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Structure theorem for five-dimensional estimation algebras

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Abstract

The problem of classification of finite-dimensional estimation algebras was formally proposed by Brockett in his lecture at International Congress of Mathematicians in 1983. Due to the difficulty of the problem, in the early 1990s Brockett suggested that one should understand the low-dimensional estimation algebras first. In this paper we give classification of estimation algebras of dimension at most five. Although the classification of finite-dimensional estimation algebra of maximal rank was completed by Yau and his coworkers Chen, Chiou, Hu, Wong and Wu; the problem of classification of non-maximal rank finite-dimensional estimation algebra is still wide open except for the case of state space dimension 2. Hopefully, the result of this paper will shed some light on the non-maximal rank estimation algebras.

Keywords: Nonlinear filter; Estimation algebras; Kalman-Bucy filter

1. Introduction

Kalman, and Kalman-Bucy published two historically important mathematical papers in ASME's Journal of Basic Engineering in 1960 and 1961, respectively. Kalman-Bucy filter has been used in many areas such as navigational and guidance systems, radar tracking, solar mapping, and satellite orbit determination. However, Kalman-Bucy filter has limited applicability because of the linearity assumptions of the drift term and observation term as well as the Gaussian assumption of initial value. In the late seventies, Brockett and Clark [2], Brockett [1], and Mitter [12,13] proposed the idea of using estimation algebras to construct finite-dimensional nonlinear filters.

At the International Congress of Mathematicians in 1983, Brockett proposed to classify all finite-dimensional estimation algebras. Since then, the concept of estimation algebra has been proven to be an invaluable tool in the study of nonlinear filtering and stochastic control problems. Wong [17] introduced a fundamental notion of Wong matrix which plays an important role in classification of finite-dimensional estimation algebras. In [18], Wong gave a new light on the classification problem of finite-dimensional estimation algebras for the first time. Specifically, under the hypothesis that the drift term $f(x)$ in Eq. (2.1) of Section 2 is real analytic and its first, second and third order partial derivatives are bounded functions, Wong proved that all finite-dimensional estimation algebras of (2.1) are solvable and the observation $h(x)$ in (2.1) is a polynomial of degree one. Moreover, he was able to give a structural description of finite-dimensional estimation algebras under these conditions. Yau [21] has studied the general class of nonlinear filtering systems which include both Kalman-Bucy and Benés filtering systems as special cases. He gave necessary and sufficient conditions for an estimation algebra of such a filtering system to be finite-dimensional. Using the Wei–Norman approach, he constructed explicitly finite-dimensional recursive filters for such nonlinear filtering systems. In [9], the concept of an estimation algebra of maximal rank was introduced. Chiou and Yau were able to classify all finite-dimensional estimation algebras of maximal rank with state

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space dimension at most two. Later Chen et al. [5,6] succeeded in classifying all finite-dimensional estimation algebras of maximal rank with state space dimension equal to 3 and 4, respectively. The novelty of their theorem is that there is no assumption on the drift term of the nonlinear filtering system. On the other hand, if the drift term has a potential function (i.e. the drift term is a gradient vector field), the corresponding estimation algebra is called exact. In [16], Tam et al. have classified all finite-dimensional exact estimation algebras of maximal rank with arbitrary state space dimension [10]. Recently Yau and his co-workers Chen, Chiou, Hu, Wong and Wu in a series of paper [21,3,4,11,20,25] and [23] have classified all finite-dimensional estimation algebras with maximal rank. The self-contained proof of the-classification of finite-dimensional estimation algebras can be found in [22].

Despite the success of the classification of finite-dimensional estimation algebras with maximal rank, the problem of classification of non-maximal rank finite-dimensional estimation algebras is still wide open except for the case of state space dimension 2 which was finished by Wu and Yau [19] and some construction of non-maximal rank finite-dimensional algebras by Rasoulian and Yau [15]. There are two reasons to tackle the problem of classification of non-maximal rank finite-dimensional estimation algebras. The first reason is that the problem was attempted by many well-known engineers without great success in the eighties of the last century. The second reason is that the solution of this problem may give us many new classes of finite-dimensional estimation algebras and hence many new classes of finite-dimensional filters. Due to the difficulty of the problem, Brockett suggested that one should understand the low-dimensional estimation algebras first. In [24], Yau and Rasoulian have classified estimation algebras of dimension at most four. In this paper, we give a structure theorem for estimation algebras of dimension five. Using this structure theorem, we have found a new class of finite-dimensional estimation algebras. Hopefully, the result of this paper will shed some light on the non-maximal rank finite-dimensional estimation algebras.

