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Wavelet-Galerkin Method for the Kolmogorov Equation

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Abstract—It is well known that the Kolmogorov equation plays an important role in applied science. For example, the nonlinear filtering problem, which plays a key role in modern technologies, was solved by Yau and Yau [1] by reducing it to the off-line computation of the Kolmogorov equation. In this paper, we develop a theorical foundation of using the wavelet-Galerkin method to solve linear parabolic P.D.E. We apply our theory to the Kolmogorov equation. We give a rigorous proof that the solution of the Kolmogorov equation can be approximated very well in any finite domain by our wavelet-Galerkin method. An example is provided by using Daubechies D_4 scaling functions.

Keywords—Nonlinear filtering, Kolmogorov equation, Wavelet-Galerkin method, Daubechies scaling function, Pyramid algorithm.

1. INTRODUCTION

Despite its usefulness, the Kalman-Bucy filter is not perfect. One of its weaknesses is that it needs a Gaussian assumption on the initial data. The situation is more complex when the statistics of the initial condition are modeled by an arbitrary distribution. As observed by Makowski [2], in that event, the filtering question is genuinely nonlinear, and few results have been obtained. Notable exceptions are the works of Benes and Karatzas [3], Ocone [4], and Makowski [2]. In [2], simple and direct probabilistic arguments are developed for evaluating the conditional expectation $\pi_t(\varphi(x_t))$ of the state density $\varphi(x_t)$ given the observations $\{y_s \mid 0 \leq s \leq t\}$. It was shown as in [3,4] that there always exists a set of sufficient statistics that can be recursively computed as outputs of a finite-dimensional dynamic system. In contrast with previous results, the sufficient statistics generated in [2] can be termed "universal" in the sense that they are independent of the initial state distribution. Furthermore, no assumptions on the moments of this initial state distribution or its absolute continuity are made in [2], as was the case in [3,4].

However, Makowski's method has a major disadvantage. Let n be the dimension of the state space. The number of sufficient statistics in order to compute the conditional expectation $\pi_t(\varphi(x_t))$ of $\varphi(x_t)$ in Makowski's method is a polynomial of degree two in n, while for the classical Kalman-Bucy filter, the number of sufficient statistics is only a polynomial of degree one

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in n. In the case where the linear filter system is completely reachable and completely observable, Hazewinkel observed in [5, p. 115] that the estimation algebra E is the $2n+2$-dimensional Lie algebra with an explicitly given basis. Even in this case, the Wei-Norman approach of finding an explicit filter is more complicated than the method of Yau and Yau [6]. Not only must one solve a finite system of ordinary differential equations and a Kolmogorov equation, but one also has to integrate n partial differential equations corresponding to operators $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$. More important, if the Kalman-Bucy system is not completely reachable or completely observable, then the basis of the estimation algebra is not explicitly known (although it can be computed). As a result, there is an additional disadvantage of the Wei-Norman approach: one cannot write down the finite system of ordinary differential equations explicitly.

The novelty of the method of Yau and Yau [6] is that their finite system of ordinary differential equations is explicitly written down and only n sufficient statistics are needed in order to compute the conditional expectation. The problem of computing the conditional probability density is factored into two parts:

(1) the on-line solution of a finite system of ordinary differential equations, and
(2) the off-line calculation of the Kolmogorov equation, which does not depend on observations.

Hence, any method to solve the Kolmogorov equation provides the fundamental step of the Wei-Norman's approach to solving DMZ equation, which is the central problem of nonlinear filtering.

With the appearance of wavelet functions, especially Daubechies' wavelets, people recognized that wavelets provide a powerful tool that can be applied in the finite-element method. There are two reasons that one wants to use wavelets in the finite-element method. First, the orthonormal bases of the compact supported wavelets constructed by Daubechies [7] are unconditional bases for Sobolev spaces, and therefore, provide accurate approximations to PDEs' solutions. Furthermore, the multiresolution analysis properties of these bases, described in [8,9], work well with multigrid methods and adaptive grid refinement methods. Thus, they perform well even for PDEs with initial ill-conditions. Second, the locality of the Daubechies functions and the pyramid algorithm of Mallat described in [10] are extremely efficient for adaptive finite-element methods.

Adaptive finite-element methods have been proposed by Brandt [11] for the elliptic problem and developed by Bank [12] and others. More recently, Berger and Oliger [13] studied and implemented an adaptive mesh-refining method for a hyperbolic partial differential equation, which was successful in solving previously intractable problems [14]. They used a sequence of progressively finer nested grids in space. An automatic error estimation step determines locally whether the current resolution of the numerical solution was sufficient or a finer grid was needed. The main difficulty was to find stable and accurate difference approximations of the differential operators at the interfaces between grids of different sizes.

Wavelets orthogonal bases are excellent examples of hierarchical bases. Liandrat and Tchamitchian [15] have shown that the multiresolution structure of wavelets orthonormal bases is a simple and effective framework for spatial adaptive algorithms. Instead of refining the computations through nested grids of successively finer meshes, as in the algorithm of Berger and Oliger [13], wavelet orthonormal bases implement adaptive refinement by successively adding layers of details that increase the resolution of the approximation locally. Communication between the different layers is regulated automatically by the orthogonality of the basis functions and the pyramid algorithm.

In this paper, we solve the Kolmogorov equation by the wavelet-Galerkin method.

In Section 3, we will discuss the use of this finite-element method in solving partial differential equations. Here, we use Daubechies' wavelets as basis functions because of their ability to approximate many functions and because of the pyramid algorithm.

In order to apply the wavelet-Galerkin method to the Kolmogorov equation, first, we will show that we can use the solution of the initial-boundary value problem

$$\begin{aligned} \frac{\partial \bar{u}}{\partial t} &= \mathbf{A}\bar{u}, & x \in \Omega, \\ \bar{u}(x,0) &= u_0(x), & x \in \Omega, \\ \bar{u}(a,t) &= \bar{u}(b,t) = 0, \end{aligned} \tag{1.1}$$

as an approximation to the solution of the Kolmogorov equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \mathbf{A}u, \\ u(x,0) &= u_0(x), \end{aligned} \tag{1.2}$$

where $\mathbf{A}$ is a differential operator with respect to x in either form

$$\mathbf{A} = \frac{1}{2}\sum_i \frac{\partial^2}{\partial x_i^2} - \sum_i \left(f_i(x) + \frac{\partial F}{\partial x_i}(x)\right)\frac{\partial}{\partial x_i} - \sum_i \left(\frac{\partial f_i}{\partial x_i} + \frac{\partial^2 F}{\partial x_i^2}(x)\right) \tag{1.3}$$

or form

$$\mathbf{A} = \frac{1}{2}\sum_i \frac{\partial^2}{\partial x_i^2} - \sum_i f_i(x)\frac{\partial}{\partial x_i} - \sum_i \left(\frac{\partial f_i}{\partial x_i} + \frac{1}{2}h_i^2(x)\right). \tag{1.4}$$

Then, second, we apply the wavelet-Galerkin method to equation (1.1) to find the numerical approximation to its solution in form $u_n(x,t) = \sum_{k=2^a-R+1}^{2^b-1} \lambda_{n,k}\varphi_{n,k}$. In view of the results from the first part, this is also the numerical approximation of the Kolmogorov equation (1.2).

In Section 4, we will discuss how we select an interval based upon which we will calculate the numerical solution for the Kolmogorov equation. We derive a method called the time-dependent-boundary wavelet-Galerkin method. With this method, we first determine an initial interval Ω for the boundary value problem (1.1) based upon the initial condition of the Kolmogorv equation (1.2). Then, during the computation, we adjust this interval based upon the numerical solution of (1.1). Thus, we prove that the solution of the time-dependent-boundary wavelet-Galerkin method is really the numerical approximation to the solution of the Kolmogorov equation (1.2).

2. APPROXIMATION PROPERTIES OF DAUBECHIES' FUNCTIONS

Daubechies' functions are one type of wavelet. The advantage of Daubechies' functions is that they are compact supported, orthonormal, and easy to construct.

For every integer $N \geq 1$, let $R = 2N-1$. From [7], we have the Daubechies' scaling function φ and wavelet ψ satisfying

$$\varphi(x) = \sum_{k=0}^{R} h(k)\sqrt{2}\varphi(2x-k)$$

and

$$\psi(x) = \sum_{k=0}^{R} g(k)\sqrt{2}\varphi(2x-k),$$

where

1. $h(k) \neq 0$, $0 \leq k \leq R$, and $g(k) = (-1)^k h(R-k)$,
2. $\sum_k h(k) = \sqrt{2}$,
3. $\sum_k h(k)h(k+2m) = \delta_{0m}$, for every integer m,
4. $\sum_k g(k)k^m = 0$, whenever $0 \leq m \leq N-1$, and
5. $\mathrm{Supp}(\varphi) = [0,R]$, $\mathrm{Supp}(\psi) = [0,R]$.

Define $\varphi_{n,k} = 2^{n/2}\varphi(2^n x - k)$ and $\psi_{n,k} = 2^{n/2}\psi(2^n x - k)$. Then,

$$\forall\, j,k, \qquad \int_{-\infty}^{\infty} x^m \psi_{j,k}(x)\,dx = 0, \qquad 0 \le m \le N-1. \tag{2.1}$$

Let

$$V_n = \text{ closure of } \left\{ \sum_k a_k \varphi_{n,k} : k \text{ an integer} \right\} \subset L^2(\mathbf{R}), \tag{2.2}$$

$$W_n = \text{ closure of } \left\{ \sum_k b_k \psi_{n,k} : k \text{ an integer} \right\} \subset L^2(\mathbf{R}). \tag{2.3}$$

From [7], we know the following.

(i) W_n is the orthogonal complement of V_n in V_{n+1}, or $V_{n+1} = V_n \oplus W_n$.
(ii) Closure $(\bigcup_n V_n) = L^2(\mathbf{R})$, where $\mathbf{R} = (-\infty, \infty)$.
(iii) $\{\varphi_{n,k} : k \text{ is an integer}\}$, $\{\psi_{n,k} : k \text{ is an integer}\}$ are orthonormal bases for V_n and W_n, respectively.
(iv) $\mathrm{Supp}(\varphi_{n,k}) = \mathrm{supp}(\psi_{n,k}) = [k/2^n, (k+2N-1)/2^n] = [k/2^n, (k+R)/2^n]$.

PROPERTY 2.1.

$$\begin{gathered} \varphi_{n,k} \text{ and } \psi_{n,k} \in C^{\lambda(N)} = \text{space of Hölder continuous functions} \\ \text{with exponent } \lambda(N), \text{ where } \lambda(N) \approx 0.3485N. \end{gathered} \tag{2.4}$$

Let Ω be either $\mathbf{R}$ or a closed interval $[a,b]$, where a, b are rational numbers. We use notation from [16] as follows.

1. $H^0(\Omega) = L^2(\Omega)$ with the standard Hilbert space inner product$\langle \cdot, \cdot \rangle$.
2. $H^m(\Omega) = \{f \in H^{m-1}(\Omega) \mid f' \in H^{m-1}(\Omega)\}$ with Hilbert space inner product $(\cdot,\cdot)_{m,\Omega}$ defined inductively by $(\cdot,\cdot)_{0,\Omega} = \langle \cdot, \cdot \rangle$ and $(f,g)_{m,\Omega} = \langle f, g\rangle + (f', g')_{m-1,\Omega}$.
3. The associated norm $\|\cdot\|_{m,\Omega}$ is given by $\|f\|_{m,\Omega} = \sqrt{(f,f)_{m,\Omega}}$, for $f \in H^m(\Omega)$.
4. Also for $f \in H^m(\Omega)$, we define $|f|_{m,\Omega} = \|D^m f\|_{0,\Omega}$, where $D = \frac{d}{dx}$.
5. $H_0^m(\Omega) = \{f \in H^m(\Omega) \mid f(a) = f(b) = 0\}$.

In what follows, we use the following notation. Suppose $\Omega = [a,b]$ and p is a positive integer. Then,

$$I_n^\Omega = \{k \in \mathbf{Z} \mid \mathrm{supp}(\varphi_{n,k}) \cap \Omega \neq \emptyset\} = \{k \in \mathbf{Z} \mid 2^n a - R < k < 2^n b\},$$

$$V_n^\Omega = \left\{ \sum_{k \in I_n^\Omega} \lambda_k \varphi_{n,k} \,\middle|\, \lambda_k \in \mathbf{R} \right\}.$$

For $f \in H^r(\mathbf{R})$,

$$P_n^\Omega(f) = \left(\sum_{k \in I_n^\Omega} \langle f|_\Omega, \varphi_{n,k} \rangle \varphi_{n,k} \right) \in V_n^\Omega.$$

It is clear that $V_n^\Omega \subseteq V_n = V_n^{\mathbf{R}}$. In order to derive the approximation property of Daubechies' functions, we need to state the following lemma. The first is proved in [17].

LEMMA 2.1. *If $f \in H_0^1(\Omega)$, where $\Omega = [a,b]$ and $-\infty < a < b < \infty$, then*

$$\pi \|f\|_{0,\Omega} \le (b-a) \|D(f)\|_{0,\Omega} \qquad \textit{(Rayleigh-Ritz inequality).}$$

The next lemma is straightforward. This is easily proved by induction on m.

LEMMA 2.2. *If $f \in H^m(\Omega)$, there is a unique polynomial P of degree $\leq m-1$ (or $=0$), such that*

$$\int_\Omega D^\alpha (f-P)\,dx = 0,$$

for all $0 \leq \alpha \leq m-1$.

Using Lemma 2.2, we can prove the following lemma.

