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CLASSIFICATION OF ESTIMATION ALGEBRAS WITH STATE DIMENSION 2*

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Abstract. This paper considers general finite-dimensional estimation algebras associated with nonlinear filtering systems. General considerations and approaches toward the classification of finite-dimensional estimation algebras are proposed. Some structural results are obtained. The properties of Euler operator and the solutions to an underdetermined partial differential equation, which inevitably arise in an estimation algebra, are studied. These tools and techniques are applied to the study of finite-dimensional estimation algebras with state dimension 2 to obtain a complete classification result. It is shown that a finite-dimensional estimation algebra with state dimension 2 can only have dimension less than or equal to 6. Moreover, the Mitter conjecture and the Levine conjecture hold for finite-dimensional estimation algebras with state dimension 2.

Key words. finite-dimensional filter, estimation algebra, nonlinear drift

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1. Introduction.

The Lie algebraic method, pioneered by Brockett [3], Brockett and Clark [5], and Mitter [15], provides an important research direction for nonlinear filtering theory. By interpreting the Duncan–Mortensen–Zakai equation or its robust form as a partial differential equation with time varying parameters, one derives an approach to filtering based on Lie algebra as well as the theory of linear differential operators. The search and construction of the finite-dimensional filter are turned into the study of the structure of the estimation algebra. In return, the theory of estimation algebra provides a systematic tool to deal with questions concerning finite-dimensional filters. It has led to a number of new results concerning finite-dimensional filters and to a deeper understanding of the structure of nonlinear filtering in general. It explains convincingly in [6] and [11] why it is easy to find exact recursive filters for linear dynamical systems, while it is very hard to handle the cubic sensor problem. Some new filters have been discovered using the estimation algebra [1], [16], [20], [21], [24]. More importantly, the finite-dimensionality of the estimation algebra guarantees the explicit construction of the finite-dimensional recursive filter, and the filter so constructed is universal in the sense of [7]. However, as an inherited difficulty from the nonlinear filtering, many questions are not satisfyingly answered and still open. Two excellent survey papers [10], [14] provide the detailed material on the Wei–Norman approach, the connections between estimation algebra and nonlinear filtering, the application of the Lie algebra method to a variety of specific nonlinear filtering problems, and the construction of the approximate filters, as well as sufficient reviews of the requisite background and a sufficient number of examples. For more contemporary results, see [23].

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At the International Congress of Mathematics in 1983, Brockett [4] proposed classifying all the finite-dimensional estimation algebras. In [26], the concept of estimation algebra of maximal rank was introduced by Yau and Chiou. The complete classification of such a subclass finite-dimensional estimation algebra was done by Yau and his coworkers in a series of papers, and the final results were reported in [25]. In [27], the conditions by which a general estimation algebra has dimension less than or equal to 4 were discussed. In [17], a new class of finite-dimensional estimation algebras were constructed when the state dimension $n \geq 3$. In this paper, the finite-dimensional estimation algebras that can arise when the underlying stochastic systems have state dimension 2 are completely classified. It is shown that an estimation algebra with state dimension 2 can have only 1, 2, 4, 5, or 6 dimensions. The finite-dimensional estimation algebras with state dimension 1 were considered in [5] and [16].

This paper is organized as follow: Section 2 introduces some basic concepts of estimation algebra for a nonlinear filtering system and gives the classification result on the finite-dimensional estimation algebra of maximal rank. In section 3, some tools and techniques are discussed for the classification of general estimation algebras. The complete classification of estimation algebras with state dimension 2 is given in section 4.

2. Basic concepts. In this paper, the filtering system to be considered is based on the following continuous signal-observation model:

$$\begin{cases} dx(t) = f\big(x(t)\big)dt + g\big(x(t)\big)dv(t), & x(0) = x_0, \\ dy(t) = h\big(x(t)\big)dt + dw(t), & y(0) = 0. \end{cases} \tag{2.1}$$

Here x, v, y, and w are, respectively, R^n, R^n, R^m, and R^m valued processes, and v and w have components which are independent, standard Brownian processes. Moreover, $f = (f_1, \dots, f_n)$ and $h = (h_1, \dots, h_m)$ are assumed to be C^∞ smooth and g is an orthogonal matrix. $x(t)$ is referred to as the state of the system at time t and $y(t)$ as the observation at time t.

Let $\rho(t,x)$ denote the conditional probability density of the state $x(t)$ given the observation $\{y(s) : 0 \leq s \leq t\}$. $\rho(t,x)$ is governed by the well-known Duncan–Mortensen–Zakai (DMZ) equation, which is a stochastic partial differential equation in terms of an unnormalized version of $\rho(t,x)$, denoted by $\sigma(t,x)$ (see [13], for example). Under the Stratonovich calculus, the DMZ equation for the filtering system (2.1) can be written as

$$\begin{cases} d\sigma(t,x) = L_0\sigma(t,x)dt + \displaystyle\sum_{i=1}^m h_i\sigma(t,x)dy_i(t), \\ \sigma(0,x) = \sigma_0, \end{cases} \tag{2.2}$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^n f_i \frac{\partial}{\partial x_i} - \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^m h_i^2. \tag{2.3}$$

The term σ_0 is the probability density of the initial state x_0.

The Wei–Norman result [19] on constructing explicit solutions to complex partial differential equations suggests a formal approach to constructing a finite-dimensional filter by defining the estimation algebra as a Lie algebra generated by the operators occurring in the DMZ equation.

Definition 2.1. *If X and Y are differential operators, the Lie bracket of X and Y, $[X,Y]$, is defined by $[X,Y]\phi = X(Y\phi) - Y(X\phi)$ for any C^∞-function ϕ.*

Definition 2.2. *The estimation algebra E of a filtering system* (2.1) *is defined as the Lie algebra generated by $\{L_0, h_1, \dots, h_m\}$, i.e., $E = \langle L_0, h_1, \dots, h_m\rangle_{L.A.}$, where L_0 is defined in* (2.3).

The following theorem due to Ocone [16] is the first result to show what kind of functions can appear in a finite-dimensional estimation algebra.

Theorem 2.3 (see [16]). *Let E be a finite-dimensional estimation algebra. If ϕ is a function in E, then ϕ is a polynomial of degree at most* 2.

In fact, Mitter and Levine have conjectured the following.

Mitter conjecture. *Let E be a finite-dimensional estimation algebra. If ϕ is a function in E, then ϕ is a polynomial of degree at most* 1.

Levine conjecture (see [12]). *Let E be a finite-dimensional estimation algebra. The differential operators in E have orders at most* 2.

A fundamental step in the Lie algebraic approach as introduced in [21] is to represent the elliptic operator L_0 in (2.3) in a more compact form by defining

$$D_i = \frac{\partial}{\partial x_i} - f_i, \tag{2.4}$$

$$\eta = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^n f_i^2 + \sum_{i=1}^m h_i^2. \tag{2.5}$$

Then

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^n D_i^2 - \eta\right). \tag{2.6}$$

Definition 2.4. *The Ω-matrix of a filtering system* (2.1) *is an $n \times n$ matrix $\Omega = (\omega_{ij})$ defined by*

$$\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j} \qquad \forall\ 1 \le i, j \le n. \tag{2.7}$$

When $\Omega = O$, i.e., f is a gradient vector field in view of the Poincaré lemma, the filtering system and the corresponding estimation algebra are called exact. In [9] and [18], the finite-dimensional exact estimation algebras are completely classified. In [24], Yau classified the finite-dimensional estimation algebras when the Ω-matrix has constant entries.

Theorem 2.5 (see [24]). *Let E be an estimation algebra of* (2.1) *whose Ω-matrix has constant entries.*

(1) *If η (in* (2.5)*) is a polynomial of degree at most* 2 *and $h_1, \dots, h_m$ are affine in x, then E is finite-dimensional and has a basis consisting of $E_0 = L_0$, $E_1, \dots, E_p, E_{p+1}, \dots, E_q, 1$ (for some $p < q$). The differential operators $E_1, \dots, E_p$ have the form*

$$\sum_{j=1}^n \alpha_{ij} D_j + \beta_j, \qquad 1 \le i \le p,$$

where α_{ij}'s are constants and β_i's are affine in x, and the differential operators $E_{p+1}, \dots, E_q$ are affine in x. Moreover, the quadratic part of $\eta - \sum_{i=1}^m h_i^2$ is positive semidefinite.

(2) *Conversely, if E is finite-dimensional, then $h_1, \dots, h_m$ are affine in x, i.e., the observation matrix $H = [\nabla h_1, \dots, \nabla h_m]$ is a constant matrix. Furthermore, if the observation matrix has rank n, then η is a polynomial of degree at most 2 and E is of dimension $2n+2$ with a basis given by $1, x_1, \dots, x_n, D_1, \dots, D_n$, and L_0.*

Definition 2.6. *An estimation algebra E is said to be of maximal rank if, for any $1 \le i \le n$, there exists a constant c_i, such that $x_i + c_i \in E$.*

By direct calculation, if E is of maximal rank, $[L_0, x_i + c_i] = D_i \in E$, and $[D_i, x_i + c_i] = 1 \in E$. Thus, $\langle 1, x_1, \dots, x_n \rangle$ form a vector space of E. Any degree 1 polynomial is an element of E.

Theorem 2.7 (see [25]). *Let E be an estimation algebra of (2.1). If E is of maximal rank and finite-dimensional, then*

(1) $\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j} = c_{ij}$ $\forall 1 \le i, j \le n$, *where c_{ij}'s are constants;*

(2) *$h_1, \dots, h_m$ are affine in x;*

(3) *η is a polynomial of degree 2 and the quadratic part of $\eta - \sum_{i=1}^m h_i^2$ is positive semidefinite.*

Moreover, E is of dimension $2n+2$ with a basis given by $1, x_1, \dots, x_n, D_1, \dots, D_n$, and L_0.

For the convenience of the reader, the following elementary lemma on the calculations of the Lie bracket of the estimation algebra is listed without proof.

Lemma 2.8 (see [8], [24]). *Let E be an estimation algebra for the filtering system (2.1). $\Omega = (\omega_{ij})$ is defined as in (2.7). $X, Y, Z \in E$ and $g, h \in C^\infty(R^n)$. Then*

(1) $[XY, Z] = X[Y, Z] + [X, Z]Y$;

(2) $[gD_i, h] = g\frac{\partial h}{\partial x_i}$;

(3) $[gD_i, hD_j] = -gh\omega_{ij} + g\frac{\partial h}{\partial x_i}D_j - h\frac{\partial g}{\partial x_j}D_i$;

(4) $[gD_i^2, h] = 2g\frac{\partial h}{\partial x_i}D_i + g\frac{\partial^2 h}{\partial x_i^2}$;

(5) $[D_i^2, hD_j] = 2\frac{\partial h}{\partial x_i}D_iD_j - 2h\omega_{ij}D_i + \frac{\partial^2 h}{\partial x_i^2}D_j - h\frac{\partial \omega_{ij}}{\partial x_i}$;

(6) $[D_i^2, D_j^2] = 4\omega_{ji}D_jD_i + 2\frac{\partial \omega_{ji}}{\partial x_j}D_i + 2\frac{\partial \omega_{ji}}{\partial x_i}D_j + \frac{\partial^2 \omega_{ji}}{\partial x_i \partial x_j} + 2\omega_{ji}^2$;

(7) $[D_k^2, hD_iD_j] = 2\frac{\partial h}{\partial x_k}D_kD_iD_j + 2h\omega_{jk}D_iD_k + 2h\omega_{ik}D_kD_j + \frac{\partial^2 h}{\partial x_k^2}D_iD_j + 2h\frac{\partial \omega_{jk}}{\partial x_i}D_k + h\frac{\partial \omega_{jk}}{\partial x_k}D_i + h\frac{\partial \omega_{ik}}{\partial x_k}D_j + h\frac{\partial^2 \omega_{jk}}{\partial x_i \partial x_k}$;

(8) $[gD_iD_j, hD_k] = g\frac{\partial h}{\partial x_j}D_iD_k + g\frac{\partial h}{\partial x_i}D_jD_k + gh\omega_{kj}D_i + gh\omega_{ki}D_j + g\frac{\partial^2 h}{\partial x_i \partial x_j}D_k + gh\frac{\partial \omega_{kj}}{\partial x_i} - h\frac{\partial g}{\partial x_k}D_iD_j$.

3. Preliminary results.

3.1. Differential operator. Let U be the set of differential operators in the form

$$A = \sum_{(i_1,\dots,i_n) \in I_A} a_{i_1,\dots,i_n} D_1^{i_1} \cdots D_n^{i_n}, \tag{3.1}$$

where nonzero functions $a_{i_1,\dots,i_n} \in C^\infty(R^n)$ and I_A is the finite index set of A. Each element of the index set is an n-tuple $(i_1, \dots, i_n)$ of nonnegative integers. The norm of an index $i = (i_1, \dots, i_n)$ is defined to be $|i| = \sum_{l=1}^n i_l$. The order of A is denoted by $ord\ A = \max_{i=(i_1,\dots,i_n) \in I_A} |i|$. If $A = 0$, $ord\ A$ is defined to be $-\infty$. It is clear that for $A, B \in U$

$$ord\ (AB) = ord\ (BA) = ord\ A + ord\ B, \tag{3.2}$$

$$ord\ (A \pm B) \le \max(ord\ A, ord\ B). \tag{3.3}$$

U is a Lie algebra under the Lie bracket $[\cdot,\cdot]$ defined earlier. Two differential operators A and B in U are equal if they have the identical index sets $I_A = I_B$ and $a_{i_1,\ldots,i_n} = b_{i_1,\ldots,i_n} \forall (i_1,\ldots,i_n) \in I_A$. Let U_k denote the subspace of U consisting of elements with order less than or equal to k. In particular, $U_0 = C^\infty(R^n)$. As usual, *mod* is used to denote the equivalence class, i.e., if V is a subspace of U,

$$A = B, \qquad \text{mod } V \Longleftrightarrow A - B \in V. \tag{3.4}$$

If $A, B \in U$, define

$$Ad_A B = [A,B], \qquad Ad_A^l B = [A, Ad_A^{l-1} B], \quad l \geq 1, \tag{3.5}$$

where Ad_A^0 is the identity operator by standard convention.

Recall that

$$D_i = \frac{\partial}{\partial x_i} - f_i, \qquad 1 \leq i \leq n, \tag{3.6}$$

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^n D_i^2 - \eta\right), \qquad \eta = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^n f_i^2 + \sum_{i=1}^m h_i^2. \tag{3.7}$$

The estimation algebra $E = \langle L_0, h_1, \ldots, h_m\rangle_{L.A.}$. Since $L_0 \in U_2$ and $h_i \in U_0, 1 \leq i \leq m$, $E \subset U$. Every element of E will have representation in the form of (3.1).

The finite-dimensionality of E can be measured in terms of the finite order of the elements in E. If E is finite-dimensional, then the orders of its element will have an upper bound. In particular, if there exists a sequence of elements $A_j \in E$ such that the orders of A_j's are strictly increasing, E is not finite-dimensional.

The following is a very useful lemma in the computation of the orders of the differential operators generated under the Lie bracket.