2. Filtering model and basic theorems

The filtering problem considered here is based on the following signal observation model:

$$\begin{cases} \mathrm{d}x(t) = f(x(t))\,\mathrm{d}t + g(x(t))\,\mathrm{d}v(t), & x(0) = x_0, \\ \mathrm{d}y(t) = h(x(t))\,\mathrm{d}t + \mathrm{d}w(t), & y(0) = 0 \end{cases} \tag{2.1}$$

in which x, v, y and w are respectively $\mathbb{R}^n$, $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^m$ valued processes, and v and w have components that are independent, standard Brownian processes. We further assume that f, h are C^∞ smooth, and that g is an orthogonal matrix.

Let $\rho(t, x)$ denote the conditional density of the state given the observation $\{y(s) : 0 \leqslant s \leqslant t\}$. It is well known that $\rho(t, x)$ is given by normalizing a function $\sigma(t, x)$ which satisfies the Duncan–Mortensen–Zakai equation:

$$\mathrm{d}\sigma(t, x) = L_0\sigma(t, x)\,\mathrm{d}t + \sum_{i=1}^{m} L_i\sigma(t, x)\,\mathrm{d}y_i(t),$$
$$\sigma(0, x) = x_0, \tag{2.2}$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i} - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2$$

and for $i = 1, \ldots, m$, L_i is the zero degree differential operator of multiplication by h_i. Here σ_0 is the probability density of the initial point x_0. Let

$$D_i = \frac{\partial}{\partial x_i} - f_i,$$
$$\eta = \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{m} h_i^2. \tag{2.3}$$

Then L_0 can be written as $L_0 = \frac{1}{2}(\sum_{i=1}^{n} D_i^2 - \eta)$.

Definition 2.1. The estimation algebra E of a filtering model (2.1) is defined to be the Lie algebra generated by $L_0, L_1, \ldots, L_m$ or $E = \langle L_0, L_1, \ldots, L_m\rangle_{\mathrm{LA}}$. If $x_i \in E$ for every $1 \leqslant i \leqslant n$, then E is called an estimation algebra of maximal rank.

Definition 2.2. Wong matrix $\Omega = (\omega_{ij})$ is an $n \times n$ skew-symmetric matrix with $\omega_{ij} = (\partial f_j/\partial x_i) - (\partial f_i/\partial x_j)$.

Theorem 2.1 (*Ocone [14]*). *Any function in a finite-dimensional estimation algebra is a polynomial of degree at most two.*

The following theorem in [21] is useful in the classification of finite-dimensional estimation algebras.

Theorem 2.2 (*Yau [21]*). *Let $F(x_1, \ldots, x_n)$ be a polynomial on $\mathbb{R}^n$. Suppose there exists a polynomial path $c : \mathbb{R} \to \mathbb{R}^n$ such that $\lim_{t\to\infty} c(t) = \infty$ and $\lim_{t\to\infty} F(c(t)) = -\infty$. Then there are no C^∞ function $f_1, \ldots, f_n$ on $\mathbb{R}^n$ such that*

$$\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = F.$$

Let U_k be the vector space of differential operators of order up to and including k. Assume that the coefficients of these differential operators are C^∞ functions. Wong [17] proved the following theorem.

Theorem 2.3 (*Wong [17]*). *If $Y = \sum_{i=1}^{n}\gamma_i D_i$ mod U_0 is an element in a finite-dimensional estimation algebra, then γ_i are polynomials of $x_1, \ldots, x_n$ for all i.*

3. Five-dimensional estimation algebras

In this section, we recall some basic results proved in our previous paper [8]. The following lemma plays an important role in [8] and will also be useful in this paper.