LEMMA 2.3. *Let $\Omega = [a,b]$ be a finite interval. For every $f \in H^m(\Omega)$, such that $\int_\Omega D^\alpha(f)\,dx = 0$, for all $0 \leq \alpha \leq m-1$,*

$$\left\|D^j f\right\|_{0,\Omega} \leq \left(\frac{b-a}{\pi}\right)^{m-j} \|D^m f\|_{0,\Omega}, \qquad 0 \leq j \leq m-1.$$

PROOF. Let $f_j(x) = \int_a^x D^j(f)(s)\,ds$, where $0 \leq j \leq m-1$. Then, $D(f_j) = D^j(f)$. From $\int_\Omega D^\alpha(f)\,dx = 0$, for all $0 \leq \alpha \leq m-1$, we know that $f_j \in H_0^1(\Omega)$. Then, from Lemma 2.1

$$\|f_j\|_{0,\Omega} \leq \frac{b-a}{\pi}\|D(f_j)\|_{0,\Omega} \leq \frac{b-a}{\pi}\left\|D^j(f)\right\|_{0,\Omega}.$$

Doing the integration by part, we get

$$\begin{aligned}\int_\Omega \left(D^j(f)\right)^2 dx &= f_j(b)D^j(f)(b) - f_j(a)D^j(f)(a) - \int_\Omega D^{j+1}(f) f_j\,dx \\ &\leq \left\|D^{j+1}(f)\right\|_{0,\Omega}\|f_j\|_{0,\Omega} \\ &\leq \frac{b-a}{\pi}\left\|D^{j+1}(f)\right\|_{0,\Omega}\|D(f_j)\|_{0,\Omega} \qquad \text{(by Lemma 2.1)} \\ &= \frac{b-a}{\pi}\left\|D^{j+1}(f)\right\|_{0,\Omega}\left\|D^j(f)\right\|_{0,\Omega}.\end{aligned}$$

Therefore,

$$\left\|D^j(f)\right\|_{0,\Omega} \leq \frac{b-a}{\pi}\left\|D^{j+1}(f)\right\|_{0,\Omega}.$$

By induction on j, we get

$$\left\|D^j f\right\|_{0,\Omega} \leq \left(\frac{b-a}{\pi}\right)^{m-j} \|D^m f\|_{0,\Omega}, \qquad 0 \leq j \leq m-1. \qquad \blacksquare$$

LEMMA 2.4. *For $\Omega = [a,b]$ and $-\infty < a < b < \infty$, then*

$$\inf_q \left\{\left\|D^j(f-q)\right\|_{0,\Omega}\right\} \leq \left(\frac{b-a}{\pi}\right)^{m-j} \|D^m f\|_{0,\Omega}, \qquad 0 \leq j \leq m-1,$$

for every $f \in H^m(\Omega)$, where the $\inf_q$ is taken over all degree $\leq m-1$ polynomials q.

PROOF. From Lemma 2.2, there exists a polynomial q_0 of degree $\leq m-1$, such that $\int_\Omega D^j(f-q_0)\,dx = 0$, for all $0 \leq j \leq m-1$. Then, from Lemma 2.3,

$$\begin{aligned}\inf_q \left\{\left\|D^j(f-q)\right\|_{0,\Omega}\right\} &\leq \left\|D^j(f-q_0)\right\|_{0,\Omega} \leq \left(\frac{b-a}{\pi}\right)^{m-j} \|D^m(f-q_0)\|_{0,\Omega} \\ &= \left(\frac{b-a}{\pi}\right)^{m-j} \|D^m(f)\|_{0,\Omega}. \qquad \blacksquare\end{aligned}$$

The next lemma describes the approximation properties of Daubechies' functions.

LEMMA 2.5. *For $n > 0$ and $N \geq 1$, let $f \in H^N(\mathbf{R})$. Then,*

$$|\langle f, \psi_{n,k}\rangle| \leq 2^{-np}\frac{R^p}{\pi^p}\,|f|_{p,S_{n,k}}, \qquad 0 \leq p \leq N,$$

where $S_{n,k} = \operatorname{supp}\psi_{n,k}$.

PROOF. For any polynomial $q(x)$ of degree $\leq p-1$, in view of (2.1), we have

$$\begin{aligned}\langle f, \psi_{n,k}\rangle &= \int_{-\infty}^{\infty}(f-q)\psi_{n,k}\,dx = \int_{S_{n,k}}(f-q)\psi_{n,k}\,dx\\ &\leq \|f-q\|_{0,S_{n,k}}\|\psi_{n,k}\| = \|f-q\|_{0,S_{n,k}}.\end{aligned}$$

Since $|S_{n,k}| = 2^{-n}R$, in view of Lemma 2.4,

$$|\langle f, \psi_{n,k}\rangle| \leq \inf_q \|f-q\|_{0,S_{n,k}} \leq 2^{-np}\frac{R^p}{\pi^p}|f|_{p,S_{n,k}}.$$

∎

COROLLARY 2.1. *For $n > 0$ and $N \geq 1$, let $f \in H^N(\Omega)$, where Ω is a finite interval $[a,b]$. Then, for a fixed value of x_0 where $f(x_0)$ has definition,*

$$\left|\frac{d^l}{dx^l}\left(f - P_n^{\Omega}(f)\right)(x_0)\right| \leq \frac{\Psi R^{p+1}}{\pi^p}\max_{j\geq n, k\in I_j^{\Omega}}\left\{|f|_{p,S_{j,k}}\right\}\frac{2^{-n(p-l-1/2)}}{1-2^{-(p-l-1/2)}}, \qquad l \leq p \leq N,$$

where $\Psi = \sup_{-\infty<x<\infty, 0\leq r\leq l}\frac{d^r\psi}{dx^r}(x)$.

PROOF. In the interval $\Omega = [a,b]$, let $I_\psi^j(x_0) = \{k \mid \psi_{j,k}(x_0) \neq 0\}$. Then, there are only R integers in $I_\psi^j(x_0)$. They satisfy that $2^j x_0 - R < k < 2^j x_0$.

$$\frac{d^l}{dx^l}\left(f - P_n^{\Omega}(f)\right)(x_0) = \sum_{j\geq n}\sum_{k\in I_j^{\Omega}}\langle f|_{\Omega}, \psi_{j,k}\rangle\frac{d^l\psi_{j,k}}{dx^l}(x_0),$$

$$\begin{aligned}\left|\frac{d^l}{dx^l}\left(f - P_n^{\Omega}(f)\right)(x_0)\right| &= \left|\sum_{j\geq n}\sum_{k\in I_\psi^j(x_0)}\langle f|_{\Omega}, \psi_{j,k}\rangle\frac{d^l\psi_{j,k}}{dx^l}(x_0)\right|\\ &\leq \sum_{j\geq n}\sum_{k\in I_\psi^j(x_0)}\left|\langle f|_{\Omega}, \psi_{j,k}\rangle\frac{d^l\psi_{j,k}}{dx^l}(x_0)\right|\\ &\leq \Psi\sum_{j\geq n}2^{jl+j/2}\sum_{k\in I_\psi^j(x_0)}|\langle f, \psi_{j,k}\rangle| \\ &\leq \Psi\sum_{j\geq n}\frac{R^p}{\pi^p}\sum_{k\in I_\psi^j(x_0)}2^{jl+j/2}2^{-jp}|f|_{p,S_{j,k}}\\ &= \frac{\Psi R^p}{\pi^p}\sum_{j\geq n}2^{-j(p-l)+j/2}\sum_{k\in I_\psi^j(x_0)}|f|_{p,S_{j,k}}.\end{aligned} \tag{2.5}$$

Then,

$$\begin{aligned}\left|f(x_0) - P_n^{\Omega}(f)(x_0)\right| &\leq \frac{\Psi R^p}{\pi^p}\sum_{j\geq n}2^{-j(p-l-1/2)}R\max_{j\geq n, k\in I_j^{\Omega}}\{|f|_{p,S_{j,k}}\}\\ &\leq \frac{\Psi R^{p+1}}{\pi^p}\max_{j\geq n, k\in I_j^{\Omega}}\{|f|_{p,S_{j,k}}\}\sum_{j\geq n}2^{-j(p-l-1/2)}\\ &\leq \frac{\Psi R^{p+1}}{\pi^p}\max_{j\geq n, k\in I_j^{\Omega}}\{|f|_{p,S_{j,k}}\}\frac{2^{-n(p-l-1/2)}}{1-2^{-(p-l-1/2)}}.\end{aligned} \tag{2.6}$$

∎

We now derive the main result of this section, which provides the mathematical justification for wavelet-based Galerkin methods applying to the Kolmogorov equation.

THEOREM 2.1. *Let $N \geq 2$ and $\Omega = [a,b]$ is a finite interval, where a, b are integers. If $f \in H^{2p}(\mathbf{R})$, where $1 \leq p \leq N/2$, then*

1.
$$\left\|f - P_n^\Omega(f)\right\|_{p,\mathbf{R}} \leq C(\Omega,p)|f|_{2p,\Omega}2^{-np/2} + \|f\|_{p,\mathbf{R}-\Omega},$$

where
$$C(\Omega,p) = \left(\frac{1}{1-2^{-p/2}}\right)\frac{R^{2p+1}}{\pi^{2p}}\sqrt{b-a}\|\psi\|_p;$$

2.
$$\left\|f - P_n^\Omega(f)\right\|_{0,\mathbf{R}} \leq C(\Omega,p)|f|_{2p,\Omega}2^{-3np/2} + \|f\|_{0,\mathbf{R}-\Omega}.$$

PROOF. Let $p \geq 1$, where

$$f = f_1 + f_2, \qquad f_1(x) = \begin{cases} f(x), & x \in \Omega, \\ 0, & x \in \mathbf{R} - \Omega, \end{cases} \qquad \text{and} \qquad f_2(x) = f(x) - f_1(x).$$

Then, $P_n^\Omega(f_1) = P_n^\Omega f$ and $\|f_2\|_{p,\mathbf{R}} = \|f\|_{p,\mathbf{R}-\Omega}$. From $H^p(\Omega) \subseteq L^2(\mathbf{R}) = V_n \oplus W_n \oplus W_{n+1} \oplus \cdots$, so $f_1 = P_n^\Omega(f_1) + \sum_{j\geq n}\sum_{k\in I_j^\Omega}\langle f_1, \psi_{j,k}\rangle\psi_{j,k}$,

$$\|\psi_{j,k}\|_p^2 = |\psi|_0^2 + 2^{2j}|\psi|_1^2 + \cdots + 2^{2jp}|\psi|_p^2 \leq 2^{2jp}\|\psi\|_p^2.$$

Therefore,

$$\begin{aligned}\left\|f - P_n^\Omega(f)\right\|_{p,\mathbf{R}} &\leq \left\|f_1 - P_n^\Omega(f_1)\right\|_{p,\mathbf{R}} + \|f_2\|_{p,\mathbf{R}} \\ &= \left\|f_1 - P_n^\Omega(f_1)\right\|_{p,\mathbf{R}} + \|f\|_{p,\mathbf{R}-\Omega},\end{aligned}$$

$$\begin{aligned}\left\|f_1 - P_n^\Omega(f_1)\right\|_{p,\mathbf{R}} &= \left\|\sum_{j\geq n}\sum_{k\in I_j^\Omega}\langle f_1,\psi_{j,k}\rangle\psi_{j,k}\right\|_{p,\mathbf{R}} \\ &\leq \sum_{j\geq n}\sum_{k\in I_j^\Omega}|\langle f_1,\psi_{j,k}\rangle|\|\psi_{j,k}\|_p \\ &\leq \frac{R^{2p}}{\pi^{2p}}\|\psi\|_p\sum_{j\geq n}\sum_{k\in I_j^\Omega}2^{-2jp}|f_1|_{2p,S_{j,k}}2^{jp} \\ &\leq \frac{R^{2p}}{\pi^{2p}}\|\psi\|_p\sum_{j\geq n}2^{-jp}\sum_{k\in I_j^\Omega}|f_1|_{2p,S_{j,k}}.\end{aligned}$$

For $\Omega = [a,b]$, we let $\bar{I}_{j,i} = [a + i/2^j, a + (i+1)/2^j]$. Then, $\Omega = \bigcup_{i=0}^{2^j(b-a)-1}\bar{I}_{j,i}$. Let

$$\chi_{j,i}(x) = \begin{cases} 1, & x \in \bar{I}_{j,i}, \\ 0, & \text{otherwise}. \end{cases}$$

Observe that

$$\begin{aligned}S_{j,k} &= \left[\frac{k}{2^j}, \frac{k+R}{2^j}\right] = \bigcup_{v=0}^{R-1}\left[\frac{k+v}{2^j}, \frac{k+v+1}{2^j}\right] \\ &= \bigcup_{i=k-2^ja}^{k+R-1-2^ja}\left[a + \frac{i}{2^j}, a + \frac{i+1}{2^j}\right] = \bigcup_{i=k-2^ja}^{k+R-1-2^ja}\bar{I}_{j,i}.\end{aligned}$$

Hence,

$$|f_1|_{2p,S_{j,k}} = \left| \sum_{i=k-2^j a}^{k+R-1-2^j a} f_1\chi_{j,i} \right|_{2p,S_{j,k}} \le \sum_{i=k-2^j a}^{k+R-1-2^j a} |f_1|_{2p,\bar{I}_{j,i}}.$$

If

$$\left[a+\frac{i}{2^j}, a+\frac{i+1}{2^j}\right] = \left[\frac{k+v}{2^j}, \frac{k+v+1}{2^j}\right],$$

then $k+v = i+2^j a$. Hence, $k = 2^j a + i - v$, where v can go from 0 to $R-1$. It follows that every interval of form

$$\left[a+\frac{i}{2^j}, a+\frac{i+1}{2^j}\right]$$

is contained in at most R cases of $S_{j,k}$, for all possible j and k. Hence, every $|f_1|_{2p,\bar{I}_{j,i}}$ appears at most R times in the following summation:

$$\sum_{k\in I_j^\Omega} |f_1|_{2p,S_{j,k}} \le \sum_{k\in I_j^\Omega} \sum_{i=k-2^j a}^{k+R-1-2^j a} |f_1|_{2p,\bar{I}_{j,i}} \le R \sum_{i=0}^{2^j(b-a)-1} |f_1|_{2p,\bar{I}_{j,i}}. \tag{2.7}$$

But

$$\begin{aligned}
\left(\sum_{i=0}^{2^j(b-a)-1} |f_1|_{2p,\bar{I}_{j,i}}\right)^2 &= \sum_{k,l=0}^{2^j(b-a)-1} |f_1|_{2p,\bar{I}_{j,k}} |f_1|_{2p,\bar{I}_{j,l}} \\
&\le \frac{1}{2} \sum_{k,l=0}^{2^j(b-a)-1} \left(|f_1|^2_{2p,\bar{I}_{j,k}} + |f_1|^2_{2p,\bar{I}_{j,l}}\right) \\
&= \frac{1}{2} \sum_{k=0}^{2^j(b-a)-1} \sum_{l=0}^{2^j(b-a)-1} |f_1|^2_{2p,\bar{I}_{j,k}} + \frac{1}{2} \sum_{k=0}^{2^j(b-a)-1} \sum_{l=0}^{2^j(b-a)-1} |f_1|^2_{2p,\bar{I}_{j,l}} \\
&= \frac{1}{2} 2^j(b-a) \sum_{k=0}^{2^j(b-a)-1} |f_1|^2_{2p,\bar{I}_{j,k}} + \sum_{l=0}^{2^j(b-a)-1} \frac{1}{2} 2^j(b-a) |f_1|^2_{2p,\bar{I}_{j,l}} \\
&= 2^j(b-a)|f_1|^2_{2p,\Omega},
\end{aligned}$$

$$\begin{aligned}
\sum_{k\in I_j^\Omega} |f_1|_{2p,S_{j,k}} &\le R \sum_{i=0}^{2^j(b-a)-1} |f_1|_{2p,\bar{I}_{j,i}} \\
&\le 2^{j/2} R\sqrt{b-a}|f_1|_{2p,\Omega} \le 2^{jp/2} R\sqrt{b-a}|f|_{2p,\Omega}.
\end{aligned} \tag{2.8}$$