LEMMA 3.1. *Let $g, h \in C^\infty(R^n)$ and let $i_1,\ldots,i_n, j_1,\ldots,j_n$ be nonnegative integers with $\sum_{l=1}^n i_l = r$, $\sum_{l=1}^n j_l = s$, and $r+s \geq 2$. Let δ_{ij} be the Kronecker symbol, i.e., $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$. Then*

$$\begin{aligned}&[gD_1^{i_1}\cdots D_n^{i_n}, hD_1^{j_1}\cdots D_n^{j_n}]\\ &\quad = \sum_{l=1}^n \left(i_l g \frac{\partial h}{\partial x_l} - j_l h \frac{\partial g}{\partial x_l}\right) D_1^{i_1+j_1-\delta_{1l}} \cdots D_n^{i_n+j_n-\delta_{nl}}, \quad \text{mod } U_{r+s-2}.\end{aligned} \tag{3.8}$$

Proof. If $r+s=2$, there are two cases:

$$[gD_iD_j, h] = -[h, gD_iD_j] = g\frac{\partial h}{\partial x_j}D_i + g\frac{\partial h}{\partial x_i}D_j + g\frac{\partial^2 h}{\partial x_i \partial x_j}, \tag{3.9}$$

$$[gD_i, hD_j] = -[hD_j, gD_i] = gh\omega_{ij} + g\frac{\partial h}{\partial x_i}D_j - h\frac{\partial g}{\partial x_j}D_i. \tag{3.10}$$

Clearly, (3.8) holds for these two cases. Induction on $r+s$ is used to prove (3.8).

If $r+s > 2$, at least one of r and s is greater than 0. Without loss of generality, assume $r > 0$. (In the case where $r = 0$ and $s > 0$, by using the property $[A,B] = -[B,A]$, the same procedure applies below.) Since $\sum_{l=1}^n i_l = r > 0$, at least one i_l is no less than 1. Assume $i_k \geq 1$. By Lemma 2.8, $gD_1^{i_1}\cdots D_k^{i_k}\cdots D_n^{i_n} = D_k g D_1^{i_1}\cdots D_k^{i_k-1}\cdots D_n^{i_n}$, mod U_{r-1}. Hence,

$$gD_1^{i_1}\cdots D_k^{i_k}\cdots D_n^{i_n} = D_k g D_1^{i_1}\cdots D_k^{i_k-1}\cdots D_n^{i_n} + A, \tag{3.11}$$

where $A \in U_{r-1}$. By the assumption of the induction, $[A, hD_1^{j_1} \cdots D_n^{j_n}] \in U_{r+s-2}$, since each term of A has order less than or equal to $r-1$. Therefore,

$$
\begin{aligned}
&[gD_1^{i_1} \cdots D_n^{i_n}, hD_1^{j_1} \cdots D_n^{j_n}] \\
&= [D_k g D_1^{i_1} \cdots D_k^{i_k - 1} \cdots D_n^{i_n}, hD_1^{j_1} \cdots D_n^{j_n}] \\
&= D_k [g D_1^{i_1} \cdots D_k^{i_k - 1} \cdots D_n^{i_n}, hD_1^{j_1} \cdots D_n^{j_n}] \\
&\qquad + [D_k, hD_1^{j_1} \cdots D_n^{j_n}] g D_1^{i_1} \cdots D_k^{i_k - 1} \cdots D_n^{i_n} \\
&= D_k \sum_{l=1}^n \left((i_l - \delta_{lk}) g \frac{\partial h}{\partial x_l} - j_l h \frac{\partial g}{\partial x_l} \right) D_1^{i_1 + j_1 - \delta_{1l}} \cdots D_k^{i_k + j_k - 1 - \delta_{kl}} \cdots D_n^{i_n + j_n - \delta_{nl}} \\
&\qquad + \frac{\partial h}{\partial x_k} D_1^{j_1} \cdots D_k^{j_k} \cdots D_n^{j_n} g D_1^{i_1} \cdots D_k^{i_k - 1} \cdots D_n^{i_n} \\
&= \sum_{l=1}^n \left((i_l - \delta_{lk}) g \frac{\partial h}{\partial x_l} - j_l h \frac{\partial g}{\partial x_l} \right) D_1^{i_1 + j_1 - \delta_{1l}} \cdots D_k^{i_k + j_k - \delta_{kl}} \cdots D_n^{i_n + j_n - \delta_{nl}} \\
&\qquad + g \frac{\partial h}{\partial x_k} D_1^{i_1 + j_1} \cdots D_k^{i_k + j_k - 1} \cdots D_n^{i_n + j_n} \\
&= \sum_{l=1}^n \left(i_l g \frac{\partial h}{\partial x_l} - j_l h \frac{\partial g}{\partial x_l} \right) D_1^{i_1 + j_1 - \delta_{1l}} \cdots D_n^{i_n + j_n - \delta_{nl}}, \quad \mod U_{r+s-2}. \qquad \square
\end{aligned}
$$

In fact, (3.8) is still valid for $r + s = 0$ or 1 if the definition of U_k is extended to when $k < 0$, $U_k = \{0\}$, i.e., U_k has just one element 0 when $k < 0$.

A direct result from Lemma 3.1 is that for $A, B \in U$,

$$
ord\ [A, B] \le ord\ A + ord\ B - 1. \tag{3.12}
$$

Moreover, if $ord\ [A, B] = ord\ A + ord\ B - 1$, the highest order terms of $[A, B]$ are uniquely determined in (3.8) by those of A and B.

The following is a very important theorem on the polynomial structure of coefficient functions of the differential operators in E which generalizes Wong's [22] result.

Theorem 3.2. *Let E be a finite-dimensional estimation algebra and let the D_i's be defined as in* (3.6). *If $l \ge 0$ and*

$$
A = \sum_{|(i_1, \ldots, i_n)| = l+1} a_{i_1, \ldots, i_n} D_1^{i_1} \cdots D_n^{i_n}, \quad \mod U_l,
$$

is in E, then $a_{i_1, \ldots, i_n}$'s are polynomials.

Proof. If not all the $a_{i_1, \ldots, i_n}$'s are polynomials, at least one, say, $a_{j_1, \ldots, j_n}$, is transcendental. Then there exists a variable x_k in which $a_{j_1, \ldots, j_n}$ is transcendental, i.e., $\frac{\partial^s a_{j_1, \ldots, j_n}}{\partial x_k^s} \neq 0 \ \forall s \ge 0$.

Now, among the indices $(j_1, \ldots, j_n)$ of all $a_{j_1, \ldots, j_n}$ that are transcendental in x_k, there exists one $(j_1, \ldots, j_n)$ having the largest kth index j_k. Without loss of generality, assume $k = 1$ and $a_{j_1, \ldots, j_n}$ is transcendental in x_1 and j_1 is the largest among the first indices whose coefficient functions are transcendental in x_1, i.e., if $a_{i_1, \ldots, i_n}$ is transcendental in x_1 as well, then $j_1 \geq i_1$. Let

$$A_1 = [L_0, A] = \left[L_0, \sum_{|(i_1,\ldots,i_n)|=l+1} a_{i_1,\ldots,i_n} D_1^{i_1} \cdots D_n^{i_n} \right]$$

$$= \sum_{|(i_1,\ldots,i_n)|=l+1} \sum_{r=1}^{n} \frac{\partial a_{i_1,\ldots,i_n}}{\partial x_r} D_1^{i_1} \cdots D_{r-1}^{i_{r-1}} D_r^{i_r+1} D_{r+1}^{i_{r+1}} \cdots D_n^{i_n}, \quad \text{mod } U_{l+1}.$$

The coefficient function of $D_1^{j_1+1} D_2^{j_2} \cdots D_n^{j_n}$ in A_1 is

$$\frac{\partial a_{j_1,j_2,\ldots,j_n}}{\partial x_1} + \frac{\partial a_{j_1+1,j_2-1,\ldots,j_n}}{\partial x_2} + \cdots + \frac{\partial a_{j_1+1,j_2,\ldots,j_n-1}}{\partial x_n}. \tag{3.13}$$

Since j_1 is the largest first index such that a is transcendental in x_1, except that the first term in (3.13) is transcendental in x_1, the other $(n-1)$ terms are all polynomials in x_1. Hence, the coefficient function (3.13) of $D_1^{j_1+1} D_2^{j_2} \cdots D_n^{j_n}$ is transcendental in x_1.

Similarly, let $|(i_1, \ldots, i_n)| = l+2$, $i_1 \geq j_1+2$, and consider the coefficient functions of $D_1^{i_1} D_2^{i_2} \cdots D_n^{i_n}$ in A_1. Their coefficient functions are the sum of $\frac{\partial a_{p_1,p_2,\ldots,p_n}}{\partial x_q}$, where $|(p_1, \ldots, p_n)| = l+1$ and $p_1 \geq j_1 + 1$. By assumption on the a's, these functions are all polynomials in x_1.

Thus, from A to A_1, the differential orders increase by 1, while the transcendental structure on x_1 remains unchanged. The coefficient function of $D_1^{j_1+1} D_2^{j_2} \cdots D_n^{j_n}$ in A_1 is transcendental in x_1 and $j_1 + 1$ is the largest first indices whose coefficient functions are transcendental in x_1. (Note that there may exist other a's having this property, but they must have different numbers in another index, say the nth index. These terms are linearly independent.)

By keeping this process, $A_2 = [L_0, A_1], A_3 = [L_0, A_2], \ldots$. A_k has order of $l + k + 1 \to \infty$. Contradiction! ∎

3.2. Linear rank and quadratic rank. Unlike estimation algebras of maximal rank (cf. Definition 2.6), it is not clear whether x_i's are elements of a general estimation algebra E. Moreover, 1 may not be in E either. To simplify the situation, the concept of linear rank is introduced in this section together with quadratic rank to explore the structure of the function elements in E. In what follows, dim E is denoted the dimension of the estimation algebra E. For a polynomial ϕ, $\phi^{(k)}$ denotes its homogeneous degree k part.

THEOREM 3.3. *Let $E = \langle L_0, h_1, \ldots, h_m \rangle_{L.A.}$ and $\bar{E} = \langle 1, L_0, h_1, \ldots, h_m \rangle_{L.A.}$. Then E is finite-dimensional if and only if $\bar{E}$ is finite-dimensional.*

Proof. Since $E \subset \bar{E}$, it suffices to show dim $\bar{E}$ − dim $E \leq 1$ if dim $E < \infty$.

Start from the generators of $\bar{E}$ to construct increasing subsets A_k of $\bar{E}$, as follows:

- $A_0 = \{1, L_0, h_1, \ldots, h_m\}$;
- for $k \geq 1$, $A_k = \{a_1 B_1 + a_2 B_2 + a_3 [B_3, B_4] : a_i \in R, B_j \in A_{k-1}, i = 1, 2, 3, j = 1, 2, 3, 4\}$.

Clearly $A_k \nearrow \bar{E}$ as $k \to \infty$. Now, assume that dim $E < \infty$. Let vector space $E^* = E + 1 := \{aX + b : X \in E, a, b \in R\}$. It is easy to show by induction that

$A_k \subset E^*$ for $k \geq 0$. Thus,

$$\bar{E} = \cup_{k \geq 0} A_k \subset E^*,$$

which means

$$\dim \bar{E} \leq \dim E^* \leq \dim E + 1. \qquad \square$$

Thus, the estimation algebra E of a filtering problem being finite-dimensional is equivalent to the finite-dimensionality of $\langle E, 1\rangle_{L.A.}$. In the discussion of the finite-dimensionality of E, 1 can always be assumed an element of E.

Under the assumption that $1 \in E$, any degree 1 polynomial in $E \Longrightarrow$ its homogeneous degree 1 part is in E.

DEFINITION 3.4. *Let $L(E) \subset E$ be the vector space (Lie subalgebra of E) consisting of all the homogeneous degree* 1 *polynomials in E. Then $\nu(E) := \dim L(E)$ is called the linear rank of the estimation algebra E.*

From this definition, an estimation algebra of maximal rank is in fact an estimation algebra with linear rank n.

Note that an estimation algebra is associated with a filtering system and is coordinate-dependent. The recognition of the structurally equivalent estimation algebras is very important in the classification problem. Since an estimation algebra is essentially a Lie algebra, the definitions of homomorphism and isomorphism of estimation algebras follow from those of Lie algebras. It is shown in [2], [8], and [14] that orthogonal variable transformation and affine transformations extend to estimation algebra isomorphisms.

Let $\nu(E) = r$. Clearly $r \leq n$ and there exist r independent linear functions (the basis of L) $l_1(x), \ldots, l_r(x)$ such that

$$(l_1(x), \ldots, l_r(x))^T = Ax,$$

where A is an $r \times n$ matrix with rank r. From the singular value decomposition theorem, there exist orthogonal matrices $U(r \times r)$, $V(n \times n)$ such that

$$A = U[D \quad 0]V^T,$$

where $D = \mathrm{diag}(d_1, \ldots, d_r)$, with $d_1, \ldots, d_r \neq 0$ the singular values of A. Thus,

$$U^T Ax = U^T AV(V^T x) = [D \quad 0]V^T x.$$

After an orthogonal change of the variable $y = V^T x$, it is easy to see that for $1 \leq i \leq r$, y_i is a linear combination of $l_j(x)$ $(1 \leq j \leq r)$. In view of $\nu(E) = r$, $\{y_i, 1 \leq i \leq r\}$ is the basis of $L(E)$.

Hence, by an orthogonal variable transformation, if necessary, a linear function l is in E if and only if

$$l(x) \in L(E) := span\{x_1, \ldots, x_r\}.$$

DEFINITION 3.5. *For a given function $h \in E$, its quadratic part $h^{(2)} = x^T Ax$ for a symmetric matrix A. The quadratic rank $\lambda(h)$ of h is defined as the rank of A. The quadratic rank $\lambda(E)$ of E is the greatest quadratic rank of $h \in E$, i.e., $\lambda(E) = \max_{h \in E} \lambda(h)$.*

LEMMA 3.6. *Let E be an estimation algebra with linear rank r and $\phi \in E$. Then the quadratic part of ϕ is*

$$\phi^{(2)} = x^T \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} x,$$

where A_1 and A_2 are symmetric matrices with dimensions $r \times r$ and $(n-r)\times(n-r)$.

Proof. $\phi^{(2)}$ is a degree 2 homogeneous polynomial; therefore it can be written as $\phi^{(2)} = x^T A x$ for a symmetric matrix $A = (a_{ij})$ of dimension $n \times n$.

