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Classification of finite-dimensional estimation algebras of maximal rank with arbitrary state–space dimension and mitter conjecture

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In the late seventies, the concept of the estimation algebra of a filtering system was introduced. It was proven to be an invaluable tool in the study of non-linear filtering problems. In the early eighties, Brockett proposed to classify finite dimensional estimation algebras and Mitter conjectured that all functions in finite dimensional estimation algebras are necessarily polynomials of total degree at most one. Despite the massive effort in understanding the finite dimensional estimation algebras, the 20 year old problem of Brockett and Mitter conjecture remains open. In this paper, we give a classification of finite dimensional estimation algebras of maximal rank and solve the Mitter conjecture affirmatively for finite dimensional estimation algebras of maximal rank. In particular, for an estimation algebra E of maximal rank, we give a necessary and sufficient conditions for E to be finite dimensional in terms of the drift $f_i(x)$ and observation $h_j(x)$. As an important corollary, we show that the number of statistics needed to compute the conditional density of the state given the observation $\{y(s) : 0 \leq s \leq t\}$ by the algebraic method is n where n is the dimension of the state.

1. Introduction

In 1960 and 1961, ASME's *Journal of Basic Engineering* published two historically important mathematics papers on filtering by Kalman (1960), and Kalman and Bucy (1961), respectively. Despite the usefulness of the Kalman–Bucy filter, however, it is not perfect. One of its weaknesses is that it needs a Gaussian assumption for the initial data. Another weakness is that it is restricted to linear dynamical systems. (In 1986, Makowski (1986) showed that the initial data need not be Gaussian, and what is essential is that the system is linear and driven by Brownian motion.) In view of these weaknesses, Brockett and Clark (1980), Brockett (1981) and Mitter (1939) proposed independently the idea of using estimation algebras to construct finite-dimensional non-linear filters. The idea is to imitate the approaches Wei–Norman (1964), using the Lie algebraic method to solve the Duncan–Mortensen–Zakai (DMZ) equation, which the unnormalized conditional probability density of the state $x(t)$ must satisfy. The advantage of this approach is that as long as the estimation algebra is finite dimensional, one gets a finite-dimensional recursive filter and there is no need to make any assumption on the initial date. Moreover, the approach applies well to non-linear dynamical systems. More importantly, the number of sufficient statistics in the Lie algebra method in computing the conditional probability density is linear in n, where n is the dimension of the state space. In fact, the finite dimensional filters constructed by the Lie algebraic method are universal in the sense of Chaleyat-Maurel and Michel (1984) (see Corollary 5). Hence the Lie algebraic method is of practical importance. For more detail, we refer the readers to an excellent survey article written by Marcus (1984). However, in the Wei–Norman approach, one has to know explicitly the basis of the estimation algebra in order to reduce the DMZ equation to a finite system of ordinary differential equations driven by $y(t)$ and a Kolmogorov equation that is independent of $y(t)$. Therefore it is very important to find out the basis of finite dimensional estimation algebra.

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In 1983, Brockett proposed to classify all finite-dimensional estimation algebras. The fundamental breakthrough in Brockett's problem was due to Wong (1987). In a series of papers Wong (1987a, b), gave a new light on the classification problem of finite-dimensional estimation algebras for the first time. Specifically, under the hypothesis that the drift f in (1) is real analytic and its first, second and third order partial derivatives are bounded functions, Wong proved that all finite-dimensional estimation algebras of (1) are solvable and the observation $h(x)$ in (1) is a polynomial of degree one. Most strikingly, he was able to give a structural description of finite dimensional estimation algebras under these conditions. Nevertheless, without the assumptions of the real analyticity of f and the bounds of the first, second and third order partial derivatives of f, the structure and classification of finite-dimensional exact estimation algebras were studied in detail only in the early 1990s (Chen and Yau 1996, 1997, Chen *et al.* 1996, 1997, Chiou and Yau 1994, Tam *et al.* 1990, Yau 1994). In Wong (1987b) the antisymmetric matrix Ω was introduced, which is defined as the matrix whose (i, j) element is $\omega_{ij} = \partial f_j/\partial x_i - \partial f_i/\partial x_j$, where f is the drift term of the state evolution equation. If the drift term has potential (i.e., if the drift term is a gradient vector field), the corresponding estimation algebra is called exact. For an exact filtering system, Ω is zero. Tam *et al.* (1990) and Dong *et al.* (1991) classified all finite-dimensional exact estimation algebras of maximal rank with arbitrary state space dimension. Chiou and Yau (1994) introduced the concept of a general estimation algebra of maximal rank. They were able to classify all finite-dimensional estimation algebras of maximal rank with state space dimension less than or equal to two. Chen *et al.* (1996, 1997) classified all finite-dimensional estimation algebras of maximal rank with state space dimension three and four respectively. Recently Hu *et al.* (2000), by using the quadratic theory developed in Chen and Yau (1996) and partial constant structure of Ω matrix proved in Wu *et al.* (2002) and Yau *et al.* (1999), have classified all finite dimensional estimation algebras of maximal rank with state space dimension less than or equal to five. The novelty of their theorem is that there is no assumption on the drift term of the non-linear filtering system.

Despite the massive effort in understanding the finite dimensional estimation algebras, the 20 years old problem of Brockett remains open. The purpose of this paper is to solve the Brockett's problem on classification of finite dimensional estimation algebra of maximal rank. Yau's program of classifying finite dimensional estimation algebras of maximal rank consists of four crucial steps.

Step 1. In 1990, Yau first observed that Wong's Ω-matrix plays an important role. As the first crucial step, Yau (1994) classified all finite dimensional estimation algebras of maximal rank if Wong's matrix has entries in constant coefficients.

Step 2. The second crucial step was due to Chen and Yau (1996). They developed quadratic structure theory for finite dimensional estimation algebra. They also laid down all the ingredients needed to give the classification of finite dimensional estimation algebras of maximal rank. In particular, they introduced the notion of quadratic rank k. In this way, the Wong's Ω-matrix is divided into 3 parts

(1) (ω_{ij}), $1 \le i,\ j \le k$;
(2) (ω_{ij}), $k+1 \le i,\ j \le n$; and
(3) (ω_{ij}), $1 \le i \le k$, $k+1 \le j \le n$, or $k+1 \le i \le n$, $1 \le j \le n$.

Chen and Yau (1996) proved among many other things that part (1) (ω_{ij}), $1 \le i, j \le k$, is a matrix with constant coefficients.

Step 3. Chen *et al.* (1997) proved the weak Hessian matrix non-decomposition theorem for $n \le 4$. As a result, part (2), (ω_{ij}), $k+1 \le i, j \le n$ is a matrix with constant coefficients. Wu *et al.* (2002) proved the weak Hessian matrix non-decomposition theorem for general n. Yau *et al.* (1999) proved the strong Hessian matrix non-decomposition theorem for general n. Thus part (2), $(\omega_{ij}), k+1 \le i,j \le n$ is also a matrix with constant coefficients.

Step 4. This paper uses the full power of the quadratic structure theory developed by Chen and Yau (1996) to prove that the matrix (ω_{ij}), $1 \le i \le k$, $k+1 \le j \le n$ and the matrix (ω_{ij}), $k+1 \le i \le n$, $1 \le j \le k$ are with the constant coefficients. This finishes the 20 years old classification problem of finite dimensional estimation algebras of maximal rank.

Theorem 1: *Suppose that the state space of the filtering system* (1) *is of dimension n. If E is the finite-dimensional estimation algebra with maximal rank, then E is a real vector space of dimensional* $2n+2$ *with basis given by* $1, x_1, \ldots, x_n, D_1, \ldots, D_n$ *and* L_0.

Mitter conjectured a long time ago that all the functions in finite dimensional estimation algebras are polynomials of degree one. As an immediate consequence of the above Main Theorem, following corollary applies.

Corollary 1 (Mitter Conjecture)**:** *Suppose that E is the finite-dimensional estimation algebra with maximal rank corresponding to the filtering system* (1). *Then any function in E is a polynomial of degree one.*

The following corollary is an immediate consequence of Theorem 1 above and Theorem 7 of Yau (1994).

Corollary 2: *Suppose that the state space of the filtering system* (1) *is of dimension n. If E is the finite-dimensional estimation algebra with maximal rank, then the number of statistics in order to compute the conditional density by Lie algebraic method is n.*

The following corollary is an immediate consequence of Theorem 1 above and Theorem 5 of Yau (1994).

Corollary 3: *Suppose that E is the estimation algebra with maximal rank corresponding to the filtering system* (1). *Then E is finite dimensional if and only if*

(1) $(\partial f_j/\partial x_i) - (\partial f_i/\partial x_j) = c_{ij}$ *where* c_{ij} *are constants for all* $1 \le i, j \le n$,
(2) $h_1, \ldots, h_m$ *are affine in x* (*i.e. polynomials of total degree at most one*),
(3) $\sum_{i=1}^{n} (\partial f_i/\partial x_i) + \sum_{i=1}^{n} {f_i}^2 + \sum_{i=1}^{m} h_i^2$ *is a polynomial of total degree at most two.*

This paper is, in essence, a continuation of previous papers (Chen and Yau 1996, Chen *et al.* 1996, 1997, Chiou and Yau 1994, Hu *et al.* 2000, Wu *et al.* 2002, Yau 1994, Yau *et al.* 1999). It is strongly recommend that readers familiarize themselves with the results in Chen and Yau (1996) and Yau and Wong (1999). However, every effort is made to have this paper as self-contained as possible without too much duplication of the previous paper.

In § 2, some basic concepts and results are recalled. § 3 classifies finite-dimensional estimation algebras of maximal rank with arbitrary state space dimension. Section 4 gives a general construction of finite dimensional estimation algebras beyond Kalman or Benés types. Construction of finite dimensional filters from finite dimensional estimation algebras are also given. These finite dimensional filters turn out to be universal in the sense of Chaleyat-Maurel and Michel (1984).

2. Some basic concepts and results

In this section, some basic concepts and results are recalled (Chen and Yau 1996, Chiou and Yau 1994, Wu *et al.* 2002, Yau 1994, Yau *et al.* 1999). The filtering problem considered here is based on the signal observation model

$$\begin{cases} dx(t) = f(x(t))\,dt + g(x(t))\,dv(t) & x(0) = x_0 \\ dy(t) = h(x(t)\,dt + dw(t) & y(0) = 0 \end{cases} \tag{1}$$

in which x, v, y and w are respectively R^n, R^p, R^m, and R^m valued processes and v and w have components that are independent, standard Brownian processes. It is further assumed that $n = p$; f, g and h are vector-valued, orthogonal matrix-valued and vector-field C^∞ smooth functions; $y(t)$ is referred to as the observation at time t.

Let $\rho(t, x)$ denote the conditional probability density of the state given the observation $\{y(s) : 0 \le s \le t\}$. It is well known (see Davis and Marcus (1981), for example) that $\rho(t, x)$ is given by normalizing a function $\sigma(t, x)$ that satisfies the following DMZ equations.

$$\begin{cases} d\sigma(t,x) = L_0\sigma(t,x)\,dt + \sum_{i=1}^{m} L_i\sigma(t,x)\,dy_i(t), \\ \sigma(0,x) = \sigma_0(x) \end{cases} \tag{2}$$

where $L_0 = \frac{1}{2}\sum_{i=1}^{n} \partial^2/\partial x_i^2 - \sum_{i=1}^{n} f_i\partial/\partial x_i - \sum_{i=1}^{n} \partial f_i/\partial x_i - \frac{1}{2}\sum_{i=1}^{m} h_i^2$ for $i = 1, \ldots, m$, L_i is the zero-degree differential operator of multiplication by h_i, and σ_0 is the probability density of the initial point x_0.

Equation (2) is a stochastic partial differential equation. In real applications, one is interested in constructing robust state estimators from observed sample paths with some property of robustness. Davis in (1980) studied this problem and proposed some robust algorithms. In this case, his basic idea reduces to defining a new unnormalized density

$$u(t,x) = \exp\left(-\sum_{i=1}^{m} h_i(x)y_i(t)\right)\sigma(t,x).$$

Davis reduced (2) to the following time-varying partial differential equation, which is called the robust DMZ equation:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = L_0u(t,x) + \sum_{i=1}^{m} y_i(t)[L_0, L_i]u(t,x) \\ \qquad + \frac{1}{2}\sum_{i,j=1}^{m} y_i(t)y_j(t)[[L_0, L_i], L_j]u(t,x), \\ u(0,x) = \sigma_0(x), \end{cases} \tag{3}$$

where $[\cdot,\cdot]$ is the Lie bracket.

Definition 1: The estimation algebra E of a filtering problem (1) is defined as the Lie algebra generated by $\{L_0, L_1, \ldots, L_m\}$. E is said to be an estimation algebra of maximal rank if, for any $1 \le i \le n$, there exists a constant c_i such that $x_i + c_i$ is in E.

Definition 2: Define matrix $\Omega = (\omega_{ij})$, where

$$\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}, \quad \forall\ 1 \le i, j \le n. \tag{4}$$

Clearly Ω is skew symmetric and $(\partial\omega_{jk}/\partial x_i) + (\partial\omega_{ki}/\partial x_j) + (\partial\omega_{ij}/\partial x_k) = 0$ for every $1 \le i, j, k \le n$.