Hence,

$$\begin{aligned}
\left\|f_1 - P_n^\Omega(f_1)\right\|_{p,\mathbf{R}} &\le \frac{R^{2p}}{\pi^{2p}} \|\psi\|_p \sum_{j\ge n} 2^{-jp} \sum_{k\in I_j^\Omega} |f_1|_{2p,S_{j,k}} \\
&\le \frac{R^{2p+1}}{\pi^{2p}} \|\psi\|_p \sum_{j\ge n} 2^{-jp} \left(\sqrt{b-a}|f|_{2p,\Omega} 2^{jp/2}\right) \\
&\le \frac{R^{2p+1}}{\pi^{2p}} \sqrt{b-a}\|\psi\|_p |f|_{2p,\Omega} \sum_{j\ge n} 2^{-jp/2} \\
&\le \frac{R^{2p+1}}{\pi^{2p}} \sqrt{b-a}\|\psi\|_p |f|_{2p,\Omega} \left(\frac{1}{1-2^{-p/2}}\right) 2^{-np/2}.
\end{aligned}$$

This completes the proof of (1). For (2), as before, because of $\|\psi_{j,k}\|_0 = \|\psi\|_0$ from Lemma 2.5, (2.7), and (2.8), we have

$$\begin{aligned}
\left\|f - P_n^{\Omega}(f)\right\|_{0,\mathbf{R}} &\le \left\|f_1 - P_n^{\Omega}(f_1)\right\|_{0,\mathbf{R}} + \|f\|_{0,\mathbf{R}-\Omega} \\
&\le \sum_{j\ge n} \sum_{k\in I_j^{\Omega}} |\langle f_1, \psi_{j,k}\rangle| \|\psi_{j,k}\|_0 + \|f\|_{0,\mathbf{R}-\Omega} \\
&\le \|\psi\|_0 \sum_{j\ge n} \sum_{k\in I_j^{\Omega}} 2^{-2jp} \frac{R^{2p}}{\pi^{2p}} |f_1|_{2p,S_{j,k}} + \|f\|_{0,\mathbf{R}-\Omega} \\
&\le \frac{R^{2p}}{\pi^{2p}} \|\psi\|_0 \sum_{j\ge n} 2^{-2jp} \sum_{k\in I_j^{\Omega}} |f_1|_{2p,S_{j,k}} + \|f\|_{0,\mathbf{R}-\Omega} \\
&\le \frac{R^{2p}}{\pi^{2p}} \|\psi\|_0 \sum_{j\ge n} 2^{-2jp} \left(R \sum_{i=0}^{2^j(b-a)-1} |f_1|_{2p,\bar{I}_{j,i}} \right) + \|f\|_{0,\mathbf{R}-\Omega} \\
&\le \frac{R^{2p+1}}{\pi^{2p}} \|\psi\|_0 \sum_{j\ge n} 2^{-2jp} \left(\sqrt{b-a} |f|_{2p,\Omega} 2^{jp/2} \right) + \|f\|_{0,\mathbf{R}-\Omega} \\
&\le \sqrt{b-a} \frac{R^{2p+1}}{\pi^{2p}} \|\psi\|_0 |f|_{2p,\Omega} \sum_{j\ge n} 2^{-3jp/2} + \|f\|_{0,\mathbf{R}-\Omega} \\
&= \frac{1}{1-2^{-3p/2}} \frac{R^{2p+1}}{\pi^{2p}} \sqrt{b-a} \|\psi\|_0 |f|_{2p,\Omega} 2^{-3np/2} + \|f\|_{0,\mathbf{R}-\Omega} \\
&\le C(\Omega,p) |f|_{2p,\Omega} 2^{-3np/2} + \|f\|_{0,\mathbf{R}-\Omega}.
\end{aligned}$$

Thus, the theorem is proven. ∎

COROLLARY 2.2. *Let $N \ge 2$ and let $\Omega = [a,b]$ be a finite interval, where a, b are integers. If $f \in H^{2p}(\mathbf{R})$, where $1 \le p \le N/2$, then*

1. $$\left\|f - P_n^{\Omega}(f)\right\|_{p,\Omega} \le C(\Omega,p) |f|_{2p,\Omega} 2^{-np/2},$$

   where

   $$C(\Omega,p) = \left(\frac{1}{1-2^{-p/2}}\right) \frac{R^{2p+1}}{\pi^{2p}} \sqrt{b-a} \|\psi\|_p;$$

2. $$\left\|f - P_n^{\Omega}(f)\right\|_{0,\Omega} \le C(\Omega,p) |f|_{2p,\Omega} 2^{-3np/2},$$

   where p satisfy $1 \le p \le N/2$.

PROOF. For the first part, let

$$f_1(x) = \begin{cases} f(x), & x \in \Omega, \\ 0, & x \in \mathbf{R} - \Omega. \end{cases}$$

Then, $P_n^{\Omega}(f_1) = P_n^{\Omega}(f)$ and $\|f - P_n^{\Omega}(f)\|_{p,\Omega} = \|f_1 - P_n^{\Omega}(f_1)\|_{p,\Omega}$ and $|f|_{2p,\Omega} = |f_1|_{2p,\Omega}$. Hence, it is clear that

$$\begin{aligned}
\left\|f - P_n^{\Omega}(f)\right\|_{p,\Omega} &= \left\|f - P_n^{\Omega}(f_1)\right\|_{p,\Omega} \\
&= \left\|f_1 - P_n^{\Omega}(f_1)\right\|_{p,\Omega} \\
&\le C(\Omega,p) |f_1|_{2p,\Omega} 2^{-np/2} + \|f_1\|_{p,\mathbf{R}-\Omega} \\
&= C(\Omega,p) |f|_{2p,\Omega} 2^{-np/2}.
\end{aligned}$$

The first part is proven. We can similarly prove the second part. ∎

For PDE problems, we usually deal with function u, which depends on t (time variable) and x (spatial variable). Let $\langle u(x,t), \psi_{n,k}\rangle$ be the inner product taken with respect to spatial variable x. Then, it must be a function of t, and $\frac{\partial}{\partial t}\langle u(x,t), \psi_{n,k}\rangle = \langle \frac{\partial u}{\partial t}(x,t), \psi_{n,k}\rangle$. Suppose $\frac{\partial u}{\partial t} \in H^p(\mathbf{R})$. Then, the next corollary follows from Theorem 2.1.

COROLLARY 2.3. *For $N \geq 2$, suppose u and $\frac{\partial u}{\partial t} \in H^{2p}(\mathbf{R})$, where $1 \leq p \leq N/2$. For a fixed t, then*

1.
$$\left\| \frac{\partial}{\partial t}\left(u - P_n^\Omega(u)\right)\right\|_{p,\mathbf{R}} \leq \left(\frac{1}{1-2^{-p/2}}\right)\sqrt{(b-a)}\|\psi\|_p \frac{R^{2p+1}}{\pi^{2p}}\left|\frac{\partial u}{\partial t}\right|_{2p,\Omega} 2^{-np/2} + \left\|\frac{\partial u}{\partial t}\right\|_{p,\mathbf{R}-\Omega};$$

2.
$$\left\| \frac{\partial}{\partial t}\left(u - P_n^\Omega(u)\right)\right\|_{0,\mathbf{R}} \leq \left(\frac{1}{1-2^{-p/2}}\right)\sqrt{(b-a)}\|\psi\|_p \frac{R^{2p+1}}{\pi^{2p}}\left|\frac{\partial u}{\partial t}\right|_{2p,\Omega} 2^{-3np/2} + \left\|\frac{\partial u}{\partial t}\right\|_{0,\mathbf{R}-\Omega},$$

where $1 \leq p \leq N/2$.

PROOF. Because $\frac{\partial}{\partial t}\langle u, \psi_{n,k}\rangle = \langle \frac{\partial u}{\partial t}, \psi_{n,k}\rangle$, we have

$$\frac{\partial}{\partial t} P_n^\Omega(u) = P_n^\Omega\left(\frac{\partial u}{\partial t}\right).$$

From Part (1) of Theorem 2.1, we have

$$\begin{aligned}\left\| \frac{\partial}{\partial t}\left(u - P_n^\Omega(u)\right)\right\|_{p,\mathbf{R}} &= \left\|\frac{\partial u}{\partial t} - P_n^\Omega\left(\frac{\partial u}{\partial t}\right)\right\|_{p,\mathbf{R}} \\ &\leq \left(\frac{1}{1-2^{-p/2}}\right)\sqrt{(b-a)}\|\psi\|_p \frac{R^{2p+1}}{\pi^{2p}}\left|\frac{\partial u}{\partial t}\right|_{2p,\Omega} 2^{-np/2} + \left\|\frac{\partial u}{\partial t}\right\|_{p,\mathbf{R}-\Omega}.\end{aligned}$$

On the other hand, Part (2) of Theorem 2.1 implies

$$\begin{aligned}\left\| \frac{\partial}{\partial t}\left(u - P_n^\Omega(u)\right)\right\|_{0,\mathbf{R}} &= \left\|\frac{\partial u}{\partial t} - P_n^\Omega\left(\frac{\partial u}{\partial t}\right)\right\|_{0,\mathbf{R}} \\ &\leq \left(\frac{1}{1-2^{-p/2}}\right)\sqrt{(b-a)}\|\psi\|_p \frac{R^{2p+1}}{\pi^{2p}}\left|\frac{\partial u}{\partial t}\right|_{2p,\Omega} 2^{-np} + \left\|\frac{\partial u}{\partial t}\right\|_{0,\mathbf{R}-\Omega}. \quad \blacksquare\end{aligned}$$

COROLLARY 2.4. *For $N \geq 2$, suppose u and $\frac{\partial u}{\partial t} \in H^{2p}(\mathbf{R})$, where $1 \leq p \leq N/2$. For a fixed t, then*

1.
$$\left\| \frac{\partial}{\partial t}\left(u - P_n^\Omega(u)\right)\right\|_{p,\Omega} \leq \left(\frac{1}{1-2^{-p/2}}\right)\sqrt{(b-a)}\,\|\psi\|_p \frac{R^{2p+1}}{\pi^{2p}}\left|\frac{\partial u}{\partial t}\right|_{2p,\Omega} 2^{-np/2};$$

2.
$$\left\| \frac{\partial}{\partial t}\left(u - P_n^\Omega(u)\right)\right\|_{0,\Omega} \leq \left(\frac{1}{1-2^{-p/2}}\right)\sqrt{(b-a)}\|\psi\|_p \frac{R^{2p+1}}{\pi^{2p}}\left|\frac{\partial u}{\partial t}\right|_{2p,\Omega} 2^{-3np/2},$$

where $1 \leq p \leq N/2$.

3. WAVELET-GALERKIN METHOD AND ITS APPLICATION TO THE KOLMOGOROV EQUATION

In this section, we first discuss the basic idea of the Galerkin method (a finite-element method) with Daubechies' functions as base functions. We then apply this method to the Kolmogorov equation. We also give a theoretical verification that the solution of the Kolmogorov equation can be approximated in a finite domain.

The following notation is used when we consider the solution $u(x,t)$ of a partial differential equation:

$$L_T^2(H^p(\mathbf{\Omega})) = \left\{ u(x,t) \;\middle|\; u(x,t) \in H^p(\mathbf{\Omega}) \text{ for any fixed } t, \left(\int_0^T \left(\|u(x,t)\|_{p,\Omega} \right)^2 dt \right)^{1/2} < \infty \right\}.$$

Now let $u(x,t)$ be solution of the Kolmogorov equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \mathbf{A}u, \\ u(x,0) &= u_0(x), \end{aligned} \tag{3.1}$$

where $\mathbf{A}$ is a differential operator with respect to x in either form

$$\mathbf{A} = \frac{1}{2}\sum_i \frac{\partial^2}{\partial x_i^2} - \sum_i \left(f_i(x) + \frac{\partial F}{\partial x_i}(x) \right) \frac{\partial}{\partial x_i} - \sum_i \left(\frac{\partial f_i}{\partial x_i} + \frac{\partial^2 F}{\partial x_i^2}(x) \right) \tag{3.2}$$

or form

$$\mathbf{A} = \frac{1}{2}\sum_i \frac{\partial^2}{\partial x_i^2} - \sum_i f_i(x) \frac{\partial}{\partial x_i} - \sum_i \left(\frac{\partial f_i}{\partial x_i} + \frac{1}{2} h_i^2(x) \right). \tag{3.3}$$

In [18], it is shown that (3.1) has solution in $\mathcal{S}$ for any fixed t, if $u_0(x) \in \mathcal{S}$. For simplicity, here we consider the case that the x variable is one dimensional.

In the following discussion, we always let T be a fixed positive value. In Section 5.4, we prove that if $u(x,t)$ and $\frac{\partial u}{\partial x}$ are continuous with respect to t, and $u(x,t) \in \mathcal{S}$ for the variable x, then for a small enough positive number ϵ, there is an interval $\Omega = [a,b]$, such that

$$|u(x,t)| \leq \epsilon, \quad |u(a,t)| \leq \epsilon, \quad |u(b,t)| \leq \epsilon, \qquad x \in \mathbf{R} - \Omega, \tag{3.4}$$

where $t \in [0,T]$.

So when we fix this interval Ω, which satisfies the above condition, it is reasonable to use the solution of following boundary-value problem as an approximation to the solution of problem (3.1)

$$\begin{aligned} \frac{\partial \bar{u}}{\partial t} &= \mathbf{A}\bar{u}, && x \in \Omega, \\ \bar{u}(x,0) &= u_0(x), && x \in \Omega, \\ \bar{u}(a,t) &= \bar{u}(b,t) = 0, && 0 \leq t \leq T. \end{aligned} \tag{3.5}$$

Practically, the finite interval Ω for (3.5) is found as follows. We first find a finite interval to make the initial condition small enough outside the interval, and thus, solve (3.5) based upon this interval. Then, we adjust the interval during the processing of solving boundary-value problem (3.1), and formulate a new boundary-value problem (3.5).

We would like to approximate the solution of (3.5) by the wavelet-Galerkin method. In view of Theorem 2.1, we would like to approximate u in Ω by functions in $V_n^\Omega = \{u_n = \sum_{k \in I_n^\Omega} \lambda_{n,k}\varphi_{n,k},\ \lambda_{n,k}$ are functions of $t\}$. Here $\varphi_{n,k}$ are sometimes called basis functions. The degree to which u_n fails to satisfy (3.1) is expressed by *an equation residual*

$$\Re = \frac{\partial u_n}{\partial t} - \mathbf{A}u_n.$$

The smaller $\Re$ is, the better a good approximation u_n is.