Since $\nu(E) = r$, $x_i \in E$ if and only if $1 \le i \le r$. For $1 \le i \le r$, $[L_0, x_i] = D_i \in E$, and

$$[D_i, \phi] = \frac{\partial \phi}{\partial x_i} = 2\sum_{j=1}^{n} a_{ij} x_j \in E, \qquad \bmod R.$$

Hence, $a_{ij} = 0$ for $r+1 \le j \le n$, $1 \le i \le r$. By the symmetry of A, the lemma follows. ∎

Now, consider an estimation algebra E with linear rank r and quadratic rank k. From the definition of the quadratic rank of E, there exists $p_0 \in E$ such that the quadratic rank of p_0 is k. By Lemma 3.6, $p_0^{(2)}(x) = x^T \left(\begin{smallmatrix} A_1 & 0 \\ 0 & A_2 \end{smallmatrix}\right) x$. Let $k_1 = \operatorname{rank}(A_1)$, $k_2 = \operatorname{rank}(A_2)$. Then $k = k_1 + k_2$, $0 \le k_1 \le r$, and $0 \le k_2 \le n-r$. Since A_1 and A_2 are real symmetric, there are orthogonal matrices U_1 and U_2 such that $A_1 = U_1 \left(\begin{smallmatrix} D_1 & 0 \\ 0 & 0 \end{smallmatrix}\right) U_1^T$ and $A_2 = U_2 \left(\begin{smallmatrix} 0 & 0 \\ 0 & D_2 \end{smallmatrix}\right) U_2^T$, where D_1 and D_2 are nonsingular diagonal matrices with dimensions $k_1 \times k_1$ and $k_2 \times k_2$. By taking the orthogonal variable transformation $T = \left(\begin{smallmatrix} U_1 & 0 \\ 0 & U_2 \end{smallmatrix}\right)$, $p_0^{(2)} = \sum_{i=1}^{k_1} d_i x_i^2 + \sum_{i=n-k_2+1}^{n} d_i x_i^2$, where $d_i \neq 0$. Moreover, by an affine variable transformation, if necessary,

$$p_0 = \sum_{i=1}^{k_1} d_i x_i^2 + \sum_{i=n-k_2+1}^{n} d_i x_i^2 + \sum_{i=k_1+1}^{n-k_2} c_i x_i + c_0 \in E \tag{3.14}$$

$$\Longrightarrow [[L_0, p_0], p_0] = \sum_{i=1}^{k_1} 4 d_i^2 x_i^2 + \sum_{i=n-k_2+1}^{n} 4 d_i^2 x_i^2 + \sum_{i=k_1+1}^{n-k_2} c_i^2 \in E$$

$$\Longrightarrow q_0 := \sum_{i=1}^{k_1} d_i^2 x_i^2 + \sum_{i=n-k_2+1}^{n} d_i^2 x_i^2 \in E, \tag{3.15}$$

$$\Longrightarrow q_j := [[L_0, q_{j-1}], q_{j-1}] = \sum_{i=1}^{k_1} 4^j d_i^{2j+2} x_i^2 + \sum_{i=n-k_2+1}^{n} 4^j d_i^{2j+2} x_i^2 \in E, \quad j \ge 1. \tag{3.16}$$

If no d_i^2's are equal, then the coefficient matrix of (3.16) for x_i^2 forms a Vandermonde matrix. By the invertibility of the Vandermonde matrix, x_i^2 can be represented as a linear combination of q_j's, and therefore $x_i^2 \in E$. Hence,

$$\sum_{i=1}^{k_1} x_i^2 + \sum_{i=n-k_2+1}^{n} x_i^2 \in E. \tag{3.17}$$

If some d_i^2's are equal, for example, d_i^2's equal for $l_1 \le i \le l_2$ in (3.14), they can be grouped to be solved as one variable. Instead of individual $x_i^2 \in E$ for $l_1 \le i \le l_2$, $x_{l_1}^2 + \cdots + x_{l_2}^2 \in E$ is obtained as a group under the above Vandermonde argument.

In any case, (3.17) can be constructed as long as the quadratic rank of E is k. An important observation is that both orthogonal variable transformation so used and affine variable transformation do not change the basis of $L(E)$. In summary, we have the following.

THEOREM 3.7. *Let E be a finite-dimensional estimation algebra with linear rank r and quadratic rank k.*

There exists $p_0 = \sum_{i=1}^{k_1} x_i^2 + \sum_{i=n-k_2+1}^{n} x_i^2 \in E$, where $k_1 + k_2 = k$, $k_1 \leq r$, and $k_2 \leq n - r$.

If $\phi \in E$ is a degree 1 polynomial, then ϕ is independent of $x_{r+1}, \ldots, x_n$.

If $\phi \in E$ is a degree 2 polynomial, then $\phi^{(2)}$ is independent of $x_{k_1+1}, \ldots, x_{n-k_2}$.

3.3. Euler operator. As discussed in section 3.1, the finite-dimensionality of an estimation algebra relies on the order of the differential operators as well as the coefficient functions. The conditions on the constructible differential operators with bounded order are shifted to those on the coefficient functions to ensure the finite-dimensionality. Thus, it is important to extract the conditions on some primary characteristics such as ω_{ij}'s from some transformations. The Euler operator (see Definition 3.8) is one of such transformations that occur frequently, especially when a quadratic polynomial is assumed to be an element of the estimation algebra. This section studies the properties of the inverse image of the Euler operators. The results obtained in this section generalize the results discussed in [18] and [27].

DEFINITION 3.8. *Let l be a positive integer such that $l \leq n$. The Euler operator $E_l(\cdot)$ is defined to be a differential operator such that*

$$E_l(\phi) = \sum_{i=1}^{l} x_i \frac{\partial \phi}{\partial x_i}$$

for any $\phi \in C^\infty(R^n)$.

In the case $l = n$, $E(\cdot)$ is used instead of $E_n(\cdot)$.

LEMMA 3.9. *Let $m > 0$ be a positive integer and $\zeta \in C^\infty(R^n)$. If $E_l(\zeta) + m\zeta = 0$, then $\zeta = 0$.*

Proof. Let $\phi = x_1^m \zeta$. Then

$$E_l(\phi) = \sum_{i=1}^{l} x_i \frac{\partial}{\partial x_i}(x_1^m \zeta) = m x_1^m \zeta + x_1^m \sum_{i=1}^{l} x_i \frac{\partial \zeta}{\partial x_i} = x_1^m (m\zeta + E_l(\zeta)) = 0, \tag{3.18}$$

$$\begin{aligned}
&\phi(x_1, \ldots, x_l, \ldots, x_n) - \phi(\epsilon x_1, \ldots, \epsilon x_l, x_{l+1}, \ldots, x_n) \\
&= \int_\epsilon^1 \frac{d\phi}{dt}(t x_1, \ldots, t x_l, x_{l+1}, \ldots, x_n) dt \\
&= \int_\epsilon^1 \sum_{i=1}^{l} x_i \frac{\partial \phi}{\partial x_i}(t x_1, \ldots, t x_l, x_{l+1}, \ldots, x_n) dt
\end{aligned}$$

$$= \int_\epsilon^1 \frac{1}{t}(E_l(\phi))(t x_1, \ldots, t x_l, x_{l+1}, \ldots, x_n) dt = \int_\epsilon^1 \frac{0}{t} dt = 0. \tag{3.19}$$

Hence, $\phi(x_1, \ldots, x_l, \ldots, x_n) = \epsilon^m x_1^m \zeta(\epsilon x_1, \ldots, \epsilon x_l, x_{l+1}, \ldots, x_n)$. Since $\zeta \in C^\infty(R^n)$, the limit $\lim_{\epsilon \to 0} \zeta(\epsilon x_1, \ldots, \epsilon x_l, x_{l+1}, \ldots, x_n)$ exists. By letting $\epsilon \to 0$, one has that $\phi = 0 \Longrightarrow \zeta = 0$. ∎

LEMMA 3.10. *Let $\zeta \in C^\infty(R^n)$. If $E_l(\zeta) = 0$, then ζ is a C^∞-function in $x_{l+1}, \ldots, x_n$ variables.*

Proof. For $1 \leq j \leq l$,

$$\frac{\partial}{\partial x_j} E_l(\zeta) = \frac{\partial \zeta}{\partial x_j} + E_l\left(\frac{\partial \zeta}{\partial x_j}\right) = 0. \tag{3.20}$$

By Lemma 3.9, $\frac{\partial \zeta}{\partial x_j} = 0$. Therefore, ζ is independent of $x_1, \ldots, x_l$. ∎

LEMMA 3.11. *Let $m > 0$ be a positive integer and $\zeta \in C^\infty(R^n)$. If $E_l(\zeta) - m\zeta = 0$, then ζ is either 0 or a degree m homogeneous polynomial in $x_1, \ldots, x_l$ variables with coefficients in C^∞-functions of $x_{l+1}, \ldots, x_n$ variables.*

Proof. Let $D = (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_l})^{\alpha_l}$ with $\sum_{i=1}^l \alpha_i = m$.

By induction,

$$D(E_l(\zeta)) = E_l(D\zeta) + mD\zeta. \tag{3.21}$$

Hence, $E_l(D\zeta) = D(E_l(\zeta) - m\zeta) = 0$. By Lemma 3.10 $D\zeta$ is a C^∞-function in $x_{l+1}, \ldots, x_n$ variables. This means that ζ is a polynomial of degree $s \leq m$ in $x_1, \ldots, x_l$ variables with coefficients in C^∞-functions of $x_{l+1}, \ldots, x_n$ variables.

Let $\zeta = \sum_{0 \leq |i| \leq s} a_{i_1 \cdots i_l}(x_{l+1}, \ldots, x_n) x_1^{i_1} \cdots x_l^{i_l}$. Then

$$\begin{aligned} E_l(\zeta) - m\zeta &= E_l\left(\sum_{0 \leq |i| \leq s} a_{i_1 \cdots i_l}(x_{l+1}, \ldots, x_n) x_1^{i_1} \cdots x_l^{i_l}\right) \\ &\qquad - m \sum_{0 \leq |i| \leq s} a_{i_1 \cdots i_l}(x_{l+1}, \ldots, x_n) x_1^{i_1} \cdots x_l^{i_l} \\ &= \sum_{0 \leq |i| \leq s} |i| a_{i_1 \cdots i_l}(x_{l+1}, \ldots, x_n) x_1^{i_1} \cdots x_l^{i_l} \\ &\qquad - m \sum_{0 \leq |i| \leq s} a_{i_1 \cdots i_l}(x_{l+1}, \ldots, x_n) x_1^{i_1} \cdots x_l^{i_l} \\ &= \sum_{0 \leq |i| \leq s} (|i| - m) a_{i_1 \cdots i_l}(x_{l+1}, \ldots, x_n) x_1^{i_1} \cdots x_l^{i_l} \\ &= 0. \end{aligned} \tag{3.22}$$

Thus, every term $(|i| - m) a_{i_1 \cdots i_l}(x_{l+1}, \ldots, x_n)$ must be 0. If $\zeta \neq 0$, s must be equal to m while $a_{i_1 \cdots i_l}(x_{l+1}, \ldots, x_n) = 0$ for $|i| < m$. ∎

LEMMA 3.12. *Let m be a constant integer and $\zeta \in C^\infty(R^n)$ such that $E_l(\zeta) + m\zeta$ is a polynomial of degree k (≥ 0) in $x_1, \ldots, x_l$ variables with coefficients in C^∞-functions of $x_{l+1}, \ldots, x_n$ variables. If ζ is a polynomial of degree s in $x_1, \ldots, x_l$ variables with coefficients in C^∞-functions of $x_{l+1}, \ldots, x_n$, then $s \geq k$.*

Proof. Let $\zeta = \sum_{0 \leq |i| \leq s} a_{i_1 \cdots i_l}(x_{l+1}, \ldots, x_n) x_1^{i_1} \cdots x_l^{i_l}$. By repeating the same process in (3.22),

$$E_l(\zeta) + m\zeta = \sum_{0 \leq |i| \leq s} (|i| + m) a_{i_1 \cdots i_l}(x_{l+1}, \ldots, x_n) x_1^{i_1} \cdots x_l^{i_l}. \tag{3.23}$$

For $E_l(\zeta) + m\zeta$ to be a polynomial of degree k in $x_1, \ldots, x_l$ variables with coefficients in C^∞-functions of $x_{l+1}, \ldots, x_n$ variables, s must be no less than k. ∎

THEOREM 3.13. *Let m be a constant integer and $\zeta \in C^\infty(R^n)$ such that $E_l(\zeta) + m\zeta$ is a polynomial of degree k (≥ 0) in $x_1, \ldots, x_l$ variables with coefficients in C^∞-functions of $x_{l+1}, \ldots, x_n$ variables.*

If $m + k + 1 > 0$, ζ is a polynomial of degree k in $x_1, \ldots, x_l$ variables with coefficients in C^∞-functions of $x_{l+1}, \ldots, x_n$.

If $m + k + 1 \leq 0$, ζ is a polynomial of degree k or degree $-m$ $(\geq k + 1 > 0)$ in $x_1, \ldots, x_l$ variables with coefficients in C^∞-functions of $x_{l+1}, \ldots, x_n$.

Proof. Let $D = (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_l})^{\alpha_l}$ with $\sum_{i=1}^{l} \alpha_i = k + 1 \geq 1$. Since $E_l(\zeta) + m\zeta$ is a polynomial of degree k in $x_1, \ldots, x_l$ variables with coefficients in C^∞-functions of $x_{l+1}, \ldots, x_n$ variables,

$$D(E_l(\zeta) + m\zeta) = D(E_l(\zeta)) + mD\zeta = 0. \tag{3.24}$$

On the other hand, by (3.21),

$$D(E_l(\zeta)) = E_l(D\zeta) + (k + 1)D\zeta. \tag{3.25}$$

By (3.24) and (3.25),

$$E_l(D\zeta) + (m + k + 1)D\zeta = 0. \tag{3.26}$$

Now, there are three cases:

(i) $m + k + 1 > 0$. By Lemma 3.9, $D\zeta = 0$. This means that ζ is a polynomial of degree $\leq k$ in $x_1, \ldots, x_l$ variables with coefficients in C^∞-functions of $x_{l+1}, \ldots, x_n$ variables. By Lemma 3.12, ζ is a polynomial of degree k in $x_1, \ldots, x_l$ variables with coefficients in C^∞-functions of $x_{l+1}, \ldots, x_n$ variables.

(ii) $m + k + 1 = 0$. By Lemma 3.10, $D\zeta$ is 0 or a nonzero C^∞-function in $x_{l+1}, \ldots, x_n$. Thus, ζ is a polynomial of degree $\leq k + 1$ in $x_1, \ldots, x_l$ variables with coefficients in C^∞-functions of $x_{l+1}, \ldots, x_n$. By Lemma 3.12, ζ is a polynomial of degree k or $k + 1$ $(= -m)$ in $x_1, \ldots, x_l$ variables with coefficients in C^∞-functions of $x_{l+1}, \ldots, x_n$ variables.

(iii) $m + k + 1 < 0$ $(m < -k - 1 < 0)$. Let $m' = -(m + k + 1) > 0$. Then (3.26) becomes $E_l(D\zeta) - m'D\zeta = 0$. By Lemma 3.11, $D\zeta$ is either 0 or a degree m' homogeneous polynomial in $x_1, \ldots, x_l$ variables with coefficients in C^∞-functions of $x_{l+1}, \ldots, x_n$ variables. If it is the latter case, ζ is a polynomial of degree $(k+1)+m' = -m > 0$ in $x_1, \ldots, x_l$ variables with coefficients in C^∞-functions of $x_{l+1}, \ldots, x_n$. If $D\zeta = 0$, ζ is a polynomial of degree at most k in $x_1, \ldots, x_l$ variables with coefficients in C^∞-functions of $x_{l+1}, \ldots, x_n$ variables. By combining these two situations with Lemma 3.12, ζ is a polynomial of degree k or degree $-m$ in $x_1, \ldots, x_l$ variables with coefficients in C^∞-functions of $x_{l+1}, \ldots, x_n$ variables since $-m > k$. ∎

THEOREM 3.14. *Let m be a constant integer and $\zeta \in C^\infty(R^n)$ such that $E_l(\zeta) + m\zeta = \varphi$, where φ is a C^∞-function in $x_{l+1}, \ldots, x_n$ variables.*

If $m > 0$, then $\zeta = \frac{1}{m}\varphi$.

