frac{\partial v}{\partial x_j}(t,B(t)x+b(t))b_{ji}(t)\right) \\ &\quad - \left(\frac{1}{2}\Delta\phi(x) + \frac{1}{2}|\nabla\phi(x)|^2 + \operatorname{tr}L + (Lx+\ell)\right. \\ &\quad \left.\cdot\nabla\phi(x) + \frac{1}{2}\sum_{i=1}^m h_i^2(x)\right) v(t,B(t)x+b(t)) \\ &= \frac{1}{2}\Delta v(t,B(t)x+b(t)) - \sum_{i,j=1}^n \frac{\partial v}{\partial x_i}(t,B(t)x+b(t)) \\ &\quad \times \sum_{k=1}^n b_{ik}(t)\ell_{kj}x_j - \sum_{i=1}^n \frac{\partial v}{\partial x_i}(t,B(t)x+b(t))\sum_{j=1}^n b_{ij}(t)\ell_j \\ &\quad - \left(\frac{1}{2}\Delta\phi(x) + \frac{1}{2}|\nabla\phi(x)|^2 + \operatorname{tr}L + (Lx+\ell)\cdot\nabla\phi(x)\right. \\ &\quad \left. + \frac{1}{2}\sum_{i=1}^m h_i^2(x)\right) v(t,B(t)x+b(t)). \end{aligned} \tag{27}$$

Comparing (26) and (27) and using (21), we get the following equation. For $\tau_{i-1} \le t \le \tau_i$:

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,B(t)x+b(t)) = \dfrac{1}{2}\Delta v(t,B(t)x+b(t)) \\ \qquad - \left(\dfrac{1}{2}\Delta\phi(x) + \dfrac{1}{2}|\nabla\phi|^2(x) + (Lx+\ell)\right. \\ \qquad \left.\cdot\nabla\phi(x) + \operatorname{tr}L + \dfrac{1}{2}\displaystyle\sum_{i=1}^m h_i^2(x)\right) v(t,B(t)x+b) \\ v(\tau_{i-1},x) = \sigma_i(x)e^{-\phi(x)}. \end{cases} \tag{28}$$

Thus we have proved the following theorem.

THEOREM 2 *In order to solve the nonlinear filtering problem with nonlinear observations as in* (12)*,* (13)*, and* (14)*, it suffices to solve the Schrödinger equation* (28)*, which is equivalent to the following equation* (29)*. For $\tau_{i-1} \le t \le \tau_i$:*

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,\tilde{x}) = \dfrac{1}{2}\Delta v(t,\tilde{x}) - \dfrac{1}{2}q(B^{-1}(t)\tilde{x} - B^{-1}(t)b(t))v(t,\tilde{x}) \\ v(\tau_{i-1},\tilde{x}) = \sigma_i(\tilde{x})e^{-\phi(\tilde{x})}. \end{cases} \tag{29}$$

V. FILTERING PROBLEM WITH NONLINEAR OBSERVATIONS

In this section we assume that $q(x)$ in (14) is a quadratic polynomial, i.e.,

$$q(x) = x^TQx + P^Tx + r \tag{30}$$

where $Q = Q^T = (q_{ij})$, $1 \le i, j \le n$, $P^T = (p_1,\ldots,p_n)$ and r is a scalar. Observe here that $h_i(x)$ may not be linear (i.e., degree one polynomial). Since $q(x)$ is quadratic, $h_i(x)$, $1 \le i \le m$, are of linear growths, i.e., $h_i^2(x) \le M(1+|x|^2)$, $1 \le i \le m$, for some constant M in view of (14).

DEFINITION 1 $K(t,x,y)$ is said to be the fundamental solution of the parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = L_x u(t,x), \qquad 0 \le t < \infty, \quad x \in \mathbb{R}^n \\ u(0,x) = \phi(x) \end{cases} \tag{31}$$

if $\partial K/\partial t(t,x,y) = L_x K(t,x,y)$ and $\lim_{t\to 0}\int_{\mathbb{R}^n} K(t,x,y) \cdot\phi(y)dy = \phi(x)$.

In view of the above definition, solution of (31) can be written as $u(t,x) = \int_{\mathbb{R}^n} K(t,x,y)\phi(y)dy$.

We are now going to solve (29). For simplicity, we treat (29) for $0 \le t \le \tau$.