The essence of Galerkin's method is to require that this residual be orthogonal to the set of basis functions $\varphi_{n,k}$. That is,

$$\langle \varphi_{n,k}, \Re \rangle \equiv 0, \qquad k \in I_n^{\Omega}, \tag{3.6}$$

where the inner product is taken in space $L^2(\mathbf{R})$ over $\mathbf{R}$. Rewriting condition (3.6), we get

$$\begin{gathered} \left\langle \frac{\partial u_n}{\partial t} - \mathbf{A} u_n, \varphi_{n,k} \right\rangle = 0, \quad \text{in } V_n^{\Omega}, \qquad 0 \le t \le T, \\ u_n(x,0) = P_n^{\Omega}(u_0). \end{gathered} \tag{3.7}$$

Here $\varphi_{n,k}$, $2^n a + 1 - R \le k \le 2^n b - 1$, are basis functions of V_n^{Ω}.

In summary, we can use Galerkin's method to get an approximation solution $u_n \in V_n$ in interval Ω, where Ω satisfies condition (3.4), for problem (3.1) by solving (3.7).

In Theorem 3.5, we give a theoretical justification that the solution of boundary value problem (3.5) by the wavelet-Galerkin method is the approximation of solution (3.1) in Ω. We first recall the maximum principles for parabolic equations and the Riemann-Lebesgue lemma for Fourier transform, which are needed in the proof of Theorem 3.5. In Theorem 3.6, we prove the convergency of the wavelet-Galerkin method applied to the Kolmogorov equation.

We need to recall the maximum principle for parabolic equation and several concepts from [19, pp. 159–177].

Let D be the open domain in n-dimensional space. Then,

$$E = \{(x_1, x_2, \ldots, x_n, t) : (x_1, x_2, \ldots, x_n) \in D,\ 0 < t < \infty\}$$

is the $n+1$-dimensional region. We define region

$$E_T = \{(x_1, x_2, \ldots, x_n, t) \in E : t \le T\}.$$

The operator

$$L \equiv \sum_{i,j=1}^{n} a_{i,j}(\mathbf{x}, t) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(\mathbf{x}, t) \frac{\partial}{\partial x_i} - \frac{\partial}{\partial t} \tag{3.8}$$

is said to be *parabolic* at $(\mathbf{x}, t) \equiv (x_1, x_2, \ldots, x_n, t)$ if for a fixed t the operator consisting of the first sum is elliptic at $(\mathbf{x}, t)$. That is, L is parabolic if there is a number $\mu > 0$, such that

$$\sum_{i,j=1}^{n} a_{i,j}(\mathbf{x}, t) \xi_i \xi_j \ge \mu \sum_{i=1}^{n} \xi_i^2, \tag{3.9}$$

for all n-tuples of real numbers $(\xi_1, \xi_2, \ldots, \xi_n)$. Operator L is *uniformly parabolic* in a region E_T if (3.9) holds with the same number $\mu > 0$, for all $(\mathbf{x}, t)$ in E_T. The following is Theorem 5 in [19, p. 173].

THEOREM 3.1. *Let u satisfy the uniformly parabolic differential inequality*

$$(L)[u] \equiv a(x,t) \frac{\partial^2 u}{\partial x^2} + b(x,t) \frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} \ge 0 \tag{3.10}$$

in a region $E_T = \{(x_1, x_2, \ldots, x_n, t) \in E : t \le T\}$, where E is an open domain, and suppose the coefficients of L are bounded. Suppose that the maximum of u in E_T is M and that it is attained at a point $P(x,t)$ of E_T. Thus, if Q is a point of E that can be connected to P by a path in E consisting only of horizontal segments and upward vertical segments, then $u(Q) = M$.

The following is Theorem 7 in [19, p. 174].

THEOREM 3.2. *The conclusions of Theorem 3.1 remain valid if u is a solution of $(L + h)[u] \geq 0$, provided $h \leq 0$ and $M \geq 0$.*

REMARK. The change of variable $v = ue^{-\lambda t}$ replaces the inequality $(L + h)[u] \geq 0$ by $(L + h - \lambda)[v] \geq 0$. If h is bounded above, we can choose λ so large that $h - \lambda \leq 0$, so that a maximum principle applies to v.

The following lemma is the special case of the Riemann-Lebesgue lemma [20, p. 246].

LEMMA 3.1.

(1) *Suppose that f is continuous over $[a, b]$. Then,*

$$\lim_{c\to\infty} \int_a^b f(x)\cos(cx)\,dx = \lim_{c\to\infty} \int_a^b f(x)\sin(cx)\,dx = 0. \tag{3.11}$$

(2) *Suppose that f is continuous and absolutely integrable over $(-\infty, \infty)$. Then, $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i2\pi x\xi} \to 0$ as $|\xi| \to +\infty$.*

Let us recall two theorems in [21]. The first is Theorem 2.2 in [21, p. 25]. The second is Theorem 2.11 in [21, p. 32].

THEOREM 3.3. *Let $f(x) \in L^1(\mathbf{R})$ Then, its Fourier transform $\hat{f}$ satisfies the following.*

(i) *$\hat{f} \in L^\infty(\mathbf{R})$ with $\sup_\xi |\hat{f}| \leq \int_{-\infty}^{\infty} |f|$.*
(ii) *$\hat{f}$ is uniformly continuous on $\mathbf{R}$.*
(iii) *If the derivative f' of f also exists and is in $L^1(\mathbf{R})$, then*

$$\hat{f'}(\xi) = i\xi\hat{f}(\xi); \qquad \textit{and} \tag{3.12}$$

(iv) *$\hat{f}(\xi) \to 0$, as $\xi \to \infty$ or $-\infty$.*

THEOREM 3.4. *Let $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. Then, the Fourier transform of $\hat{f}$ of f is in $L^2(\mathbf{R})$, and satisfies the following "Parseval identity":*

$$\int_{-\infty}^{\infty} \left|\hat{f}\right|^2 d\xi = 2\pi \int_{-\infty}^{\infty} |f|^2\,d\xi. \tag{3.13}$$

The following is a fundamental step in the numerical solution to the Kolmogorov equation.

LEMMA 3.2. *Let T be fixed. We have that $u(x,t)$ and $\frac{\partial u}{\partial x}$ are continuous respect to t, $u(x,t) \in \mathcal{S}$ for every fixed t. Then, for any $\epsilon > 0$, there exists a constant number $X_0 > 0$, such that when $|x| > X_0$, $|u(x,t)| < 2\epsilon$ for $0 < t \leq T$.*

PROOF. We have $u(x,t) \in \mathcal{S}$. Hence, for any fixed t, $u(x,t), \frac{\partial u}{\partial x} \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. By Theorem 3.4, $\hat{u} \in L^2(\mathbf{R})$ and $\widehat{\frac{\partial u}{\partial x}}(\xi) = \xi\hat{u}(\xi) \in L^2(\mathbf{R})$. Thus, $g = (1 + |\xi|)\hat{u} \in L^2(\mathbf{R})$. Note that $(1/(1+|\xi|))g = \hat{u}$. Therefore,

$$\int |\hat{u}| = \int \frac{|g|}{1+|\xi|}\,d\xi \leq \left[\int \frac{1}{(1+|\xi|)^2}\,d\xi\right]^{1/2} \left[\int |g|^2\,d\xi\right]^{1/2}.$$

From $u(x,t) \in \mathcal{S}$, we know that $u(x,t) = \int_{-\infty}^{\infty} \hat{u}(\xi,t)e^{2\pi i x\xi}\,d\xi$.

On the other hand, according to the "Parseval identity",

$$\begin{aligned}
\left[\int_{-\infty}^{\infty} |g|^2 d\xi\right]^{1/2} &= \left[\int_{-\infty}^{\infty} (\hat{u} + |\xi|\hat{u})^2\,d\xi\right]^{1/2} \\
&\leq \left[\int_{-\infty}^{\infty} |\hat{u}|^2\,d\xi\right]^{1/2} + \left[\int_{-\infty}^{\infty} |\xi\hat{u}|^2\,d\xi\right]^{1/2} \\
&= \sqrt{2\pi}\left[\int_{-\infty}^{\infty} |u|^2\,dx\right]^{1/2} + \sqrt{2\pi}\left[\int_{-\infty}^{\infty} \left|\frac{\partial u}{\partial x}\right|^2\,dx\right]^{1/2}.
\end{aligned}$$

According to the Fatou theorem, both $[\int_{-\infty}^{\infty}|u|^2\,dx]^{1/2}$ and $[\int_{-\infty}^{\infty}|\frac{\partial u}{\partial x}|^2\,dx]^{1/2}$ are continuous with respect to t. Hence, they are bounded in $[0,T]$, that is,

$$\sup_{0\le t\le T}\left\{\left[\int_{-\infty}^{\infty}|u|^2\,dx\right]^{1/2}+\left[\int_{-\infty}^{\infty}\left|\frac{\partial u}{\partial x}\right|^2\,dx\right]^{1/2}\right\}$$

is finite. Thus,

$$C_0:=\sup_{0\le t\le T}\left\{\left[\int_{-\infty}^{\infty}|g|^2\,d\xi\right]^{1/2}\right\}$$

is finite.

For any $\epsilon>0$, there exists a number $K>0$, such that

$$\left|\int_{|\xi|\ge K}\frac{1}{(1+|\xi|)^2}\,d\xi\right|^{1/2}\le\epsilon.$$

From the Schwartz inequality,

$$\int_{|\xi|\ge K}|\hat{u}(\xi,t)|\,d\xi\le\left[\int_{|\xi|\ge K}\frac{1}{(1+|\xi|)^2}\,d\xi\right]^{1/2}\left[\int_{-\infty}^{\infty}|g|^2\,d\xi\right]^{1/2}\le C_0\epsilon, \tag{3.14}$$

for $0\le t\le T$.

For $|\int_{-K}^{K}\hat{u}(\xi,t)e^{2\pi i\xi x}\,d\xi|$, we first need to prove that $\hat{u}(\xi,t)$ is continuous with respect to (ξ,t) in $[-K,K]\times[0,T]$. Let $\Delta\xi$ and Δt be positive. Then,

$$\begin{aligned}
&|\hat{u}(\xi+\Delta\xi,t+\Delta t)-\hat{u}(\xi,t)|\\
&\quad\le|\hat{u}(\xi+\Delta\xi,t+\Delta t)-\hat{u}(\xi,t+\Delta t)|+|\hat{u}(\xi,t+\Delta)-\hat{u}(\xi,t)|\\
&\quad=\left|\int_{-\infty}^{\infty}u(x,t+\Delta t)e^{-2\pi i\xi x}\left(e^{-2\pi i\Delta\xi x}-1\right)\,dx\right|+|\hat{u}(\xi,t+\Delta t)-\hat{u}(\xi,t)|\\
&\quad\le\left|\int_{-\infty}^{\infty}(u(x,t+\Delta t)-u(x,t))\left(e^{-2\pi i\Delta\xi x}-1\right)e^{-2\pi i\xi x}\,dx\right|\\
&\qquad+\left|\int_{-\infty}^{\infty}u(x,t)\left(e^{-2\pi i\Delta\xi x}-1\right)e^{-2\pi i\xi x}\,dx\right|+|\hat{u}(\xi,t+\Delta t)-\hat{u}(\xi,t)|\\
&\quad=\left|\int_{-\infty}^{\infty}(u(x,t+\Delta t)-u(x,t))\left(e^{-2\pi i\Delta\xi x}-1\right)e^{-2\pi i\xi x}\,dx\right|\\
&\qquad+|\hat{u}(\xi+\Delta\xi,t)-\hat{u}(\xi,t)|+|\hat{u}(\xi,t+\Delta t)-\hat{u}(\xi,t)|.
\end{aligned}$$

According to Lemma 3.3, we know $\hat{u}(\xi,t)$ is continuous in $[-K,K]$ with respect to x. And according to the Fatou theorem, $\hat{u}(\xi,t)$ is continuous with respect to t in $[0,T]$. Also,

$$\left|\int_{-\infty}^{\infty}(u(x,t+\Delta t)-u(x,t))\left(e^{-2\pi i\Delta\xi x}-1\right)e^{-2\pi i\xi x}\,dx\right|\le 2\int_{-\infty}^{\infty}|u(x,t+\Delta t)-u(x,t)|\,dx.$$

So there exists a positive number δ, such that when $|\Delta\xi|+|\Delta t|\le\delta$,

$$\left|\int_{-\infty}^{\infty}(u(x,t+\Delta t)-u(x,t))\left(e^{-2\pi i\Delta\xi x}-1\right)e^{-2\pi i\xi x}\,dx\right|\le 2\int_{-\infty}^{\infty}|u(x,t+\Delta t)-u(x,t)|\,dx\le\epsilon,$$

$$|\hat{u}(\xi+\Delta\xi,t)-\hat{u}(\xi,t)|\le\epsilon,$$

and

$$|\hat{u}(\xi,t+\Delta t)-\hat{u}(\xi,t)|\le\epsilon.$$

Then, $\hat{u}(\xi,t)$ is continuous in $[-K,K]\times[0,T]$. Then, it is uniformly continuous in $[-K,K]\times[0,T]$.