Define

$$D_i = \frac{\partial}{\partial x_i} - f_i, \quad \eta = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} {f_i}^2 + \sum_{i=1}^{m} h_i^2. \tag{5}$$

Since $D_i^2 = (\partial^2/\partial x_i^2) - (\partial/\partial x_i) f_i - f_i(\partial/\partial x_i) + f_i^2 = (\partial^2/\partial x_i^2) - f_i(\partial/\partial x_i) - (\partial f_i/\partial x_i) - f_i(\partial/\partial x_i) + f_i^2 = (\partial^2/\partial x_i^2) - 2f_i(\partial/\partial x_i) - (\partial f_i/\partial x_i) + f_i^2$,

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^{n} D_i^2 - \eta\right). \tag{6}$$

The following basic results which play a fundamental role in the classification of finite dimensional estimation algebras are needed.

Theorem 2 (Ocone 1980): *Let E be a finite-dimensional estimation algebra. If a function ψ is in E, then ψ is a polynomial of total degree ≤ 2.*

Theorem 3 (Yau 1994): *Let E be a finite-dimensional estimation algebra of* (1) *such that ω_{ij} are constant functions. If E is of maximal rank, then E is a real vector space of dimension $2n+2$ with basis given by $1, x_1, \ldots, x_n$, $D_1, \ldots, D_n$ and L_0.*

Corollary 4 (Yau 1994): *Let E be a finite-dimensional estimation algebra with maximal rank. Then E contains the real vector space spanned by $1, x, \ldots, x_n$, $D_1 \ldots, D_n$ and L_0.*

Let Q be the space of quadratic forms in n variables, namely, real vector space spanned by $x_i x_j$ with $1 \leq i \leq j \leq n$. Let $X = (x_1, \ldots, x_n)^T$. For any quadratic form $p \in Q$, there exists a symmetric matrix A such that $p(x) = X^T AX$. The rank of the quadratic form p is denoted by $r(p)$ and is defined to be the rank of the matrix A. A fundamental quadratic form of the estimation algebra E is an element $p_0 \in E \cap Q$ with the greatest positive rank, that is, $r(p_0) \geq r(p)$ for any $p \in E \cap Q$. The maximal rank of quadratic form in the estimation algebra E is defined to be $k = r(p_0)$ and is called the quadratic rank of E.

After an orthogonal transformation on x, p_0 can be written as

$$p_0(x) = c_1 x_1^2 + c_2 x_2^2 + \cdots + c_k x_k^2, c_i \neq 0, \quad 0 \leq k \leq n. \tag{7}$$

Notice that k is the quadratic rank defined right after Corollary 4 and is fixed. A priori, k can be any number from 0 to n. Following Wong (1987b) from $p_0(x)$, we can construct a sequence of quadratic forms in $E \cap Q$ as follows:

$$q_0(x) = p_0(x) \tag{8}$$

$$q_j(x) = \big[[L_0, q_{j-1}], q_0\big] = \sum_{i=1}^{k} 4^j c_i^{j+1} x_i^2. \tag{9}$$

In view of the invertibility of the Vandermonde matrix, it can be assumed that

$$p_0(x) = x_1^2 + x_2^2 + \cdots + x_k^2 \in E.$$

Remark 1: A theorem of Brockett (Brockett 1979) states that the estimation algebras are isomorphic under a smooth non-singular change of coordinates. Therefore for the problem of classification of finite dimensional estimation algebras, one is free to change coordinates.

Lemma 1 (Chen and Yau 1996): *If p is a quadratic form in estimation algebra E, then p is independent of x_j for $j > k$, where $k = r(p_0)$ is the quadratic rank of E. In other words, $\partial p/\partial x_j = 0$ for $k+1 \leq j \leq n$.*

Let $p_1 \in E \cap Q$ be an element with least positive rank, that is, $0 < r(p_1) \leq r(q)$ for any non-zero $q \in E \cap Q$. After an orthogonal transform that fixes $x_{k+1}, \ldots, x_n$ variables (i.e. an orthogonal transform on $x_1, x_2, \ldots, x_k$), and the Vandermonde matrix procedure as above, we can assume

$$p_1 = \sum_{i=1}^{k_1} x_i^2 \in E, \quad 1 \leq k_1 \leq k. \tag{10}$$

Note that the orthogonal transform on $x_1, \ldots, x_k$ leaves p_0 invariant. In summary, we deduce that $p_0 = \sum_{i=1}^{k} x_i^2$ has the greatest positive rank and $p_1 = \sum_{i=1}^{k_1} x_i^2$ has the least positive rank. Define

$$S_1 = \{1, 2, \ldots, k_1\} \subseteq S = \{1, 2, \ldots, k\} \tag{11}$$

and Q_1 = real vector space spanned by $\{x_i x_j : k_1 + 1 \leq i \leq j \leq k\} \subseteq Q$.

If $k_1 < k$, then $Q_1 \cap E$ is a non-trivial space, since $p - p_0 \in E \cap Q$. In a similar procedure as above, there exists

$$p_2 = \sum_{i=k_1+1}^{k_2} x_i^2 \in E \cap Q_1 \tag{12}$$

with the least positive rank in $E \cap Q_1$. By induction, one can construct a series of S_i, Q_i and p_i such that

$$S_i = \{k_{i-1}+1, \ldots, k_i\}, k_0 = 0, k_i \leq k, \tag{13}$$

Q_i = real vector space spanned by $\{x_l x_j : k_i + 1 \leq l \leq j \leq k\}$,

$$p_i = \sum_{j=k_{i-1}+1}^{k_i} x_j^2 = \sum_{j \in S_i} x_j^2, i > 0 \tag{14}$$

and p_i has the least positive rank in $E \cap Q_{i-1}$ for $i > 0$.

Lemma 2 (Chen and Yau 1996): *If $p \in E \cap Q$, then*

$$p(0, \ldots, 0, x_{k_{i-1}+1}, \ldots, x_{k_i}, 0, \ldots, 0) = \lambda\, p_i, \quad \text{for } i > 0,$$

where λ is a real constant.

Lemma 3 (Chen and Yau 1996): *If $p \in E \cap Q$, then*

$$p(x_1, \ldots, x_{k_{i-1}}, 0, \ldots, 0, x_{k_i+1}, \ldots, x_n) \in E \quad \text{for } i > 0$$

Lemma 4 (Chiou and Yau 1994, Wong 1987, Yau 1994): *Let α, β be differentiable functions defined on $\mathbb{R}^n$. Then*

(i) $[XY, Z] = X[Y, Z] + [X, Z]Y$, *where X, Y and Z are differential operators*;
(ii) $[\alpha D_i, \beta] = \alpha(\partial\beta/\partial x_i)$, where $D_i = (\partial/\partial x_i) - f_i$;
(iii) $[\alpha D_i, \beta D_j] = -\alpha\beta\omega_{ij} + \alpha(\partial\beta/\partial x_i)D_j - \beta(\partial\alpha/\partial x_j)D_i$, where $\omega_{ji} = [D_i, D_j] = (\partial f_i/\partial x_j) - (\partial f_j/\partial x_i)$;
(iv) $[\alpha D_i^2, \beta] = 2\alpha(\partial\beta/\partial x_i)D_i + \alpha(\partial^2\beta/\partial x_i^2)$;
(v) $[D_i^2, \beta D_j] = 2(\partial\beta/\partial x_i)D_iD_j - 2\beta\omega_{ij}D_i + (\partial^2\beta/\partial x_i^2)D_j - \beta(\partial\omega_{ij}/\partial x_i)$;
(vi) $[D_i^2, D_j^2] = 4\omega_{ji}D_jD_i + 2(\partial\omega_{ji}/\partial x_j)D_i + 2(\partial\omega_{ji}/\partial x_i)D_j + 2(\partial\omega_{ji}/\partial x_i\partial x_j) + 2\omega_{ji}^2$.

Lemma 5 (Yau 1994): *Let E be a finite dimensional estimation algebra with maximal rank. Then the vector space spanned by $< 1, x_1, \ldots x_n,\ D_1, \ldots, D_n > \subseteq E$.*

Before going to the next section, we summary an important Theorem from previous papers (Wu *et al.* 2002, Yau *et al.* 1999) that will appear frequently in our subsequent discussion.

Theorem 4: *Suppose that the estimation algebra associated to* (1) *is finite dimensional with maximal rank. Let k be the quadratic rank of E. Then ω_{ij}, for $1 \le i, j \le k$* or *$k+1 \le i, j \le n$, are constants, and ω_{ij}, for $1 \le i \le k, k+1 \le j \le n$* or *$k+1 \le i \le n$, $1 \le j \le k$, are polynomials of degree one in $x_1, \ldots, x_k$. Furthermore, $\alpha_j = \sum_{l=1}^{k} x_l\omega_{jl} \in E$, $\forall k+1 \le j \le n$.*

3. Proof of the main theorem

In the subsequent discussion, we shall denote k to be the quadratic rank of the estimation algebra E, $k_1, k_2, \ldots$ be those sequences defined in (11) and (13), and $p_i(x) = \sum_{j=k_{i-1}+1}^{k_i} x_j^2$ in (14) be the quadratic polynomial of E in $x_{k_{i-1}+1}, \ldots, x_{k_i}$ variables with minimal rank.

We now sketch the idea of proving our Main Theorem. In view of Theorem 3, we only need to prove that ω_{ij}, $1 \le i, j \le n$, are constant functions. If the quadratic rank k is equal to zero, then our Main Theorem follows from Theorem 4. Therefore it remains to prove ω_{ij}, for $1 \le i \le k$, $k+1 \le j \le n$ or $k+1 \le i \le n$, $1 \le j \le k$, are constant functions if $k > 0$ by Theorem 4. Since (ω_{ij}) is skew symmetric, it suffices to prove that ω_{ij}, for $1 \le i \le k$, $k+1 \le j \le n$, are constant functions. We are going to prove that these ω_{ij} functions are independent of $x_1, \ldots, x_k$ variables. On the other hand, Theorem 4 says that these ω_{ij} are polynomials of degree one in $x_1, \ldots, x_k$. This of course implies that ω_{ij}, for $1 \le i \le k$, $k+1 \le j \le n$, are constant functions.

Now we describe briefly our strategy to prove that these $\omega_{j\ell}$, $1 \le j \le k$, $k+1 \le \ell \le n$ functions are independent of $x_1, \ldots, x_k$ variables. The proof is highly nontrivial. It took us several years to work it out! We first observe that if $x_{k_{r-1}+1}^2 + \cdots + x_{k_r}^2$ is a basic quadratic form in E, then $\partial\omega_{j\ell}/\partial x_i = 0$ for all $k+1 \le \ell \le n$, $k_{r-1}+1 \le i, j \le k_r$ and $i \ne j$. This is Lemma 14 below and it is not difficult to prove. The difficult part is to prove Lemma 9 which asserts that $\partial\omega_{j\ell}/\partial x_i = 0$ even for $i = j$. The idea is to prove by contradiction in the following manner: if $\partial\omega_{i\ell}/\partial x_i \ne 0$, then one can construct an infinite sequence linearly independent elements in E.

We next prove the following fact. Let $x_{k_{r-1}}^2 + \cdots + x_{k_r}^2$ and $x_{k_{s-1}}^2 + \cdots + x_{k_s}^2$ be two basic quadratic forms in E, where $k_{r-1} < k_r \le k_{s-1} < k_s$. Then $\partial\omega_{j\ell}/\partial x_i = 0$ for all $k+1 \le \ell \le n$, $k_{r-1}+1 \le i \le k_r$ and $k_{s-1}+1 \le j \le k_s$. This is Lemma 13 below and is the most difficult part of our paper. We need to establish Lemma 10, Lemma 4 and Lemma 12 in order to prove it. The idea is to construct an infinite sequence of independent elements in E. This time the construction is even harder than the previous one because the appearance of even and odd numbers of the infinite sequence are different.

Lemma 6: *Any quadratic form $q(x)$ of E in $x_{k_{i-1}+1}, \ldots, x_{k_i}$ variables is a constant multiple of $p_i(x)$.*

Proof: This follows immediately from Lemma 2 Q.E.D

Let $\omega_{ij} = (\partial f_j/\partial x_i) - (\partial f_i/\partial x_j)$.

Lemma 7: *Then $\partial\omega_{il}/\partial x_j = \partial\omega_{jl}/\partial x_i$ for all $k+1 \le l \le n$, and $1 \le i, j \le k$.*

Proof: It is clear that for any i, j, l, we have

$$\frac{\partial\omega_{il}}{\partial x_j} + \frac{\partial\omega_{ji}}{\partial x_l} + \frac{\partial\omega_{lj}}{\partial x_i} = 0.$$

Now suppose $1 \le i,\ j \le k$ and $k+1 \le l \le n$. By Theorem 4, ω_{ij} is constant. Hence the above equality is reduced to the following equality.

$$\frac{\partial\omega_{il}}{\partial x_j} + \frac{\partial\omega_{lj}}{\partial x_i} = 0, \text{ i.e., } \frac{\partial\omega_{il}}{\partial x_j} = \frac{\partial\omega_{jl}}{\partial x_i}. \qquad \text{Q.E.D.}$$

Let U_i be the space of differential operators with order at most i. The following lemmas will facilitate the proof of our main theorem.