THEOREM 3 *Let $K(t,\tilde{x},\tilde{y})$ be the fundamental solution of*

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,\tilde{x}) = \dfrac{1}{2}\Delta v(t,\tilde{x}) - \dfrac{1}{2}q(B^T(t)\tilde{x} - B^T(t)b(t))v(t,\tilde{x}) \\ v(0,\tilde{x}) = \sigma_1(\tilde{x})e^{-\phi(\tilde{x})} \end{cases} \tag{32}$$

where

$$\begin{aligned} q(B^T(t)\tilde{x} - B^T(t)b(t)) &= \tilde{x}^T B(t)QB^T(t)\tilde{x} - [2(b^T(t)B(t)QB^T(t)) \\ &\quad - P^T B^T(t)]\tilde{x} + b^T(t)B(t)QB^T(t)b(t) \\ &\quad - P^T B^T(t)b(t) + r. \end{aligned} \tag{33}$$

Assume that the fundamental solution $K(t,\tilde{x},\tilde{y})$ can be written as

$$\begin{aligned} K(t,\tilde{x},\tilde{y}) = (2\pi t)^{-n/2}\exp\{&\tilde{x}^T\tilde{A}(t)\tilde{x} + \tilde{x}^T\tilde{B}(t)\tilde{y} + \tilde{y}^T\tilde{C}(t)\tilde{y} \\ &+ \tilde{D}^T(t)\tilde{x} + \tilde{E}^T(t)\tilde{y} + s(t)\} \end{aligned} \tag{34}$$

where $\tilde{A}(t) = (\tilde{a}_{ij}(t))$, $\tilde{B}(t) = (\tilde{b}_{ij}(t))$, $\tilde{C}(t) = (\tilde{c}_{ij}(t))$, $1 \le i,\ j \le n$, $\tilde{D}^T(t) = (\tilde{d}_1(t),\ldots,\tilde{d}_n(t))$, $\tilde{E}^T(t) = (\tilde{e}_1(t),\ldots,\tilde{e}_n(t))$. Then

$$\frac{d}{dt}(\tilde{A}(t) + \tilde{A}^T(t)) = (\tilde{A}(t) + \tilde{A}^T(t))^2 - B(t)QB^T(t) \tag{35}$$

$$\frac{d\tilde{B}}{dt}(t) = (\tilde{A}(t) + \tilde{A}^T(t))\tilde{B}(t) \tag{36}$$

$$\frac{d}{dt}(\tilde{C}(t) + \tilde{C}^T(t)) = \tilde{B}^T(t)\tilde{B}(t) \tag{37}$$

$$\begin{aligned} \frac{d\tilde{D}}{dt}(t) &= (\tilde{A}(t) + \tilde{A}^T(t))\tilde{D}^T(t) \\ &\quad + B(t)QB^T(t)b(t) - \tfrac{1}{2}B(t)P \end{aligned} \tag{38}$$

$$\frac{d\tilde{E}}{dt}(t) = \tilde{B}^T(t)\tilde{D}(t) \tag{39}$$

$$\begin{aligned} \frac{ds}{dt}(t) &= \frac{1}{2}\tilde{D}^T(t)\tilde{D}(t) + \operatorname{tr}\tilde{A}(t) \\ &\quad - \frac{1}{2}[b^T(t)B(t)QB^T(t)b(t) \\ &\qquad - P^T B^T(t)b(t) + r)] + \frac{n}{2t}. \end{aligned} \tag{40}$$

PROOF

$$\begin{aligned} \frac{\partial K}{\partial t}(t,\tilde{x},\tilde{y}) = \Bigg[&\tilde{x}^T\frac{d\tilde{A}}{dt}(t)\tilde{x} + \tilde{x}^T\frac{d\tilde{B}}{dt}(t)\tilde{y} + \tilde{y}^T\frac{d\tilde{C}}{dt}(t)\tilde{y} + \frac{d\tilde{D}^T}{dt}(t)\tilde{x} \\ &+ \frac{d\tilde{E}^T}{dt}(t)\tilde{y} + \frac{ds}{dt}(t) - \frac{n}{2t}\Bigg]K(t,\tilde{x},\tilde{y}) \end{aligned} \tag{41}$$

$$\nabla_{\tilde{x}}K(t,\tilde{x},\tilde{y}) = [(\tilde{A}(t) + \tilde{A}^T(t))\tilde{x} + \tilde{B}(t)\tilde{y} + \tilde{D}(t)]K(t,\tilde{x},\tilde{y}) \tag{42}$$