In the following, we just choose $x > 0$, such that Kx an integer. The proof is similar for the case that Kx is not integer. Then,

$$\int_{-K}^{K} \hat{u}(\xi,t)\sin(2\pi\xi x)\,d\xi = \sum_{l=-Kx}^{(K-1)x} \left(\int_{l/x}^{(2l+1)/2x} \hat{u}(\xi,t)\sin(2\pi\xi x)\,d\xi + \int_{(2l+1)/2x}^{(l+1)/x} \hat{u}(\xi,t)\sin(2\pi\xi x)\,d\xi \right), \tag{3.15}$$

$$\int_{l/x}^{(2l+1)/2x} \hat{u}(\xi,t)\sin(2\pi\xi x)\,d\xi = \hat{u}(\xi_{l1},t)\int_{l/x}^{(2l+1)/2x} \sin(2\pi\xi x)\,d\xi = \frac{2\hat{u}(\xi_{l1},t)}{2\pi x},$$

and

$$\int_{(2l+1)/2x}^{(l+1)/x} \hat{u}(\xi,t)\sin(2\pi\xi x)\,d\xi = \hat{u}(\xi_{l2},t)\int_{(2l+1)/2x}^{(l+1)/x} \sin(2\pi\xi x)\,d\xi = -\frac{2\hat{u}(\xi_{l2},t)}{2\pi x},$$

where $\xi_{l1} \in [l/x, (2l+1)/2x]$ and $\xi_{l2} \in [(2l+1)/2x, (l+1)/x]$. Then, $|\xi_{l1} - \xi_{l2}| \le 1/x$. Because $\hat{u}$ is uniformly continuous. We can make x large enough, such that $|\hat{u}(\xi_{l1},t) - \hat{u}(\xi_{l2},t)| \le \epsilon/K$. Hence, (3.15) becomes

$$\begin{aligned}
\left|\int_{-K}^{K} \hat{u}(\xi,t)\sin(2\pi\xi x)\,d\xi\right| &\le \sum_{l=-Kx}^{(K-1)x} \left|\int_{l/x}^{(2l+1)/2x} \hat{u}(\xi,t)\sin(2\pi\xi x)\,d\xi + \int_{(2l+1)/2x}^{(l+1)/x} \hat{u}(\xi,t)\sin(2\pi\xi x)\,d\xi\right| \\
&\le \sum_{l=-Kx}^{(K-1)x} \frac{|\hat{u}(\xi_{l1},t) - \hat{u}(\xi_{l2},t)|}{2\pi x} \\
&\le \sum_{l=-Kx}^{(K-1)x} \frac{\epsilon}{2\pi K x} \\
&= \frac{\epsilon}{\pi}.
\end{aligned} \tag{3.16}$$

Similar, we can prove that for large enough x,

$$\left|\int_{-K}^{K} \hat{u}(\xi,t)\cos(2\pi\xi x)\,d\xi\right| \le \frac{\epsilon}{\pi}. \tag{3.17}$$

Hence, from the above, we know that we can find a x_0 when $|x| \ge x_0$, for any $0 \le t \le T$,

$$\begin{aligned}
|u(x,t)| &= \left|\int_{-\infty}^{\infty} \hat{u}(\xi,t)e^{2\pi i\xi x}\,d\xi\right| \le \left|\int_{-K}^{K} \hat{u}(\xi,t)e^{2\pi i\xi x}\,d\xi\right| + \int_{|\xi|\ge K} |\hat{u}(\xi,t)|\,d\xi \\
&\le \left|\int_{-\infty}^{\infty} \hat{u}(\xi,t)\sin(2\pi\xi x)\,d\xi\right| + \left|\int_{-\infty}^{\infty} \hat{u}(\xi,t)\cos(2\pi\xi x)\,d\xi\right| + \int_{|\xi|\ge K} |\hat{u}(\xi,t)|\,d\xi \\
&\le \frac{2\epsilon}{\pi} + \epsilon \le \left(\frac{2}{\pi} + 1\right)\epsilon.
\end{aligned} \tag{3.18}$$

∎

The following theorem tells us that we can use a boundary value problem to approximate the Kolmogorov equation.

THEOREM 3.5. *Let $u(x,t)$ be the solution of the Kolgomorov equation (3.1),*

$$\frac{\partial u}{\partial t} = \mathbf{A}u, \qquad u(x,0) = u_0(x), \tag{3.19}$$

where $\mathbf{A}$ *is in either form (3.2)*

$$\mathbf{A} = \frac{1}{2}\frac{\partial^2}{\partial x^2} - \left(\frac{\partial F}{\partial x} + f\right)\frac{\partial}{\partial x} - \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial f}{\partial x}\right)$$

or form (3.3)

$$\mathbf{A} = \frac{1}{2}\frac{\partial^2}{\partial x^2} - f(x)\frac{\partial}{\partial x} - \left(\frac{h^2}{2} + \frac{\partial f}{\partial x}\right).$$

There exists a fixed number $\Lambda < 0$, such that $\frac{\partial^2 F}{\partial x^2} + \frac{\partial f}{\partial x} \geq \Lambda$ for $\mathbf{A}$ in form (3.2), or $h^2/2 + \frac{df}{dx} \geq \Lambda$ for $\mathbf{A}$ in form (3.3), and $u_0(x) \in \mathcal{S}$. Then, for every $\epsilon > 0$, there exists a finite interval $\Omega = [a,b]$, such that the following is true.

(1)
$$|u(a,t)| \leq \epsilon, \qquad |u(b,t)| \leq \epsilon, \qquad |u(x,t)| \leq \epsilon, \tag{3.20}$$
where $x \in \mathbf{R} - \Omega$, $t \in [0,T]$.

(2) *The solution of equation*
$$\begin{aligned} \frac{\partial \bar{u}}{\partial t} &= \mathbf{A}\bar{u}, && x \in \Omega, \\ \bar{u}(x,0) &= u_0(x), && x \in \Omega, \\ \bar{u}(a,t) &= \bar{u}(b,t) = 0, && t \in [0,T], \end{aligned} \tag{3.21}$$
in the interval Ω approximates the solution of (3.19) in the following manner:
$$\begin{aligned} |u(x,t) - \bar{u}(x,t)| &\leq e^{-\Lambda T}\epsilon, && x \in \Omega, \\ |u(x,t)| &\leq \epsilon, && x \in \mathbf{R} - \Omega, \quad t \in [0,T]. \end{aligned} \tag{3.22}$$

PROOF. For (1), because u is the solution of the Kolmogorov equation (3.1), $u(x,t) \in \mathcal{S}$, and $u(x,t)$ and $\frac{\partial u}{\partial x}$ are continuous with respect to t. From Lemma 3.2, we can find a finite interval $[a_1, b_1]$, such that $|u(x,t)| < \epsilon$ when $x \notin [a_1, b_1]$ and $0 \leq t \leq T$. We can take a larger interval $\Omega = [a,b] \supset [a_1, b_1]$. Then, for $\Omega = [a,b]$, Item (1) is proven.

For the chosen interval $\Omega = [a,b]$, let $v = ue^{\Lambda t}$ and $\bar{v} = \bar{u}e^{\Lambda t}$. Then, both v and $\bar{v}$ satisfy

$$\frac{\partial v}{\partial t} = (\mathbf{A} - \Lambda)v, \tag{3.23}$$

in Ω. So does $v - \bar{v}$. We assume $\sup_{x\in\Omega}(v - \bar{v}) \geq 0$ (if not, we can consider $\bar{v} - v$, and get the same result). Then, $v - \bar{v}$ satisfies the condition of Theorem 3.2 in domain $\Omega \times (0,T)$. Hence, it obtains the maximum and minimum values at the boundary of $\Omega \times (0,T]$. At boundary $t = 0$, $(v - \bar{v})(x,0) = u_0(x) - u_0(x) = 0$. At boundary $x = a$ and $x = b$,

$$\begin{aligned} |v(x,t) - \bar{v}(x,t)| &= \left|e^{\Lambda t}\right| |u(x,t) - \bar{u}(x,t)| \\ &= \left|e^{\Lambda t}\right| |u(x,t)| \\ &\leq |u(x,t)| \\ &\leq \epsilon. \end{aligned}$$

So for any $x \in \Omega$, $|v(x,t) - \bar{v}(x,t)| \le \epsilon$. But

$$
\begin{aligned}
|u(x,t) - \bar{u}(x,t)| &= e^{-\Lambda t}|v(x,t) - \bar{v}(x,t)| \\
&\le e^{-\lambda T}\epsilon.
\end{aligned}
\tag{3.24}
$$

When $x \in \mathbf{R} - \Omega$, from (1),

$$
|u(x,t)| \le \epsilon. \tag{3.25}
$$

Then, (2) holds. ∎

We are going to find a numerical solution of (3.21). In view of Theorem 3.5, we can see that it is also the approximation of solution (3.1).

In the following, we assume that all the Kolmogorov equations are of form (3.1). First, we show that the solution of boundary value problem (3.5) can be approximated by

$$
u_n = \sum_{k=2^n a+1-R}^{2^n b-1} \lambda_{n,k}\varphi_{n,k},
$$

where u_n must satisfy (3.7) according to the principle (3.6).

For $\mathbf{A}$ with form (3.2), u_n must satisfy

$$
\begin{aligned}
\left\langle \frac{\partial u_n}{\partial t}, \varphi_{n,j} \right\rangle_{\mathbf{R}} &= \frac{1}{2}\left\langle \frac{\partial^2 u_n}{\partial x^2}, \varphi_{n,j} \right\rangle_{\mathbf{R}} - \left\langle \left(\frac{dF}{dx}(x) + f(x)\right)\frac{\partial u_n}{\partial x}, \varphi_{n,j} \right\rangle_{\mathbf{R}} \\
&\quad - \left\langle \left(\frac{d^2F}{dx^2}(x) + \frac{df}{dx}(x)\right) u_n, \varphi_{n,j} \right\rangle_{\mathbf{R}}, \\
u_n(x,0) &= P_n^{\Omega} u_0(x), \qquad x \in (a,b), \\
u_n(a,t) &= u_n(b,t) = 0, \qquad t \in [0,T],
\end{aligned}
\tag{3.26}
$$

where $\Omega = [a,b]$.

Rewriting (3.26), we get

$$
\begin{aligned}
&\sum_{j=2^n a+1-R}^{2^n b-1} \frac{d\lambda_j}{dt}\langle \varphi_{n,j}, \varphi_{n,i}\rangle_{\mathbf{R}} \\
&\quad = \sum_{j=2^n a+1-R}^{2^n b-1} \lambda_j \left[\frac{1}{2}\left\langle \frac{d^2\varphi_{n,j}}{dx^2}, \varphi_{n,i} \right\rangle_{\mathbf{R}} - \left\langle \left(\frac{d^2F}{dx^2}(x) + \frac{df}{dx}(x)\right)\varphi_{n,j}, \varphi_{n,i} \right\rangle_{\mathbf{R}} \right. \\
&\qquad \left. - \left\langle \left(\frac{dF}{dx}(x) + f(x)\right)\frac{d\varphi_{n,j}}{dx}, \varphi_{n,i} \right\rangle_{\mathbf{R}} \right], \qquad \text{for } 2^n a + 1 - R \le i \le 2^n b - 1, \\
&\sum_{j=2^n a+1-R}^{2^n b-1} \lambda_j(0)\varphi_{n,j} = P_n^{\Omega} u_0(x), \\
&\sum_{k=2^n a+1-R}^{2^n a-1} \lambda_k \varphi(2^n a - k) = 0, \\
&\sum_{k=2^n b-R+1}^{2^n b-1} \lambda_k \varphi(2^n b - k) = 0,
\end{aligned}
\tag{3.27}
$$

$t \in [0,T]$.

For $\mathbf{A}$ with form (3.3), u_n must satisfy

$$\begin{aligned}
\left\langle \frac{\partial u_n}{\partial t}, \varphi_{n,j} \right\rangle_{\mathbf{R}} &= \frac{1}{2} \left\langle \frac{\partial^2 u_n}{\partial x^2}, \varphi_{n,j} \right\rangle_{\mathbf{R}} - \left\langle f(x) \frac{\partial u_n}{\partial x}, \varphi_{n,j} \right\rangle_{\mathbf{R}} \\
&\quad - \left\langle \left(\frac{h^2(x)}{2} + \frac{df}{dx}(x) \right) u_n, \varphi_{n,j} \right\rangle_{\mathbf{R}}, \\
u_n(x,0) &= P_n^{\Omega} u_0(x), \qquad x \in (a,b), \\
u_n(a,t) &= u_n(b,t) = 0, \qquad t \in [0,T],
\end{aligned} \tag{3.28}$$

where $\Omega = [a,b]$.

Rewriting (3.28), we get

$$\begin{aligned}
&\sum_{j=2^n a+1-R}^{2^n b-1} \frac{d\lambda_j}{dt} \langle \varphi_{n,j}, \varphi_{n,i} \rangle_{\mathbf{R}} \\
&= \sum_{j=2^n a+1-R}^{2^n b-1} \lambda_j \left[\frac{1}{2} \left\langle \frac{d^2 \varphi_{n,j}}{dx^2}, \varphi_{n,i} \right\rangle_{\mathbf{R}} - \left\langle \left(\frac{h^2(x)}{2} + \frac{df}{dx}(x) \right) \varphi_{n,j}, \varphi_{n,i} \right\rangle_{\mathbf{R}} \right. \\
&\quad \left. - \left\langle f(x) \frac{d\varphi_{n,j}}{dx}, \varphi_{n,i} \right\rangle_{\mathbf{R}} \right], \qquad \text{for } 2^n a + 1 - R \le i \le 2^n b - 1, \\
&\qquad \sum_{j=2^n a+1-R}^{2^n b-1} \lambda_j(0) \varphi_{n,j} = P_n^{\Omega} u_0(x), \\
&\qquad \sum_{k=2^n a+1-R}^{2^n a-1} \lambda_k \varphi(2^n a - k) = 0, \\
&\qquad \sum_{k=2^n b-R+1}^{2^n b-1} \lambda_k \varphi(2^n b - k) = 0,
\end{aligned} \tag{3.29}$$

$t \in [0,T]$.

In the rest of the section, we prove that u_n converges to the solution of (3.5) as n go to infinity. In view of last section, then, it is reasonable to use u_n as the approximation of the solution of (3.1).

First, let us recall the following lemma. A special case of this lemma can be found in [22, p. 35], where it is called a fundamental lemma (or Bellman-Gronwall lemma). The proof of this lemma here is essentially the same as in [22].

LEMMA 3.3. *Let f, g, h, and l be piecewise continuous nonnegative functions defined on an interval $a \le t \le b$, g being nondecreasing. If, for each $t \in [a,b]$,*

$$f(t) + h(t) \le g(t) + c \int_a^t f(s) l(s)\, ds, \tag{3.30}$$

where c is a constant, then

$$f(t) + h(t) \le g(t) \exp\left(c \int_a^t l(s)\, ds \right). \tag{3.31}$$

PROOF. We first assume $h(t) = 0$. We need to prove that if

$$f(t) \le g(t) + c \int_a^t f(s) l(s)\, ds,$$

then $f(t) \le g(t) \exp(c \int_a^t l(s)\, ds)$.