Lemma 8: *Let $D_i = (\partial/\partial x_i) - f_i$ and a, b be C^∞ differentiable functions defined on $\mathbb{R}^n$. Then*

$$[aD_1^{i_1}\cdots D_s^{i_s}, bD_1^{j_1}\cdots D_t^{j_t}]$$
$$= i_1 a\frac{\partial b}{\partial x_1}D_1^{i_1-1}D_2^{i_2},\ldots,D_s^{i_s}D_1^{j_1}\cdots D_t^{j_t}+\cdots$$
$$+ i_s a\frac{\partial b}{\partial x_s}D_1^{i_1}\cdots D_{s-1}^{i_{s-1}}D_s^{i_s-1}D_1^{j_1}\cdots D_t^{j_t}$$
$$- j_1 b\frac{\partial a}{\partial x_1}D_1^{i_1}\cdots D_s^{i_s}D_1^{j_1-1}D_2^{j_2}\cdots D_t^{j_t}-\cdots$$
$$- j_t b\frac{\partial a}{\partial x_t}D_1^{i_1}\cdots D_s^{i_s}D_1^{j_1}\cdots D_{t-1}^{j_{t-1}}D_t^{j_t-1}.$$
$$(\text{Mod } U_{i_1+\cdots+i_s+j_1+\cdots j_t-2})$$

Proof: We shall prove this by induction on the total order of $gD_1^{i_1}\cdots D_s^{i_s}$ and $hD_1^{j_1}\cdots D_t^{j_t}$.

$$[aD_1^{i_1}\cdots D_s^{i_s}, bD_1^{j_1}\cdots D_t^{j_t}]$$
$$=[D_1 aD_1^{i_1-1}D_2^{i_2}\cdots D_s^{i_s}, bD_1^{j_1}\cdots D_t^{j_t}] \quad (\text{Mod } U_{i_1+\cdots+i_s+j_1+\cdots j_t-2})$$
$$= D_1[aD_1^{i_1-1}D_2^{i_2}\cdots D_s^{i_s}, bD_1^{j_1}\cdots D_t^{j_t}]$$
$$+[D_1, bD_1^{j_1}\cdots D_t^{j_t}]aD_1^{i_1-1}D_2^{i_2}\cdots D_s^{i_s} \quad \text{by Lemma 2.4 (i)} \quad (\text{Mod } U_{i_1+\cdots+i_s+j_1+\cdots j_t-2})$$
$$= D_1\Big\{(i_1-1)a\frac{\partial b}{\partial x_1}D_1^{i_1-2}D_2^{i_2}\cdots D_s^{i_s}D_1^{j_1}\cdots D_t^{j_t}+\cdots$$
$$+ i_s a\frac{\partial b}{\partial x_s}D_1^{i_1-1}D_2^{i_2}\cdots D_{s-1}^{i_{s-1}}D_s^{i_s-1}D_1^{j_1}\cdots D_t^{j_t}$$
$$- j_1 b\frac{\partial a}{\partial x_1}D_1^{i_1-1}D_2^{i_2}\cdots D_s^{i_s}D_1^{j_1}\cdots D_t^{j_t}-\cdots$$
$$- j_t b\frac{\partial a}{\partial x_t}D_1^{i_1-1}D_2^{i_2}\cdots D_s^{i_s}D_1^{j_1}\cdots D_t^{j_t-1}\Big\}$$
$$+\Big[\frac{\partial b}{\partial x_1}D_1^{j_1}\cdots D_t^{j_t}\Big]aD_1^{i_1-1}D_2^{i_2}\cdots D_s^{i_s}$$
by induction hypothesis
$$(\text{Mod } U_{i_1+\cdots+i_s+j_1+\cdots j_t-2})$$
$$= i_1 a\frac{\partial b}{\partial x_1}D_1^{i_1-1}D_2^{i_2}\cdots D_s^{i_s}D_1^{j_1}\cdots D_t^{j_t}+\cdots$$
$$+ i_s a\frac{\partial b}{\partial x_s}D_1^{i_1}\cdots D_s^{i_s-1}D_1^{j}\cdots D_t^{j_t}$$
$$- j_1 b\frac{\partial a}{\partial x_1}D_1^{i_1}\cdots D_s^{i_s}D_1^{j_1-1}D_2^{j_2}\cdots D_t^{j_t}-\cdots$$
$$- j_t b\frac{\partial a}{\partial x_t}D_1^{i_1}\cdots D_s^{i_s}D_1^{j_1}\cdots D_t^{j_t-1}.$$
$$(\text{Mod } U_{i_1+\cdots+i_s+j_1+\cdots j_t-2})$$

Q.E.D.

Lemma 9: *If $x_{k_{p-1}+1}^2+\cdots+x_{k_p}^2$ is a basic quadratic form in E constructed in §2 and $\partial\omega_{jl}/\partial x_i = 0$ for all $k+1 \le l \le n$, $k_{p-1}+1 \le i, j \le k_p$ and $i \ne j$, then $\partial\omega_{il}/\partial x_i = 0$ for all $k_{p-1}+1 \le i \le k_p$.*

Proof: We shall construct a sequence of elements in E in the following manner

$$Z_1 = [L_0, x_{k_{p-1}+1}^2+\cdots+x_{k_p}^2]$$
$$= \sum_{i=k_{p-1}+1}^{k_p} x_i D_i + k_p - k_{p-1}$$
$$Z_2 = [L_0, Z_1] = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=k_{p-1}+1}^{k_p}[D_i^2, x_j D_j]$$
$$= \sum_{i=1}^{n}\sum_{j=k_{p-1}+1}^{k_p}\left(\frac{\partial x_j}{\partial x_i}D_i D_j - x_j\omega_{ij}D_i\right) \quad (\text{Mod } U_0)$$
$$= \sum_{j=k_{p-1}+1}^{k_p} D_j^2 + \sum_{i=1}^{n}\sum_{j=k_{p-1}+1}^{k_p} x_j\omega_{ji}D_i \quad (\text{Mod } U_0)$$
$$Z_3 = [L_0, Z_2] = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=k_{p-1}+1}^{k_p}[D_i^2, D_j^2]$$
$$+\frac{1}{2}\sum_{i=1}^{n}\sum_{l=1}^{n}\sum_{j=k_{p-1}+1}^{k_p}[D_i^2, x_j\omega_{jl}D_l] \quad (\text{Mod } U_1)$$
$$= 2\sum_{i=1}^{n}\sum_{j=k_{p-1}+1}^{k_p}\omega_{ji}D_j D_i$$
$$+\sum_{i=1}^{n}\sum_{l=1}^{n}\sum_{j=k_{p-1}+1}^{k_p}\frac{\partial(x_j\omega_{jl})}{\partial x_i}D_i D_l \quad (\text{Mod } U_1)$$
$$= 2\sum_{i=1}^{n}\sum_{j=k_{p-1}+1}^{k_p}\omega_{ji}D_j D_i + \sum_{l=1}^{n}\sum_{j=k_{p-1}+1}^{k_p}\omega_{jl}D_j D_l$$
$$+\sum_{i=1}^{k}\sum_{l=k+1}^{n}\sum_{j=k_{p-1}+1}^{k_p}x_j\frac{\partial\omega_{jl}}{\partial x_i}D_i D_l$$
$$(\text{Mod } U_1) \text{ by Theorem 2.3}$$
$$= 3\sum_{i=1}^{n}\sum_{j=k_{p-1}+1}^{k_p}\omega_{ji}D_j D_i$$
$$+\sum_{i=1}^{k}\sum_{l=k+1}^{n}\sum_{j=k_{p-1}+1}^{k_p}x_j\frac{\partial\omega_{jl}}{\partial x_i}D_i D_l \quad (\text{Mod } U_1)$$

$$
\begin{aligned}
&[Z_3,Z_1] \\
&=3\sum_{i=1}^{n}\sum_{j=k_{p-1}+1}^{k_p}\sum_{l=k_{p-1}+1}^{k_p}[\omega_{ji}D_jD_i,x_lD_l] \\
&\quad+\sum_{i=1}^{k}\sum_{l=k+1}^{n}\sum_{j=k_{p-1}+1}^{k_p}\sum_{q=k_{p-1}+1}^{k_p}\Big[x_j\frac{\partial\omega_{jl}}{\partial x_i}D_iD_l,x_qD_q\Big] \qquad (\mathrm{Mod}\,U_1) \\
&=3\sum_{i=1}^{n}\sum_{j=k_{p-1}+1}^{k_p}\sum_{l=k_{p-1}+1}^{k_p}\Big(\omega_{ji}\frac{\partial x_l}{\partial x_i}D_jD_l+\omega_{ji}\frac{\partial x_l}{\partial x_j}D_iD_l-x_l\frac{\partial\omega_{ji}}{\partial x_l}D_jD_i\Big) \\
&\quad+\sum_{i=1}^{k}\sum_{l=k+1}^{n}\sum_{j=k_{p-1}+1}^{k_p}\sum_{q=k_{p-1}+1}^{k_p}\Big(x_j\frac{\partial\omega_{jl}}{\partial x_i}\frac{\partial x_q}{\partial x_i}D_lD_q+x_j\frac{\partial\omega_{jl}}{\partial x_i}\frac{\partial x_q}{\partial x_l}D_iD_q \\
&\qquad\qquad-x_q\frac{\partial x_j}{\partial x_q}\frac{\partial\omega_{jl}}{\partial x_i}D_iD_l\Big) \qquad (\mathrm{Mod}\,U_1)
\end{aligned}
$$

$$
\begin{aligned}
&=3\sum_{j=k_{p-1}+1}^{k_p}\sum_{l=k_{p-1}+1}^{k_p}\omega_{jl}D_jD_l+3\sum_{i=1}^{n}\sum_{j=k_{p-1}+1}^{k_p}\omega_{ji}D_iD_j \\
&\quad-3\sum_{i=k+1}^{n}\sum_{j=k_{p-1}+1}^{k_p}\sum_{l=k_{p-1}+1}^{k_p}x_l\frac{\partial\omega_{ji}}{\partial x_l}D_jD_i \\
&\quad+\sum_{i=k_{p-1}+1}^{k_p}\sum_{l=k+1}^{n}\sum_{j=k_{p-1}+1}^{k_p}x_j\frac{\partial\omega_{jl}}{\partial x_i}D_lD_i \\
&\quad-\sum_{i=1}^{k}\sum_{l=k+1}^{n}\sum_{j=k_{p-1}+1}^{k_p}x_j\frac{\partial\omega_{jl}}{\partial x_i}D_iD_l \qquad (\mathrm{Mod}\,U_1) \\
&=3\sum_{i=1}^{n}\sum_{j=k_{p-1}+1}^{k_p}\omega_{ji}D_iD_j-2\sum_{i=k_{p-1}+1}^{k_p}\sum_{l=k+1}^{n}\sum_{j=k_{p-1}+1}^{k_p}x_j\frac{\partial\omega_{jl}}{\partial x_i}D_iD_l \\
&\quad-\sum_{i=1}^{k}\sum_{l=k+1}^{n}\sum_{j=k_{p-1}+1}^{k_p}x_j\frac{\partial\omega_{jl}}{\partial x_i}D_iD_l \qquad (\mathrm{Mod}\,U_1)\ \text{by Lemma 7}
\end{aligned}
$$

$$
\begin{aligned}
Z_4&=\frac{1}{2}([Z_3,Z_1]+Z_3)=3\sum_{i=1}^{n}\sum_{j=k_{p-1}+1}^{k_p}\omega_{ji}D_iD_j \\
&\quad-\sum_{i=k_{p-1}+1}^{k_p}\sum_{l=k+1}^{n}\sum_{j=k_{p-1}+1}^{k_p}x_j\frac{\partial\omega_{jl}}{\partial x_i}D_iD_l \qquad (\mathrm{Mod}\,U_1)
\end{aligned}
$$