$$\begin{aligned} \frac{1}{2}\Delta_{\tilde{x}}K(t,\tilde{x},\tilde{y}) = \Bigg[&\frac{1}{2}\tilde{x}^T(\tilde{A}(t) + \tilde{A}^T(t))^2\tilde{x} + \tilde{x}^T(\tilde{A}(t) + \tilde{A}^T(t))\tilde{B}(t)\tilde{y} \\ &+ \frac{1}{2}\tilde{y}^T\tilde{B}^T(t)\tilde{B}(t)\tilde{y} + \tilde{x}^T(\tilde{A}(t) + \tilde{A}^T(t))\tilde{D}(t) \\ &+ \tilde{y}^T\tilde{B}^T(t)\tilde{D}(t) + \frac{1}{2}\tilde{D}^T(t)\tilde{D}(t) + \operatorname{tr}\tilde{A}(t)\Bigg]K(t,\tilde{x},\tilde{y}). \end{aligned} \tag{43}$$

For $K(t,\tilde{x},\tilde{y})$ to satisfy (32), it is easy to see that we need (35)–(40) by putting (41) and (43) in (32).

Q.E.D.

We rewrite Theorem 3 in the following form.

THEOREM 4 *Let $K(t,\tilde{x},\tilde{y})$ be the fundamental solution of*

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,\tilde{x}) = \dfrac{1}{2}\Delta v(t,\tilde{x}) - \dfrac{1}{2}q(B^T(t)\tilde{x} - B^T(t)b(t))v(t,\tilde{x}) \\ v(0,\tilde{x}) = \sigma_1(\tilde{x})e^{-\phi(\tilde{x})} \end{cases} \tag{44}$$

where

$$\begin{aligned} q(B^T(t)\tilde{x} - B^T(t)b(t)) &= \tilde{x}^T B(t)QB^T(t)\tilde{x} - [2(b^T(t)B(t)QB^T(t)) - P^T B^T(t)]\tilde{x} \\ &\quad + b^T(t)B(t)QB^T(t)b(t) - P^T B^T(t)b(t) + r. \end{aligned} \tag{45}$$

Assume that the fundamental solution $K(t,\tilde{x},\tilde{y})$ is written as

$$\begin{aligned} K(t,\tilde{x},\tilde{y}) = (2\pi t)^{-n/2}\exp\{&\tilde{x}^T\tilde{A}(t)\tilde{x} + \tilde{x}^T\tilde{B}(t)\tilde{y} + \tilde{y}^T\tilde{C}(t)\tilde{y} \\ &+ \tilde{D}^T(t)\tilde{x} + \tilde{E}^T(t)\tilde{y} + s(t)\} \end{aligned} \tag{46}$$

where $\tilde{A}(t) = \tilde{A}^T(t) = (\tilde{a}_{ij}(t))$, $\tilde{B}(t) = (\tilde{b}_{ij}(t))$, $\tilde{C}(t) = \tilde{C}^T(t) = (\tilde{c}_{ij}(t))$, $1 \le i,\ j \le n$, $\tilde{D}^T(t) = (\tilde{d}_1(t),\ldots,\tilde{d}_n(t))$, $\tilde{E}^T(t) = (\tilde{e}_1(t),\ldots,\tilde{e}_n(t))$. Then

$$\frac{d}{dt}\tilde{A}(t) = 2\tilde{A}^2(t) - \frac{1}{2}B(t)QB^T(t) \tag{47}$$

$$\frac{d\tilde{B}}{dt}(t) = 2\tilde{A}(t)\tilde{B}(t) \tag{48}$$

$$\frac{d}{dt}\tilde{C}(t) = \frac{1}{2}\tilde{B}^T(t)\tilde{B}(t) \tag{49}$$

$$\frac{d\tilde{D}}{dt}(t) = 2\tilde{A}(t)\tilde{D}(t) + B(t)QB^T(t)b(t) - \frac{1}{2}B(t)P \tag{50}$$

$$\frac{d\tilde{E}}{dt}(t) = \tilde{B}^T(t)\tilde{D}(t) \tag{51}$$

$$\begin{aligned} \frac{ds}{dt} &= \frac{1}{2}\tilde{D}(t)^T\tilde{D}(t) + \operatorname{tr}\tilde{A}(t) \\ &\quad - \frac{1}{2}[b^T(t)B(t)QB^T(t)b(t) - P^T B^T(t)b(t) + r] + \frac{n}{2t}. \end{aligned} \tag{52}$$

PROPOSITION 1 *Suppose that*

$$\tilde{A}(t) = \sum_{n=-1}^{\infty} \tilde{A}_n t^n, \qquad \tilde{B}(t) = \sum_{n=-1}^{\infty} \tilde{B}_n t^n, \qquad \tilde{C}(t) = \sum_{n=-1}^{\infty} \tilde{C}_n t^n \tag{53}$$

$$\tilde{D}(t) = \sum_{n=-1}^{\infty} \tilde{D}_n t^n, \qquad \tilde{E}(t) = \sum_{n=-1}^{\infty} \tilde{E}_n t^n, \qquad s(t) = \sum_{n=-1}^{\infty} s_n t^n \tag{54}$$

$$b(t) = \sum_{n=0}^{\infty} b^n t^n, \qquad B(t) = \sum_{n=0}^{\infty} B_n t^n. \tag{55}$$

Then the following holds.