Lemma 3.1 (*Chiou et al. [8]*). *For any* $1 \leqslant \ell \leqslant n$, if γ_i, $i = 1, \ldots, \ell$, *are polynomials in* $x_1, \ldots, x_\ell$ *with coefficients in* C^∞ *functions of* $x_{\ell+1}, \ldots, x_n$ *satisfying*

$$\frac{\partial \gamma_j}{\partial x_i} + \frac{\partial \gamma_i}{\partial x_j} = 0 \quad \text{for all } 1 \leqslant i, j \leqslant \ell, \tag{3.1}$$

then each γ_i *is necessary of the form*

$$\gamma_i = \sum_{1 \leqslant j \leqslant \ell} c_i^j(x_{\ell+1}, \ldots, x_n) x_j + d_i(x_{\ell+1}, \ldots, x_n), \tag{3.2}$$

where $c_i^j(x_{\ell+1}, \ldots, x_n)$ *and* $d_i(x_{\ell+1}, \ldots, x_n)$ *are* C^∞ *functions and* $c_i^j = -c_j^i$.

Lemma 3.2 (*Chiou et al. [8]*). *If* dim $E = 5$, *then E cannot contain two linear independent degree one polynomials.*

Theorem 3.3 (*Chiou et al. [8]*). *If* dim $E = 5$, *then E cannot contain any degree two polynomial.*

Theorem 3.3 above together with the result of [24] assert the following Mitter conjecture is true for estimation algebras with dimension at most five.

Mitter Conjecture. *Let E be a finite-dimensional estimation algebra. Then any function in E is a polynomial of degree at most one.*

4. Structure theorem of five-dimensional estimation algebras

In this section, we shall prove the structure theorem of five-dimensional estimation algebras. By Lemma 3.2 and Theorem 3.3, we have $m = 1$ (hence $\sum_{i=1}^m h_i^2 = h_1^2$) and we may assume that $h_1 = x_1$. Observe that

$$[L_0, x_1] = D_1, \quad [D_1, x_1] = 1. \tag{4.1}$$

We have $\{1, x_1, D_1, L_0\} \subseteq E$. Consider

$$Y_1 := [L_0, D_1] = \frac{1}{2}\sum_{i=1}^n [D_i^2, D_1] - \frac{1}{2}[\eta, D_1] = \sum_{i=1}^n \omega_{1i} D_i + \frac{1}{2}\sum_{i=1}^n \frac{\partial \omega_{1i}}{\partial x_i} + \frac{1}{2}\frac{\partial \eta}{\partial x_1}. \tag{4.2}$$

By Theorem 2.3, each ω_{1i} is a polynomial. Suppose $\omega_{1i} = 0$ for $2 \leqslant i \leqslant n$. Then in view of (4.2), Lemma 3.2 and Theorem 3.3, $\frac{1}{2}(\partial \eta / \partial x_1)$ is a linear combination of 1 and x_1. This implies that E is a four-dimensional estimation algebra, which leads to a contradiction. Therefore, $\omega_{1i} \neq 0$ for some $2 \leqslant i \leqslant n$. It follows that $1, x_1, D_1, L_0$ and Y_1 are five linearly independent elements in E. Consider

$$\begin{aligned} Y_2 &:= [L_0, Y_1] \\ &= \sum_{i=1}^n \sum_{j=1}^n \frac{\partial \omega_{1j}}{\partial x_i} D_i D_j + \sum_{i=1}^n \sum_{j=1}^n \omega_{1j}\omega_{ji} D_i \\ &\quad + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 \omega_{1j}}{\partial x_i^2} D_j + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \omega_{ij}\frac{\partial \omega_{ji}}{\partial x_i} \\ &\quad + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 \omega_{1j}}{\partial x_i \partial x_j} D_i + \frac{1}{4}\sum_{i=1}^n \sum_{j=1}^n \frac{\partial^3 \omega_{1j}}{\partial^2 x_i \partial x_j} \\ &\quad + \frac{1}{2}\sum_{i=1}^n \frac{\partial^2 \eta}{\partial x_i \partial x_1} D_i + \frac{1}{4}\sum_{i=1}^n \frac{\partial^3 \eta}{\partial x_1 \partial x_i^2} \\ &\quad + \frac{1}{2}\sum_{j=1}^n \omega_{1j}\frac{\partial \eta}{\partial x_j}. \end{aligned} \tag{4.3}$$