Let $k(t) = f(t) - c\int_a^t f(s)l(s)\,ds$. Then, $k(t) \le g(t)$. Let $F(t) = \int_a^t f(s)l(s)\,ds$. Then,

$$\frac{dF}{dt} = k(t)l(t) + cF(t)l(t).$$

It is clear that $F(t) = \int_a^t k(\xi)l(\xi)\exp(c\int_\xi^t l(s)\,ds)\,d\xi$ is a solution of the above differential equation with initial condition $F(a) = 0$. Hence, we have

$$\begin{aligned}\int_a^t f(s)l(s)\,ds &= \int_a^t k(\xi)l(\xi)\exp\left(c\int_\xi^t l(s)\,ds\right)\,d\xi\\ &\le g(t)\int_a^t l(\xi)\exp\left(c\int_\xi^t l(s)\,ds\right)\,d\xi\\ &= \frac{g(t)}{c}\int_a^t cl(\xi)\exp\left(c\int_\xi^t l(s)\,ds\right)\,d\xi\\ &= \frac{g(t)}{c}\left(\exp\left(c\int_a^t l(s)\,ds\right) - \exp\left(c\int_t^t l(s)\,ds\right)\right)\\ &= \frac{g(t)}{c}\left(\exp\left(c\int_a^t l(s)\,ds\right) - 1\right).\end{aligned}$$

Thus,

$$\begin{aligned}f(t) &\le g(t) + c\int_a^t f(s)l(s)\,ds\\ &\le g(t) + g(t)\left(\exp\left(c\int_a^t l(s)\,ds\right) - 1\right) = g(t)\exp\left(c\int_a^t l(s)\,ds\right).\end{aligned}$$

If $h(t) \ne 0$, then

$$f(t) + h(t) \le g(t) + c\int_a^t f(s)l(s)\,ds \le g(t) + c\int_a^t (f(s)+h(s))l(s)\,ds.$$

Hence, $f(t) + h(t) \le g(t)\exp(c\int_a^t l(s)\,ds)$.

The lemma is proven. ∎

Now for $\Omega = [a,b]$, where p is an integer, the following theorem says that the estimate of the difference between $P_n^\Omega\bar{u}$ and u_n is bounded above by the multiple of the $L_T^2(H^1(\mathbf{\Omega}))$-norms of difference between $\bar{u}$ and $P_n^\Omega\bar{u}$ and the difference of $\frac{\partial\bar{u}}{\partial t}$ and $\frac{\partial}{\partial t}(P_n^\Omega\bar{u})$.

THEOREM 3.6. *For a fixed value of T, let $\bar{u}$ be the solution of (3.5) with $\bar{u} \in L_T^2(H^2(\Omega))$ and $\frac{\partial\bar{u}}{\partial t} \in L_T^2(H^1(\Omega))$, where $\Omega = [a,b]$ with a,b is in the form of $k/2^m$ with k,m integers. Let $u_n = \sum_{k=2^na+1-R}^{2^nb-1}\lambda_{n,k}\varphi_{n,k}$, satisfy (3.7). Then,*

$$\begin{aligned}&\left\|P_n^\Omega\bar{u} - u_n\right\|_0^2(T) + \frac{1}{2}\int_0^T \left\|P_n^\Omega\bar{u} - u_n\right\|_1^2\,dt\\ &\le e^{(2+8C_1^2+6C_2+2C_1)T}\left\{\int_0^T \left\|\frac{\partial}{\partial t}\left(\bar{u} - P_n^\Omega\bar{u}\right)\right\|_0^2\,dt + 2C_3\int_0^T \left\|\bar{u} - P_n^\Omega\bar{u}\right\|_1\,dt\right\},\end{aligned} \tag{3.32}$$

where

(1) *$C_1 = \sup_{x\in[a-R,b+R]}|\frac{dF}{dx}(x) + f(x)|$ is finite, $C_2 = \sup_{x\in[a-R,b+R]}|\frac{d^2F}{dx^2}(x) + \frac{df}{dx}(x)|$ is finite, $C_3 = \max\{1/2 + C_1/2, C_2/2\}$, if $\mathbf{A}$ is in form (3.2);*

(2) *$C_1 = \sup_{x\in[a-R,b+R]}|f(x)|$ is finite, $C_2 = \sup_{x\in[a-R,b+R]}|(h^2/2)(x) + \frac{df}{dx}(x)|$ is finite, $C_3 = \max\{1/2 + C_1/2, C_2/2\}$, if $\mathbf{A}$ is in form (3.3).*

PROOF. For simplicity, we use $\langle\cdot\rangle$ to represent $\langle\cdot\rangle_\Omega$. Let $H_1(x) = \frac{dF}{dx} + f(x)$ if $\mathbf{A}$ is in form (3.2), or $f(x)$ if $\mathbf{A}$ in form (3.3). Let $H_2(x) = \frac{d^2F}{dx^2} + \frac{df}{dx}(x)$ if $\mathbf{A}$ is in form (3.2), or $\frac{df}{dx}(x) + (h^2/2)(x)$ if $\mathbf{A}$ in form (3.3).

Let $\eta_n = P_n^\Omega \bar{u} - u_n$, $w_n = \bar{u} - P_n^\Omega \bar{u}$. For any $v \in V_n^\Omega$.

$$\left\langle \frac{\partial}{\partial t}(\bar{u} - u_n), v \right\rangle = \frac{1}{2}\left\langle \frac{\partial^2(\bar{u} - u_n)}{\partial x^2}, v \right\rangle - \left\langle H_1(x)\frac{\partial(\bar{u} - u_n)}{\partial x}, v \right\rangle - \langle H_2(x)(\bar{u} - u_n), v\rangle . \tag{3.33}$$

Thus,

$$\begin{aligned} \left\langle \frac{\partial}{\partial t}\left(P_n^\Omega \bar{u} - u_n\right), v \right\rangle + \left\langle \frac{\partial w_n}{\partial t}, v \right\rangle &= \frac{1}{2}\left\langle \frac{\partial^2\left(P_n^\Omega \bar{u} - u_n\right)}{\partial x^2}, v \right\rangle + \frac{1}{2}\left\langle \frac{\partial^2 w_n}{\partial x^2}, v \right\rangle \\ &\quad + \left\langle H_1(x)\frac{\partial\left(P_n^\Omega \bar{u} - u_n\right)}{\partial x}, v \right\rangle - \left\langle H_1(x)\frac{\partial w_n}{\partial x}, v \right\rangle \\ &\quad - \left\langle H_2(x)\left(P_n^\Omega \bar{u} - u_n\right), v \right\rangle + \langle H_2(x) w_n, v\rangle . \end{aligned} \tag{3.34}$$

Moving $\langle \frac{\partial w_n}{\partial t}, v\rangle$ to the right, we get

$$\begin{aligned} \left\langle \frac{\partial \eta_n}{\partial t}, v \right\rangle = &-\left\langle \frac{\partial w_n}{\partial t}, v \right\rangle + \frac{1}{2}\left\langle \frac{\partial^2 \eta_n}{\partial x^2}, v \right\rangle - \left\langle H_1(x)\frac{\partial \eta_n}{\partial x}, v \right\rangle - \langle H_2(x)\eta_n, v\rangle \\ &+ \frac{1}{2}\left\langle \frac{\partial^2 w_n}{\partial x^2}, v \right\rangle - \left\langle H_1(x)\frac{\partial w_n}{\partial x}, v \right\rangle - \langle H_2(x) w_n, v\rangle , \end{aligned} \tag{3.35}$$

$$\begin{aligned} \left\langle \frac{\partial \eta_n}{\partial t}, v \right\rangle = &-\left\langle \frac{\partial w_n}{\partial t}, v \right\rangle - \frac{1}{2}\left\langle \frac{\partial \eta_n}{\partial x}, \frac{\partial v}{\partial x} \right\rangle - \left\langle H_1(x)\frac{\partial \eta_n}{\partial x}, v \right\rangle - \langle H_2(x)\eta_n, v\rangle \\ &- \frac{1}{2}\left\langle \frac{\partial w_n}{\partial x}, \frac{\partial v}{\partial x} \right\rangle - \left\langle H_1(x)\frac{\partial w_n}{\partial x}, v \right\rangle - \langle H_2(x) w_n, v\rangle + \frac{1}{2} v(\bar{u} - u_n)\Big|_{a-R}^{b+R} . \end{aligned} \tag{3.36}$$

By moving $-(1/2)\langle \frac{\partial \eta_n}{\partial x}, \frac{\partial v}{\partial x}\rangle$ in (3.36) to the left, it becomes

$$\begin{aligned} \left\langle \frac{\partial \eta_n}{\partial t}, v \right\rangle + \frac{1}{2}\left\langle \frac{\partial \eta_n}{\partial x}, \frac{\partial v}{\partial x} \right\rangle = &-\left\langle \frac{\partial w_n}{\partial t}, v \right\rangle - \left\langle H_1(x)\frac{\partial \eta_n}{\partial x}, v \right\rangle - \langle H_2(x)\eta_n, v\rangle \\ &- \frac{1}{2}\left\langle \frac{\partial w_n}{\partial x}, \frac{\partial v}{\partial x} \right\rangle - \left\langle H_1(x)\frac{\partial w_n}{\partial x}, v \right\rangle \\ &- \langle H_2(x) w_n, v\rangle + \frac{1}{2} v(\bar{u} - u_n)\Big|_{a-R}^{b+R} . \end{aligned} \tag{3.37}$$

Now we replace v with η_n and get

$$\begin{aligned} \left\langle \frac{\partial \eta_n}{\partial t}, \eta_n \right\rangle + \frac{1}{2}\left\langle \frac{\partial \eta_n}{\partial x}, \frac{\partial \eta_n}{\partial x} \right\rangle = &-\left\langle \frac{\partial w_n}{\partial t}, \eta_n \right\rangle - \left\langle H_1(x)\frac{\partial \eta_n}{\partial x}, \eta_n \right\rangle - \frac{1}{2}\left\langle \frac{\partial w_n}{\partial x}, \frac{\partial \eta_n}{\partial x} \right\rangle \\ &- \langle H_2(x)\eta_n, \eta_n\rangle - \left\langle H_1(x)\frac{\partial w_n}{\partial x}, \eta_n \right\rangle \\ &- \langle H_2(x) w_n, \eta_n\rangle + \frac{1}{2}\eta_n(\bar{u} - u_n)\Big|_{a-R}^{b+R} . \end{aligned} \tag{3.38}$$

Now $\bar{u}(a-R,t) = \bar{u}(b+R,t) = u_n(a-R,t) = u_n(b+R,t) = 0$.

Let $C_1 = \sup_{x\in[a-R,b+R]} |H_1(x)|$, $C_2 = \sup_{x\in[a-R,b+R]} |H_2(x)|$

$$\begin{aligned}\left|\left\langle\left(\frac{dF}{dx}+f\right)\frac{\partial\eta_n}{\partial x},\eta_n\right\rangle\right| &\le C_1\frac{1}{2}\left(4C_1\|\eta_n\|^2_{0,\Omega}+\frac{1}{4C_1}\left\|\frac{\partial\eta_n}{\partial x}\right\|^2_{0,\Omega}\right)\\ &= 2C_1^2\|\eta_n\|^2_{0,\Omega}+\frac{1}{8}\left\|\frac{\partial\eta_n}{\partial x}\right\|^2_{\Omega},\end{aligned}\tag{3.39}$$

$$\left|\left\langle\left(\frac{d^2F}{dx^2}+\frac{df}{dx}\right)\eta_n,\eta_n\right\rangle\right|\le C_2\|\eta_n\|^2_{0,\Omega},\tag{3.40}$$

$$\frac{1}{2}\left|\left\langle\frac{\partial w_n}{\partial x},\frac{\partial\eta_n}{\partial x}\right\rangle\right|\le\frac{2}{4}\left\|\frac{\partial w_n}{\partial x}\right\|^2_{0,\Omega}+\frac{1}{8}\left\|\frac{\partial\eta_n}{\partial x}\right\|^2_{0,\Omega},\tag{3.41}$$

$$\left|\left\langle\left(\frac{dF}{dx}+f\right)\frac{\partial w_n}{\partial x},\eta_n\right\rangle\right|\le\frac{C_1}{2}\left(\|\eta_n\|^2_{0,\Omega}+\left\|\frac{\partial w_n}{\partial x}\right\|^2_{0,\Omega}\right),\tag{3.42}$$

$$\left|\left\langle\left(\frac{d^2F}{dx^2}+\frac{df}{dx}\right)w_n,\eta_n\right\rangle\right|\le\frac{C_2}{2}\|w_n\|^2_{0,\Omega}+\frac{C_2}{2}\|\eta_n\|^2_{0,\Omega},\tag{3.43}$$

$$\left|\left\langle\frac{\partial w_n}{\partial t},\eta_n\right\rangle\right|\le\frac{1}{2}\left\|\frac{\partial w_n}{\partial t}\right\|^2_{0,\Omega}+\frac{1}{2}\|\eta_n\|^2_{0,\Omega}.\tag{3.44}$$

Hence,

$$\begin{aligned}\left\langle\frac{\partial\eta_n}{\partial t},\eta_n\right\rangle+\frac{1}{2}\left\langle\frac{\partial\eta_n}{\partial x},\frac{\partial\eta_n}{\partial x}\right\rangle &\le\left|\left\langle\frac{\partial w_n}{\partial t},\eta_n\right\rangle\right|+\left|\left\langle\left(\frac{dF}{dx}+f\right)\frac{\partial\eta_n}{\partial x},\eta_n\right\rangle\right|\\ &\quad+\left|\left\langle\left(\frac{d^2F}{dx^2}\right)\eta_n,\eta_n\right\rangle\right|+\left|\left\langle\frac{\partial w_n}{\partial x},\frac{\partial\eta_n}{\partial x}\right\rangle\right|\\ &\quad+\left|\left\langle\left(\frac{dF}{dx}+f\right)\frac{\partial w_n}{\partial x},\eta_n\right\rangle\right|\\ &\quad+\left|\left\langle\left(\frac{d^2F}{dx^2}+\frac{df}{dx}\right)w_n,\eta_n\right\rangle\right|\\ &\le\frac{1}{2}\left\|\frac{\partial w_n}{\partial t}\right\|^2_{0,\Omega}+\frac{1}{2}\|\eta_n\|^2_{0,\Omega}+2C_1^2\|\eta_n\|^2_{0,\Omega}+\frac{1}{8}\left\|\frac{\partial\eta_n}{\partial x}\right\|^2_{0,\Omega}\\ &\quad+C_2\|\eta_n\|^2_{0,\Omega}+\frac{1}{2}\left\|\frac{\partial w_n}{\partial x}\right\|^2_{0,\Omega}+\frac{1}{8}\left\|\frac{\partial\eta_n}{\partial x}\right\|^2_{0,\Omega}\\ &\quad+\frac{C_1}{2}\left(\|\eta_n\|^2_{0,\Omega}+\left\|\frac{\partial w_n}{\partial x}\right\|^2_{0,\Omega}\right)\frac{C_2}{2}\|w_n\|^2_{0,\Omega}\\ &\quad+\frac{C_2}{2}\|\eta_n\|^2_{0,\Omega}.\end{aligned}\tag{3.45}$$