1) *Equation* (47) *is equivalent to*

$$-\tilde{A}_{-1} = 2\tilde{A}_{-1}^2 \tag{56}$$

$$0 = 2(\tilde{A}_{-1}\tilde{A}_0 + \tilde{A}_0\tilde{A}_{-1}) \tag{57}$$

$$\begin{aligned} n\tilde{A}_n = {} & 2(\tilde{A}_{-1}\tilde{A}_n + \tilde{A}_0\tilde{A}_{n-1} + \cdots + \tilde{A}_{n-1}\tilde{A}_0 + \tilde{A}_n\tilde{A}_{-1}) \\ & - \tfrac{1}{2}(B_0 Q B_{n-1}^T + B_1 Q B_{n-2}^T + \cdots \\ & + B_{n-2} Q B_1^T + B_{n-1} Q B_0^T), \qquad n \geq 1. \end{aligned} \tag{58}$$

2) *Equation* (48) *is equivalent to*

$$-\tilde{B}_{-1} = 2\tilde{A}_{-1}\tilde{B}_{-1} \tag{59}$$

$$0 = 2(\tilde{A}_{-1}B_0 + \tilde{A}_0\tilde{B}_{-1}) \tag{60}$$

$$\begin{aligned} n\tilde{B}_n = {} & 2(\tilde{A}_{-1}\tilde{B}_n + \tilde{A}_0\tilde{B}_{n-1} + \cdots \\ & + \tilde{A}_{n-1}\tilde{B}_0 + \tilde{A}_n\tilde{B}_{-1}), \qquad n \geq 1. \end{aligned} \tag{61}$$

3) *Equation* (49) *is equivalent to*

$$-\tilde{C}_{-1} = \tfrac{1}{2}\tilde{B}_{-1}^T\tilde{B}_{-1} \tag{62}$$

$$0 = \tfrac{1}{2}(\tilde{B}_{-1}^T\tilde{B}_0 + \tilde{B}_0^T\tilde{B}_{-1}) \tag{63}$$

$$\begin{aligned} n\tilde{C}_n = {} & \tfrac{1}{2}(\tilde{B}_{-1}^T\tilde{B}_n + \tilde{B}_0^T\tilde{B}_{n-1} + \cdots \\ & + \tilde{B}_{n-1}^T\tilde{B}_0 + \tilde{B}_n^T\tilde{B}_{-1}), \qquad n \geq 1. \end{aligned} \tag{64}$$

4) *Equation* (50) *is equivalent to*

$$-\tilde{D}_{-1} = 2\tilde{A}_{-1}\tilde{D}_{-1} \tag{65}$$

$$0 = 2(\tilde{A}_{-1}\tilde{D}_0 + \tilde{A}_0\tilde{D}_{-1}) \tag{66}$$

$$\begin{aligned} n\tilde{D}_n = {} & 2(\tilde{A}_{-1}\tilde{D}_n + \tilde{A}_0\tilde{D}_{n-1} + \cdots + \tilde{A}_{n-1}\tilde{D}_0 + \tilde{A}_n\tilde{D}_{-1}) \\ & + (B_0 Q B_{n-1}^T + B_1 Q B_{n-2}^T + \cdots \\ & \quad + B_{n-2} Q B_1^T + B_{n-1} Q B_0^T) b^0 \\ & + (B_0 Q B_{n-2}^T + B_1 Q B_{n-3}^T + \cdots \\ & \quad + B_{n-3} Q B_1^T + B_{n-2} Q B_0^T) b^1 \\ & + \cdots + (B_0 Q B_1^T + B_1 Q B_0^T) b^{n-2} + B_0 Q B_0^T b^{n-1} \\ & - \tfrac{1}{2} B_{n-1} P, \qquad n \geq 1. \end{aligned} \tag{67}$$