$$
\begin{aligned}
[Z_4,Z_1]&=3\sum_{i=1}^{n}\sum_{j=k_{p-1}+1}^{k_p}\sum_{l=k_{p-1}+1}^{k_p}[\omega_{ji}D_iD_j,x_lD_l] \\
&\quad-\sum_{i=k_{p-1}+1}^{k_p}\sum_{l=k+1}^{n}\sum_{j=k_{p-1}+1}^{k_p}\sum_{q=k_{p-1}+1}^{k_p}\Big[x_j\frac{\partial\omega_{jl}}{\partial x_i}D_iD_l,x_qD_q\Big] \\
&\qquad (\mathrm{Mod}\,U_1) \\
&=3\sum_{i=1}^{n}\sum_{j=k_{p-1}+1}^{k_p}\sum_{l=k_{p-1}+1}^{k_p}\Big[\omega_{ji}\frac{\partial x_l}{\partial x_i}D_jD_l+\omega_{ji}\frac{\partial x_l}{\partial x_j}D_iD_l \\
&\qquad-x_l\frac{\partial\omega_{ji}}{\partial x_l}D_iD_j\Big] \\
&\quad-\sum_{i=k_{p-1}+1}^{k_p}\sum_{l=k+1}^{n}\sum_{j=k_{p-1}+1}^{k_p}\sum_{q=k_{p-1}+1}^{k_p}\Big[x_j\frac{\partial\omega_{jl}}{\partial x_i}\frac{\partial x_q}{\partial x_i}D_lD_q \\
&\qquad+x_j\frac{\partial\omega_{jl}}{\partial x_i}\frac{\partial x_q}{\partial x_l}D_iD_q \\
&\qquad-x_q\frac{\partial x_j}{\partial x_q}\frac{\partial\omega_{jl}}{\partial x_i}D_iD_l\Big] \\
&=3\sum_{j=k_{p-1}+1}^{k_p}\sum_{l=k_{p-1}+1}^{k_p}\omega_{jl}D_jD_l+3\sum_{i=1}^{n}\sum_{j=k_{p-1}+1}^{k_p}\omega_{ji}D_iD_j \\
&\quad-3\sum_{i=1}^{n}\sum_{j=k_{p-1}+1}^{k_p}\sum_{l=k_{p-1}+1}^{k_p}x_l\frac{\partial\omega_{ji}}{\partial x_l}D_iD_j \\
&\quad-\sum_{i=k_{p-1}+1}^{k_p}\sum_{l=k+1}^{n}\sum_{j=k_{p-1}+1}^{k_p}x_j\frac{\partial\omega_{jl}}{\partial x_i}D_lD_i \\
&\quad+\sum_{i=k_{p-1}+1}^{k_p}\sum_{l=k+1}^{n}\sum_{j=k_{p-1}+1}^{k_p}x_j\frac{\partial\omega_{jl}}{\partial x_i}D_iD_l \qquad (\mathrm{Mod}\ U_1) \\
&=3\sum_{i=1}^{n}\sum_{j=k_{p-1}+1}^{k_p}\omega_{ji}D_iD_j \\
&\quad-3\sum_{i=k+1}^{n}\sum_{j=k_{p-1}+1}^{k_p}\sum_{l=k_{p-1}+1}^{k_p}x_l\frac{\partial\omega_{ji}}{\partial x_l}D_iD_j \qquad (\mathrm{Mod}\ U_1) \\
&=3\sum_{i=1}^{n}\sum_{j=k_{p-1}+1}^{k_p}\omega_{ji}D_iD_j \\
&\quad-3\sum_{i=k_{p-1}+1}^{k_p}\sum_{l=k+1}^{n}\sum_{j=k_{p-1}+1}^{k_p}x_j\frac{\partial\omega_{il}}{\partial x_j}D_lD_i \qquad (\mathrm{Mod}\ U_1) \\
&=3\sum_{i=1}^{n}\sum_{j=k_{p-1}+1}^{k_p}\omega_{ji}D_iD_j \\
&\quad-3\sum_{i=k_{p-1}+1}^{k_p}\sum_{l=k+1}^{n}\sum_{j=k_{p-1}+1}^{k_p}x_j\frac{\partial\omega_{jl}}{\partial x_i}D_lD_i \\
&\qquad (\mathrm{Mod}\ U_1)\quad \text{by Lemma 7}
\end{aligned}
$$

$$
\begin{aligned}
Z_5&=\frac{1}{2}(Z_4-[Z_4,Z_1]) \\
&=\sum_{i=k_{p-1}+1}^{k_p}\sum_{l=k+1}^{n}\sum_{j=k_{p-1}+1}^{k_p}x_j\frac{\partial\omega_{jl}}{\partial x_i}D_iD_l. \qquad (\mathrm{Mod}\ U_1)
\end{aligned}
$$

Since $(\partial\omega_{jl}/\partial x_i) = 0,\ \forall\ k+1 \le l \le n,\quad k_{p-1}+1 \le i,\ j \le k_p,\ i \ne j$ by hypothesis, we have

$$Z_5 = \sum_{i=k_{p-1}+1}^{k_p} \sum_{l=k+1}^{n} x_i \frac{\partial\omega_{il}}{\partial x_i} D_i D_l$$

$$= \sum_{i=k_{p-1}+1}^{k_p} x_i \left(\sum_{l=k+1}^{n} \frac{\partial\omega_{il}}{\partial x_i} D_l \right) D_i \qquad (\mathrm{Mod}\ U_1)$$

$$A^{(1)} = [L_0, Z_5]$$

$$= \frac{1}{2} \sum_{q=1}^{n} \sum_{i=k_{p-1}+1}^{k_p} \sum_{l=k+1}^{n} \left[D_q^2, x_i \frac{\partial\omega_{il}}{\partial x_i} D_i D_l \right] \qquad (\mathrm{Mod}\ U_2)$$

$$= \sum_{q=1}^{n} \sum_{i=k_{p-1}+1}^{k_p} \sum_{l=k+1}^{n} \frac{\partial(x_i(\partial\omega_{il}/\partial x_i))}{\partial x_q} D_q D_i D_l$$

$$= \sum_{i=k_{p-1}+1}^{k_p} \sum_{l=k+1}^{n} \frac{\partial\omega_{il}}{\partial x_i} D_i^2 D_l \qquad (\mathrm{Mod}\ U_2)$$

$$= \sum_{i=k_{p-1}+1}^{k_p} \left(\sum_{l=k+1}^{n} \frac{\partial\omega_{il}}{\partial x_i} D_l \right) D_i^2 \qquad (\mathrm{Mod}\ U_2)$$

$$A^{(2)}$$

$$= [A^{(1)}, Z_5]$$

$$= \sum_{i=k_{p-1}+1}^{k_p} \sum_{l=k+1}^{n} \sum_{q=k_{p-1}+1}^{k_p} \sum_{r=k+1}^{n} \left[\frac{\partial\omega_{il}}{\partial x_i} D_i^2 D_l, x_q \frac{\partial\omega_{qr}}{\partial x_q} D_q D_r \right] \qquad (\mathrm{Mod}\, U_3)$$

$$= \sum_{i=k_{p-1}+1}^{k_p} \sum_{l=k+1}^{n} \sum_{q=k_{p-1}+1}^{k_p} \sum_{r=k+1}^{n} 2 \frac{\partial\omega_{il}}{\partial x_i} \frac{\partial(x_q(\partial\omega_{qr}/\partial x_q))}{\partial x_i} D_i D_l D_q D_r \qquad (\mathrm{Mod}\, U_3)$$

$$= 2 \sum_{i=k_{p-1}+1}^{k_p} \sum_{l=k+1}^{n} \sum_{r=k+1}^{n} \frac{\partial\omega_{il}}{\partial x_i} \frac{\partial\omega_{ir}}{\partial x_i} D_l D_r D_i^2 \qquad (\mathrm{Mod}\, U_3)$$

$$= 2 \sum_{i=k_{p-1}+1}^{k_p} \left(\sum_{l=k+1}^{n} \frac{\partial\omega_{il}}{\partial x_i} D_l \right)^2 D_i^2. \qquad (\mathrm{Mod}\, U_3)$$

Let $A^{(s)} := [A^{(s-1)}, Z_5]$. We claim that

$$A^{(s)} = 2^{s-1} \sum_{i=k_{p-1}+1}^{k_p} \left(\sum_{l=k+1}^{n} \frac{\partial\omega_{il}}{\partial x_i} D_l \right)^s D_i^2. \qquad (\mathrm{Mod}\ U_{s+1})$$

This can be seen as follow

$$A^{(s)} = \left[2^{s-2} \sum_{i=k_{p-1}+1}^{k_p} \left(\sum_{l=k+1}^{n} \frac{\partial\omega_{il}}{\partial x_i} D_l \right)^{s-1} D_i^2, \right.$$

$$\left. \sum_{i=k_{p-1}+1}^{k_p} x_i \left(\sum_{l=k+1}^{n} \frac{\partial\omega_{il}}{\partial x_i} D_l \right) D_i \right] \qquad (\mathrm{Mod}\ U_{s+1})$$

by induction hypothesis

$$= 2^{s-1} \sum_{i=k_{p-1}+1}^{k_p} \left(\sum_{l=k+1}^{n} \frac{\partial\omega_{il}}{\partial x_i} D_l \right)^s D_i^2. \qquad (\mathrm{Mod}\ U_{s+1})$$

by Lemma 8

Since $\{A^{(1)}, A^{(2)}, \ldots, A^{(s)}, \ldots,\} \subseteq E$ and E is finite dimensional, we conclude that

$$\frac{\partial\omega_{il}}{\partial x_i} = 0 \quad \forall\ k+1 \le l \le n,\ k_{p-1}+1 \le i \le k_p. \quad \text{Q.E.D}$$

Lemma 10: *Let* $x_{k_{r-1}+1}^2 + \cdots + x_{k_r}^2 \in E$ *and* $x_{k_{s-1}+1}^2 + \cdots + x_{k_s}^2 \in E$ *be the basic quadratic forms constructed in* §2, *where* $k_{r-1} < k_r \le k_{s-1} < k_s$. *Let* $\xi_{ij} = \sum_{l=k+1}^{n} (\partial\omega_{jl}/\partial x_i) D_l$. *Suppose* $\sum_{j=k_{s-1}+1}^{k_s} \xi_{pj}\xi_{qj} = 0$ *for all* $k_{r-1}+1 \le p, q \le k_r,\ p \ne q$. *Then* $(\partial\omega_{jl}/\partial x_i) = 0$ *for all* $k+1 \le l \le n, k_{r-1}+1 \le i \le k_r$, *and* $k_{s-1}+1 \le j \le k_s$.

Proof: By considering $x_{k_{s-1}+1}^2 + \cdots + x_{k_s}^2 \in E$ and using the proof of Lemma 9, we have the following three elements in E

$$[Z_3, Z_1] = 3 \sum_{i=1}^{n} \sum_{j=k_{s-1}+1}^{k_s} \omega_{ji} D_i D_j$$

$$- 2 \sum_{i=k_{s-1}+1}^{k_s} \sum_{l=k+1}^{n} \sum_{j=k_{s-1}+1}^{k_s} x_j \frac{\partial\omega_{jl}}{\partial x_i} D_i D_l$$

$$- \sum_{i=1}^{k} \sum_{l=k+1}^{n} \sum_{j=k_{s-1}+1}^{k_s} x_j \frac{\partial\omega_{jl}}{\partial x_i} D_i D_l \qquad (\mathrm{Mod}\ U_1)$$

$$Z_4 = 3 \sum_{i=1}^{n} \sum_{j=k_{s-1}+1}^{k_s} \omega_{ji} D_i D_j$$

$$- \sum_{i=k_{s-1}+1}^{k_s} \sum_{l=k+1}^{n} \sum_{j=k_{s-1}+1}^{k_s} x_j \frac{\partial\omega_{jl}}{\partial x_i} D_i D_l \qquad (\mathrm{Mod}\ U_1)$$

$$Z_5 = \sum_{i=k_{s-1}+1}^{k_s} \sum_{l=k+1}^{n} \sum_{j=k_{s-1}+1}^{k_s} x_j \frac{\partial\omega_{jl}}{\partial x_i} D_i D_l. \qquad (\mathrm{Mod}\ U_1)$$

From these, we can construct more elements in E in the following manners

$$Z_4+Z_5=3\sum_{i=1}^{n}\sum_{j=k_{s-1}+1}^{k_s}\omega_{ji}D_iD_j \qquad (\text{Mod } U_1)$$

$$\bar{Z}_4=-[Z_3,Z_1]+Z_4-Z_5$$

$$=\sum_{i=1}^{k}\sum_{l=k+1}^{n}\sum_{j=k_{s-1}+1}^{k_s}x_j\frac{\partial\omega_{jl}}{\partial x_i}D_iD_l. \qquad (\text{Mod } U_1)$$

Since $[L_0,x_{k_{r-1}+1}^2+\cdots+x_{k_r}^2]=\sum_{i=k_{r-1}+1}^{k_r}x_iD_i+k_r-k_{r-1}$ is in E, we have

$$\bar{Z}_5=\left[\bar{Z}_4,\ \sum_{p=k_{r-1}+1}^{k_r}x_pD_p+k_r-k_{r-1}\right]$$

$$=\sum_{i=1}^{k}\sum_{l=k+1}^{n}\sum_{j=k_{s-1}+1}^{k_s}\sum_{p=k_{r-1}+1}^{k_r}\left[x_j\frac{\partial\omega_{jl}}{\partial x_i}D_iD_l,x_pD_p\right] \qquad (\text{Mod } U_1)$$

$$=\sum_{i=1}^{k}\sum_{l=k+1}^{n}\sum_{j=k_{s-1}+1}^{k_s}\sum_{p=k_{r-1}+1}^{k_r}\left[x_j\frac{\partial\omega_{jl}}{\partial x_i}\delta_{ip}D_lD_p+x_j\frac{\partial\omega_{jl}}{\partial x_i}\delta_{lp}D_iD_p-x_p\delta_{jp}\frac{\partial\omega_{jl}}{\partial x_i}D_iD_l\right] \qquad (\text{Mod } U_1)$$

$$=\sum_{i=k_{r-1}+1}^{k_r}\sum_{l=k+1}^{n}\sum_{j=k_{s-1}+1}^{k_s}x_j\frac{\partial\omega_{jl}}{\partial x_i}D_lD_i$$

$$=\sum_{j=k_{s-1}+1}^{k_s}x_j\sum_{i=k_{r-1}+1}^{k_r}\sum_{l=k+1}^{n}\frac{\partial\omega_{jl}}{\partial x_i}D_lD_i \qquad (\text{Mod } U_1)$$

$$=\sum_{j=k_{s-1}+1}^{k_s}x_j\left(\sum_{i=k_{r-1}+1}^{k_r}\xi_{ij}D_i\right). \qquad (\text{Mod } U_1)$$