Since Y_2 has to be a linear combination of $1, x_1, D_1, L_0$ and Y_1, we can write

$$Y_2 = 2\lambda L_0 + C_0 x_1 + C_1 Y_1 + C_2 + C_3 D_1. \tag{4.4}$$

From the above equation, we have

$$\frac{\partial \omega_{1i}}{\partial x_i} = \lambda \quad \text{for } 1 \leqslant i \leqslant n \ \text{(from } D_i^2 \text{ terms)} \tag{4.5}$$

$$\frac{\partial \omega_{1j}}{\partial x_i} + \frac{\partial \omega_{1i}}{\partial x_j} = 0 \quad \text{for } 1 \leqslant i \neq j \leqslant n \ \text{(from } D_i D_j \text{ terms)}. \tag{4.6}$$

If we take $i = 1$ in (4.5), we get $\lambda = \partial \omega_{11}/\partial x_1 = 0$. In view of (4.5), ω_{1i} is independent of x_i variable. By taking $j = 1$ in (4.6), we see that ω_{1i} is also independent of x_1 variable. In view of (4.5) with $\lambda = 0$, (4.6) and Lemma 3.1, ω_{1i}, $2 \leqslant i \leqslant n$, are polynomials of degree at most one. In fact ω_{1i} must be of the form

$$\omega_{1i} = \sum_{k=2}^n e_{ik} x_k + e_i \quad \text{with } e_{ij} = -e_{ji} \ \text{for all } 2 \leqslant i \leqslant n \tag{4.7}$$

for some constants e_{ij}'s and e_i's. Note that in (4.7), $k = 1$ is not included in the summation because ω_{1i} is independent of x_1 variable. It follows that

$$Y_1 = \sum_{i=1}^{n} \omega_{1i} D_i + \frac{1}{2}\frac{\partial \eta}{\partial x_1} \tag{4.8}$$

$$\begin{aligned} Y_2 &= \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\omega_{1j}\omega_{ji} + \frac{1}{2}\frac{\partial^2\eta}{\partial x_i \partial x_1}\right) D_i \\ &\quad + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\omega_{1j}\frac{\partial \omega_{ji}}{\partial x_i} \\ &\quad + \frac{1}{4}\sum_{i=1}^{n}\frac{\partial^3\eta}{\partial x_i^2 \partial x_1} + \frac{1}{2}\sum_{j=1}^{n}\omega_{1j}\frac{\partial\eta}{\partial x_j} \\ &= C_1\left(\sum_{i=1}^{n}\omega_{1i}D_i + \frac{1}{2}\frac{\partial\eta}{\partial x_1}\right) \\ &\quad + C_0 x_1 + C_2 + C_3 D_1. \end{aligned} \tag{4.9}$$

By comparing the coefficients of D_1 term and D_i, $2 \leqslant i \leqslant n$, terms, we have

$$-\sum_{j=1}^{n}\omega_{1j}^2 + \frac{1}{2}\frac{\partial^2\eta}{\partial x_1^2} = C_3 \tag{4.10}$$

and

$$\sum_{j=1}^{n}\omega_{1j}\omega_{ji} + \frac{1}{2}\frac{\partial^2\eta}{\partial x_i\partial x_1} = C_1\omega_{1i}, \quad 2\leqslant i \leqslant n. \tag{4.11}$$

Differentiate (4.10) with respect to x_1 and (4.11) with respect to x_i and sum over i, we get

$$\begin{aligned} &\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n}\omega_{1j}\omega_{ji} + \frac{1}{2}\frac{\partial^2\eta}{\partial x_i\partial x_1}\right) \\ &\quad = \sum_{i=1}^{n}\frac{\partial}{\partial x_i}(C_3 + C_1\omega_{1i}) = 0. \end{aligned} \tag{4.12}$$