5) *Equation* (51) *is equivalent to*

$$-\tilde{E}_{-1} = \tilde{B}_{-1}^T\tilde{D}_{-1} \tag{68}$$

$$0 = \tilde{B}_{-1}^T\tilde{D}_0 + \tilde{B}_0^T\tilde{D}_{-1} \tag{69}$$

$$\begin{aligned} n\tilde{E}_n = {} & \tilde{B}_{-1}^T\tilde{D}_n + \tilde{B}_0^T\tilde{D}_{n-1} + \cdots \\ & + \tilde{B}_{n-1}^T\tilde{D}_0 + \tilde{B}_n^T\tilde{D}_{-1}, \qquad n \geq 1. \end{aligned} \tag{70}$$

6) *Equation* (52) *is equivalent to*

$$s_{-1} = -\frac{1}{2}\tilde{D}_{-1}^T\tilde{D}_{-1} \tag{71}$$

$$0 = \frac{1}{2}(\tilde{D}_{-1}^T\tilde{D}_0 + \tilde{D}_0\tilde{D}_{-1}) + \operatorname{tr}\tilde{A}_{-1} + \frac{n}{2} \tag{72}$$

$$\begin{aligned} s_1 = {} & \frac{1}{2}(\tilde{D}_{-1}^T\tilde{D}_1 + \tilde{D}_0^T\tilde{D}_0 + \tilde{D}_1^T\tilde{D}_{-1}) + \operatorname{tr}\tilde{A}_0 \\ & - \frac{1}{2}(b^{0^T} B_0 Q B_0^T b^0 - P^T B_0^T b^0 + r) \end{aligned} \tag{73}$$

$$\begin{aligned} ns_n = {} & \frac{1}{2}(\tilde{D}_{-1}^T\tilde{D}_n + \tilde{D}_0^T\tilde{D}_{n-1} + \cdots + \tilde{D}_{n-1}^T\tilde{D}_0 + \tilde{D}_n^T\tilde{D}_{-1}) \\ & + \operatorname{tr}\tilde{A}_{n-1} - \frac{1}{2}[b^{0^T} B_0 Q (B_0^T b^{n-1} + B_1^T b^{n-2} + \cdots + B_{n-1}^T b^0) \\ & \quad + (b^{0^T} B_1 + b^{1^T} B_0) Q (B_0^T b^{n-2} + B_1^T b^{n-3} \\ & \quad + \cdots + B_{n-2}^T b^0) + \cdots + (b^{0^T} B_{n-1} + b^{1^T} B_{n-2} \\ & \quad + \cdots + b^{n-1^T} B_0) Q B_0^T b^0] \\ & + \frac{1}{2} P^T (B_0^T b^{n-1} + B_1^T b^{n-2} + \cdots + B_{n-1}^T b^0), \qquad n \geq 2. \end{aligned} \tag{74}$$

PROOF Direct computation. Q.E.D.

THEOREM 5 *The fundamental solution* $K(t,\tilde{x},\tilde{y})$ *of*

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,\tilde{x}) = \dfrac{1}{2}\Delta v(t,\tilde{x}) - \dfrac{1}{2}q(B^T(t)\tilde{x} - B^T(t)b(t))v(t,\tilde{x}), \\ \hfill 0 \leq t \leq \tau_1 \\ v(0,\tilde{x}) = \sigma_1(\tilde{x})e^{-\phi(\tilde{x})} \end{cases} \tag{75}$$

where

$$\begin{aligned} & q(B^T(t)\tilde{x} - B^T(t)b(t)) \\ & \quad = \tilde{x}^T B(t) Q B^T(t)\tilde{x} - [2(b^T(t)B(t)QB^T(t)) - P^T B^T(t)]\tilde{x} \\ & \qquad + b^T(t)B(t)QB^T(t)b(t) - P^T B^T(t)b(t) + r \end{aligned} \tag{76}$$

exists and is of the following form

$$\begin{aligned} & K(t,\tilde{x},\tilde{y}) \\ & \quad = (2\pi t)^{-n/2}\exp\left\{ -\frac{|\tilde{x}-\tilde{y}|^2}{2t} + \tilde{x}^T\tilde{A}(t)\tilde{x} + \tilde{x}^T\tilde{B}(t)\tilde{y} + \tilde{y}^T\tilde{C}(t)\tilde{y} \right. \\ & \qquad \left. + \tilde{D}^T(t)\tilde{x} + \tilde{E}^T(t)\tilde{y} + s(t) \right\} \end{aligned} \tag{77}$$

where $\tilde{A}(t) = \sum_{n=1}^{\infty}\tilde{A}_n t^n$, $\tilde{B}(t) = \sum_{n=1}^{\infty}\tilde{B}_n t^n$, $\tilde{C}(t) = \sum_{n=1}^{\infty}\tilde{C}_n t^n$, $\tilde{D}(t) = \sum_{n=1}^{\infty}\tilde{D}_n t^n$, $\tilde{E}(t) = \sum_{n=1}^{\infty}\tilde{E}_n t^n$, $s(t) = \sum_{n=1}^{\infty} s_n t^n$, $B(t) = \sum_{n=0}^{\infty} B_n t^n$, $b(t) = \sum_{n=0}^{\infty} b^n t^n$.