Similarly by considering $x_{k_{r-1}+1}^2+\cdots+x_{k_r}^2\in E$ and repeating the above procedure, we have the following element in E.

$$\bar{Z}_6=\sum_{i=k_{r-1}+1}^{k_r}\sum_{l=k+1}^{n}\sum_{j=k_{s-1}+1}^{k_s}x_i\frac{\partial\omega_{il}}{\partial x_j}D_jD_l$$

$$=\sum_{i=k_{r-1}+1}^{k_r}\sum_{l=k+1}^{n}\sum_{j=k_{s-1}+1}^{k_s}x_i\frac{\partial\omega_{jl}}{\partial x_i}D_jD_l \qquad (\text{Mod } U_1)$$

by Lemma 7

We shall construct an infinite sequence of elements in E

$$\bar{A}^{(1)}=[L_0,\bar{Z}_5]$$

$$=\frac{1}{2}\sum_{p=1}^{n}\sum_{i=k_{r-1}+1}^{k_r}\sum_{l=k+1}^{n}\sum_{j=k_{s-1}+1}^{k_s}\left[D_p^2,x_j\frac{\partial\omega_{jl}}{\partial x_i}D_iD_l\right] \qquad (\text{Mod } U_2)$$

$$=\sum_{p=1}^{n}\sum_{i=k_{r-1}+1}^{k_r}\sum_{l=k+1}^{n}\sum_{j=k_{s-1}+1}^{k_s}\frac{\partial(x_j(\partial\omega_{jl}/\partial x_i))}{\partial x_p}D_pD_iD_l \qquad (\text{Mod } U_2)$$

$$=\sum_{i=k_{r-1}+1}^{k_r}\sum_{l=k+1}^{n}\sum_{j=k_{s-1}+1}^{k_s}\frac{\partial\omega_{jl}}{\partial x_i}D_jD_iD_l$$

$$=\sum_{i=k_{r-1}+1}^{k_r}\sum_{j=k_{s-1}+1}^{k_s}\xi_{ij}D_iD_j \qquad (\text{Mod } U_2)$$

$$\bar{A}^{(2)}=[\bar{A}^{(1)},\ \bar{Z}_5]$$

$$=\left[\sum_{i=k_{r-1}+1}^{k_r}\sum_{j=k_{s-1}+1}^{k_s}\xi_{ij}D_iD_j,\ \sum_{q=k_{s-1}+1}^{k_s}\sum_{p=k_{r-1}+1}^{k_r}x_q\xi_{pq}D_p\right] \qquad (\text{Mod } U_3)$$

$$=\sum_{i=k_{r-1}+1}^{k_r}\sum_{j=k_{s-1}+1}^{k_s}\sum_{p=k_{r-1}+1}^{k_r}\sum_{q=k_{s-1}+1}^{k_s}\left[\frac{\partial x_q}{\partial x_i}\xi_{ij}\xi_{pq}D_jD_p+\frac{\partial x_q}{\partial x_j}\xi_{ij}\xi_{pq}D_iD_p\right] \qquad (\text{Mod } U_3)$$

$$=\sum_{i=k_{r-1}+1}^{k_r}\sum_{j=k_{s-1}+1}^{k_s}\sum_{p=k_{r-1}+1}^{k_r}\xi_{ij}\xi_{pj}D_iD_p \qquad (\text{Mod } U_3)$$

$$=\sum_{j=k_{s-1}+1}^{k_s}\left(\sum_{i=k_{r-1}+1}^{k_r}\xi_{ij}^2D_i^2+\sum_{i=k_{r-1}+1}^{k_r}\sum_{\substack{p=k_{r-1}+1\\ p\neq i}}^{k_r}\xi_{ij}\xi_{pj}D_iD_p\right) \qquad (\text{Mod } U_3)$$

$$=\sum_{i=k_{r-1}+1}^{k_r}\left(\sum_{j=k_{s-1}+1}^{k_s}\xi_{ij}^2\right)D_i^2+\sum_{i=k_{r-1}+1}^{k_r}\sum_{\substack{p=k_{r-1}+1\\ p\neq i}}^{k_r}\left(\sum_{j=k_{s-1}+1}^{k_s}\xi_{ij}\xi_{pj}\right)D_iD_p \qquad (\text{Mod } U_3)$$

$$=\sum_{i=k_{r-1}+1}^{k_r}\eta_iD_i^2 \quad \text{by the assumption}$$

$$\sum_{j=k_{s-1}+1}^{k_s}\xi_{ij}\xi_{pj}=0 \quad \text{for } i\neq p \qquad (\text{Mod } U_3)$$

where $\eta_i = \sum_{j=k_{s-1}+1}^{k_s} \xi_{ij}^2$.

$$\begin{aligned}\bar{A}^{(3)} &= [\bar{A}^{(2)}, \bar{Z}_6]\\ &= \left[\sum_{i=k_{r-1}+1}^{k_r} \eta_i D_i^2, \sum_{p=k_{r-1}+1}^{k_r} \sum_{j=k_{s-1}+1}^{k_s} x_p \xi_{pj} D_j\right] \quad (\text{Mod } U_4)\\ &= \sum_{i=k_{r-1}+1}^{k_r} \sum_{p=k_{r-1}+1}^{k_r} \sum_{j=k_{s-1}+1}^{k_s} 2\delta_{pi}\eta_i \xi_{pj} D_i D_j\\ &= 2\sum_{i=k_{r-1}+1}^{k_r} \sum_{j=k_{s-1}+1}^{k_s} \eta_i \xi_{ij} D_i D_j \quad (\text{Mod } U_4)\end{aligned}$$

$$\begin{aligned}&\bar{A}^{(4)}\\ &= [\bar{A}^{(3)}, \bar{Z}_5]\\ &= \left[2\sum_{i=k_{r-1}+1}^{k_r} \sum_{j=k_{s-1}+1}^{k_s} \eta_i \xi_{ij} D_i D_j, \sum_{q=k_{s-1}+1}^{k_s} \sum_{p=k_{r-1}+1}^{k_r} x_q \xi_{pq} D_p\right] \quad (\text{Mod } U_5)\\ &= 2\sum_{i=k_{r-1}+1}^{k_r} \sum_{j=k_{s-1}+1}^{k_s} \sum_{p=k_{r-1}+1}^{k_r} \sum_{q=k_{s-1}+1}^{k_s} [\eta_i \xi_{ij} D_i D_j, x_q \xi_{pq} D_p] \quad (\text{Mod } U_5)\\ &= 2\sum_{i=k_{r-1}+1}^{k_r} \sum_{j=k_{s-1}+1}^{k_s} \sum_{p=k_{r-1}+1}^{k_r} \eta_i \xi_{ij} \xi_{pj} D_i D_p \quad (\text{Mod } U_5)\\ &= 2\sum_{j=k_{s-1}+1}^{k_s} \left(\sum_{i=k_{r-1}+1}^{k_r} \eta_i \xi_{ij}^2 D_i^2 + \sum_{i=k_{r-1}+1}^{k_r} \sum_{\substack{p=k_{r-1}+1\\ p\neq i}}^{k_r} \eta_i \xi_{ij} \xi_{pj} D_i D_p\right) \quad (\text{Mod } U_5)\\ &= 2\sum_{i=k_{r-1}+1}^{k_r} \eta_i^2 D_i^2\\ &\quad + 2\sum_{\substack{i,p=k_{r-1}+1\\ i\neq p}}^{k_r} \eta_i \left(\sum_{j=k_{s-1}+1}^{k_s} \xi_{ij}\xi_{pj}\right) D_i D_p = 2\sum_{i=k_{r-1}+1}^{k_r} \eta_i^2 D_i^2. \quad (\text{Mod } U_5)\end{aligned}$$

Let $\bar{A}^{(2s+1)} = [\bar{A}^{(2s)}, \bar{Z}_6]$ and $\bar{A}^{(2s+2)} = [\bar{A}^{(2s+1)}, \bar{Z}_5]$. We claim that

$$\bar{A}^{(2s+1)} = 2^s \sum_{i=k_{r-1}+1}^{k_r} \sum_{j=k_{s-1}+1}^{k_s} \eta_i^s \xi_{ij} D_i D_j \quad (\text{Mod } U_{2s+2})$$

$$\bar{A}^{(2s+2)} = 2^s \sum_{i=k_{r-1}+1}^{k_r} \eta_i^{s+1} D_i^2 \quad (\text{Mod } U_{2s+3})$$

We shall prove this by induction

$$\begin{aligned}\bar{A}^{(2s+1)} &= [\bar{A}^{(2s)}, \bar{Z}_6]\\ &= \left[2^{s-1}\sum_{i=k_{r-1}+1}^{k_r} \eta_i^s D_i^2, \sum_{p=k_{r-1}+1}^{k_r} \sum_{j=k_{s-1}+1}^{k_s} x_p \xi_{pj} D_j\right] \quad (\text{Mod } U_{2s+2})\\ &= 2^{s-1}\sum_{i,p=k_{r-1}+1}^{k_r} \sum_{j=k_{s-1}+1}^{k_s} [\eta_i^s D_i^2, x_p \xi_{pj} D_j]\\ &= 2^s \sum_{i=k_{r-1}+1}^{k_r} \sum_{j=k_{s-1}+1}^{k_s} \eta_i^2 \xi_{ij} D_i D_j \quad (\text{Mod } U_{2s+2})\end{aligned}$$

$$\begin{aligned}&\bar{A}^{(2s+2)}\\ &= [\bar{A}^{(2s+1)}, \bar{Z}_5]\\ &= \left[2^s \sum_{i=k_{r-1}+1}^{k_r} \sum_{j=k_{s-1}+1}^{k_s} \eta_i^s \xi_{ij} D_i D_j, \sum_{q=k_{s-1}+1}^{k_s} \sum_{p=k_{r-1}+1}^{k_r} x_q \xi_{pq} D_p\right] \quad (\text{Mod } U_{2s+3})\\ &= 2^s \sum_{i=k_{r-1}+1}^{k_r} \sum_{j=k_{s-1}+1}^{k_s} \sum_{p=k_{r-1}+1}^{k_r} \sum_{q=k_{s-1}+1}^{k_s} [\eta_i^s \xi_{ij} D_i D_j, x_q \xi_{pq} D_p] \quad (\text{Mod } U_{2s+3})\\ &= 2^s \sum_{i=k_{r-1}+1}^{k_r} \sum_{j=k_{s-1}+1}^{k_s} \sum_{p=k_{r-1}+1}^{k_r} \eta_i^s \xi_{ij}\xi_{pj} D_i D_p \quad (\text{Mod } U_{2s+3})\\ &= 2^s \sum_{j=k_{s-1}+1}^{k_s} \left(\sum_{i=k_{r-1}+1}^{k_r} \eta_i^s \xi_{ij}^2 D_i^2 + \sum_{\substack{i,p=k_{r-1}+1\\ i\neq p}}^{k_r} \eta_i^s \xi_{ij}\xi_{pj} D_i D_p\right) \quad (\text{Mod } U_{2s+3})\\ &= 2^s \sum_{i=k_{r-1}+1}^{k_r} \eta_i^{s+1} D_i^2\\ &\quad + \sum_{\substack{i,p=k_{r-1}+1\\ i\neq p}}^{k_r} \eta_i^s \left(\sum_{j=k_{s-1}+1}^{k_s} \xi_{ij}\xi_{pj}\right) D_i D_p\\ &= 2^s \sum_{i=k_{r-1}+1}^{k_r} \eta_i^{s+1} D_i^2. \quad (\text{Mod } U_{2s+3})\end{aligned}$$

Since E is finite dimensional, we conclude that

$$\eta_i = \sum_{j=k_{s-1}+1}^{k_s} \xi_{ij}^2 = \sum_{j=k_{s-1}+1}^{k_s} \sum_{l_1=k+1}^{n} \sum_{l_2=k+1}^{n} \frac{\partial\omega_{jl_1}}{\partial x_i}\frac{\partial\omega_{jl_2}}{\partial x_i} D_{l_1} D_{l_2} = 0,$$

for all $k_{r-1}+1 \le i \le k_r$. By considering the coefficient of D_l^2 in η_i, we have $(\partial w_{jl}/\partial x_i) = 0$ for $k_{r-1}+1 \le i \le k_r$, $k_{s-1}+1 \le j \le k_s$, and $k+1 \le l \le n$ Q.E.D.