On the other hand, by comparing the zero order differential operators in both sides of (4.9), we get

$$\begin{aligned} &\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\omega_{1j}\frac{\partial\omega_{ji}}{\partial x_i} + \frac{1}{4}\sum_{i=1}^{n}\frac{\partial^3\eta}{\partial x_i^2\partial x_1} + \frac{1}{2}\sum_{j=1}^{n}\omega_{1j}\frac{\partial\eta}{\partial x_j} \\ &\quad = C_0x_1 + C_2\frac{\partial\eta}{\partial x_1} + C_2 \\ &\quad \Rightarrow \frac{1}{2}\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n}\omega_{1j}\omega_{ji} \right. \\ &\qquad \left. + \frac{1}{2}\frac{\partial^2\eta}{\partial x_i\partial x_1}\right) - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial\omega_{1j}}{\partial x_i}\omega_{ji} \\ &\qquad + \frac{1}{2}\sum_{j=1}^{n}\omega_{1j}\frac{\partial\eta}{\partial x_j} \\ &\quad = C_0x_1 + \frac{C_1}{2}\frac{\partial\eta}{\partial x_1} + C_2. \end{aligned} \tag{4.13}$$

In view of (4.12), (4.13) becomes

$$\begin{aligned} &-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial\omega_{1j}}{\partial x_i}\omega_{ji} + \frac{1}{2}\sum_{j=1}^{n}\omega_{1j}\frac{\partial\eta}{\partial x_j} \\ &\quad = C_0x_1 + \frac{C_1}{2}\frac{\partial\eta}{\partial x_1} + C_2. \end{aligned} \tag{4.14}$$

From (4.10), we get

$$\begin{aligned} \eta = &\left(\sum_{j=2}^{n}\omega_{1j}^2 + C_3\right)x_1^2 + \beta(x_2,\ldots,x_n)x_1 \\ &+ \gamma(x_2,\ldots,x_n). \end{aligned} \tag{4.15}$$

By (2.3) (in which $m = 1$) and (4.15), we get

$$\begin{aligned} &\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 \\ &\quad = \left(\sum_{j=2}^{n}\omega_{1j}^2 + C_3 - 1\right)x_1^2 + \beta(x_2,\ldots,x_n)x_1 \\ &\qquad + \gamma(x_2,\ldots,x_n). \end{aligned} \tag{4.16}$$

In order for (4.16) to have a smooth solution $(f_1,\ldots,f_n)$ defined on $\mathbb{R}^n$, we deduce that $C_3 \geqslant 1$ by Theorem 2.2. Finally we note that $[Y_1, h_1] = 0$ and by Eq. (4.10), $[Y_1, D_1] = [\sum_{i=1}^{n}\omega_{1i}D_i + \frac{1}{2}(\partial\eta/\partial x_1), D_1] = \sum_{i=1}^{n}\omega_{1i}^2 - \frac{1}{2}(\partial^2\eta/\partial x_1^2) = -C_3$. Summarizing what we have shown above, we have the following structure theorem for five-dimensional estimation algebras.

Theorem 4.1. *Suppose that the state space of the filtering model* (2.1) *is of dimension at least two. Then the five-dimensional estimation algebra is isomorphic to a Lie algebra generated by* L_0 *and an observation function* $h = x_1$ *with a basis given by* 1, x_1, $D_1 = (\partial/\partial x_1) - f_1(x_1,\ldots,x_n)$, $Y_1 = [L_0, D_1] = \sum_{i=1}^{n}\omega_{i1}D_i + \frac{1}{2}(\partial\eta/\partial x_1)$, $L_0 = \frac{1}{2}(\sum_{i=1}^{n}D_i^2 - \eta)$. *Moreover, the following holds*:

(1) $\omega_{1i} \neq 0$ *for some* $i = 2,\ldots,n$ *and each* ω_{1i} *is of the form*

$$\omega_{1i} = \sum_{k=2}^{n} e_{ik}x_k + e_i \quad \text{for } 2\leqslant i\leqslant n, \tag{4.17}$$

$$e_{ij} = -e_{ji} \quad 2\leqslant i, j\leqslant n, \tag{4.18}$$

where e_{ij} *and* e_i *are constants.*

(2) η *is of the form*

$$\begin{aligned} \eta = &\left(\sum_{j=2}^{n}\omega_{1j}^2 + C_3\right)x_1^2 + \beta(x_2,\ldots,x_n)x_1 \\ &+ \gamma(x_2,\ldots,x_n), \end{aligned} \tag{4.19}$$

where $C_3 \geqslant 1$ *is a constant and* $\beta(x_2,\ldots,x_n)$ *and* $\gamma(x_2,\ldots,x_n)$ *are* C^∞ *functions.*