If $m = 0$, then $\varphi = 0 \Longrightarrow \zeta$ is a C^∞-function in $x_{l+1}, \ldots, x_n$ variables.

If $m \leq -1$, then $\zeta = \xi + \frac{1}{m}\varphi$, where ξ is either 0 or a degree $-m$ homogeneous polynomial in $x_1, \ldots, x_l$ variables with coefficients in C^∞-functions of $x_{l+1}, \ldots, x_n$ variables.

Proof. (i) If $m + 1 > 0$, by Theorem 3.13, ζ is a C^∞-function in $x_{l+1}, \ldots, x_n$ variables $\Longrightarrow E_l(\zeta) = 0 \Longrightarrow m\zeta = \varphi \Longrightarrow \zeta = \frac{1}{m}\varphi$ if $m \neq 0$; $\varphi = 0$ if $m = 0$.

(ii) $m \leq -1$. Let $\xi = \zeta - \frac{1}{m}\varphi$. Then $E_l(\xi) + m\xi = 0$. By Lemma 3.11, ξ is either 0 or a degree $-m$ homogeneous polynomial in $x_1, \ldots, x_l$ variables with coefficients in C^∞-functions of $x_{l+1}, \ldots, x_n$ variables. ∎

THEOREM 3.15. *Let m be a constant integer and $\zeta \in C^\infty(R^n)$ such that $E_l(\zeta) + m\zeta \in P_k(x_1, \ldots, x_n)$, a polynomial of degree $k \geq 0$ in $x_1, \ldots, x_n$ variables.*

If $m > 0$, $\zeta \in P_k(x_1, \ldots, x_n)$.

If $m = 0$, $\zeta \in P_k(x_1, \ldots, x_n) + a(x_{l+1}, \ldots, x_n)$, where $a(x_{l+1}, \ldots, x_n)$ is a C^∞-function in $x_{l+1}, \ldots, x_n$ variables.

Proof. By Theorem 3.13, $\zeta = \sum_{0\leq|i|\leq k} a_{i_1\cdots i_l}(x_{l+1}, \ldots, x_n) x_1^{i_1} \cdots x_l^{i_l}$ for $m \geq 0$. By (3.23)

$$E_l(\zeta) + m\zeta = \sum_{0\leq|i|\leq k} (|i| + m) a_{i_1\cdots i_l}(x_{l+1}, \ldots, x_n) x_1^{i_1} \cdots x_l^{i_l}. \tag{3.27}$$

If $m > 0$, $|i| + m > 0 \Longrightarrow a_{i_1\cdots i_l}(x_{l+1}, \ldots, x_n)$ must be a polynomial of degree less than or equal to $k - \sum_{j=1}^{l} i_j$ in $x_{l+1}, \ldots, x_n$ variables since $x_1^{i_1} \cdots x_l^{i_l}$'s are linearly independent.

If $m = 0$, only when $|i| = 0$, i.e., when $i_1 = \cdots = i_l = 0$, $a_{i_1\cdots i_l}(x_{l+1}, \ldots, x_n)$ may not be a polynomial. □

3.4. Underdetermined partial differential equation. From Theorems 2.5 and 2.7, for the finite-dimensional estimation algebras of maximal rank, $F = \eta - \sum_{i=1}^{m} h_i^2$ must be a polynomial of degree at most 2. Notice that the filtering system (2.1) is completely parameterized by the pair, (f, h). It follows by (2.5) that the underdetermined partial differential equation

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = F \tag{3.28}$$

provides a complete characterization of the realization set of such systems. Therefore, it is of primary interest to investigate the solution and solution properties for this class of equations.

In (3.28), $f_1, \ldots, f_n$ and F are C^∞-functions on R^n. F is given and $f_1, \ldots, f_n$ are treated as unknown. Although there is only one equation with n unknowns, (3.28) may not have solutions. The following important result can be found in [24].

THEOREM 3.16 (see [24]). *Let $F(x_1, \ldots, x_n)$ be a C^∞-function on R^n. Suppose that there exists a path $c : R \to R^n$ and $\delta > 0$ such that $\lim_{t\to\infty} \|c(t)\| = \infty$ and $\lim_{t\to\infty} \sup_{B_\delta(c(t))} F = -\infty$, where $B_\delta(c(t)) = \{x \in R^n : \|x - c(t)\| < \delta\}$. Then there are no C^∞-functions $f_1, f_2, \ldots, f_n$ on R^n satisfying* (3.28).

The following results are applications of Theorem 3.16 that will be used in the classification problems.

THEOREM 3.17. *Let $F(x_1, \ldots, x_n)$ be a polynomial on R^n, let $c(t) = (c_1(t), \ldots, c_n(t))$ be a polynomial path (i.e., $c_i(t)$'s are polynomials in t), and let $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$. Then $F(c(t) + \epsilon)$ is a polynomial in t, i.e., $F(c(t) + \epsilon) = \sum_{j=0}^{d} a_j(\epsilon) t^j$, where $a_j(\epsilon)$'s are polynomials of $\epsilon_1, \ldots, \epsilon_n$. If there exists a path $c(t)$ such that the coefficient $a_d(\epsilon)$ of the leading term of $F(c(t) + \epsilon)$ is a negative constant and $d \geq 1$, there are no C^∞-functions $f_1, f_2, \ldots, f_n$ on R^n satisfying the equation*

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = F. \tag{3.29}$$

Proof. Suppose $a_d(\epsilon) = a_d$ is a negative constant with $d \geq 1$. It is clear that $\lim_{t\to\infty} \|c(t)\| = \infty$.

For $0 \leq j \leq d-1$, $a_j(\epsilon)$ is a polynomial in $\epsilon_1, \ldots, \epsilon_n$. By continuity, there exists a $\delta > 0$ and a ball $B_\delta(0) = \{\epsilon \in R^n \ : \ \|\epsilon\| < \delta\}$, such that for any $\epsilon \in B_\delta(0)$, the following bounds hold:

$$|a_j(\epsilon) - a_j(0)| \leq 1 \qquad \forall 0 \leq j \leq d-1. \tag{3.30}$$

It follows that for $t > 0$,

$$\sup_{B_\delta(c(t))} F(x_1, \ldots, x_n) = \sup_{\epsilon \in B_\delta(0)} F(c(t)+\epsilon) = \sup_{\epsilon \in B_\delta(0)} a_d t^d + \sum_{j=0}^{d-1} a_j(\epsilon) t^j$$

$$\leq a_d t^d + \sum_{j=0}^{d-1} (1 + a_j(0)) t^j \longrightarrow -\infty \quad \text{ as } t \to \infty,$$

since $a_d < 0$ and $d \geq 1$. The assertion follows immediately by Theorem 3.16. ∎

THEOREM 3.18. *Let $F(x_1, \ldots, x_n)$ be a degree $d \geq 1$ polynomial on R^n. The homogeneous degree d part of F is denoted by $F_d = \sum_{|i|=d} a_i x_1^{i_1} \cdots x_n^{i_n}$, where $i = (i_1, \ldots, i_n)$. If there exist n numbers $b_1, \ldots, b_n$ such that $F_d(b_1, \ldots, b_n) < 0$, there are no C^∞-functions $f_1, f_2, \ldots, f_n$ on R^n satisfying* (3.29).

Proof. By taking the polynomial path $c(t) = (b_1 t, \ldots, b_n t)$ one has that $F(c(t)+\epsilon)$ is a degree d polynomial in t and the coefficient of t^d in $F(c(t)+\epsilon)$ is a negative constant $F_d(b_1, \ldots, b_n) < 0$. The proof is concluded by using Theorem 3.17. ∎

COROLLARY 3.19. *Let $F(x_1, \ldots, x_n)$ be a polynomial on R^n. If the degree of F is odd, there are no C^∞-functions $f_1, f_2, \ldots, f_n$ on R^n satisfying* (3.29).

COROLLARY 3.20. *Let $F(x_1, \ldots, x_n)$ be a degree $d \geq 1$ polynomial on R^n. If there exist C^∞-functions $f_1, f_2, \ldots, f_n$ on R^n satisfying* (3.29)*, the homogeneous degree d part of F is a nonnegative function.*

In fact, the above theorems can be further generalized to the case when F is a polynomial in $x_1, \ldots, x_r$ variables with C^∞ coefficient functions in $x_{r+1}, \ldots, x_n$. For example, we have the following.

THEOREM 3.21. *Let d and $r \leq n$ be two positive integers and*

$$F(x_1, \ldots, x_n) = \sum_{|i| \leq d} a_i(x_{r+1}, \ldots, x_n) x_1^{i_1} \cdots x_r^{i_r},$$

where $i = (i_1, \ldots, i_r)$, and where a_i's are C^∞-functions in $x_{r+1}, \ldots, x_n$ variables. The homogeneous degree d part in $x_1, \ldots, x_r$ variables of F is denoted by $F_d = \sum_{|i|=d} a_i(x_{r+1}, \ldots, x_n) x_1^{i_1} \cdots x_r^{i_r}$. If there exist n numbers $b_1, \ldots, b_n$ such that $F_d(b_1, \ldots, b_n) < 0$, there are no C^∞-functions $f_1, f_2, \ldots, f_n$ on R^n satisfying (3.29).

Proof. Take the path $c(t) = (b_1 t, \ldots, b_r t, b_{r+1}, \ldots, b_n)$. Clearly $\lim_{t\to\infty} \|c(t)\| \to \infty$ since $F_d(b_1, \ldots, b_n) < 0$.

Let $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$. Then $F(c(t)+\epsilon)$ is a polynomial in t,

$$F(c(t)+\epsilon) = F_d(b_1, \ldots, b_r, b_{r+1}+\epsilon, \ldots, b_n+\epsilon) t^d + \sum_{j=0}^{d-1} c_j(\epsilon) t^j.$$

By continuity, there exist a $\delta > 0$ and a ball $B_\delta(0) = \{\epsilon \in R^n \ : \ \|\epsilon\| < \delta\}$, such that for any $\epsilon \in B_\delta(0)$, the following bounds hold:

$$F_d(b_1,\dots,b_r,b_{r+1}+\epsilon,\dots,b_n+\epsilon)\le\frac{1}{2}F_d(b_1,\dots,b_n)<0,$$
$$|c_j(\epsilon)-c_j(0)|\le 1\qquad\forall 0\le j\le d-1.$$

It follows that for $t>0$,

$$\begin{aligned}\sup_{B_\delta(c(t))}F(x_1,\dots,x_n)&=\sup_{\epsilon\in B_\delta(0)}F(c(t)+\epsilon)\\&=\sup_{\epsilon\in B_\delta(0)}F_d(b_1,\dots,b_r,b_{r+1}+\epsilon,\dots,b_n+\epsilon)t^d+\sum_{j=0}^{d-1}c_j(\epsilon)t^j\\&\le\frac{1}{2}F_d(b_1,\dots,b_n)t^d+\sum_{j=0}^{d-1}(1+c_j(0))t^j\longrightarrow-\infty,\qquad\text{as } t\to\infty,\end{aligned}$$

since $F_d(b_1,\dots,b_n)<0$ and $d\ge 1$. The assertion follows immediately by Theorem 3.16. ∎

4. Classification of estimation algebras $n=2$. In this section, the state dimension is assumed to be $n=2$, i.e., there are two state variables x_1 and x_2. The estimation algebra $E=\langle L_0,h_1,\dots,h_m\rangle_{L.A.}$, where

$$L_0=\frac{1}{2}(D_1^2+D_2^2-\eta),\qquad D_i=\frac{\partial}{\partial x_i}-f_i,\quad i=1,2,\tag{4.1}$$

$$\eta=\sum_{i=1}^{2}\frac{\partial f_i}{\partial x_i}+\sum_{i=1}^{2}f_i^2+\sum_{i=1}^{m}h_i^2.\tag{4.2}$$

m is assumed to be a positive integer; otherwise $E=\langle L_0\rangle_{L.A.}$ is a one-dimensional estimation algebra. The Ω-matrix is a 2×2 antisymmetric matrix. Therefore, only $\omega_{12}=-\omega_{21}$ is unknown.

By Theorem 3.3, 1 is assumed to be an element of E in the discussion.

4.1. Linear structure of the Ω-matrix.

THEOREM 4.1. *Suppose* $\dim E<\infty$ *and* $Y=p(x)D_2,\ \mathrm{mod}\ U_0\in E$. *Then p is a polynomial in x_1,x_2 of degree at most* 1.

Proof. By Theorem 3.2, p is a polynomial in x_1,x_2. Let $l=deg\ p$ and $p^{(l)}$ be the homogeneous degree l part of p. Then

$$p^{(l)}=\sum_{i=0}^{l}a_ix_1^{l-i}x_2^i=\sum_{i=s}^{t}a_ix_1^{l-i}x_2^i,$$

where a_i's are constants, $0\le s\le t\le l$, $a_s\ne 0$, $a_t\ne 0$, and $a_i=0$ for $i<s$ or $i>t$. Let $Y_k=Ad_{L_0}^kY$ for $k\ge 0$. By induction,

$$Y_k=Ad_{L_0}^kY=\sum_{j=0}^{k}\binom{k}{j}\frac{\partial^k p}{\partial x_1^{k-j}\partial x_2^j}D_1^{k-j}D_2^{j+1},\quad\mathrm{mod}\ U_k,$$

where $\binom{k}{j}$'s are binomial numbers. In particular,

$$Y_l=\sum_{j=0}^{l}\binom{l}{j}(l-j)!j!a_jD_1^{l-j}D_2^{j+1}=l!\sum_{j=s}^{t}a_jD_1^{l-j}D_2^{j+1},\quad\mathrm{mod}\ U_l,$$

and

$$Y_{l-1} = \sum_{j=0}^{l-1} \binom{l-1}{j} \frac{\partial^{l-1} p}{\partial x_1^{l-j-1} \partial x_2^j} D_1^{l-j-1} D_2^{j+1}$$

$$= \sum_{j=0}^{l-1} (l-1)! \Big((l-j) a_j x_1 + (j+1) a_{j+1} x_2 + c_j \Big) D_1^{l-j-1} D_2^{j+1}, \quad \mod U_{l-1},$$

where c_j's are constants from the $(l-1)$th partial derivatives of the homogeneous degree $l-1$ part of p.

Now, depending on whether or not $s = 0$, and in the case when $s = 0$ whether a_1, a_2 are 0, there are four cases for which similar constructions of sequences with different calculations will show that l must be less than 2 if E is finite-dimensional.

(i) *Case* 1. $s \neq 0$.

$$\begin{aligned}
A_0 &:= Y_{l-1} = (d_0 x_2 + (l-1)! c_{s-1}) D_1^{l-s} D_2^s \\
&\qquad\qquad + \text{terms with lower order in } D_1, \quad \mod U_{l-1}, \\
A_1 &:= Y_l = d_1 D_1^{l-s} D_2^{s+1} \ + \text{terms with lower order in } D_1, \quad \mod U_l, \\
A_2 &:= [A_1, A_0] = (s+1) d_1 d_0 D_1^{2(l-s)} D_2^{2s} \\
&\qquad\qquad + \text{terms with lower order in } D_1, \quad \mod U_{2l-1}, \\
&\ \vdots \\
A_{r+1} &:= [A_r, A_0] = d_{r+1} D_1^{(r+1)(l-s)} D_2^{(r+1)s-r+1} \\
&\qquad\qquad + \text{terms with lower order in } D_1, \quad \mod U_{(r+1)l-r},
\end{aligned}$$

where $d_0 = (l-1)! s a_s \neq 0$, $d_1 = l! a_s \neq 0$, and $d_{r+1} = (rs - r + 2) d_r d_0 \neq 0$ for $r \geq 1$. The orders of A_{r+1}'s are $(r+1)l - r + 1 \to \infty$ unless $l < 2$.