Lemma 11: *Let $x_{k_{r-1}+1}^2 + \cdots + x_{k_r}^2 \in E$ and $x_{k_{s-1}+1}^2 + \cdots + x_{k_s}^2 \in E$ be the basic quadratic forms constructed in § 2 where $k_{r-1} < k_r \le k_{s-1} < k_s$. Let $\xi_{ij} = \sum_{l=k+1}^{n} (\partial\omega_{jl}/\partial x_i) D_l$.*

Then $\sum_{j=k_{s-1}+1}^{k_s} \xi_{pj}\xi_{qj}=0$ *for all* $k_{r-1}+1\le p,q\le k_r, p\neq q$ *if and only if* $\sum_{j=k_{s-1}+1}^{k_s} a_{jl_1}^{p} a_{jl_2}^{q}=0$ *for all* $k+1\le l_1, l_2\le n, k_{r-1}+1\le p,q\le k_r, p\neq q$, *where* $a_{jl}^{p}=(\partial\omega_{jl}/\partial x_p)$.

Proof:

$$\sum_{j=k_{s-1}+1}^{k_s} \xi_{pj}\xi_{qj} = 0$$
$$\Leftrightarrow \sum_{j=k_{s-1}+1}^{k_s}\left(\sum_{l_1=k+1}^{n}\frac{\partial\omega_{jl_1}}{\partial x_p}D_{l_1}\right)\left(\sum_{l_2=k+1}^{n}\frac{\partial\omega_{jl_2}}{\partial x_q}D_{l_2}\right)=0$$
$$\Leftrightarrow \sum_{l_1,l_2=k+1}^{n}\left(\sum_{j=k_{s-1}+1}^{k_s} a_{jl_1}^{p} a_{jl_2}^{q}\right)D_{l_1}D_{l_2}=0$$
$$\Leftrightarrow \sum_{j=k_{s-1}+1}^{k_s} a_{jl_1}^{p} a_{jl_2}^{q}=0 \quad \text{for all } k+1\le l_1, l_2\le n \quad \text{Q.E.D.}$$

Lemma 12: *Let* $x_{k_{r-1}+1}^2+\cdots+x_{k_r}^2\in E$ *and* $x_{k_{s-1}+1}^2+\cdots+x_{k_s}^2\in E$ *be the basic quadratic forms constructed in §2, where* $k_{r-1}<k_r\le k_{s-1}<k_s$. *Assume that* $Q_l=\sum_{i=k_{r-1}+1}^{k_r}\sum_{j=k_{s-1}+1}^{k_s} a_{jl}^{i}x_ix_j\in E$ *for all* $k+1\le l\le n$ *where* $a_{jl}^{i}=\partial\omega_{jl}/\partial x_i$. *Then* $\sum_{j=k_{s-1}+1}^{k_s} a_{jl_1}^{p}a_{jl_2}^{q}=0$ *for all* $k+1\le l_1, l_2\le n, k_{r-1}+1\le p,q\le k_r, p\neq q$.

Proof:

$$\begin{aligned}R_{l_1} &= [L_0, Q_{l_1}]\\ &= \frac{1}{2}\left[\sum_{i=1}^{n}D_i^2, \sum_{p=k_{r-1}+1}^{k_r}\sum_{j=k_{s-1}+1}^{k_s} a_{jl_1}^{p}x_px_j\right]\\ &= \sum_{i=1}^{n}\sum_{p=k_{r-1}+1}^{k_r}\sum_{j=k_{s-1}+1}^{k_s} a_{jl_1}^{p}\left(x_j\frac{\partial x_p}{\partial x_i}D_i + x_p\frac{\partial x_j}{\partial x_i}D_i\right)\\ &= \sum_{p=k_{r-1}+1}^{k_r}\sum_{j=k_{s-1}+1}^{k_s} a_{jl_1}^{p}(x_jD_p+x_pD_j)\in E\end{aligned}$$

$$\begin{aligned}&[R_{l_1}, Q_{l_2}]\\ &=\left[\sum_{p=k_{r-1}+1}^{k_r}\sum_{j=k_{s-1}+1}^{k_s} a_{jl_1}^{p}(x_jD_p+x_pD_j),\right.\\ &\qquad\left.\times\sum_{u=k_{r-1}+1}^{k_r}\sum_{v=k_{s-1}+1}^{k_s} a_{vl_2}^{u}x_ux_v\right]\\ &=\sum_{p,u=k_{r-1}+1}^{k_r}\sum_{j,v=k_{s-1}+1}^{k_s}\left(a_{jl_1}^{p}x_ja_{vl_2}^{u}\frac{\partial x_u}{\partial x_p}x_v + a_{jl_1}^{p}x_ja_{vl_2}^{u}x_u\frac{\partial x_v}{\partial x_p}\right)\\ &\quad+\sum_{p,u=k_{r-1}+1}^{k_r}\sum_{j,v=k_{s-1}+1}^{k_s}\left(a_{jl_1}^{p}x_pa_{vl_2}^{u}x_v\frac{\partial x_u}{\partial x_j} + a_{jl_1}^{p}x_pa_{vl_2}^{u}x_u\frac{\partial x_v}{\partial x_j}\right)\\ &=\sum_{p=k_{r-1}+1}^{k_r}\sum_{j,v=k_{s-1}+1}^{k_s} a_{jl_1}^{p}a_{vl_2}^{p}x_jx_v\\ &\quad+\sum_{p,u=k_{r-1}+1}^{k_r}\sum_{j=k_{s-1}+1}^{k_s} a_{jl_1}^{p}a_{jl_2}^{u}x_px_u\in E.\end{aligned}$$

In view of Lemma 3, by setting $x_j=0$ for $k_{s-1}+1\le j\le k_s$ in the above expression, we get the following quadratic form in E

$$\begin{aligned}&\sum_{p,u=k_{r-1}+1}^{k_r}\sum_{j=k_{s-1}+1}^{k_s} a_{jl_1}^{p}a_{jl_2}^{u}x_px_u\\ &=\sum_{p=k_{r-1}+1}^{k_r}\sum_{j=k_{s-1}+1}^{k_s} a_{jl_1}^{p}a_{jl_2}^{p}x_p^2\\ &\quad+\sum_{\substack{p,u=k_{r-1}+1\\ p\neq u}}^{k_r}\sum_{j=k_{s-1}+1}^{k_s} a_{jl_1}^{p}a_{jl_2}^{u}x_px_u\in E.\end{aligned}$$

By Lemma 6, we know that the above quadratic form must be a constant multiple of $x_{k_{r-1}+1}^2+\cdots+x_{k_r}^2$. This implies that

$$\sum_{j=k_{s-1}+1}^{k_s} a_{jl_1}^{p}a_{jl_2}^{u}=0$$

for all $k+1\le l_1, l_2\le n, k_{r-1}+1\le p, u\le k_r, p\neq u$.

Q.E.D.

Lemma 13: *Let* $x_{k_{r-1}}^2+\cdots+x_{k_r}^2\in E$ *and* $x_{k_{s-1}+1}^2+\cdots+x_{k_s}^2\in E$ *be the basic quadratic forms constructed in* § 2, *where* $k_{r-1}<k_r\le k_{s-1}<k_s$. *Then* $a_{jl}^{i}=\partial\omega_{jl}/\partial x_i=0$ *for all* $k+1\le l\le n, k_{r-1}+1\le i\le k_r$ *and* $k_{s-1}+1\le j\le k_s$.

Proof: In view of Theorem 3, we have for $k+1\le l\le n$, $-\alpha_l(x_1,\ldots,x_k)=-\sum_{j=1}^{k}x_j\omega_{lj}=\sum_{j=1}^{k}x_j\sum_{i=1}^{k}a_{jl}^{i}x_i=\sum_{i,j=1}^{k}a_{jl}^{i}x_ix_j\in E$. Let $S_r=\{k_{r-1}+1,\ldots,k_r\}$ and $S_s=\{k_{s-1}+1,\ldots,k_s\}$. Then by Lemma 2, we have the following elements in E

$$\sum_{i,j\in S_r} a_{jl}^{i}x_ix_j=-\alpha_l(0,\ldots,0,x_{k_{r-1}+1},\ldots,x_{k_r},0,\ldots,0)$$
$$\sum_{i,j\in S_s} a_{jl}^{i}x_ix_j=-\alpha_l(0,\ldots,0,x_{k_{s-1}+1},\ldots,x_{k_s},0,\ldots,0).$$

By repeatedly using Lemma 3, we have

$$\begin{aligned}\sum_{i,j\in S_r\cup S_s} a_{jl}^{i}x_ix_j &= -\alpha_l(0,\ldots,0,x_{k_{r-1}+1},\ldots,x_{k_r},0,\ldots,0,\\ &\qquad x_{k_{s-1}+1},\ldots,x_{k_s},0,\ldots,0)\in E.\end{aligned}$$

Recall that $a_{jl}^{i}=\partial\omega_{jl}/\partial x_i=\partial\omega_{il}/\partial x_j=a_{il}^{j}$ for $1\le i,j\le k$ and $k+1\le l\le n$ by Lemma 3.2 Since $Q_l=\sum_{i=k_{r-1}+1}^{k_r}\sum_{j=k_{s-1}+1}^{k_s} a_{jl}^{i}x_ix_j$ is a linear combination of the above three elements ($Q_l=\frac{1}{2}$ ($\sum_{i,j\in S_r\cup S_s} a_{jl}^{i}x_ix_j-\sum_{i,j\in S_r} a_{jl}^{i}x_ix_j-\sum_{i,j\in S_s} a_{jl}^{i}x_ix_j$)), we have $Q_l\in E$.

Our Lemma now follows immediately from Lemma 10, Lemma 11 and Lemma 12 Q.E.D.

Lemma 14: *If $x_{k_{r-1}+1}^2+\cdots+x_{k_r}^2$ is a basic quadratic form in E constructed in §2, then $(\partial\omega_{jl}/\partial x_i)=0$ for all $k+1\le l\le n, k_{r-1}+1\le i,j\le k_r$ and $i\neq j$.*

Proof: In view of Theorem 4, we have for $k+1\le l\le n$, $-\alpha_l(x_1,\ldots,x_k)=-\sum_{j=1}^k x_j\omega_{lj}$ $=\sum_{j=1}^k x_j\sum_{i=1}^k(\partial\omega_{jl}/\partial x_i)x_i=\sum_{i,j=1}^k(\partial\omega_{jl}/\partial x_i)x_ix_j$. By Lemma 2, we have the following element in E

$$\sum_{i,j=k_{r-1}+1}^{k_r}\frac{\partial\omega_{jl}}{\partial x_i}x_ix_j=-\alpha_l(0,\ldots,0,x_{k_{r-1}+1},\ldots,x_{k_r},0,\ldots,0).$$

Recall that $\partial\omega_{jl}/\partial x_i=\partial\omega_{il}/\partial x_j$ for $k_{r-1}+1\le i,j\le k_r$ by Lemma 7. Therefore the quadratic form $\sum_{i,j=k_{r-1}+1}^{k_r}(\partial\omega_{jl}/\partial x_i)x_ix_j$ is a constant multiple of $x_{k_{r-1}+1}^2+\cdots+x_{k_r}^2$ by Lemma 6. Therefore $\partial\omega_{jl}/\partial x_i=0$ for all $k+1\le l\le n, k_{r-1}+1\le i,j\le k_r$ and $i\neq j$. Q.E.D.

Proof of Theorem 1: By Theorem 3, we only need to prove that ω_{ij} are constant functions for all $1\le i,j\le n$. In view of Theorem 3, we only need to prove that $\omega_{jl}, 1\le j\le k, k+1\le l\le n$, are constants. By Lemma 9, Lemma 14 and Lemma 13, we know that $\omega_{jl}, 1\le j\le k, k+1\le l\le n$ are independent of $x_1,\ldots,x_k$ variables. On the other hand Theorem 4 says that $\omega_{jl}, 1\le j\le k, k+1\le l\le n$, depend only on $x_1,\ldots,x_k$ variables. Therefore $\omega_{jl}, 1\le j\le k, k+1\le l\le n$, are constants Q.E.D.

4. Finite dimensional estimation algebra beyond Kalman or Benés types and implementation of finite dimensional filter

For the sake of convenience to the readers, we include a construction of a large class of finite dimensional estimation algebras which are neither of Kalman types nor of Benés types. We also show how a finite-dimensional filter can be implemented from a finite-dimensional estimation algebra. Most of the materials in this section can be found in Dong *et al*. (1997) and yau (1994). For practical applications, it is important that the finite-dimensional filter is universal in the sense that Chaleyat-Maurel and Michel (1984). It turns out that all finite-dimensional filters constructed from finite-dimensional estimation algebras in this section are necessarily universal. Let ϕ be a C^∞ function defined on $\mathbb{R}^n$. For any $(x_1,\ldots,x_n)$ in $\mathbb{R}^n$, let $r=\sqrt{x_1^2+\cdots+x_n^2}$. r is called a radial function.

Lemma 15: *Let a, c be positive real numbers. Then*

$$\Delta\phi+|\nabla\phi|^2=ar^2-c \tag{15}$$

has a radical solution (i.e. ϕ is a function of r) if $c<2\sqrt{a}$.

Proof: In view of Theorem 12 of Dong *et al*. (1991), there exists a solution g of the equation (15). It is an easy exercise to show that $u=e^g$ satisfies the following linear partial differential equation

$$\Delta u=(ar^2-c)u. \tag{16}$$

Let $G(x)=\int_{\tau\in O}e^{g(\tau\cdot x)}$ where the integral takes place over O the group of all orthogonal transformations. G is still a positive solution which depends only on r. The radial function $\phi=\ln G$ solves (15). Q.E.D.