(3) *There exists a constant* C_1 *such that*

$$\sum_{j=1}^{n} \omega_{1j}\omega_{ji} + \frac{1}{2}\frac{\partial^2 \eta}{\partial x_i \partial x_1} = C_1 \omega_{1i}, \quad 2 \leqslant i \leqslant n. \tag{4.20}$$

(4) *There exist constants* C_0 *and* C_2 *such that*

$$-\frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial \omega_{1j}}{\partial x_i}\omega_{ji} + \frac{1}{2}\sum_{j=1}^{n} \omega_{1j}\frac{\partial \eta}{\partial x_j} = C_0 x_1 + \frac{C_1}{2}\frac{\partial \eta}{\partial x_1} + C_2. \tag{4.21}$$

In particular, $f_1, \ldots, f_n$ *have to satisfy the following equation*:

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = \left(\sum_{j=2}^{n} \omega_{1j}^2 + C_3 - 1\right) x_1^2 + \beta(x_2, \ldots, x_n)x_1 + \gamma(x_2, \ldots, x_n). \tag{4.22}$$

Moreover, this five-dimensional estimation algebra has the following multiplication table

E	1	x_1	D_1	Y_1	L_0
1	0	0	0	0	0
x_1	0	0	-1	0	$-D_1$
D_1	0	1	0	C_3	$-Y_1$
Y_1	0	0	$-C_3$	0	$-C_0x_1 - C_1Y_1 - C_2 - C_3D_1$
L_0	0	D_1	Y_1	$C_0x_1 + C_1Y_1 + C_2 + C_3D_1$	0

Corollary 4.2. *Suppose that the state space of the filtering model* (2.1) *is of dimension at least two. If* ω_{1j}, $2 \leqslant j \leqslant n$, *are constants, then the five-dimensional estimation algebra is isomorphic to a Lie algebra generated by* L_0 *and* x_1 *with a basis given by* 1, x_1, $D_1 = (\partial/\partial x_1) - f_1(x_1, \ldots, x_n)$, $Y_1 = [L_0, D_1] = \sum_{i=1}^{n} \omega_{i1} D_i + \frac{1}{2}(\partial\eta/\partial x_1)$, $L_0 = \frac{1}{2}(\sum_{i=1}^{n} D_i^2 - \eta)$. *Moreover, the following holds*:

(1) η *is of the form*

$$\eta = ax_1^2 + \beta(x_2, \ldots, x_n)x_1 + \gamma(x_2, \ldots, x_n) \tag{4.23}$$

for some constant $a > 1$ *and* C^∞ *functions* $\beta(x_2, \ldots, x_n)$ *and* $\gamma(x_2, \ldots, x_n)$.

(2) *There exists constant* d_1 *such that*

$$\frac{1}{2}\frac{\partial \beta}{\partial x_i}(x_2, \ldots, x_n) = d_1 \omega_{1i} + \sum_{j=1}^{n} \omega_{1j}\omega_{ij} \quad \text{for } 2 \leqslant i \leqslant n. \tag{4.24}$$

(3) *There exist constants* d_2 *and* d_3 *such that*

$$\sum_{j=1}^{n} \omega_{1j} \cdot \frac{\partial \beta}{\partial x_j}(x_2, \ldots, x_n) = d_2, \tag{4.25}$$

$$\sum_{j=1}^{n} \omega_{1j}\frac{\partial \gamma}{\partial x_j}(x_2, \ldots, x_n) = d_1 \beta(x_2, \ldots, x_n) + d_3. \tag{4.26}$$

Remark 4.3. In [7], Chiou considered a filtering model with $g(x(t))$ in (2.1) an invertible constant matrix instead of an orthogonal matrix. By Lemma A in [7], the result of our Main Theorem is true under the filtering model considered by Chiou.

We shall give some examples of five-dimensional estimation algebras by using Theorem 4.1. Example 4.1 is a new class of finite-dimensional estimation algebras.