Moreover, $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$, $\tilde{D}_n$, $\tilde{E}_n$ and s_n can be computed by the following formulas:

$$\tilde{A}_1 = -\frac{1}{6}B_0QB_0^T \tag{78}$$

$$\tilde{A}_2 = -\frac{1}{8}(B_0QB_1^T + B_1QB_0^T) \tag{79}$$

$$\tilde{A}_n = \frac{2}{n+2}(\tilde{A}_1\tilde{A}_{n-2} + \tilde{A}_2\tilde{A}_{n-3} + \cdots + \tilde{A}_{n-2}\tilde{A}_1) - \frac{1}{2(n+2)} \times (B_0QB_{n-1}^T + B_1QB_{n-2}^T + \cdots + B_{n-2}QB_1^T + B_{n-1}QB_0^T), \qquad n \geq 3 \tag{80}$$

$$\tilde{B}_1 = \tilde{A}_1 \tag{81}$$

$$\tilde{B}_2 = \frac{2}{3}\tilde{A}_2 \tag{82}$$

$$\tilde{B}_n = \frac{2}{n+1}(\tilde{A}_1\tilde{B}_{n-2} + \tilde{A}_2\tilde{B}_{n-3} + \cdots + \tilde{A}_{n-2}\tilde{B}_1 + \tilde{A}_n), \qquad n \geq 3 \tag{83}$$

$$\tilde{C}_1 = \frac{1}{2}(\tilde{B}_1 + \tilde{B}_1^T) \tag{84}$$

$$\tilde{C}_2 = \frac{1}{4}(\tilde{B}_2 + \tilde{B}_2^T) \tag{85}$$

$$\tilde{C}_n = \frac{1}{2n}(\tilde{B}_n + \tilde{B}_1^T\tilde{B}_{n-2} + \tilde{B}_2^T\tilde{B}_{n-3} + \cdots + \tilde{B}_{n-2}^T\tilde{B}_1 + \tilde{B}_n^T), \qquad n \geq 3 \tag{86}$$

$$\tilde{D}_1 = \frac{1}{2}(B_0QB_0^T)b^0 - \frac{1}{4}B_0P \tag{87}$$

$$\tilde{D}_2 = \frac{1}{3}B_0QB_1^Tb^0 + B_1QB_0^T)b^0 + \frac{1}{3}B_0QB_0^Tb^1 - \frac{1}{6}B_1P \tag{88}$$

$$\begin{aligned}\tilde{D}_n = {} & \frac{2}{n+1}(\tilde{A}_1\tilde{D}_{n-2} + \tilde{A}_2\tilde{D}_{n-3} + \cdots + \tilde{A}_{n-2}\tilde{D}_1) \\ & + \frac{1}{n+1}(B_0QB_{n-1}^T + B_1QB_{n-2}^T + \cdots + B_{n-2}QB_1^T + B_{n-1}QB_0^T)b^0 \\ & + \frac{1}{n+1}(B_0QB_{n-2}^T + B_1QB_{n-3}^T + \cdots + B_{n-3}QB_1^T + B_{n-2}QB_0^T)b^1 \\ & + \cdots + \frac{1}{n+1}(B_0QB_1^T + B_1QB_0^T)b^{n-2} + \frac{1}{n+1}B_0QB_0^Tb^{n-1} \\ & - \frac{1}{2(n+1)}B_{n-1}P, \qquad n \geq 3\end{aligned} \tag{89}$$

$$\tilde{E}_1 = \tilde{D}_1 \tag{90}$$

$$\tilde{E}_2 = \frac{1}{2}\tilde{D}_2 \tag{91}$$

$$\tilde{E}_n = \frac{1}{n}(\tilde{D}_n + \tilde{B}_1^T\tilde{D}_{n-2} + \tilde{B}_2^T\tilde{D}_{n-3} + \cdots + \tilde{B}_{n-2}^T\tilde{D}_1), \qquad n \geq 3 \tag{92}$$