Example 1: In this example, we give a class of finite-dimensional estimation algebras which are neither of Kalman types nor of Benés types. We shall take $n=2$ in equation (1), and let

$$\begin{aligned}
f_1&=\frac{\partial\phi}{\partial x_1}+\lambda x_2\\
f_2&=\frac{\partial\phi}{\partial x_2}-\lambda x_1\\
h_1&=a_{11}x_1+a_{12}x_2+b_1\\
h_2&=a_{21}x_1+a_{22}x_2+b_2
\end{aligned}$$

where $\lambda, a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$ are real constants, $a_{11}a_{22}-a_{21}a_{12}\neq 0$, and ϕ is the radial solution of the equation (15) in Lemma 15. Then

$$\begin{aligned}
(1)\ \frac{\partial f_2}{\partial x_1}-\frac{\partial f_1}{\partial x_2}&=\frac{\partial}{\partial x_1}\left[\frac{\partial\phi}{\partial x_2}-\lambda x_1\right]-\frac{\partial}{\partial x_2}\left[\frac{\partial\phi}{\partial x_1}+\lambda x_2\right]\\
&=\frac{\partial^2\phi}{\partial x_1\partial x_2}-\lambda-\frac{\partial^2\phi}{\partial x_2\partial x_1}+\lambda=0
\end{aligned}$$

$$\begin{aligned}
(2)\ &\frac{\partial f_1}{\partial x_1}+\frac{\partial f_2}{\partial x_2}+f_1^2+f_2^2+h_1^2+h_2^2\\
&=\frac{\partial}{\partial x_1}\left[\frac{\partial\phi}{\partial x_1}+\lambda x_2\right]+\frac{\partial}{\partial x_2}\left[\frac{\partial\phi}{\partial x_2}-\lambda x_1\right]\\
&\quad+\left(\frac{\partial\phi}{\partial x_1}+\lambda x_2\right)^2+\left(\frac{\partial\phi}{\partial x_2}-\lambda x_1\right)^2\\
&\quad+(a_{11}x_1+a_{12}x_2+b_1)^2+(a_{21}x_1+a_{22}x_2+b_2)^2\\
&=\Delta\phi+|\nabla\phi|^2+\lambda^2(x_1^2+x_2^2)+2\lambda\left(x_2\frac{\partial\phi}{\partial x_1}-x_1\frac{\partial\phi}{\partial x_2}\right)\\
&\quad+(a_{11}x_1+a_{12}x_2+b_1)^2+(a_{21}x_1+a_{22}x_2+b_2)^2.
\end{aligned} \tag{17}$$

Recall that ϕ is a radial solution of (15). We have

$$\Delta\phi + |\nabla\phi|^2 = a(x_1^2 + x_2^2) - c \qquad (18)$$

$$x_2\frac{\partial\phi}{\partial x_1} - x_1\frac{\partial\phi}{\partial x_2} = x_2\frac{\partial\phi}{\partial r}\frac{\partial r}{\partial x_1} - x_1\frac{\partial\phi}{\partial r}\frac{\partial r}{\partial x_2} = \frac{x_1x_2}{r}\frac{\partial\phi}{\partial r} - \frac{x_1x_2}{r}\frac{\partial\phi}{\partial r} = 0. \qquad (19)$$

Put (18) and (19) into (17). We have deduced that

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + {f_1}^2 + {f_2}^2 + h_1^2 + h_2^2 = (a+\lambda^2)(x_1^2 + x_2^2) - C + (a_{11}x_1 + a_{12}x_2 + b_1)^2 + (a_{21}x_1 + a_{22}x_2 + b_2)^2,$$

which is a polynomial of total degree at most two. Thus by Corollary 3 the estimation algebra E is of finite dimensional.

From what follows, we shall show how a finite-dimensional filter can be constructed from a finite-dimensional estimation algebra. We shall begin with the following Baker–Campbell–Hausdorff type relations.

Lemma: 16 *Consider the filtering system* (1) *with* $\partial f_j/\partial x_j = c_{ij}$ *where* c_{ij} *are constants for all* $1 \le i, j \le n$. *Suppose* $\eta = \sum_{i,j=1}^{\infty}\eta_{ij}x_ix_j + \sum_{i=1}^{n}\eta_i x_i + \eta_0$ *where* η_{ij}, η_i *and* η_0 *are constants for all* $1 \le i, j \le n$. *Then the following relations hold.*

(1) *For* $1 \le i \le n$,

$$e^{s_i(t)D_i}L_0 = \left[L_0 - s_i(t)\sum_{j=1}^{n}c_{ij}D_j - \frac{s_i(t)}{2} \times \left(\sum_{j=1}^{n}(\eta_{ij}+\eta_{ji})x_j + \eta_i\right) + \frac{s_i^2(t)}{2}\left(\sum_{j=1}^{n}c_{ij}^2 - \eta_{ii}\right)\right]e^{s_i(t)D_i}.$$

(2) *For* $1 \le i, k \le n$,

$$e^{s_k(t)D_k}M_i = \left[M_i - s_i(t)s_k(t) \times \left(\sum_{j=1}^{n}c_{ij}c_{jk} + \frac{1}{2}(\eta_{ik}+\eta_{ki})\right)\right]e^{s_k(t)D_k},$$

where

$$M_i = -s_i(t)\sum_{j=1}^{n}c_{ij}D_j - \frac{s_i(t)}{2}\left(\sum_{j=1}^{n}(\eta_{ij}+\eta_{ji})x_j + \eta_i\right) + \frac{s_i^2(t)}{2}\left(\sum_{j=1}^{n}c_{ij}^2 - \eta_{ii}\right).$$

(3) *For* $1 \le i \le n$,

$$e^{r_i(t)x_i}L_0 = \left[L_0 - r_i(t)D_i + \frac{r_i^2(t)}{2}\right]e^{r_i(t)x_i}.$$

(4) *For* $1 \le i \le n$,

$$e^{r_i(t)x_i}N = \left[N + r_i(t)\sum_{k=1}^{n}s_k(t)c_{ki}\right]e^{r_i(t)x_i}$$

where

$$N = -\sum_{i,j=1}^{n}s_i(t)c_{ij}D_j - \frac{1}{2}\sum_{i=1}^{n}s_i(t)\left(\sum_{j=1}^{n}(\eta_{ij}+\eta_{ji})x_j + \eta_i\right) + \frac{1}{2}\sum_{i=1}^{n}s_i^2(t)\left(\sum_{j=1}^{n}c_{ij}^2 - \eta_{ii}\right) - \sum_{1\le i<k\le n}s_i(t)s_k(t)\left(\sum_{j=1}^{n}c_{ij}c_{jk} + \frac{1}{2}(\eta_{ik}+\eta_{ki})\right).$$

(5) *For* $1 \le i, k \le n$,

$$e^{r_k(t)x_k}D_i = [D_i - r_k(t)\delta_{ik}]e^{r_k(t)x_k}.$$

(6) *For* $1 \le i, k \le n$,

$$e^{s_k(t)D_k}D_i = [D_i + s_k(t)c_{ik}]e^{s_k(t)D_k}.$$

Proof: The following computations using the adjoint representation formula are legitimate by the argument given in Proposition 1 of Tam *et al.* (1990)

(1) $e^{s_i(t)D_i}L_0$

$$= \left\{L_0 + s_i(t)[D_i, L_0] + \frac{s_i^2(t)}{2}[D_i,[D_i,L_0]] + \frac{s_i^3(t)}{3!}[D_i[D_i[D_iL_0]]] + \cdots\right\}e^{s_i(t)D_i}$$

$$= \left\{L_0 + s_i(t)\left(-\sum_{j=1}^{n}c_{ij}D_j - \frac{1}{2}\sum_{j=1}^{n}(\eta_{ij}+\eta_{ji})x_j - \frac{1}{2}\eta_i\right) + \frac{s_i^2(t)}{2}\left(\sum_{j=1}^{n}c_{ij}^2 - \eta_{ii}\right)\right\}e^{s_i(t)D_i}$$

$$= \left[L_0 - s_i(t)\sum_{j=1}^{n}c_{ij}D_j - \frac{s_i(t)}{2}\left(\sum_{j=1}^{n}(\eta_{ij}+\eta_{ji})x_j + \eta_i\right) + \frac{s_i^2(t)}{2}\left(\sum_{j=1}^{n}c_{ij}^2 - \eta_{ii}\right)\right]e^{s_i(t)D_i}.$$

(2) $e^{s_k(t)D_k} M_i$

$$= \left\{ M_i + s_k(t)[D_k, M_i] + \frac{s_k^2(t)}{2}[D_k,[D_k,M_i]] + \frac{s_i^3(t)}{3!}[D_k,[D_k,[D_k,M_i]]] + \cdots \right\} e^{s_k(t)D_k}$$

$$= \left\{ m_i + s_k(t)\left[-s_i(t)\sum_{j=1}^n c_{ij}c_{jk} - \frac{s_i(t)}{2}(\eta_{ik}+\eta_{ki}) \right] \right\} e^{s_k(t)D_k}$$

$$= \left\{ M_i - s_i(t)s_k(t)\left(\sum_{j=1}^n c_{ij}c_{jk} + \frac{1}{2}(\eta_{ik}+\eta_{ki}) \right) \right\} e^{s_k(t)D_k}.$$

The proofs of (3)–(6) are similar. Q.E.D.

We are now ready to construct finite-dimensional filter from finite-dimensional estimation algebra.

Theorem 5: *Let E be a finite-dimensional estimation algebra with maximal rank of* (1) *satisfying* $(\partial f_j/\partial x_i) - (\partial f_i/\partial x_j) = c_{ij}$ *with* $\eta(x) = \sum_{i,j=1}^n \eta_{ij}x_ix_j + \sum_{i=1}^n \eta_i x_i + \eta_0$ *where $c_{ij}, \eta_{ij}, \eta_i$, and η_0 are constants. Then the robust Duncan–Morten–Zakai equation* (3) *has a solution for all $t \geq 0$ of the following form*

$$u(t,x) = e^{T(t)}e^{r_n(t)x_n}\ldots e^{r_1(t)x_1}e^{s_n(t)D_n}\ldots e^{s_1(t)D_1}e^{tL_0}\sigma_0 \quad (20)$$

where $T(t), r_1(t), \ldots, r_n(t), s_1(t), \ldots, s_n(t)$ satisfy the following ordinary differential equations (21), (22) *and* (23).

For $1 \leq i \leq n$,

$$\frac{ds_i}{dt}(t) = r_i(t) + \sum_{j=1}^n s_j(t)c_{ji} + \sum_{k=1}^m h_{ki}y_k(t), \qquad (21)$$

where $h_k(x) = \sum_{j=1}^n h_{kj}x_j + e_k$, *for* $1 \leq k \leq m$. *Here h_{kj} and e_k are constants.*

For $1 \leq j \leq n$,

$$\frac{dr_j}{dt}(t) = \frac{1}{2}\sum_{i=1}^n s_i(t)(\eta_{ij}+\eta_{ji}), \qquad (22)$$

and

$$\begin{aligned}\frac{dT}{dt}(t) = &-\frac{1}{2}\sum_{i=1}^n r_i^2(t) - \frac{1}{2}\sum_{i=1}^n s_i^2(t)\left(\sum_{j=1}^n c_{ij}^2 - \eta_{ii}\right) \\ &+ \sum_{1\leq i<k\leq n} s_i(t)s_k(t)\left(\sum_{j=1}^n c_{ij}c_{jk} + \frac{1}{2}(\eta_{ik}+\eta_{ki})\right) \\ &+ \frac{1}{2}\sum_{i=1}^n s_i(t)b_i + \sum_{i=1}^n r_i(t)\frac{ds_i}{dt}(t) \\ &- \sum_{1\leq i<j\leq n}\frac{ds_i}{dt}(t)s_j(t)c_{ij} - \sum_{i,j=1}^n s_i(t)r_j(t)c_{ij} \\ &+ \frac{1}{2}\sum_{i,j=1}^m y_i(t)y_j(t)\left(\sum_{k=1}^n h_{ik}h_{jk}\right). \end{aligned} \qquad (23)$$

Proof: Since L_0 is uniformly elliptic, for any $t > 0$, $e^{tL_0}\sigma_0$ is C^∞. By differentiating $\xi(t,x)$, we have

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x) &= e^{T(t)}e^{r_n(t)x_n}\ldots e^{r_1(t)x_1}e^{s_n(t)D_n}\ldots e^{s_1(t)D_1}L_0e^{tL_0}\sigma_0 \\ &+ \frac{ds_1}{dt}(t)e^{T(t)}e^{r_n(t)x_n}\ldots e^{r_1(t)x_1}e^{s_n(t)D_n}\ldots e^{s_2(t)D_2} \\ &\times D_1e^{s_1(t)D_1}e^{tL_0}\sigma_0 + \cdots \\ &+ \frac{ds_n}{dt}(t)e^{T(t)}e^{r_n(t)x_n}\ldots e^{r_1(t)x_1}D_ne^{s_n(t)D_n}\ldots e^{s_1(t)D_1}e^{tL_0}\sigma_0 \\ &+ \frac{dr_1}{dt}(t)e^{T(t)}e^{r_n(t)x_n}\ldots e^{r_2(t)x_2}x_1e^{r_1(t)x_1}e^{s_n(t)D_n}\ldots \\ &\times e^{s(t)D_1}e^{tL_0}\sigma_0 + \cdots \\ &+ \frac{dr_n}{dt}(t)e^{T(t)}x_ne^{r_n(t)x_n}\ldots e^{r_1(t)x_1}e^{s_n(t)D_n}\ldots e^{s_1(t)D_1}e^{tL_0}\sigma_0 \\ &+ \frac{dT}{dt}(t)e^{T(t)}e^{r_n(t)x_n}\ldots e^{r_1(t)x_1}e^{s_n(t)D_n}\ldots e^{s_1(t)D_1}e^{tL_0}\sigma_0.\end{aligned} \qquad (24)$$