(ii) *Case* 2. $s = 0$ and $a_1 \neq 0$.

$$\begin{aligned}
A_0 &:= Y_{l-1} = (l! a_0 x_1 + d_0 x_2 + (l-1)! c_0) D_1^{l-1} D_2 \\
&\qquad\qquad + \text{terms with lower order in } D_1, \quad \mod U_{l-1}, \\
A_1 &:= Y_l = d_1 D_1^l D_2 \ + \text{terms with lower order in } D_1, \quad \mod U_l, \\
A_2 &:= [A_1, A_0] = d_1 d_0 D_1^{2l-1} D_2 \\
&\qquad\qquad + \text{terms with lower order in } D_1, \quad \mod U_{2l-1}, \\
&\ \vdots \\
A_{r+1} &:= [A_r, A_0] = d_{r+1} D_1^{(r+1)l-r} D_2 \\
&\qquad\qquad + \text{terms with lower order in } D_1, \quad \mod U_{(r+1)l-r},
\end{aligned}$$

where $d_0 = (l-1)! a_1 \neq 0$, $d_1 = l! a_0 \neq 0$, and $d_{r+1} = d_r d_0 \neq 0$ for $r \geq 1$. The orders of A_{r+1}'s are $(r+1)l - r + 1 \to \infty$ unless $l < 2$.

(iii) *Case* 3. $s = 0$ and $a_1 = a_2 = 0$.

$$\begin{aligned}
A_0 &:= Y_{l-1} = (d_0 x_1 + (l-1)! c_0) D_1^{l-1} D_2 + (l-1)! c_2 D_1^{l-2} D_2^2 \\
&\qquad + \text{degree 1 coeff. terms with lower order in } D_1, \quad \text{mod } U_{l-1}, \\
A_1 &:= Y_l = d_1 D_1^l D_2 \\
&\qquad + \text{constant coeff. terms with lower order in } D_1, \quad \text{mod } U_l, \\
A_2 &:= [A_1, A_0] = l d_1 d_0 D_1^{2l-2} D_2^2 \\
&\qquad + \text{constant coeff. terms with lower order in } D_1, \quad \text{mod } U_{2l-1}, \\
&\ \vdots \\
A_{r+1} &:= [A_r, A_0] = d_{r+1} D_1^{(r+1)l-2r} D_2^{r+1} \\
&\qquad + \text{constant coeff. terms with lower order in } D_1, \quad \text{mod } U_{(r+1)l-r},
\end{aligned}$$

where $d_0 = l! a_0 \neq 0$, $d_1 = l! a_0 \neq 0$, and $d_{r+1} = (rl - 2r + 2) d_r d_0 \neq 0$ for $r \geq 1$ and $l \geq 2$. The orders of A_{r+1}'s are $(r+1)l - r + 1 \to \infty$ unless $l < 2$.

(iv) *Case* 4. $s = 0$, $a_1 = 0$, but $a_2 \neq 0$.

$$Y_l = d_1 D_1^l D_2 + d_3 D_1^{l-2} D_2^3 \ + \text{terms with lower order in } D_1, \quad \text{mod } U_l,$$

where $d_1 = l! a_0 \neq 0$ and $d_3 = l! a_2 \neq 0$. If $l \geq 2$, consider $Z = [Y_l, pD_2]$ and $A_0 = Ad_{L_0}^{l-2} Z$:

$$\begin{aligned}
Z = [Y_l, pD_2] &= d_1 \frac{\partial p}{\partial x_2} D_1^l D_2 + l d_1 \frac{\partial p}{\partial x_1} D_1^{l-1} D_2^2 \\
&\qquad + \text{terms with lower order in } D_1, \quad \text{mod } U_l, \\
A_0 &= Ad_{L_0}^{l-2} Z \\
&= d_1 \frac{\partial^{l-1} p}{\partial x_1^{l-2} \partial x_2} D_1^{2l-2} D_2 \ + \text{terms with lower order in } D_1, \quad \text{mod } U_{2l-2}.
\end{aligned}$$

Since $p^{(l)} = a_0 x_1^l + a_2 x_1^{l-2} x_2^2$ + terms with lower degree in x_1, $\frac{\partial^{l-1} p}{\partial x_1^{l-2} \partial x_2} = 2(l-2)! a_2 x_2 + c_2$, where c_2 is $(l-2)!$ multiplied by the coefficient of $x_1^{l-2} x_2$ in p. Hence,

$$\begin{aligned}
A_0 &= (e_0 x_2 + d_1 c_2) D_1^{2l-2} D_2 \ + \ \text{terms with lower order in } D_1, \quad \text{mod } U_{2l-2}, \\
A_1 &:= Y_l = d_1 D_1^l D_2 + d_3 D_1^{l-2} D_2^3 \\
&\qquad + \text{terms with lower order in } D_1, \quad \text{mod } U_l, \\
A_2 &:= [A_1, A_0] = d_1 e_0 D_1^{3l-2} D_2 \\
&\qquad + \text{terms with lower order in } D_1, \quad \text{mod } U_{3l-2}, \\
&\ \vdots \\
A_{r+1} &:= [A_r, A_0] = e_{r+1} D_1^{(2r+1)l-2r} D_2 \\
&\qquad + \text{terms with lower order in } D_1, \quad \text{mod } U_{(2r+1)l-2r},
\end{aligned}$$

where $e_0 = 2(l-2)! a_2 d_1 \neq 0$, $e_1 = d_1 \neq 0$, and $e_{r+1} = e_r e_0 \neq 0$. The orders of A_{r+1}'s are $(2r+1)l - 2r + 1 \to \infty$ unless $l < 2$. ∎

THEOREM 4.2. *Suppose* dim $E < \infty$. $x_1 + c \in E \Rightarrow \omega_{12}$ *is a polynomial of degree at most* 1.

Proof. Clearly, $L_0, x_1, D_1 \in E$.

$$H_0 := [L_0, D_1] = \omega_{12} D_2 + \frac{1}{2}\frac{\partial \omega_{12}}{\partial x_2} + \frac{1}{2}\frac{\partial \eta}{\partial x_1} \in E. \tag{4.3}$$

By Theorem 4.1, ω_{12} is a polynomial of degree at most 1. ◻

THEOREM 4.3. *Suppose* dim $E < \infty$. $x_1^2 + c \in E \Rightarrow \omega_{21}$ *is a constant.*

Proof.

$$K_0 = \left[L_0, \frac{1}{2}(x_1^2 + c)\right] - \frac{1}{2} = x_1 D_1 \in E,$$

$$K_1 = [L_0, K_0] = D_1^2 - \alpha_2 D_2 + \frac{1}{2}E_1(\eta) - \frac{1}{2}\frac{\partial \alpha_2}{\partial x_2} \in E,$$

$$K_2 = [K_1, K_0] = 2D_1^2 + E_1(\alpha_2)D_2 + \alpha_2^2 - \frac{1}{2}E_1^2(\eta) + \frac{1}{2}E_1\left(\frac{\partial \alpha_2}{\partial x_2}\right) \in E,$$

$$Z_0 = K_2 - 2K_1 = (E_1(\alpha_2) + 2\alpha_2)D_2 + \gamma(x) \in E, \tag{4.4}$$

where $\alpha_2 = x_1\omega_{21}$, $E_1(\cdot) = x_1\frac{\partial}{\partial x_1}$, and $\gamma(x) = \alpha_2^2 - E_1(\eta) - \frac{1}{2}E_1^2(\eta) + \frac{\partial \alpha_2}{\partial x_2} + \frac{1}{2}E_1(\frac{\partial \alpha_2}{\partial x_2})$.

By Theorem 4.1, $E_1(\alpha_2) + 2\alpha_2$ is a polynomial of at most 1, and so is $\alpha_2 = x_1\omega_{21}$ by Theorem 3.15. Since $\omega_{21} \in C^\infty(R^2)$, ω_{21} must be a constant. ◻

LEMMA 4.4. *Suppose* $K_0 = x_1 D_1 + x_2 D_2 \in E$ *and* $Y = \sum_{i=0}^k b_i(x) D_1^{k-i} D_2^i$, mod $U_{k-1} \in E$. *Let* $b_i^{(r)}$ *denote the homogeneous degree* r *part of* b_i *for* $0 \le i \le k$. *Then* $\sum_{i=0}^k b_i^{(r)} D_1^{k-i} D_2^i$, mod $U_{k-1} \in E$, *for* $r \ge 0$.

Proof. Let $E(\cdot) = \sum_{i=1}^2 x_i \frac{\partial}{\partial x_i}$.

$$\left[K_0, \sum_{i=0}^k b_i D_1^{k-i} D_2^i\right] = \sum_{i=0}^k (E(b_i) - k b_i) D_1^{k-i} D_2^i, \qquad \text{mod } U_{k-1}. \tag{4.5}$$

Let $l = \max_{0 \le i \le k} deg\ b_i$. Use (4.5) to construct a sequence of elements

$$Z_r = \sum_{j=0}^{l-r} c_{rj} \sum_{i=0}^k b_i^{(j)} D_1^{k-i} D_2^i \in E, \qquad \text{mod } U_{k-1}, \tag{4.6}$$

in E as follows:

(1) $Z_0 = Y$, i.e., $c_{0j} = 1, 0 \le j \le l$;

(2) $Z_{r+1} = (l - r - k)Z_r - [K_0, Z_r] \Longrightarrow c_{(r+1)j} = (l - r - j)c_{rj}, 0 \le j \le l - r - 1$.

Note that in (4.6) $c_{rj} \ne 0$ for $r \le l$. Starting from $Z_l = c_{l0} \sum_{i=0}^k b_i^{(0)} D_1^{k-i} D_2^i$, mod U_{k-1}, $Z_{l-1} = c_{(l-1)0} \sum_{i=0}^k b_i^{(0)} D_1^{k-i} D_2^i + c_{(l-1)1} \sum_{i=0}^k b_i^{(1)} D_1^{k-i} D_2^i$, mod $U_{k-1}, \dots,$ $Z_r, \dots,$ one can solve $\sum_{i=0}^k b_i^{(r)} D_1^{k-i} D_2^i$ successively. This means

$$\sum_{i=0}^k b_i^{(r)} D_1^{k-i} D_2^i, \ \text{ mod } U_{k-1} \in E. \qquad ◻$$

LEMMA 4.5. *Suppose* dim $E < \infty$. *Let* $Y = \sum_{i=i_0}^{i_k} b_i(x) D_1^{k-i} D_2^i$ mod $U_{k-1} \in E$ *be a differential operator with the highest order* k *in* E *(obviously* $k \ge 2$*), where* $0 \le i_0 \le i_k \le k$. *Then*

$$\frac{\partial b_{i_0}}{\partial x_1} = \frac{\partial b_{i_k}}{\partial x_2} = 0, \qquad \frac{\partial b_i}{\partial x_1} + \frac{\partial b_{i-1}}{\partial x_2} = 0, \quad i_0 + 1 \le i \le i_k. \tag{4.7}$$

Proof.

$$[L_0, Y] = \sum_{i=i_0}^{i_k} \frac{\partial b_i}{\partial x_1} D_1^{k+1-i} D_2^i + \sum_{i=i_0}^{i_k} \frac{\partial b_i}{\partial x_2} D_1^{k-i} D_2^{i+1}$$
$$= \frac{\partial b_{i_0}}{\partial x_1} D_1^{k+1} + \sum_{i=i_0+1}^{i_k} \left(\frac{\partial b_i}{\partial x_1} + \frac{\partial b_{i-1}}{\partial x_2} \right) D_1^{k+1-i} D_2^i + \frac{\partial b_{i_k}}{\partial x_2} D_2^{k+1}, \text{ mod } U_k. \quad (4.8)$$

Since E's elements have the highest differential order k, all the coefficient functions of the order $k+1$ terms in the above equation must be 0. ∎

LEMMA 4.6. *Suppose* $\dim E < \infty$ *and* $Y = p_1 D_1 + p_2 D_2$, mod $U_0 \in E$, *where* $p_1 = \sum_{j=1}^{l} a_j x_1^{l-j} x_2^j$ *and* $p_2 = -\sum_{j=1}^{l} a_j x_1^{l+1-j} x_2^{j-1}$, a_j*'s are constants and* $a_j \neq 0$ *for some* j. *Then* $l \leq 2$.

Proof. Since $\dim E < \infty$, there exists an element $P_0 = \sum_{i=i_0}^{i_k} b_i(x) D_1^{k-i} D_2^i$, mod $U_{k-1} \in E$, that has the highest differential order k, where $0 \leq i_0 \leq i_k \leq k$, $b_{i_0} \neq 0$, and $b_{i_k} \neq 0$. By Lemma 4.4, b_i's can be assumed to be homogeneous polynomials. More precisely, P_0 can be assumed to have the following properties:

(C1) $P_0 = \sum_{i=i_0}^{i_k} b_i(x) D_1^{k-i} D_2^i$, mod $U_{k-1} \in E$, $0 \leq i_0 \leq i_k \leq k$, $b_{i_0} \neq 0$, and $b_{i_k} \neq 0$.

(C2) P_0 has the highest differential order in E, i.e., if $A \in E$, then *ord* $A \leq k$.

(C3) $\frac{\partial b_{i_0}}{\partial x_1} = \frac{\partial b_{i_k}}{\partial x_2} = 0$, $\frac{\partial b_i}{\partial x_1} + \frac{\partial b_{i-1}}{\partial x_2} = 0$, $i_0 + 1 \leq i \leq i_k$.

(C4) b_i's are homogeneous degree r polynomials.

Moreover, among the elements in E satisfying (C1) and (C2), P_0 can be chosen such that b_i's have the highest degree:

(C5) Among all elements in E with order k, b_i's have the highest degree.

Assume $l \geq 3$.

Assume either $i_0 > 0$ or $i_k < k$. Without loss of generality, assume $i_0 > 0$.

$$[P_0, Y] = \left[P_0, \sum_{i=1}^{2} p_i D_i \right] = \left[b_{i_0} D_1^{k-i_0} D_2^{i_0} + \cdots \sum_{i=1}^{2} p_i D_i \right]$$
$$= i_0 b_{i_0} \frac{\partial p_1}{\partial x_2} D_1^{k+1-i_0} D_2^{i_0-1}$$
$$+ \text{ terms with lower order in } D_1, \qquad \text{mod } U_{k-1}.$$

Since $\frac{\partial p_1}{\partial x_2} \neq 0$ is a degree $l-1 \geq 2$ polynomial, while b_{i_0} is a polynomial independent of x_1 by (C3), $i_0 b_{i_0} \frac{\partial p_1}{\partial x_2}$ is a polynomial of degree higher than b_{i_0}. This contradicts with (C5). Thus,

(C6) $i_0 = 0$ and $i_k = k$, i.e., $P_0 = \sum_{i=0}^{k} b_i(x) D_1^{k-i} D_2^i$, mod U_{k-1}, where $b_0 \neq 0$ and $b_k \neq 0$.