Example 4.1. Consider the filtering model

$$\begin{cases} dx(t) = f(x(t))\,dt + g(x(t))\,dv(t), \\ dy(t) = h(x(t))\,dt + dw(t), \end{cases}$$

where

$$f_1 = ax_1,$$
$$f_2 = bx_1x_3,$$
$$f_3 = -bx_1x_2,$$
$$f_i = g_i(x_4, \ldots, x_n), \quad 4 \leqslant i \leqslant n,$$
$$h(x) = x_1,$$

and a, b are nonzero constants. Then

$$\omega_{12} = \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} = bx_3,$$

$$\omega_{13} = \frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3} = -bx_2,$$

$$\omega_{23} = \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} = -2bx_1,$$

$$\omega_{1j} = \frac{\partial f_j}{\partial x_1} - \frac{\partial f_1}{\partial x_j} = 0, \quad 4 \leqslant i \leqslant n,$$

$$\sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$$
$$= a^2x_1^2 + b^2x_1^2x_3^2 + b^2x_1^2x_2^2$$
$$+ \sum_{i=4}^{n} g_i^2(x_4, \ldots, x_n) + a$$
$$+ \sum_{i=4}^{n} \frac{\partial g_i}{\partial x_i}(x_4, \ldots, x_n)$$
$$= \left(\sum_{i=1}^{n} \omega_{1i}^2 + a^2\right) x_1^2 + \sum_{i=4}^{n} g_i^2(x_4, \ldots, x_n) + a$$
$$+ \sum_{i=4}^{n} \frac{\partial g_2}{\partial x_i}(x_4, \ldots, x_n).$$

It is easy to check that ω_{1i}, $1 \leqslant i \leqslant n$, satisfy (4.17), η is of the form (4.19) with $C_3 = a^2 + 1$ satisfying (4.20) and (4.21) with $C_0 = 2b^2$, $C_1 = C_2 = 0$. The estimation algebra E is five-dimensional with basis $\{1, x_1, D_1, Y_1 = bx_3D_2 - bx_2D_3 + (a^2 + 1 + b^2x_2^2 + b^2x_3^2)x_1, L_0\}$.

Example 4.2. Consider the filtering model

$$\begin{cases} \mathrm{d}x(t) = f(x(t))\,\mathrm{d}t + g(x(t))\,\mathrm{d}v(t), \\ \mathrm{d}y(t) = h(x(t))\,\mathrm{d}t + \mathrm{d}w(t), \end{cases}$$

where

$$f_1 = a_1x_1 + \frac{a_1^2}{a_2}x_2 + \sum_{i=3}^{n} a_ix_i + e,$$
$$f_2 = a_2x_1 + a_1x_2 - \frac{a_1}{a_2}\sum_{i=3}^{n} a_ix_i + c,$$
$$f_i = a_ix_1 + a_i\frac{a_1}{a_2}x_2 + g_i(x_3, \ldots, x_n), \quad 3 \leqslant i \leqslant n,$$
$$h(x) = x_1,$$
$$a_1^2 \neq a_2^2, \quad \sum_{i=1}^{n} a_i^2 > 0,$$
$$\left(a_2 - \frac{a_1^2}{a_2}\right)^2 = a_1^2 + a_2^2 + \sum_{i=3}^{n} a_i^2,$$
$$\sum_{i=3}^{n} a_ig_i(x_3, \ldots, x_n) = 0.$$

Then

$$\omega_{12} = \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} = a_2 - \frac{a_1^2}{a_2} \neq 0,$$
$$\omega_{kj} = \frac{\partial f_j}{\partial x_k} - \frac{\partial f_k}{\partial x_j} = 0 \quad \text{for } k = 1, 2, \ 3 \leqslant j \leqslant n,$$

$$\sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} = \sum_{i=1}^{n} a_i^2x_1^2 + \beta(x_2)x_1 + \gamma(x_2, \ldots, x_n),$$

where

$$\beta(x_2) = 2\left(\frac{a_1^3}{a_2} + a_2a_1 + \frac{a_1}{a_2}\sum_{i=3}^{n} a_i^2\right)x_2 + 2(a_1e + a_2c),$$

$$\gamma(x_2, \ldots, x_n) = \left[\left(\frac{a_1^2}{a_2}\right)^2 + a_1^2 + \left(\frac{a_1}{a_2}\right)^2\left(\sum_{i=3}^{n} a_i^2\right)\right]x_2^2$$
$$+ \left[1 + \left(\frac{a_1}{a_2}\right)^2\right]\left(\sum_{i=3}^{n} a_ix_i\right)^2$$
$$+ \sum_{i=3}^{n} g_i^2 + e^2 + c^2 + 2\left(\frac{a_1^2}{a_2}e + a_1c\right)x_2$$
$$+ 2\left(e - c\frac{a_1}{a_2}\right)\sum_{i=3}^{n} a_ix_i + 2a_1 + \sum_{i=3}^{n} \frac{\partial g_i}{\partial x_i}.$$