$$s_1 = -\frac{1}{2}(b^{0^T}B_0QB_0^Tb^0 - P^TB_0^Tb^0 + r) \tag{93}$$

$$\begin{aligned}s_2 = {} & \frac{1}{2}\mathrm{tr}\tilde{A}_1 - \frac{1}{4}[b^{0^T}B_0Q(B_0^Tb^1 + B_1^Tb^0) + (b^{0^T}B_1 + b^{1^T}B_0)QB_0^Tb^0] \\ & + \frac{1}{4}P^T(B_0^Tb^1 + B_1^Tb^0)\end{aligned} \tag{94}$$

$$\begin{aligned}s_n = {} & \frac{1}{2n}(\tilde{D}_1^T\tilde{D}_{n-2} + \tilde{D}_2^T\tilde{D}_{n-3} + \cdots + \tilde{D}_{n-2}^T\tilde{D}_1) + \frac{1}{n}\mathrm{tr}\tilde{A}_{n-1} \\ & - \frac{1}{2n}[b^{0^T}B_0Q(B_0^Tb^{n-1} + B_1^Tb^{n-2} + \cdots + B_{n-1}^Tb^0) \\ & + (b^{0^T}B_1 + b^{1^T}B_0)Q(B_0^Tb^{n-2} + B_1^Tb^{n-3} + \cdots + B_{n-2}^Tb^0) \\ & + \cdots + (b^{0^T}B_{n-1} + b^{1^T}B_{n-2} + \cdots + b^{n-1T}B_0)QB_0^Tb^0] \\ & + \frac{1}{2n}P^T(B_0^Tb^{n-2} + B_1^Tb^{n-1} + \cdots + B_{n-1}^Tb^0), \qquad n \geq 3.\end{aligned} \tag{95}$$

PROOF Observe that if we let

$$\tilde{A}_{-1} = -\tfrac{1}{2}I = \tilde{C}_{-1}, \qquad \tilde{B}_{-1} = I$$
$$\tilde{A}_0 = 0 = \tilde{B}_0 = \tilde{C}_0 = \tilde{D}_{-1} = \tilde{D}_0 = \tilde{E}_{-1} = \tilde{E}_0$$
$$s_0 = 0 = s_{-1}$$

and $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$, $\tilde{D}_n$, $\tilde{E}_n$, s_n, $n \geq 1$ as in (78)–(95), then (56)–(74) are satisfied. By Proposition 1, (47)–(52) are satisfied. In order to show that $K(t,\tilde{x},\tilde{y})$ in (77) is a fundamental solution of (75), it remains to show that

$$\lim_{t\to 0}\int_{\mathbb{R}^n} K(t,\tilde{x},\tilde{y})v(0,\tilde{y})d\tilde{y} = v(0,\tilde{x}). \tag{96}$$

By replacing $\tilde{y}$ by $\tilde{x} - \tilde{y}$, we see that

$$\begin{aligned}&\lim_{t\to 0}\int_{\mathbb{R}^n} K(t,\tilde{x},\tilde{y})v(0,\tilde{y})d\tilde{y} \\ &\quad = \lim_{t\to 0}\int_{\mathbb{R}^n} K(t,\tilde{x},\tilde{x}-\tilde{y})v(0,\tilde{x}-\tilde{y})d\tilde{y} \\ &\quad = \lim_{t\to 0}\int_{\mathbb{R}^n} (2\pi t)^{-n/2}\exp\left\{-\frac{|\tilde{y}|^2}{2t} + \tilde{x}^T\tilde{A}(t)\tilde{x} + \tilde{x}^T\tilde{B}(t)(\tilde{x}-\tilde{y}) + (\tilde{x}-\tilde{y})^T\tilde{C}(t)(\tilde{x}-\tilde{y}) + \tilde{D}^T(t)\tilde{x} + \tilde{E}^T(t)(\tilde{x}-\tilde{y}) + s(t)\right\} \\ &\qquad \times v(0,\tilde{x}-\tilde{y})d\tilde{y}.\end{aligned}$$