Again, consider $[P_0, Y]$,

$$[P_0, Y] = \left[P_0, \sum_{i=1}^{2} p_i D_i \right] = \left[b_0 D_1^k + b_1 D_1^{k-1} D_2 + \cdots \sum_{i=1}^{2} p_i D_i \right]$$
$$= \left(k b_0 \frac{\partial p_1}{\partial x_1} - p_2 \frac{\partial b_0}{\partial x_2} + b_1 \frac{\partial p_1}{\partial x_2} \right) D_1^k$$
$$+ \text{ terms with lower order in } D_1, \qquad \text{mod } U_{k-1}. \quad (4.9)$$

By (C3), (C4), and (C6), one can assume that

$$b_0 = x_2^r, \quad \text{where } r \geq 0, \tag{4.10}$$

$$b_1 = -rx_1x_2^{r-1} + cx_2^r, \quad \text{where } c \text{ is an unknown constant.} \tag{4.11}$$

Using (4.10) and (4.11), we obtain the coefficient function of the D_1^k term in $[P_0, \sum_{i=1}^2 p_iD_i]$ given in (4.9):

$$\begin{aligned} &kb_0\frac{\partial p_1}{\partial x_1} - p_2\frac{\partial b_0}{\partial x_2} + b_1\frac{\partial p_1}{\partial x_2} \\ &= (ca_1 - ra_2)x_1^{l-1}x_2^r + \sum_{j=2}^{l-1}(-jra_{j+1} + jca_j + k(l-j+1)a_{j-1})x_1^{l-j}x_2^{r+j-1} \\ &\qquad + (lca_l + ka_{l-1})x_2^{l+r-1}. \end{aligned} \tag{4.12}$$

Equation (4.12) is a degree $l+r-1 > r$ homogeneous polynomial. To meet condition (C5), every term in (4.12) must be 0. That is, $a_1, \ldots, a_l$ must satisfy

$$\begin{aligned} &ca_1 - ra_2 = 0, \qquad lca_l + ka_{l-1} = 0, \\ &-jra_{j+1} + jca_j + k(l-j+1)a_{j-1} = 0 \quad \text{for } 2 \leq j \leq l-1. \end{aligned} \tag{4.13}$$

The above relations can be summarized as a matrix equation:

$$\begin{pmatrix} c & -r & & & & \\ k(l-1) & 2c & -2r & & & \\ & k(l-2) & 3c & -3r & & \\ & & \ddots & \ddots & \ddots & \\ & & & 2k & (l-1)c & -(l-1)r \\ & & & & k & lc \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{l-1} \\ a_l \end{pmatrix} = 0. \tag{4.14}$$

Let A denote the $l \times l$ tridiagonal matrix in the above equation. Since all the $A(i, i+1)$ entries are nonpositive while all $A(i, i-1)$ entries are positive, it is easy by induction that $c = 0$ is a necessary condition for the determinant of A to be 0. More specifically, let B_i denote an $i \times i$ matrix having the entries $B_i(s,t) = A(s,t)$ for $1 \leq s, t \leq i$, i.e., B_i is the upper left block of A with size $i \times i$. The determinants of B_i's have the following recursion:

$$|B_1| = c, \qquad |B_2| = 2c^2 + d_1, \tag{4.15}$$

$$|B_{i+1}| = (i+1)c|B_i| + d_i|B_{i-1}|, \tag{4.16}$$

where $d_i = (l-i)ikr > 0$. If l is odd, $|A| = |B_l| = c \cdot q(c^2)$, where q is a nondegenerate polynomial of degree $(l-1)/2$ with nonnegative coefficients. If l is even, $|A| = |B_l| = q(c^2)$, where q is a nondegenerate polynomial of degree $l/2$ with nonnegative coefficients. Thus, to ensure the existence of nontrivial solutions of $a_1, \ldots, a_l$, c must be 0.

Similarly, by assuming $b_k = x_1^r$ and repeating the same computation for the coefficient function of D_2^k term in $[P, \sum_{i=1}^2 p_iD_i]$, one will get

$$\begin{pmatrix} c_1 & -r & & & & \\ k(l-1) & 2c_1 & -2r & & & \\ & k(l-2) & 3c_1 & -3r & & \\ & & \ddots & \ddots & \ddots & \\ & & & 2k & (l-1)c_1 & -(l-1)r \\ & & & & k & lc_1 \end{pmatrix} \begin{pmatrix} a_l \\ a_{l-1} \\ a_{l-2} \\ \vdots \\ a_2 \\ a_1 \end{pmatrix} = 0, \tag{4.17}$$

and c_1 must be 0 as well.

This first observation from (4.14) and (4.17) is that $r \neq 0$. Otherwise, one will have $a_1 = \cdots = a_{l-1} = 0$ from (4.14) and $a_l = \cdots = a_2 = 0$ from (4.17) since $c = c_1 = 0$.

Now, $r \neq 0$ and $c = c_1 = 0$, again by solving (4.14) and (4.17),

$$(4.18) \quad a_j = \begin{cases} 0, & j = 2, l-1, \\ \frac{k(l-j+2)}{r(j-1)} a_{j-2}, & 3 \leq j \leq l, \end{cases} \qquad a_j = \begin{cases} 0, & j = 2, l-1, \\ \frac{r(l-j+2)}{k(j-1)} a_{j-2}, & 3 \leq j \leq l. \end{cases}$$

Hence, l must be an odd number and $k = r$ so that (4.18) holds. Without loss of generality, a_1 can be assumed to be 1. That is,

$$(4.19) \quad \begin{gathered} l \text{ is an odd number } \geq 3, \quad a_1 = a_l = 1, \quad a_2 = 0, \\ a_j = \frac{l-j+2}{j-1} a_{j-2} \ \text{ for } 3 \leq j \leq l. \end{gathered}$$

(C7) $k = r$. This means the degree of the coefficient function b_i in P_0 must be the same as the order of P_0. That is,

$$(4.20) \quad P_0 = x_2^k D_1^k - k x_1 x_2^{k-1} D_1^{k-1} D_2 \ + \text{terms with lower order in } D_1, \ \text{mod } U_{k-1}.$$

Consider any $N = \sum_{j=0}^t n_j(x) D_1^{t-j} D_2^j$, mod $U_{t-1} \in E$,

$$\begin{aligned} (4.21) \qquad [P_0, N] &= \left[\sum_{i=0}^k b_i(x) D_1^{k-i} D_2^i, \sum_{j=0}^t n_j(x) D_1^{t-j} D_2^j \right] \\ &= \left(k b_0 \frac{\partial n_0}{\partial x_1} - t n_0 \frac{\partial b_0}{\partial x_1} + b_1 \frac{\partial n_0}{\partial x_2} - n_1 \frac{\partial b_0}{\partial x_2} \right) D_1^{k+t-1} \\ &\qquad + \text{terms with lower order in } D_1 \\ &= k x_2^{k-1} \left(x_2 \frac{\partial n_0}{\partial x_1} - x_1 \frac{\partial n_0}{\partial x_2} - n_1 \right) D_1^{k+t-1} \\ &\qquad + \text{terms with lower order in } D_1, \ \text{mod } U_{k+t-2}. \end{aligned}$$

If $t \geq 2$, then

$$(4.22) \qquad n_1 = x_2 \frac{\partial n_0}{\partial x_1} - x_1 \frac{\partial n_0}{\partial x_2}.$$

Otherwise, $[P_0, N]$ will have order $k + t - 1 > k$, contradicting (C2).

In particular, by (4.22),

$$(4.23) \qquad \text{if } n_0 = x_2^s \Longrightarrow n_1 = -s x_1 x_2^{s-1} \quad \text{ for } s \geq 0.$$

Now, suppose that $N = x_2^s D_1^t - s x_1 x_2^{s-1} D_1^{t-1} D_2$ + terms with lower order in D_1, mod $U_{t-1} \in E$. By using (4.18)

$$\begin{aligned} (4.24) \qquad [N, Y] &= [\ x_2^s D_1^t - s x_1 x_2^{s-1} D_1^{t-1} D_2 + \cdots p_1 D_1 + p_2 D_2\ ] \\ &= \left(t x_2^s \frac{\partial p_1}{\partial x_1} - s x_2^{s-1} p_2 - s x_1 x_2^{s-1} \frac{\partial p_1}{\partial x_2} \right) D_1^t \\ &\qquad + \text{terms with lower order in } D_1 \\ &= (t-s) \sum_{j=1}^{l-2} (l-j) a_j x_1^{l-j-1} x_2^{j+s} D_1^t \\ &\qquad + \text{terms with lower order in } D_1, \ \text{mod } U_{t-1}, \end{aligned}$$

$$\begin{aligned} Ad_{L_0}^{l-2}[N, Y] &= (t-s)(l-1)! x_2^{s+1} D_1^{t+l-2} \\ (4.25) \qquad &\qquad + \text{terms with lower order in } D_1, \ \text{mod } U_{t+l-1}. \end{aligned}$$

Equations (4.23), (4.24), and (4.25) show that if $s < t$,

$$x_2^s D_1^t \ + \text{terms with lower order in } D_1, \ \text{mod } U_{t-1} \in E$$

$$\text{(4.26)} \quad \Longrightarrow x_2^{s+1} D_1^{t+l-2} \ + \text{terms with lower order in } D_1, \ \text{mod } U_{t+l-3} \in E.$$

Since $l \geq 3$, $s+1 < t+l-2$, and repeating (4.26) will make E not finite-dimensional. Now, choose $N = 2L_0$. Then the above procedure (4.23)–(4.26) applies. Contradiction! □

THEOREM 4.7. *Suppose* dim $E < \infty$. $x_1^2 + x_2^2 + c \in E \Rightarrow \omega_{12}$ *is a polynomial of degree at most* 1.

Proof.

$$K_0 = \left[L_0, \frac{1}{2}(x_1^2 + x_2^2 + c)\right] - 1 = x_1 D_1 + x_2 D_2 \in E,$$

$$K_1 = [L_0, K_0] = \sum_{i=1}^{2} D_i^2 - \sum_{i=1}^{2} \alpha_i D_i + \frac{1}{2} E(\eta) - \frac{1}{2} \sum_{i=1}^{2} \frac{\partial \alpha_i}{\partial x_i} \in E,$$

$$K_2 = 2L_0 - K_1 = \sum_{i=1}^{2} \alpha_i D_i + \frac{1}{2} \sum_{i=1}^{2} \frac{\partial \alpha_i}{\partial x_i} - \eta - \frac{1}{2} E(\eta) \in E,$$

where $\alpha_1 = x_2 \omega_{12}$, $\alpha_2 = x_1 \omega_{21}$, $E(\cdot) = \sum_{i=1}^{2} x_i \frac{\partial}{\partial x_i}$. By Theorem 3.2, α_i is a polynomial. Since ω_{12} is smooth, ω_{12} must be a polynomial as well.

Let $\alpha_i^{(r)}$ be the homogeneous degree r part of α_i. By Lemma 4.4,

$$\sum_{i=1}^{2} \alpha_i^{(r)} D_i \in E, \qquad \text{mod } U_0.$$

On the other hand,

$$\sum_{i=1}^{2} x_i \alpha_i = \sum_{i,j=1}^{2} x_i x_j \omega_{ij} = 0 \Longrightarrow x_1 \alpha_1^{(r)} + x_2 \alpha_2^{(r)} = 0.$$

By Lemma 4.6, $\alpha_i^{(r)} = 0$ for $r > 2$. Therefore, ω_{12} is a polynomial of degree at most 1. □

THEOREM 4.8. *Suppose* dim $E < \infty$. h_i*'s are polynomials at most of degree* 1.

Proof. Without loss of generality, h_1 is assumed to be a polynomial of degree 2. By orthogonal and affine transformations, h_1 is either $ax_1^2 + cx_2 + d$ or $ax_1^2 + bx_2^2 + d$, where $a, b \neq 0$.

(i) If $h_1 = ax_1^2 + cx_2 + d$, then $[[L_0, h_1], h_1] = 4a^2 x_1^2 + c^2 \in E$. By Theorem 4.3, ω_{21} is a constant. Hence, E is not finite-dimensional according to Theorem 2.5.

(ii) Assume $h_1 = ax_1^2 + bx_2^2 + d$. $[[L_0, h_1], h_1] = 4a^2 x_1^2 + 4b^2 x_2^2 \in E$. If $a \neq b$, both $x_1^2 + c$ and $x_2^2 + c$ are in E. From the discussion (i), E is not finite-dimensional. Hence, $h_1 = ax_1^2 + ax_2^2 + c$. By Theorem 4.7, ω_{12} is a polynomial of degree at most 1. Moreover, ω_{12} must be a nondegenerate degree 1 polynomial by Theorem 2.5, i.e., $\omega_{12} = c_1 x_1 + c_2 x_2 + c_0$, where $c_1 \neq 0$ or $c_2 \neq 0$.

From $K_0, K_2 \in E$ in Theorem 4.7 and by the construction of (4.6), let $Z_0 = K_2$ and $Z_{r+1} = (1-r)Z_r - [K_0, Z_r]$ for $r = 0, 1$,

$$\begin{aligned} Z_0 &= \sum_{i=1}^{2} \alpha_i D_i + q_0 \in E, \\ Z_1 &= c_0 x_2 D_1 - c_0 x_1 D_2 + q_0 - E(q_0) - \sum_{i=1}^{2} \alpha_i^2 \in E, \\ Z_2 &= -c_0 x_2 \alpha_1 + c_0 x_1 \alpha_2 + E\left(\sum_{i=1}^{2} \alpha_i^2\right) - E(q_0 - E(q_0)) \in E, \end{aligned}$$

where $q_0 = \frac{1}{2}\sum_{i=1}^{2} \frac{\partial \alpha_i}{\partial x_i} - \eta - \frac{1}{2}E(\eta)$. Since Z_2 must be a polynomial of degree less than or equal to 2, regardless of whether c_0 is 0 or not, $q_0 - E(q_0)$ or $E(q_0 - E(q_0))$ is a degree 4 polynomial by Theorem 3.15. So η is a degree 4 polynomial (by Theorems 3.13 and 3.15). Moreover, the homogeneous degree 4 part of η is

$$\eta^{(4)} = c\left(\sum_{i=1}^{2} \alpha_i^2\right)^{(4)} = c(x_1^2 + x_2^2)(c_1 x_1 + c_2 x_2)^2.$$

Hence, the homogeneous degree 4 part of $\eta - \sum h_i^2$ is

$$c(x_1^2 + x_2^2)(c_1 x_1 + c_2 x_2)^2 - a^2(x_1^2 + x_2^2)^2 - \sum_{i>1}^{m} (h_i^{(2)})^2.$$

By taking $x_1 = -c_2$ and $x_2 = c_1$, the above expression results in a negative number. By Theorem 3.18, there are no smooth solutions in f_i. □

THEOREM 4.9. *Suppose* dim $E < \infty$. *If* $\phi \in E$, ϕ *is a polynomial of degree at most* 1.