Then,

$$\begin{aligned}\left\langle\frac{\partial\eta_n}{\partial t},\eta_n\right\rangle+\frac{1}{2}\left\langle\frac{\partial\eta_n}{\partial x},\frac{\partial\eta_n}{\partial x}\right\rangle &=\frac{1}{2}\frac{d}{dt}\|\eta_n\|_0^2+\frac{1}{2}\left\|\frac{\partial\eta_n}{\partial x}\right\|_0^2\\ &\le\frac{1}{2}\left\|\frac{\partial w_n}{\partial t}\right\|_0^2+\left(\frac{1}{2}+2C_1^2+\frac{3C_2}{2}+\frac{C_1}{2}\right)\|\eta_n\|_0^2\\ &\quad+\frac{C_2}{2}\|w_n\|_0^2+\left(\frac{1}{2}+\frac{C_1}{2}\right)\left\|\frac{\partial w_n}{\partial x}\right\|_0^2+\frac{1}{4}\left\|\frac{\partial\eta_n}{\partial x}\right\|_0^2.\end{aligned}\tag{3.46}$$

Let $C_3=\max\{1/2+C_1/2, C_2/2\}$. We get

$$\frac{1}{2}\frac{d}{dt}\|\eta_n\|_0^2+\frac{1}{4}\left\|\frac{\partial\eta_n}{\partial x}\right\|_0^2\le\frac{1}{2}\left\|\frac{\partial w_n}{\partial t}\right\|_0^2+C_3\|w_n\|_1+\left(\frac{1}{2}+2C_1^2+\frac{3C_2}{2}+\frac{C_1}{2}\right)\|\eta_n\|_0^2\tag{3.47}$$

or

$$\frac{1}{2}\frac{d}{dt}\|\eta_n\|_0^2 + \frac{1}{4}\left\|\frac{\partial \eta_n}{\partial x}\right\|_1^2 \le \frac{1}{2}\left\|\frac{\partial w_n}{\partial t}\right\|_0^2 + C_3\|w_n\|_1 + \left(\frac{1}{2} + 2C_1^2 + \frac{3C_2}{2} + \frac{C_1}{2}\right)\|\eta_n\|_0^2. \tag{3.48}$$

Multiplying both sides by 2, we obtain

$$\frac{d}{dt}\left(\|\eta_n\|_0^2\right) + \frac{1}{2}\|\eta_n\|_1^2 \le \left\|\frac{\partial w_n}{\partial t}\right\|_0^2 + \left(1 + 4C_1^2 + 3C_2 + C_1\right)\|\eta_n\|_0^2 + 2C_3\|w_n\|_1. \tag{3.49}$$

Taking the integral, we have

$$\begin{aligned} &\|\eta_n\|_0^2(T) + \frac{1}{2}\int_0^T \|\eta_n\|_1^2\,dt \\ &\le \int_0^T \left\|\frac{\partial w_n}{\partial t}\right\|_0^2 dt + \left(1 + 4C_1^2 + 3C_2 + C_1\right)\int_0^T \|\eta_n\|_0^2\,dt + 2C_3\int_0^T \|w_n\|_1\,dt. \end{aligned} \tag{3.50}$$

From Lemma 3.3, by letting

$$f(T) = \|\eta_n\|_0^2(T), \qquad h(T) = \frac{1}{2}\int_0^T \|\eta_n\|_1^2\,dt,$$

$$g(T) = \int_0^T \left\|\frac{\partial w_n}{\partial t}\right\|_0^2 dt + 2C_3\int_0^T \|w_n\|_1\,dt,$$

we obtain

$$\|\eta_n\|_0^2(T) + \frac{1}{4}\int_0^T \|\eta_n\|_1^2\,dt \le e^{(2+8C_1^2+6C_2+2C_1)T}\left\{\int_0^T \left\|\frac{\partial w_n}{\partial t}\right\|_0^2 dt + 2C_3\int_0^T \|w_n\|_1\,dt\right\}. \tag{3.51}$$

The theorem is proven. ∎

From the above theorem, we know that we can approximate the solution of (3.5) in a given interval for a given period of time.

4. PYRAMID ALGORITHM IN THE ADAPTIVE WAVELET-GALERKIN METHOD

Here we discuss the Kolmogorov equation for a one-dimensional spatial variable x. The key element we need to consider carefully in the wavelet-Galerkin method for the Kolmogorov equation is how to determine the finite interval Ω over which the condition of boundary-value problem (3.5) is satisfied. We can choose the approximation interval from the initial condition. Then, as time increases, there should be some method to determine whether the interval needs to increase, decrease, or be retained for a certain period of time. Here we introduce a method called the time-dependent boundary wavelet-Galerkin method. The main point here is that we need to change the boundary from time to time according to the calculation.

First, we derive a time-dependent boundary value problem by making the interval Ω time-dependent in equation (3.26) (or (3.28)). First, let $\Omega(t) := [a(t), b(t)]$, $u_n = \sum_{k=2^n a(t)+1-R}^{2^n b(t)-1} \lambda_{n,k}\varphi_{n,k}$ in $\Omega(t) = [a(t), b(t)]$. Consider

$$\begin{aligned} \left\langle \frac{\partial u_n}{\partial t}, \varphi_{n,j}\right\rangle_{\mathbf{R}} &= \frac{1}{2}\left\langle \frac{\partial^2 u_n}{\partial x^2}, \varphi_{n,j}\right\rangle_{\mathbf{R}} \\ &\quad - \left\langle \left(\frac{dF}{dx}(x) + f(x)\right)\frac{\partial u_n}{\partial x}, \varphi_{n,j}\right\rangle_{\mathbf{R}} \\ &\quad - \left\langle \left(\frac{d^2F}{dx^2}(x) + \frac{df}{dx}(x)\right)u_n, \varphi_{n,j}\right\rangle_{\mathbf{R}}, \\ u_n(x,0) &= P_n^{[a(t),b(t)]}u_0(x), \qquad x \in (a(t), b(t)), \\ \sum_{k=2^n a(t)-R+1}^{2^n a(t)-1} \lambda_{n,k}\varphi(2^n a(t) - k) &= \sum_{k=2^n b(t)-R+1}^{2^n b(t)-1} \lambda_{n,k}\varphi(2^n b(t) - k) = 0, \qquad t \in [0,T], \end{aligned} \tag{4.1}$$

or

$$
\begin{aligned}
\left\langle \frac{\partial u_n}{\partial t}, \varphi_{n,j} \right\rangle_{\mathbf{R}} &= \frac{1}{2}\left\langle \frac{\partial^2 u_n}{\partial x^2}, \varphi_{n,j} \right\rangle_{\mathbf{R}} \\
&\quad - \left\langle f(x)\frac{\partial u_n}{\partial x}, \varphi_{n,j} \right\rangle_{\mathbf{R}} \\
&\quad - \left\langle \left(\frac{h^2}{2}(x) + \frac{df}{dx}(x)\right) u_n, \varphi_{n,j} \right\rangle_{\mathbf{R}}, \qquad (4.2) \\
u_n(x,0) &= P_n^{[a(t),b(t)]} u_0(x), \qquad x \in (a(t), b(t)), \\
\sum_{k=2^n a(t)-R+1}^{2^n a(t)-1} \lambda_{n,k}\varphi(2^n a(t) - k) &= \sum_{k=2^n b(t)-R+1}^{2^n b(t)-1} \lambda_{n,k}\varphi(2^n b(t) - k) = 0, \qquad t \in [0,T],
\end{aligned}
$$

where $2^n a(t) - R + 1 \leq j \leq 2^n b(t) - 1$, and $P_n^{[a(t),b(t)]} u_0 = \sum_{j=2^n a(t)-R+1}^{2^n b(t)-1} \langle u_0, \varphi_{n,j} \rangle \varphi_{n,j}$. $a(t), b(t)$ are the functions of t, and derived by the following procedure.

When $t = 0$, we choose the interval $\Omega = [a', b']$, such that $|u_0(x)| < \epsilon$, $x \in \mathbf{R} - \Omega$. Then, we enlarge the interval $[a', b']$ to $[a' - (R-1)/2^n, b' + (R-1)/2^n]$ and extend $u_n(x,0)$ to $[a' - (R-1)/2^n, b' + (R-1)/2^n]$ as follows:

$$
u_n(x,0) = \sum_{i=2^n a'-2(R-1)}^{2^n a'-R} \lambda_i \varphi_{n,i} + P_n^{[a',b']}(u_0) + \sum_{j=2^n b'}^{2^n b'+R-2} \lambda_j \varphi_{n,j},
$$

where let $\lambda_i = 0$ for $2^n a' - 2(R-1) \leq i \leq 2^n a' - R$, $\lambda_j = 0$ for $2^n b' \leq j \leq 2^n b' + R - 2$.

$[a(0), b(0)] := [a' - (R-1)/2^n, b' + (R-1)/2^n]$ is the initial interval over which we solve (4.1). After each step of the iteration in time when solving (4.1), we are at time t and the time for previous step is $t - h$. Then, we need to check $\lambda_{n,i}(t)$. There are two cases.

(1) If $\lambda_{n,i}(t) = 0$ for $2^n a(t-h) - (R-1) \leq i \leq 2^n a(t-h) - 1$ and $2^n b(t-h) - (R-1) \leq i \leq 2^n b(t-h) - 1$, then let $\Omega(t) = [a(t), b(t)] := [a(t-h), b(t-h)]$.

(2) If $\lambda_{n,i}(t) \neq 0$, for some $2^n a(t-h) - (R-1) \leq i \leq 2^n a(t-h) - 1$ and some $2^n b(t-h) - (R-1) \leq i \leq 2^n b(t-h) - 1$, then go back to the previous time stage, expand the interval $\Omega(t-h) = [a(t-h), b(t-h)]$ to interval $[a(t-h) - (R-1)/2^n, b(t-h) + (R-1)/2^n]$, and extend $u_n(x, t-h)$ to $[a(t-h) - (R-1)/2^n, b(t-h) + (R-1)/2^n]$ in the following way:

$$
u_n(x, t-h) = \sum_{i=2^n a(t-h)-2(R-1)}^{2^n a(t-h)-R} \lambda_i \varphi_{n,i} + u_n(x, t-h) + \sum_{j=2^n b(t-h)}^{2^n b(t-h)+R-2} \lambda_j \varphi_{n,j},
$$

where let $\lambda_i = 0$ for $2^n a(t-h) - 2(R-1) \leq i \leq 2^n a(t-h) - R$, $\lambda_j = 0$ for $2^n b(t-h) \leq j \leq 2^n b(t-h) + R - 2$. Then, redefine interval $\Omega(t-h) := [a(t-h) - (R-1)/2^n, b(t-h) + (R-1)/2^n]$. Then, formulate the new problem (4.1) or (4.2) by using new interval $[a - (R-1)/2^n, b + (R-1)/2^n]$. Solve it at time stage $t - h$. Repeat the process again and again until Case (1) happens for time stage t.

Thus, we can see that each time we go from one time stage to the next, we can guarantee that the value of u_n, which satisfies (4.1) or (4.2), at the boundary of interval is zero. By following the procedure of selecting $\Omega(t)$ as above, we called equation (4.1) or (4.2) a time-dependent boundary wavelet-Galerkin method. In the following theorem, we prove that the solution of the time-dependent boundary wavelet-Galerkin method comes out to be an approximation of the solution of the Kolmogorov equation.

THEOREM 4.7. *There exists a finite interval Ω, such that the solution u_n of time-dependent boundary wavelet-Galerkin method is an approximation of boundary value problem (3.5) for the interval Ω. Then, it is also a approximation of the Kolmogorov equation (3.1) with form (3.2).*

PROOF. From the discussion time-dependent interval $\Omega(t)$ above, we know $\Omega(t_1) \subset \Omega(t_2)$ when $t_1 \leq t_2$. u_n satisfy that $u_n(x,t) = 0$ for $a(t) \leq x \leq a(t) + (R-1)/2^n$ and $b(t) - (R-1)/2^n \leq$

$x \le b(t)$. Thus, u_n is the solution of problem (3.26) over the interval $[a(t), b(t)]$. (Alternative, it is the approximation to the solution of problem (3.5) with the interval $[a(t), b(t)]$.) Now consider interval $[a(T), b(T)]$.

It may not be correct that when $x \in \mathbf{R} - [a(T), b(T)]$,

$$|u(a(T),t)| \le \epsilon, \qquad |u(b(T),t)| \le \epsilon, \qquad |u(x,t)| \le \epsilon, \tag{4.3}$$

where $0 \le t \le T$. But by Lemma 3.2, at least that we can find an interval $[a,b] \supset [a(T), b(T)]$ that, when $x \in \mathbf{R} - [a,b]$, $u(x,t)$ satisfies (4.3). Now for any t_0, the solution u_n at time t_0 can be extended to $[a,b]$ by letting $u_n(x,t) = 0$ when $x \in [a,b] - [a(t_0), b(t_0)]$. From the procedure of creating an interval $[a(t_0), b(t_0)]$ above, we know that $u_n(x,t) = 0$ for $a(t_0) \le x \le a(t_0) + (R-1)/2^n$ and $b(t_0) - (R-1)/2^n \le x \le b(t_0)$. Hence, $\frac{\partial^2 u_n}{\partial x^2}$ and $\frac{\partial u_n}{\partial t}$ exist in $(a,b) - (a(t_0), b(t_0))$, and $\frac{\partial^2 u_n}{\partial x^2} = \frac{\partial u_n}{\partial x} = \frac{\partial u_n}{\partial t} = 0$ in $(a,b) - (a(t_0), b(t_0))$. Then, u_n also satisfies equation (3.26) with the initial condition

$$u_n(x,0) = \begin{cases} P_n^{[a(t_0),b(t_0)]}(u_0), & x \in [a(t_0), b(t_0)], \\ 0, & x \in [a,b] - [a(t_0), b(t_0)], \end{cases}$$

in the interval $[a,b]$. Hence, u_n is the approximation of the solution $u_1(x,t)$ of the following boundary value problem:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \\ &\quad - \left(\frac{dF}{dx} + f(x)\right)\frac{\partial u}{\partial x} - \left(\frac{d^2F}{dx^2} + \frac{df}{dx}\right) u(x,t), \\ u(x,0) &= \begin{cases} u_0(x), & x \in [a(t_0), b(t_0)], \\ 0, & x \in [a,b] - [a(t_0), b(t_0)], \end{cases} \\ u(a,t) &= u(b,t) = 0, \qquad\qquad 0 \le t \le T, \end{aligned} \tag{4.4}$$

or

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - f(x)\frac{\partial u}{\partial x} - \left(\frac{h^2}{2} + \frac{df}{dx}\right) u(x,t), \\ u(x,0) &= \begin{cases} u_0(x), & x \in [a(t_0), b(t_0)], \\ 0, & x \in [a,b] - [a(t_0), b(t_0)], \end{cases} \\ u(a,t) &= u(b,t) = 0, \qquad\qquad 0 \le t \le T. \end{aligned} \tag{4.5}$$

Now suppose the solution of (3.26) with interval $\Omega = [a,b]$ and initial condition $u_0(x)$ in Ω is $\tilde{u}_n$. Then, $\tilde{u}_n$ is the approximation of the $u_2(x,t)$ of the following boundary value problem:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \\ &\quad - \left(\frac{dF}{dx} + f(x)\right)\frac{\partial u}{\partial x} - \left(\frac{d^2F}{dx^2} + \frac{df}{dx}\right) u(x,t), \\ u(x,0) &= u_0(x), \qquad\qquad x \in [a,b], \\ u(a,t) &= u(b,t) = 0, \qquad\qquad 0 \le t \le T, \end{aligned} \tag{4.6}$$

or

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \\ &\quad - f(x)\frac{\partial u}{\partial x} - \left(\frac{h^2}{2} + \frac{df}{dx}\right) u(x,t), \\ u(x,0) &= u_0(x), \qquad\qquad x \in [a,b], \\ u(a,t) &= u(b,t) = 0, \qquad\qquad 0 \le t \le T. \end{aligned} \tag{4.7}$$

Then, $u_1 - u_2$ have to attach the maximum value at the boundary of domain $[a,b] \times [0,T)$. When $x = a$ and $x = b$, $u_1 - u_2$ equal to zero. When $a(t_0) \le x \le b(t_0)$, $(u_1 - u_2)(x, 0) = 0$. When $x \in [a,b] - [a(t_0), b(t_0)]$, $|(u_1 - u_2)(x,0)| = |u_0(x,0)| \le \epsilon$.