Let $\tilde{y} = \sqrt{2t}z$ where $z = (z_1,\ldots,z_n)$. Then

$$\begin{aligned}&\lim_{t\to 0}\int_{\mathbb{R}^n} K(t,\tilde{x},\tilde{y})v(0,\tilde{y})d\tilde{y} \\ &\quad = \lim_{t\to 0}\int_{\mathbb{R}^n} (\pi)^{-n/2}\exp\{-|z|^2 + \tilde{x}^T\tilde{A}(t)\tilde{x} \\ &\qquad + \tilde{x}^T\tilde{B}(t)(\tilde{x}-\sqrt{2t}z) + (\tilde{x}-\sqrt{2t}z)^T\tilde{C}(t)(\tilde{x}-\sqrt{2t}z) \\ &\qquad + \tilde{D}^T(t)\tilde{x} + \tilde{E}^T(t)(\tilde{x}-\sqrt{2t}z) + s(t)\}v(0,\tilde{x}-\sqrt{2t}z)dz \\ &\quad = \int_{\mathbb{R}^n}(\pi)^{-n/2}e^{-|z|^2}v(0,\tilde{x})dz \\ &\quad = v(0,\tilde{x}).\end{aligned}$$

Q.E.D.

THEOREM 6 *The solution of*

$$\begin{cases}\dfrac{\partial v}{\partial t}(t,\tilde{x}) = \dfrac{1}{2}\Delta v(t,\tilde{x}) - \dfrac{1}{2}q(B^T(t)\tilde{x} - B^T(t)b(t))v(t,\tilde{x}), \\ \qquad\qquad \tau_{i-1} \leq t \leq \tau_i \\ v(\tau_{i-1},\tilde{x}) = \sigma_i(\tilde{x})e^{-\phi(\tilde{x})}\end{cases} \tag{97}$$

where

$$\begin{aligned}&q(B^T(t)\tilde{x} - B^T(t)b(t)) \\ &\quad = \tilde{x}^TB(t)QB^T(t)x - [2(b^T(t)B(t)QB^T(t)) - P^TB^T(t)]\tilde{x} \\ &\qquad + b^T(t)B(t)QB^T(t)b(t) - P^TB^T(t)b(t) + r\end{aligned} \tag{98}$$

is given by

$$v(t,\tilde{x}) = \int_{\mathbb{R}^n} K(t,\tilde{x},\tilde{y})v(\tau_{i-1},\tilde{y})d\tilde{y}.$$

Here

$$\begin{aligned} K(t,\tilde{x},\tilde{y}) &= [2\pi(t-\tau_{i-1})]^{-n/2} \\ &\times \exp\left\{ -\frac{|\tilde{x}-\tilde{y}|^2}{2(t-\tau_{i-1})} + \tilde{x}^T\tilde{A}(t-\tau_{i-1})\tilde{x} \right. \\ &\quad + \tilde{x}^T\tilde{B}(t-\tau_{i-1})\tilde{y} + \tilde{y}^T\tilde{C}(t-\tau_{i-1})\tilde{y} \\ &\quad + \tilde{D}^T(t-\tau_{i-1})\tilde{x} + \tilde{E}^T(t-\tau_{i-1})\tilde{y} \\ &\quad \left. + s(t-\tau_{i-1}) \right\} \end{aligned} \tag{99}$$

where $\tilde{A}(t-\tau_{i-1}) = \sum_{n=1}^{\infty} \tilde{A}_n(t-\tau_{i-1})^n$, $\tilde{B}(t-\tau_{i-1}) = \sum_{n=1}^{\infty} \tilde{B}_n(t-\tau_{i-1})^n$, $\tilde{C}(t-\tau_{i-1}) = \sum_{n=1}^{\infty} \tilde{C}_n(t-\tau_{i-1})^n$, $\tilde{D}(t-\tau_{i-1}) = \sum_{n=1}^{\infty} \tilde{D}_n(t-\tau_{i-1})^n$, $\tilde{E}(t-\tau_{i-1}) = \sum_{n=1}^{\infty} E_n(t-\tau_{i-1})^n$, $s(t-\tau_{i-1}) = \sum_{n=1}^{n} s_n(t-\tau_{i-1})^n$, $B(t-\tau_{i-1}) = \sum_{n=0}^{\infty} B_n(t-\tau_{i-1})^n$, $b(t-\tau_{i-1}) = \sum_{n=0}^{\infty} b^n(t-\tau_{i-1})^n$ *can be computed via* (78)–(95).

VI. CONCLUSION

We show that in order to solve the nonlinear filtering problem for the Yau filtering system with arbitrary initial condition, it suffices to solve a time-varying Schrödinger equation with arbitrary initial condition. We actually solve the time-varying Schrödinger equation with arbitrary initial condition by constructing the fundamental solution explicitly in case the potential is quadratic in state variables (which include the case that the observation $h_i(x)$, $1 \le i \le m$, are nonlinear but with linear growth). The fundamental solution is constructed via a system of nonlinear ODEs. This system of nonlinear ODEs is solved explicitly by power series method.

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