Proof. (i) If E has linear rank 2, E is of maximal rank, and therefore no degree 2 polynomials are in E by Theorem 2.7.

(ii) If E has linear rank 0, then h_i's must be constants. E is finite-dimensional and $E \subset \langle L_0, 1\rangle_{L.A.}$. Any function element in E must be a constant.

(iii) Let E have linear rank 1, i.e., there exists a function $d_i x_1 + e_i \in E \Longrightarrow D_1 \in E$.

From the proof of Theorem 4.8, by orthogonal and affine transformations, a degree 2 polynomial $\phi \in E$ only has the form $ax_1^2 + ax_2^2 + d$, where $a \neq 0$. Again by Theorem 4.8, $\omega_{12} = c_1 x_1 + c_2 x_2 + c_0$, where $c_1 \neq 0$ or $c_2 \neq 0$ (otherwise ω_{12} is a constant).

By (4.3), $[L_0, D_1] = \omega_{12} D_2 + \varphi \in E$. Thus

$$[\omega_{12} D_2 + \varphi, ax_1^2 + ax_2^2 + d] = 2ax_2\omega_{12} = 2ac_1 x_1 x_2 + 2ac_2 x_2^2 + 2ac_0 x_2 \in E.$$

If $c_1 \neq 0$, $[D_1, 2ac_1 x_1 x_2 + 2ac_2 x_2^2 + 2ac_0 x_2] = 2ac_1 x_2$ is dependent on x_2. This contradicts the linear rank of E being 1. If $c_1 = 0$ and $c_2 \neq 0$, from $2ac_2 x_2^2 + 2ac_0 x_2$ and $ax_1^2 + ax_2^2 + d$, are in E, one has that $x_1^2 + c_4 x_2 + c_5 \in E$. From the discussion (i) in the proof of Theorem 4.8, E is not finite-dimensional. □

4.2. Constant structure of the Ω-matrix. In this section, E is assumed to be finite-dimensional and have linear rank 1. As discussed in the proof of Theorem 4.9, the structure of E is very clear when E's linear rank is 0 or 2.

Assume $x_1 \in E$. Then if a function $p \in E$, p is a degree 1 polynomial in x_1 as a result of Theorem 4.9 as well as the linear rank assumption. By Theorem 4.2, $\omega_{12} = c_1x_1 + c_2x_2 + c_0$. It will be shown that ω_{12} must be a constant in this section.

By assumptions, $L_0, x_1, D_1, 1 \in E$. Moreover, the following elements are in E:

$$H_0 = [L_0, D_1] = \omega_{12}D_2 + \frac{1}{2}\frac{\partial\eta}{\partial x_1} + \frac{1}{2}c_2, \tag{4.27}$$

$$H_1 = [D_1, H_0] = c_1D_2 - \omega_{12}^2 + \frac{1}{2}\frac{\partial^2\eta}{\partial x_1^2}, \tag{4.28}$$

$$H_2 = [D_1, H_1] = -3c_1\omega_{12} + \frac{1}{2}\frac{\partial^3\eta}{\partial x_1^3}, \tag{4.29}$$

$$\begin{aligned} H_3 &= [H_1, H_0] - c_2H_1 \\ &= 3c_2\omega_{12}^2 + \frac{1}{2}c_1\frac{\partial^2\eta}{\partial x_1\partial x_2} - \frac{1}{2}c_2\frac{\partial^2\eta}{\partial x_1^2} - \frac{1}{2}\omega_{12}\frac{\partial^3\eta}{\partial x_1^2\partial x_2}, \end{aligned} \tag{4.30}$$

$$\begin{aligned} X_0 &= [L_0, H_0] \\ &= c_1D_1D_2 + c_2D_2^2 + \left(-\omega_{12}^2 + \frac{1}{2}\frac{\partial^2\eta}{\partial x_1^2}\right)D_1 + \frac{1}{2}\frac{\partial^2\eta}{\partial x_1\partial x_2}D_2 \\ &\quad + \left(-\frac{1}{2}c_1\omega_{12} + \frac{1}{2}\omega_{12}\frac{\partial\eta}{\partial x_2} + \frac{1}{4}\frac{\partial^3\eta}{\partial x_1^3} + \frac{1}{4}\frac{\partial^3\eta}{\partial x_1\partial x_2^2}\right). \end{aligned} \tag{4.31}$$

LEMMA 4.10. *η is a polynomial in x_1 of degree at most 4 with coefficients being functions of x_2. Let $\eta = \sum_{i=0}^4 a_ix_1^i$, where a_i's are functions of x_2. Then*

(i) *a_4 is a constant;*

(ii) *$a_3 = c_1c_2x_2 +$ constant;*

(iii) *$(c_1^2 - 4a_4)c_2 = 0$;*

(iv) *$3c_2^3x_2^2 + 6c_0c_2^2x_2 - c_2a_2 + \frac{1}{2}c_1a_1' - c_0a_2' - c_2a_2'x_2 =$ constant.*

Proof. Since H_2 is a function $\in E$, H_2 is independent of x_2 and a degree 1 polynomial in x_1. Therefore, η is a polynomial in x_1 of degree at most 4 with coefficient functions in x_2. Let $\eta = \sum_{i=0}^4 a_ix_1^i$, where a_i's are functions of x_2.

$H_2 \in E \Longrightarrow a_4$ is a constant, and $a_3 = c_1c_2x_2$ + constant.

(iii) and (iv) follow from $H_3 \in E$ by substituting $\eta = \sum_{i=0}^4 a_ix_1^i$ into (4.30). ∎

Now, from (4.31), $X_0 \in E$.

$$\begin{aligned} X &= [L_0, X_0] \\ &= \left[L_0, c_1D_1D_2 + c_2D_2^2 + \left(-\omega_{12}^2 + \frac{1}{2}\frac{\partial^2\eta}{\partial x_1^2}\right)D_1 + \frac{1}{2}\frac{\partial^2\eta}{\partial x_1\partial x_2}D_2\right] \\ &= \left(-3c_1\omega_{12} + \frac{1}{2}\frac{\partial^3\eta}{\partial x_1^3}\right)D_1^2 + \left(-4c_2\omega_{12} + \frac{\partial^3\eta}{\partial x_1^2\partial x_2}\right)D_1D_2 \\ &\quad + \left(c_1\omega_{12} + \frac{1}{2}\frac{\partial^3\eta}{\partial x_1\partial x_2^2}\right)D_2^2 \\ &= q_0D_1^2 + q_1D_1D_2 + q_2D_2^2 \in E, \quad \mod U_1, \end{aligned} \tag{4.32}$$

where $q_0 = -3c_1\omega_{12} + \frac{1}{2}\frac{\partial^3\eta}{\partial x_1^3}$, $q_1 = -4c_2\omega_{12} + \frac{\partial^3\eta}{\partial x_1^2\partial x_2}$, $q_2 = c_1\omega_{12} + \frac{1}{2}\frac{\partial^3\eta}{\partial x_1\partial x_2^2}$. By

Lemma 4.10 (i) and (ii),

$$\frac{\partial q_0}{\partial x_1} = 12a_4 - 3c_1^2, \quad \frac{\partial q_0}{\partial x_2} = 0, \tag{4.33}$$

$$\frac{\partial q_1}{\partial x_1} = 2c_1c_2, \quad \frac{\partial q_1}{\partial x_2} = -4c_2^2 + 2a_2'', \tag{4.34}$$

$$\frac{\partial q_2}{\partial x_1} = c_1^2 + a_2'', \quad \frac{\partial q_2}{\partial x_2} = c_1c_2 + \frac{1}{2}a_1''' + a_2'''x_1. \tag{4.35}$$

LEMMA 4.11. $a_4 = \frac{1}{4}c_1^2$.

Proof. If $a_4 \neq \frac{1}{4}c_1^2$, i.e., $r := \frac{\partial q_0}{\partial x_1} = 12a_4 - 3c_1^2 \neq 0$, then from X in (4.32) and L_0 one can construct a sequence of elements in E whose orders strictly increase as follows:

$$\begin{aligned}
Y_1 &= [L_0, X] = rD_1^3 + \sum_{i=1}^{3} p_{1i} D_1^{3-i} D_2^i, \quad \text{mod } U_2,\\
Y_2 &= [Y_1, X] = [rD_1^3 + p_{11}D_1^2D_2 + \cdots q_0D_1^2 + q_1D_1D_2 + q_2D_2^2]\\
&= \left(3r\frac{\partial q_0}{\partial x_1} + p_{11}\frac{\partial q_0}{\partial x_2}\right) D_1^4 + \sum_{i=1}^{4} p_{2i} D_1^{4-i} D_2^i\\
&= 3r^2 D_1^4 + \sum_{i=1}^{4} p_{2i} D_1^{4-i} D_2^i, \quad \text{mod } U_3,\\
&\vdots
\end{aligned}$$

Assume that $Y_k = r_k D_1^{k+2} + \sum_{i=1}^{k+2} p_{ki} D_1^{k+2-i} D_2^i$, mod U_{k+1}, with $r_k \neq 0$:

$$\begin{aligned}
Y_{k+1} &= [Y_k, X] = [r_k D_1^{k+2} + p_{k1} D_1^{k+1} D_2 + \cdots q_0 D_1^2 + q_1 D_1 D_2 + q_2 D_2^2]\\
&= \left((k+2) r_k \frac{\partial q_0}{\partial x_1} + p_{k1}\frac{\partial q_0}{\partial x_2}\right) D_1^{k+3} + \sum_{i=1}^{k+3} p_{(k+1)i} D_1^{k+3-i} D_2^i\\
&= r_{k+1} D_1^{k+3} + \sum_{i=1}^{k+3} p_{(k+1)i} D_1^{k+3-i} D_2^i, \quad \text{mod } U_{k+2},
\end{aligned}$$

where $r_{k+1} = (k+2) r r_k \neq 0$. ◻

THEOREM 4.12. *If $c_1c_2 \neq 0$, E is not finite-dimensional.*

Proof. Assume that $c_1c_2 \neq 0$.

(i) From Lemma 4.11 and (4.32), (4.33), q_0 is a constant.

(ii) Let $r_1 := \frac{\partial q_1}{\partial x_1} = 2c_1c_2$ in (4.34):

$$\begin{aligned}
Z &= [L_0, X] = [L_0, q_0D_1^2 + q_1D_1D_2 + q_2D_2^2]\\
&= r_1 D_1^2 D_2 + \text{terms with lower order in } D_1, \quad \text{mod } U_2.
\end{aligned}$$

Suppose q_1 in (4.32) is a polynomial of degree k in x_2, i.e., $\frac{\partial^k q_1}{\partial x_2^k}$ is a nonzero constant.

If $k \geq 1$,

$$\begin{aligned} A_1 &= [Z, X] \\ &= [r_1 D_1^2 D_2 \ \text{ + terms with lower order in } D_1, q_0 D_1^2 + q_1 D_1 D_2 + q_2 D_2^2] \\ &= r_1 \frac{\partial q_1}{\partial x_2} D_1^3 D_2 + \text{terms with lower order in } D_1, \quad \text{mod } U_3, \\ A_2 &= [Z, A_1] = \Big[r_1 D_1^2 D_2 \ \text{ + terms with lower order in } D_1, \\ &\qquad r_1 \frac{\partial q_1}{\partial x_2} D_1^3 D_2 \ \text{ + terms with lower order in } D_1 \Big] \\ &= r_1^2 \frac{\partial^2 q_1}{\partial x_2^2} D_1^5 D_2 \ \text{ + terms with lower order in } D_1, \quad \text{mod } U_5, \\ &\ \ \vdots \\ A_k &= [Z, A_{k-1}] = r_1^k \frac{\partial^k q_1}{\partial x_2^k} D_1^{2k+1} D_2 \ \text{ + terms with lower order in } D_1, \quad \text{mod } U_{2k+1}. \end{aligned}$$

Now one can repeat the above process by letting $Z = A_k$ to construct a sequence of elements in E whose orders strictly increase.

Hence, k must be zero, which means $\frac{\partial q_1}{\partial x_2} = 0 \Longrightarrow a_2'' = 2c_2^2$.

(iii) By taking twice the derivative with respect to x_2 in Lemma 4.10 (iv) and substituting $a_2'' = 2c_2^2$, one has $c_1 a_1''' = 0 \Longrightarrow a_1''' = 0$.

Thus, $X = q_0 D_1^2 + q_1 D_1 D_2 + q_2 D_2^2$, mod U_1, is in E, with

$$\frac{\partial q_0}{\partial x_1} = \frac{\partial q_0}{\partial x_2} = 0, \tag{4.36}$$

$$\frac{\partial q_1}{\partial x_1} = 2c_1 c_2 = r_1 \neq 0, \qquad \frac{\partial q_1}{\partial x_2} = 0, \tag{4.37}$$

$$\frac{\partial q_2}{\partial x_1} = c_1^2 + 2c_2^2 = r_2 > 0, \qquad \frac{\partial q_2}{\partial x_2} = c_1 c_2 = \frac{1}{2} r_1, \tag{4.38}$$

$$\begin{aligned} Y_0 &= [L_0, X] = [L_0, q_0 D_1^2 + q_1 D_1 D_2 + q_2 D_2^2] \\ &= p_{01} D_1^2 D_2 + p_{02} D_1 D_2^2 + p_{03} D_2^3, \quad \text{mod } U_2, \end{aligned} \tag{4.39}$$

where $p_{01} = r_1$, $p_{02} = r_2$, $p_{03} = \frac{1}{2} r_1$. For $l \geq 0$, suppose $Y_l = p_{l1} D_1^2 D_2^{l+1} + p_{l2} D_1 D_2^{l+2} + p_{l3} D_2^{l+3} \in E$, mod U_{l+2}, where p_{l1}, p_{l2}, p_{l3} are constants.

$$\begin{aligned} Y_{l+1} &= [Y_l, X] = [p_{l1} D_1^2 D_2^{l+1} + p_{l2} D_1 D_2^{l+2} + p_{l3} D_2^{l+3}, q_0 D_1^2 + q_1 D_1 D_2 + q_2 D_2^2] \\ &= 2p_{l1} \frac{\partial q_1}{\partial x_1} D_1^2 D_2^{l+2} + p_{l2} \frac{\partial q_1}{\partial x_1} D_1 D_2^{l+3} + (l+1) p_{l1} \frac{\partial q_2}{\partial x_2} D_1^2 D_2^{l+2} \\ &\quad + 2p_{l1} \frac{\partial q_2}{\partial x_1} D_1 D_2^{l+3} + (l+2) p_{l2} \frac{\partial q_2}{\partial x_2} D_1 D_2^{l+3} + p_{l2} \frac{\partial q_2}{\partial x_1} D_2^{l+4} \\ &\quad + (l+3) p_{l3} \frac{\partial q_2}{\partial x_2} D_2^{l+4} \end{aligned}$$

$$
\begin{aligned}
&= \frac{l+5}{2} r_1 p_{l1} D_1^2 D_2^{l+2} + (2r_2 p_{l1} + \frac{l+4}{2} r_1 p_{l2}) D_1 D_2^{l+3} \\
&\quad + \left(r_2 p_{l2} + \frac{l+3}{2} r_1 p_{l3} \right) D_2^{l+4} \\
&= p_{(l+1)1} D_1^2 D_2^{l+2} + p_{(l+1)2} D_1 D_2^{l+3} + p_{(l+1)3} D_2^{l+4}, \quad \text{mod } U_{l+3},
\end{aligned}
$$

where $p_{(l+1)1} = \frac{l+5}{2} r_1 p_{l1}$, $p_{(l+1)2}, p_{(l+1)3}$ are constants. Since $p_{01} = r_1 \neq 0$, $p_{l1} \neq 0$ for any l. Thus, Y_l is a differential operator of degree $l+3$. Contradiction! □

LEMMA 4.13. *If* $c_1 = 0, c_2 \neq 0$, *we have* (i) $a_4 = 0$; (ii) $a_3 = 0$; (iii) $a_2 = \omega_{12}^2 +$ *constant;* (iv) $a_1''' = 0$.