So $|u_1(x,t) - u_2(x,t)| < \epsilon$. In other words, u_n is also the approximation of $u_2(x,t)$. Then, it is also the approximation of (3.5) with interval $[a,b]$. (Alternative, it is the approximation of the Kolmogorov equation (3.1).) ∎

We know that one problem in solving the partial differential equation by numerical method is the stability problem. For the conventional numerical methods, for example, the finite-difference or finite-element method, the step size in time domain depends on the resolution in spatial domain. In the forward finite-difference method, $h/d^2 < 1/2$, where h is the step size in time domain, d is the distance of two resolution points in spatial domain. When the resolution in spatial domain is higher, the step size in time domain must be smaller in order to control the culmination of error. On the other hand, if we can choose a large step size in time domain, the speed of computation can be faster. Thus, the resolution in spatial variables is a premium for efficiency of computation for a huge partial differential equation problem, for example, a Kolmogorov equation with a larger initial condition.

When a function is smooth, the Daubechies' functions approximation to it in V_n can have satisfactory accuracy even when n is relatively small compared with some functions with steep jump, because of the approximation properties of Daubechies' wavelet.

From Lemma 2.5, we know that

$$|\langle f, \psi_{n,k}\rangle| \le 2^{-np}\frac{R^p}{\pi^p}|f|_{p,S_{n,k}}, \qquad 0 \le p \le N,$$

where $S_{n,k} = \operatorname{supp}\psi_{n,k}$ and $f \in H^N(\mathbf{R})$. When f is smooth, mathematically it means that $\frac{d^p f}{dx^p}$, or $|f|_{p,S_{n,k}}$, is small for fairly large values of p. Hence, $|\langle f, \psi_{n,k}\rangle|$ can decrease very rapidly for a moderate increase in the value of n.

When we apply this principle to a numerical method for partial differential equations, we can decrease the resolution in spatial variables if the solution is smooth, without the loss of the accuracy of the approximation.

From Example 1 here, if for the Kolmogorov equation, coefficients F and f are second- and first-order polynomial, respectively, the solution will become smooth as time increases. Hence, it is ideal to dynamically adjust the resolution according to the solution.

There is one drawback for adaptive numerical methods. That is, each time the resolution changes, a complicated computation is needed to move the approximation from the old resolution to a new resolution. But for the wavelet-Galerkin method, with the help of the pyramid algorithm, it is very easy to jump between the different levels of resolution. Here, we discuss how.

Suppose in time t_0, we go with step size h to $t_0 + h$. We get an approximation in time $t_0 + h$ in the form $u_n = \sum_{i=2^n a-R+1}^{2^n b-1} \lambda_{n,i}\varphi_{n,i}$. From the properties of wavelet approximation, we use $|\langle f, \psi_{n,k}\rangle|$ to determine how close the approximation is to the real function.

The following is the pyramid algorithm:

$$\begin{aligned}\varphi_{n,j} &= \sum_k h(k-2j)\varphi_{n+1,k},\\ \psi_{n,j} &= \sum_k g(k-2j)\varphi_{n+1,k},\\ \varphi_{n+1,j} &= \sum_k h(j-2k)\varphi_{n,k} + \sum_k g(j-2k)\psi_{n,k}.\end{aligned} \tag{4.8}$$

For a function $f(x)$, let $\lambda_{n,j} = \langle f, \psi_{n,j}\rangle = \int f(x)\psi_{n,j}\,dx$. Hence, we have

$$\lambda_{n,j} = \sum_k h(k-2j)\lambda_{n+1,k}, \tag{4.9}$$

$$\mu_{n,j} = \sum_k g(k-2j)\lambda_{n+1,k}, \tag{4.10}$$

$$\lambda_{n+1,j} = \sum_k h(j-2k)\lambda_{n,k} + \sum_k g(j-2k)\mu_{n,k}. \tag{4.11}$$

We already said that $\lambda_{n,j}$ is used to determine the level of resolution. Now we have $\lambda_{n,j}$. Then, we can use (4.10) to get the $\mu_{n-1,j}$.

Numerically, two very small constants ϵ_1 and ϵ_2 are preassigned, where $\epsilon_1 > \epsilon_2$. After each step of computation, if $|\lambda_{n-1,j}| > \epsilon_1$, we need to go back one step to the previous stage t_0 and then increase the level of resolution from n to $n+1$ at that stage. We may assume $\mu_{n,j}$ is zero at that stage. Then, from (4.11), we get $\lambda_{n+1,j}$ from $\lambda_{n,j}$. On the other hand, if $|\lambda_{n-1,j}| < \epsilon_2$ at t_0+h, we can decrease the level of resolution from n to $n-1$. Using (4.9), we get the coefficients $\lambda_{n-1,j}$.

This is the procedure of dynamically changing the level of resolution in spatial variables. We apply this numerical scheme to the following example.

EXAMPLE 1.

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - \left(\frac{dF}{dx} + f\right)\frac{\partial u}{\partial x} - \left(\frac{d^2F}{dx^2} + \frac{df}{dx}\right)u, \\ u(x,0) &= u_0(x), \end{aligned} \tag{4.12}$$

where $F(x) = (3/4)x^2$, $f(x) = 11x + 5$, *and*

$$u_0(x) = \begin{cases} x, & 0 \le x \le \frac{1}{2}, \\ 1-x, & \frac{1}{2} \le x \le 1, \\ 0, & \textit{otherwise}. \end{cases}$$

We shall approximate the solution by our wavelet-Galerkin method and adaptive scheme with initial interval $[-0.4, 1.6]$, and do the computation until $t = 0.1$. For the purpose of comparison, we can also approximate the solution by applying Fourier transform to this equation, from which we can solve the Fourier transform of the solution. Then, we get the solution by applying inverse Fourier transform. Then, numerical computation is applied to it to get another approximation to the solution of equation (4.12). We shall call this approach the Fourier method.

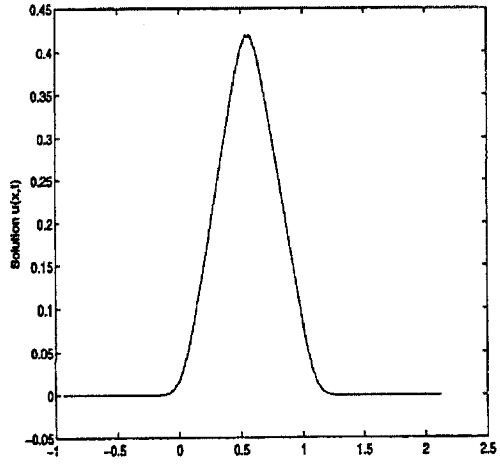

Figure 1. Wavelet-Galerkin approximation, time = 0.005.

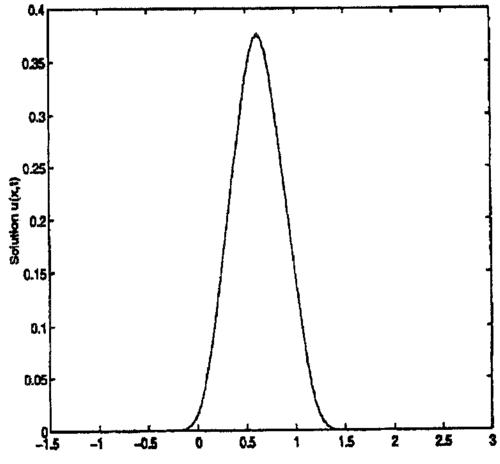

Figure 2. Wavelet-Galerkin approximation, time = 0.01.

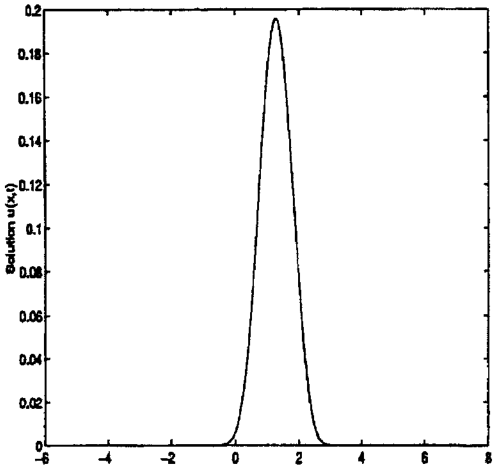

Figure 3. Wavelet-Galerkin approximation, time = 0.05.

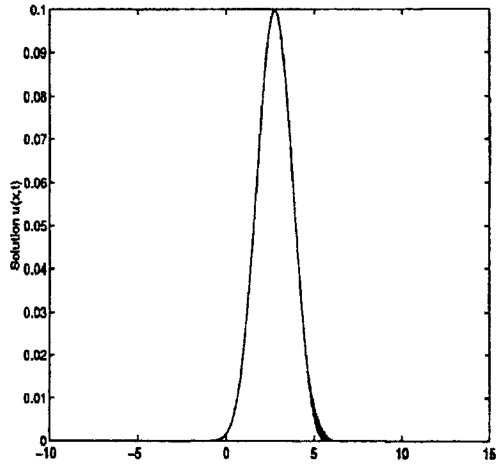

Figure 4. Wavelet-Galerkin approximation, time = 0.1.

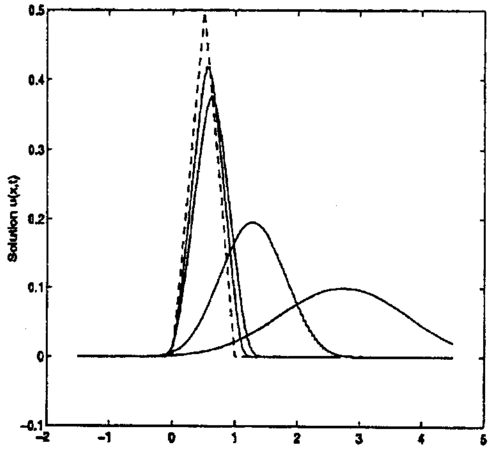

Figure 5. Wavelet-Galerkin approximation. The time for each curve is 0.005, 0.01, 0.05, 0.1.

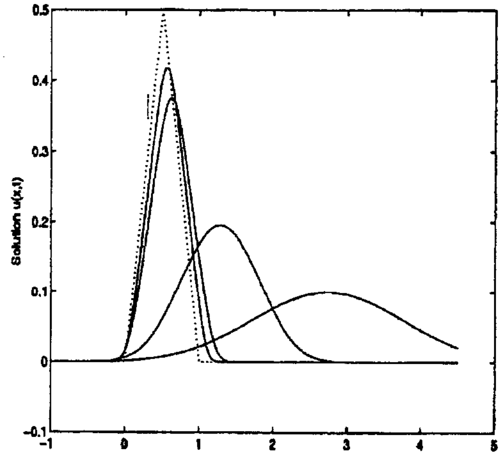

Figure 6. Approximation from Fourier method. The time for each curve is 0.005, 0.01, 0.05, 0.1.

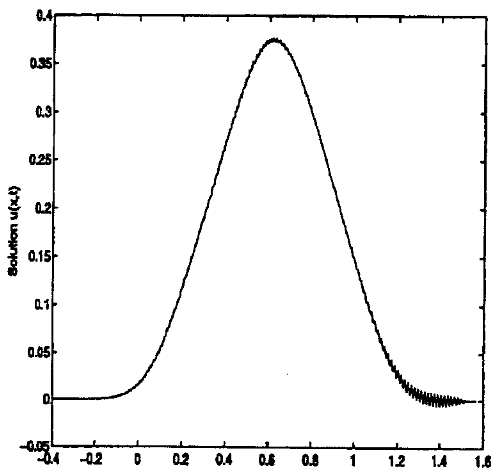

Figure 7. Wavelet-Galerkin approximation without adaptive scheme.

We compare the results from these two approximation. The approximations and comparisons are shown in the figures. In Figure 1, we compute the approximation at time $t = 0.005$. Figure 2 is the approximation at time $t = 0.01$, Figure 3 at time $t = 0.05$, Figure 4 at time $t = 0.1$. In

Figure 1, we observe that at the time 0.005, the interval in the adaptive scheme has been expanded to $[-1, 2.5]$. There are similar phenomena in Figures 2–4 that the spatial intervals expand due to the adaptive scheme. Figures 5 and 6 show the comparison of the approximation from these two different methods. Figure 5 shows the approximation by the wavelet-Galerkin method. Figure 6 shows the approximation by the Fourier method. In Figure 7, we just demonstrate that without using adaptive scheme, there will be a fluctuation in the computation.

5. CONCLUSION

It is well known that the Kolmogorov equation plays an important role in applied science. For example, the nonlinear filtering problem, which plays a key role in modern technologies, was solved by Yau and Yau [1] by reducing it to the off-line computation of the Kolmogorov equation.

In this paper, we develop a theorical foundation of using the wavelet-Galerkin method to solve linear parabolic P.D.E. We apply our theory to the Kolmogorov equation. We give a rigorous proof that the solution of the Kolmogorov can be approximated very well in any finite domain by our wavelet-Galerkin method. An example is provided by using Daubechies D_4 scaling functions.

In this example, we notice that we can do it even without any boundary condition, which is a big hurdle for many PDE numerical method. This computation is very stable. We can balance the requirements between computation stability and efficiency dynamically by using pyramid properties of Daubechies functions.

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