It is easy to check that ω_{1i}, $1 \leqslant i \leqslant n$, satisfy (4.17), η is of the form (4.19) satisfying (4.20) and (4.21). The estimation algebra E is five-dimensional with basis $\{1, x_1, D_1, Y_1 = ((a_1^2/a_2) - a_2)D_2 + (\sum_{i=1}^n a_i^2 + 1)x_1 + ((a_1^3/a_2) + a_2a_1 + (a_1/a_2)\sum_{i=3}^n a_i^2)x_2 + (a_1e + a_2c), L_0\}$.

Example 4.3. Consider the filtering model

$$\begin{cases} \mathrm{d}x(t) = f(x(t))\,\mathrm{d}t + g(x(t))\,\mathrm{d}v(t), \\ \mathrm{d}y(t) = h(x(t))\,\mathrm{d}t + \mathrm{d}w(t), \end{cases}$$

where

$$f_1 = a_1x_1 + b_1x_2 + b_1x_3 + d_1,$$
$$f_2 = a_2x_1 + d_2,$$
$$f_3 = a_2x_1 + d_3,$$
$$f_i = g_i(x_4, \ldots, x_n) \quad \text{for } 4 \leqslant i \leqslant n,$$
$$h(x) = x_1,$$

where $a_1^2 = 2(a_2 - b_1)^2 > 0$.

Then

$$\omega_{12} = \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} = a_2 - b_1 \neq 0,$$
$$\omega_{13} = \frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3} = a_2 - b_1 \neq 0, \quad \omega_{23} = 0$$
$$\omega_{kj} = \frac{\partial f_j}{\partial x_k} - \frac{\partial f_k}{\partial x_j} = 0 \quad \text{for } k = 1, 2, 3, \ 4 \leqslant j \leqslant n,$$

$$\sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} = (a_1^2 + 2a_2^2)x_1^2 + \beta(x_2, x_3)x_1 + \gamma(x_2, \ldots, x_n),$$

where

$$\beta(x_2, x_3) = 2(a_1b_1x_2 + a_1b_1x_3 + a_1d_1 + a_2d_2 + a_2d_3),$$

$$\gamma(x_2, \ldots, x_n) = 2b_1^2x_2x_3 + 2d_1b_1(x_2 + x_3) + b_1^2(x_2^2 + x_3^2) + d_1^2 + d_2^2 + d_3^2 + \sum_{i=4}^{n} g_i^2 + a_1 + \sum_{i=4}^{n} \frac{\partial g_i}{\partial x_i}.$$

It is easy to check that ω_{1i}, $1 \leqslant i \leqslant n$, satisfy (4.17), η is of the form (4.19) satisfying (4.20) and (4.21). The estimation algebra E is five-dimensional with basis $\{1, x_1, D_1, Y_1 = (b_1 - a_2)D_2 + (b_1 - a_2)D_3 + (a_1^2 + 1 + 2a_2^2)x_1 + a_1b_1x_2 + a_1b_1x_3 + a_1d_1 + a_2d_2 + a_2d_3, L_0\}$.

5. Conclusion

Despite the success of the classification of finite-dimensional estimation algebras with maximal rank, the problem of classification of non-maximal rank finite-dimensional estimation algebras is still wide open except for the case of state space dimension 2 which was finished by Wu and Yau [19] recently. Due to the difficulty of the problem, Brockett suggested that one should understand the low-dimensional estimation algebras first. In [24], Yau and Rasoulian have classified estimation algebras of dimension at most four. In this paper, we give a structure theorem for estimation algebras of dimension five. Using this structure theorem, we have constructed three families of five-dimensional estimation algebras. The first family of five-dimensional estimation algebras is a new class of finite-dimensional algebras. Hopefully, the result of this paper will shed some new light on the non-maximal rank finite-dimensional estimation algebra. In the future work, we shall investigate whether there are more new classes of finite-dimensional estimation algebras in low-dimensional similar to those in Example 4.1.

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