Let

$$M_i = -s_i(t)\sum_{j=1}^n c_{ij}D_j - \frac{s_i(t)}{2}\left(\sum_{j=1}^n(\eta_{ij}+\eta_{ji})x_j + \eta_i\right) + \frac{s_i^2(t)}{2}\left(\sum_{j=1}^n c_{ij}^2 - \eta_{ii}\right),$$

$$\begin{aligned}N = &-\sum_{i,j=1}^n s_i(t)c_{ij}D_j - \frac{1}{2}\sum_{i=1}^n s_i(t)\left(\sum_{j=1}^n(\eta_{ij}+\eta_{ji})x_j + \eta_i\right) \\ &+ \frac{1}{2}\sum_{i=1}^n s_i^2(t)\left(\sum_{j=1}^n c_{ij}^2 - \eta_{ii}\right) \\ &- \sum_{1\leq i<k\leq n} s_i(t)s_k(t)\left(\sum_{j=1}^n c_{ij}c_{jk} + \frac{1}{2}(\eta_{ik}+\eta_{ki})\right) \\ = &\sum_{i=1}^n M_i - \sum_{1\leq i<k\leq n} s_i(t)s_k(t)\left(\sum_{j=1}^n c_{ij}c_{jk} + \frac{1}{2}(\eta_{ik}+\eta_{ki})\right).\end{aligned}$$

By applying Lemma 16, we have

$$
\begin{aligned}
&e^{T(t)}e^{r_n(t)x_n}\dots e^{r_1(t)x_1}e^{s_n(t)D_n}\dots e^{s_1(t)D_1}L_0e^{tL_0}\sigma_0\\
&=e^{T(t)}e^{r_n(t)x_n}\dots e^{r_1(t)x_1}e^{s_n(t)D_n}\dots e^{s_2(t)D_2}\\
&\quad\times\{L_0+M_1\}e^{s_1(t)D_1}e^{tL_0}\sigma\\
&=e^{T(t)}e^{r_n(t)x_n}\dots e^{r_1(t)x_1}e^{s_n(t)D_n}\dots e^{s_3(t)D_3}\\
&\quad\times\Bigg\{L_0+M_2+M_1-s_1(t)s_2(t)\\
&\quad\times\left[\sum_{j=1}^{n}c_{1j}c_{j2}+\frac{1}{2}(\eta_{12}+\eta_{21})\right]\Bigg\}e^{s_2(t)D_2}e^{s_1(t)D_1}e^{tL_0}\sigma\\
&=e^{T(t)}e^{r_n(t)x_n}\dots e^{r_1(t)x_1}e^{s_n(t)D_n}\dots e^{s_4(t)D_4}\\
&\quad\times\Bigg\{L_0+\sum_{i=1}^{3}M_i-\sum_{1\le i<k\le 3}s_i(t)s_k(t)\\
&\quad\times\left[\sum_{j=1}^{n}c_{ij}c_{jk}+\frac{1}{2}(\eta_{ik}+\eta_{ki})\right]\Bigg\}e^{s_3(t)D_3}e^{s_2(t)D_2}e^{s_1(t)D_1}e^{tL_0}\sigma\\
&=e^{T(t)}e^{r_n(t)x_n}\dots e^{r_1(t)x_1}\{L_0+N\}e^{s_n(t)D_n}\dots e^{s_1(t)D_1}e^{tL_0}\sigma_0\\
&=e^{T(t)}e^{r_n(t)x_n}\dots e^{r_2(t)x_2}\Bigg\{L_0-r_1(t)D_1+\frac{r_1^2(t)}{2}+N\\
&\quad+r_1(t)\sum_{k=1}^{n}s_k(t)c_{k1}\Bigg\}e^{r_1(t)x_1}e^{s_n(t)D_n}\dots e^{s_1(t)D_1}e^{tL_0}\sigma_0\\
&=e^{T(t)}\Bigg\{L_0-\sum_{i=1}^{n}r_i(t)D_i+\sum_{i=1}^{n}\frac{r_i^2(t)}{2}+N\\
&\quad+\sum_{i,k=1}^{n}r_i(t)s_k(t)c_{ki}\Bigg\}e^{r_n(t)x_n}\dots e^{r_1(t)x_1}e^{s_n(t)D_n}\dots e^{s_1(t)D_1}e^{tL_0}\sigma_0\\
&=\Bigg\{L_0-\sum_{i=1}^{n}r_i(t)d_i+\sum_{i=1}^{n}\frac{r_i^2(t)}{2}+N\\
&\quad+\sum_{i,k=1}^{n}r_i(t)s_k(t)c_{ki}\Bigg\}u(t,x). \qquad (25)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&e^{T(t)}e^{r_n(t)x_n}\dots e^{r_1(t)x_1}e^{s_n(t)D_n}\dots e^{s_{i+1}(t)D_{i+1}}D_ie^{s_i(t)}\dots e^{s_1(t)D_1}e^{tL_0}\sigma_0\\
&=e^{T(t)}e^{r_n(t)x_n}\dots e^{r_1(t)x_1}e^{s_n(t)D_n}\dots e^{s_{i+2}(t)D_{i+2}}\\
&\quad\times[D_i+s_{i+1}(t)c_{i,i+1}]e^{s_{i+1}(t)D_{i+1}}\dots e^{s_1(t)D_1}e^{tL_0}\sigma_0\\
&=e^{T(t)}e^{r_n(t)x_n}\dots e^{r_1(t)x_1}\\
&\quad\times[D_i+s_n(t)c_{i,n}+s_{n-1}(t)c_{i,n-1}+\dots+s_{i+1}(t)c_{i,i+1}]\\
&\quad\times e^{s_n(t)D_n}\dots e^{s_1(t)D_1}e^{tL_0}\sigma_0\\
&=\left[D_i-r_i(t)+\sum_{j=i+1}^{n}s_j(t)c_{i,j}\right]u(t,x). \qquad (26)
\end{aligned}
$$

Putting (25) and (26) in (24), we have

$$
\begin{aligned}
&\frac{\partial u}{\partial t}(t,x)\\
&=\Bigg\{L_0-\sum_{i=1}^{n}r_i(t)D_i+\frac{1}{2}\sum_{i=1}^{n}r_i^2(t)\\
&\quad-\sum_{i,j=1}^{n}s_i(t)c_{ij}D_j-\frac{1}{2}\sum_{i=1}^{n}s_i(t)\\
&\qquad\times\left[\sum_{j=1}^{n}(\eta_{ij}+\eta_{ji})x_j+\eta_i\right]\\
&\quad+\frac{1}{2}\sum_{i=1}^{n}s_i^2(t)(\sum_{j=1}^{n}c_{ij}^2-\eta_{ii})\\
&\quad-\sum_{1\le i<k\le n}s_i(t)s_k(t)\left[\sum_{j=1}^{n}c_{ij}c_{jk}+\frac{1}{2}(\eta_{ik}+\eta_{ki})\right]\\
&\quad+\sum_{j,k=1}s_k(t)r_j(t)c_{kj}\Bigg\}u(t,x)\\
&\quad+\frac{ds_1}{dt}(t)[D_1-r_1(t)+s_n(t)c_{1n}\\
&\quad+s_{n-1}(t)c_{1,n-1}+\dots+s_2(t)c_{12}]u(t,x)\\
&\quad+\frac{ds_2}{dt}(t)[D_2-r_2(t)+s_n(t)c_{2n}\\
&\quad+s_{n-1}(t)c_{2,n-1}+\dots+s_3(t)c_{23}]u(t,x)\\
&\quad+\dots+\frac{ds_n}{dt}(t)[D_n-r_n(t)]u(t,x)\\
&\quad+\frac{dr_1}{dt}(t)x_1u(t,x)+\frac{dr_2}{dt}(t)x_2u(t,x)+\cdots\\
&\quad+\frac{dr_n}{dt}(t)x_nu(t,x)+\frac{dT}{dt}(t)u(t,x)\\
&=L_0u(t,x)+\sum_{i=1}^{n}\left[-r_i(t)-\sum_{j=1}^{n}s_j(t)c_{ji}+\frac{ds_i}{dt}(t)\right]\\
&\quad\times D_iu(t,x)+\sum_{j=1}^{n}\left[\frac{dr_j}{dt}(t)-\frac{1}{2}\sum_{j=1}^{n}(\eta_{ij}+\eta_{ji})s_i(t)\right]x_ju(t,x)\\
&\quad-\Bigg\{-\frac{1}{2}\sum_{i=1}^{n}r_i^2(t)-\frac{1}{2}\sum_{i=1}^{n}s_i^2(t)\left(\sum_{j=1}^{n}c_{ij}^2-\eta_{ii}\right)\\
&\quad+\sum_{1\le i<k\le n}s_i(t)s_k(t)\left[\sum_{j=1}^{n}c_{ij}c_{jk}+\frac{1}{2}(\eta_{ik}+\eta_{ki})\right]\\
&\quad-\sum_{j,k=1}^{n}s_k(t)r_j(t)c_{kj}+\frac{1}{2}\sum_{i=1}^{n}s_i(t)\eta_i\\
&\quad+\sum_{i=1}^{n}r_i(t)\frac{ds_i}{dt}(t)-\sum_{1\le i<j\le n}\frac{ds_i}{dt}(t)s_j(t)c_{ij}-\frac{dT}{dt}(t)\Bigg\}u(t,x).
\end{aligned}
\qquad (27)
$$

On the other hand, recall that for $1 \leq i \leq n$, L_i is the zero degree differential operator of multiplication by $h_i = \sum_{j=1}^{n} h_{ij} x_j + e_i$ where h_{ij} and e_i are constants. Then the right hand side of (3) is of the form

$$\begin{aligned}
&\frac{\partial u}{\partial t}(t,x) \\
&\quad = L_0 u(t,x) + \sum_{i=1}^{m} y_i(t)[L_0, L_i]u(t,x) \\
&\qquad + \frac{1}{2}\sum_{i,j=1}^{m} y_i(t)y_j(t)[[L_0, L_i], L_j]u(t,x) \\
&\quad = L_0 u(t,x) + \sum_{i=1}^{m} y_i(t)\sum_{j=1}^{n} h_{ij} D_j u(t,x) \\
&\qquad + \frac{1}{2}\sum_{i,j=1}^{n} y_i(t)y_j(t)\left[\sum_{\ell=1}^{n} h_{i\ell} D_\ell, \sum_{k=1}^{n} h_{jk} x_k + e_j\right]u(t,x) \\
&\quad = L_0 u(t,x) + \sum_{i=1}^{n}\left(\sum_{j=1}^{m} h_{ji} y_j(t)\right) D_i u(t,x) \\
&\qquad + \frac{1}{2}\sum_{i,j=1}^{n} y_i(t)y_j(t)\left(\sum_{k=1}^{n} h_{ik} h_{jk}\right)u(t,x). \qquad (28)
\end{aligned}$$

By comparing (27) and (28), it is clear that $u(t,x)$ is a solution of (3) if (21), (22) and (23) are satisfied. It is also clear that (21), (22) are (23) have solutions for all t. Q.E.D.

Corollary 5: *The finite dimensional filter construction in Theorem 5 is a universal finite dimensional filter in the sense of Chaleyat-Maurel and Michel* (1984).

Proof: ODEs (21), (22) and (23) are independent of the initial data $\sigma_0(x)$ in (2). Q.E.D.

5. Conclusion

In the early eighties, Brockett proposed to classify finite dimensional estimation algebras and Mitter conjectured that all functions in finite dimensional estimation algebras are necessarily polynomials of total degree at most one. Despite the massive effort in understanding the finite dimensional estimation algebras, the 20 years old problem of Brockett and Mitter conjecture remain open. In this paper, we give a classification of finite dimensional estimation algebras of maximal rank and solve the Mitter conjecture affirmatively for finite dimensional estimation algebra of maximal rank. In particular, for an estimation algebra E of maximal rank, we give a necessary and sufficient condition for E to be finite dimensional in terms of the drift $f_i(x)$ and observation $h_j(x)$. As an important corollary, we show that the number of statistics in order to compute the conditional density of the state given the observation $\{y(s) : 0 \leq s \leq t\}$ by Lie algebras method is n where n is the dimension of the state. We construct a large class of finite dimensional estimation algebras which are neither of Kalman types nor of Benés types. We also give explicit construction of finite dimensional filters from finite dimensional estimation algebras. For practical applications, it is important that the finite dimensional filter is universal in the sense of Chaleyat-Maurel and Michel. It turns out that our finite dimensional filters are necessarily universal.

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