Proof. (i) follows from Lemma 4.11.

By Lemma 4.10 (ii), a_3 is a constant. Since a_3 is the coefficient of an odd order term in η with the highest order in x_1, $a_3 = 0$ by Theorem 3.21.

(iii) follows from (4.28). H_1 is a function in E when $c_1 = 0$.

Now, q_0, q_1 are actually constants in (4.32), while $\frac{\partial q_2}{\partial x_1} = a_2'' = 2c_2^2$ and $\frac{\partial q_2}{\partial x_2} = \frac{1}{2} a_1'''$. Let a_1'' be a degree k polynomial in x_2, i.e., $\frac{d^{k+2} a_1}{d x_2^{k+2}}$ is a nonzero constant. If $k > 1$,

$$
\begin{aligned}
Z_0 &= Ad_{L_0}^k X = \frac{1}{2} \frac{d^{k+2} a_1}{d x_2^{k+2}} D_2^{k+2}, \quad \text{mod } U_{k+1}, \\
Y_0 &= [Z_0, X] = (k+2) \left(\frac{1}{2} \right)^2 \frac{d^{k+2} a_1}{d x_2^{k+2}} \frac{d^3 a_1}{d x_2^3} D_2^{k+3}, \quad \text{mod } U_{k+2}.
\end{aligned}
$$

By letting $Z_l = Ad_{L_0}^{k-1} Y_{l-1}$ and $Y_l = [Z_l, X]$ for $l \geq 1$, one can construct a sequence of elements in E with strictly increasing orders.

If $k = 1$, consider $Z_0 = [L_0, X] = \frac{\partial q_2}{\partial x_1} D_1 D_2^2 + \frac{\partial q_2}{\partial x_2} D_2^3$, mod U_2, and $Z_{l+1} = [Z_l, X]$ for $l \geq 0$. Assume

$$
Z_l = p_{l0} D_1 D_2^{l+2} + p_{l1} D_2^{l+3}, \qquad \text{mod } U_{l+2},
$$

where $p_{00} = \frac{\partial q_2}{\partial x_1} = 2c_2^2$ and $p_{01} = \frac{\partial q_2}{\partial x_2} = \frac{1}{2} \frac{d^3 a_1}{d x_2^3}$. Then

$$
\begin{aligned}
Z_{l+1} &= [Z_l, X] = [p_{l0} D_1 D_2^{l+2} + p_{l1} D_2^{l+3}, q_0 D_1^2 + q_1 D_1 D_2 + q_2 D_2^2] \\
&= (l+2) p_{l0} \frac{\partial q_2}{\partial x_2} D_1 D_2^{l+3} + \left(p_{l0} \frac{\partial q_2}{\partial x_1} + (l+3) p_{l1} \frac{\partial q_2}{\partial x_2} \right) D_2^{l+4}, \quad \text{mod } U_{l+3}.
\end{aligned}
$$

Therefore,

$$
p_{(l+1)0} = (l+2) p_{l0} \frac{\partial q_2}{\partial x_2} \Longrightarrow p_{l0} = (l+1)! \frac{\partial q_2}{\partial x_1} \left(\frac{\partial q_2}{\partial x_2} \right)^l \neq 0.
$$

Z_l's have strictly increasing orders. Contradiction!

Hence k must be zero $\Longrightarrow a_1''' = 0$. □

THEOREM 4.14. *If* $c_1 = 0, c_2 \neq 0$, E *is not finite-dimensional.*

Proof. In the proof of this theorem, $r_1, r_2, \ldots$ are used to denote constants. The exact values of these r_i's are not important. Some r_i's may be used repeatedly to denote different constants.

By substituting Lemma 4.13 (i)–(iv) into (4.31), we obtain

$$
\begin{aligned}
Z_0 &= X_0 - r_1 D_1 - r_2 x_1 - r_3 \\
&= c_2 D_2^2 + \frac{1}{2}\frac{\partial^2 \eta}{\partial x_1 \partial x_2} D_2 + \frac{1}{2}\omega_{12}\frac{\partial \eta}{\partial x_2}, \qquad (4.40)\\
Z_1 &= [L_0, Z_0] - r_5 D_1 \\
&= \frac{1}{2}\frac{\partial^3 \eta}{\partial x_1 \partial x_2^2} D_2^2 + \left(\frac{3}{2}c_2\frac{\partial \eta}{\partial x_2} + \frac{1}{2}\omega_{12}\frac{\partial^2 \eta}{\partial x_2^2}\right) D_2, \quad \mathrm{mod}\ U_0,\\
Z_2 &= [L_0, Z_1] = a_2'' D_1 D_2^2 + \left(\frac{3}{2}c_2\frac{\partial^2 \eta}{\partial x_1 \partial x_2} - \frac{1}{2}\omega_{12}\frac{\partial^3 \eta}{\partial x_1 \partial x_2^2}\right) D_1 D_2 \\
&\quad + \left(2c_2\frac{\partial^2 \eta}{\partial x_2^2} + \frac{1}{2}\omega_{12}\frac{\partial^3 \eta}{\partial x_2^3}\right) D_2^2, \quad \mathrm{mod}\ U_1,\\
Z_3 &= [L_0, Z_2] = 3c_2\frac{\partial^3 \eta}{\partial x_1 \partial x_2^2} D_1 D_2^2 + \left(a_2''\omega_{12} + \frac{5}{2}c_2\frac{\partial^3 \eta}{\partial x_2^3} + \frac{1}{2}\omega_{12}\frac{\partial^4 \eta}{\partial x_2^4}\right) D_2^3 \\
&= p_1 D_1 D_2^2 + p_2 D_2^3, \quad \mathrm{mod}\ U_2, \qquad (4.41)
\end{aligned}
$$

where $p_1 = 3c_2 \frac{\partial^3 \eta}{\partial x_1 \partial x_2^2} = 3c_2(a_1'' + 2a_2'' x_1)$ is degree 1 in x_1 and independent of x_2 with $\frac{\partial p_1}{\partial x_1} = 6c_2 a_2'' = 12c_2^3 \neq 0$, and $p_2 = a_2''\omega_{12} + \frac{5}{2}c_2\frac{\partial^3 \eta}{\partial x_2^3} + \frac{1}{2}\omega_{12}\frac{\partial^4 \eta}{\partial x_2^4}$ is a function of x_2 and independent of x_1. By Theorem 3.2, p_2 must be a polynomial in x_2; otherwise E is not finite-dimensional.

By (4.40), $Y_0 = [Z_0, Z_3] = 2c_2 \frac{\partial p_2}{\partial x_2} D_2^4$, mod $U_3 \in E$, $Y_1 = [Z_0, Y_0] = (2c_2)^2 \frac{\partial^2 p_2}{\partial x_2^2} D_2^5$, mod $U_4 \in E$, ..., $Y_l = Ad_{Z_0}^{l+1} Z_3 = (2c_2)^{l+1} \frac{\partial^{l+1} p_2}{\partial x_2^{l+1}} D_2^{l+4}$, mod $U_{l+3} \in E$. If $deg\ p_2 = k \geq 1$, one can construct an infinite sequence of elements in E whose orders strictly increase as follows:

$$
\begin{aligned}
W_0 &= Ad_{Z_0}^k Z_3 = (2c_2)^k \frac{\partial^k p_2}{\partial x_2^k} D_2^{k+3} \\
&= d_0 D_2^{e_0}, \quad \mathrm{mod}\ U_{e_0 - 1}, \quad \text{where } d_0 = (2c_2)^k \frac{\partial^k p_2}{\partial x_2^k} \neq 0, e_0 = k+3,\\
W_{l+1} &= Ad_{Z_0}^{k-1}[W_l, Z_3] = Ad_{Z_0}^{k-1}[d_l D_2^{e_l}, p_1 D_1 D_2^2 + p_2 D_2^3] \\
&= Ad_{Z_0}^{k-1} e_l d_l \frac{\partial p_2}{\partial x_2} D_2^{e_l + 2} \\
&= e_l d_l (2c_2)^{k-1} \frac{\partial^k p_2}{\partial x_2^k} D_2^{e_l + k + 1} \\
&= d_{l+1} D_2^{e_{l+1}}, \quad \mathrm{mod}\ U_{e_{l+1} - 1},
\end{aligned}
$$

where $d_{l+1} = e_l d_l (2c_2)^{k-1} \frac{\partial^k p_2}{\partial x_2^k} \neq 0$, and $e_{l+1} = e_l + k + 1 > e_l$. Therefore, p_2 is a constant.

Now, p_1 is a degree 1 polynomial in x_1, and p_2 is a constant in (4.41).

$$V_0 = [L_0, Z_3] = \frac{\partial p_1}{\partial x_1} D_1^2 D_2^2, \quad \text{mod } U_3,$$

$$V_1 = [V_0, Z_3] = 2\left(\frac{\partial p_1}{\partial x_1}\right)^2 D_1^2 D_2^4, \quad \text{mod } U_5,$$

$$V_{l+1} = [V_l, Z_3] = \left[2^l \left(\frac{\partial p_1}{\partial x_1}\right)^{l+1} D_1^2 D_2^{2(l+1)}, p_1 D_1 D_2^2 + p_2 D_2^3\right]$$

$$= 2^{l+1}\left(\frac{\partial p_1}{\partial x_1}\right)^{l+2} D_1^2 D_2^{2(l+2)}, \quad \text{mod } U_{2l+5}.$$

Again, V_l's have increasing orders. Hence, E is not finite-dimensional. □

LEMMA 4.15. *If $c_1 \neq 0, c_2 = 0$, we have* (i) $a_4 = \frac{1}{4}c_1^2$; (ii) a_3 *is a constant;* (iii) $a_2'' = 0$; (iv) $a_1'' = 0$.

Proof. (i) follows from Lemma 4.11.

(ii) follows from Lemma 4.10 (ii).

Substituting $c_2 = 0$ into Lemma 4.10 (iv), $c_1 a_1' - 2c_0 a_2' =$ constant. Hence,

$$a_1'' = 2\frac{c_0}{c_1} a_2''. \tag{4.42}$$

From (4.32)–(4.35), q_0 is a constant, $\frac{\partial q_1}{\partial x_1} = 0$, $\frac{\partial q_1}{\partial x_2} = 2a_2''$. Consider

$$Y_0 = [L_0, [X, x_1]] = [L_0, 2q_0 D_1 + q_1 D_2] = \frac{\partial q_1}{\partial x_2} D_2^2 = 2a_2'' D_2^2, \quad \text{mod } U_1. \tag{4.43}$$

Let a_2'' be a degree k polynomial in x_2, i.e., $\frac{d^{k+2} a_2}{dx_2^{k+2}}$ is a nonzero constant. Then the orders of the elements $Z_0 = Ad_{L_0}^k Y_0$, $Y_l = [Z_{l-1}, Y_0]$, $Z_{l+1} = Ad_{L_0}^{k-1} Y_l$ for $l \geq 1$ will be strictly increasing unless $k = 0$. Hence, a_2'' must be a constant.

By (4.35) and (4.42), $\frac{\partial q_2}{\partial x_2} = 0$. Thus, among q_0, q_1, and q_2 in X in (4.32), q_0 is a constant, $\frac{\partial q_1}{\partial x_1} = \frac{\partial q_2}{\partial x_2} = 0$, $\frac{\partial q_1}{\partial x_2} = 2a_2''$, and $\frac{\partial q_2}{\partial x_1} = c_1^2 + a_2''$ are constants. Consider $Y_l = Ad_{-X}^l Y_0$, where Y_0 is in (4.43).

$$Y_1 = [Y_0, X] = \left[\frac{\partial q_1}{\partial x_2} D_2^2, q_0 D_1^2 + q_1 D_1 D_2 + q_2 D_2^2\right] = 2\left(\frac{\partial q_1}{\partial x_2}\right)^2 D_1 D_2^2, \quad \text{mod } U_2,$$

$$Y_2 = [Y_1, X] = \left[2\left(\frac{\partial q_1}{\partial x_2}\right)^2 D_1 D_2^2, q_0 D_1^2 + q_1 D_1 D_2 + q_2 D_2^2\right]$$

$$= 4\left(\frac{\partial q_1}{\partial x_2}\right)^3 D_1^2 D_2^2 \ + \text{ terms with lower order in } D_1$$

$$\text{with constant coefficients}, \quad \text{mod } U_3,$$

$$\vdots$$

$$
\begin{aligned}
Y_{l+1} &= [Y_l, X] \\
&= \Bigg[2^l \left(\frac{\partial q_1}{\partial x_2}\right)^{l+1} D_1^l D_2^2 \ + \text{ terms with lower order in } D_1 \text{ with constant coefficients}, \\
&\qquad\qquad q_0 D_1^2 + q_1 D_1 D_2 + q_2 D_2^2 \Bigg] \\
&= 22^l \left(\frac{\partial q_1}{\partial x_2}\right)^{l+1} \frac{\partial q_1}{\partial x_2} D_1^{l+1} D_2^2 \ + \text{ terms with lower order in } D_1 \\
&\qquad\qquad \text{with constant coefficients} \\
&= 2^{l+1} \left(\frac{\partial q_1}{\partial x_2}\right)^{l+2} D_1^{l+1} D_2^2 \ + \text{ terms with lower order in } D_1 \\
&\qquad\qquad \text{with constant coefficients } \mathrm{mod} U_{l+2}.
\end{aligned}
$$

Thus, the order of Y_l goes to infinity unless $\frac{\partial q_1}{\partial x_2} = 0 \Longrightarrow a_2'' = 0 \Longrightarrow a_1'' = 0$ by (4.42). □

LEMMA 4.16. *If $c_1 \neq 0, c_2 = 0$, then η is a degree 4 polynomial in x_1, x_2 with its principal part $\eta^{(4)} = \frac{1}{4}c_1^2 x_1^4$.*

Proof. Again, $r_1, r_2, \ldots$ are used to denote constants. The exact values of these r_i's are not important. Some r_i's may be used repeatedly to denote different constants.

By substituting Lemma 4.15 (i)–(iv) into (4.31),

$$
\begin{aligned}
Z_0 &= X_0 - r_1 x_1 - r_2 \\
&= c_1 D_1 D_2 + \left(-\omega_{12}^2 + \frac{1}{2}\frac{\partial^2 \eta}{\partial x_1^2}\right) D_1 + \frac{1}{2}\frac{\partial^2 \eta}{\partial x_1 \partial x_2} D_2 + \frac{1}{2}\omega_{12}\frac{\partial \eta}{\partial x_2}, \\
Z_1 &= [L_0, Z_0] - r_3 D_1 \\
&= r_4 D_1^2 + c_1 \omega_{12} D_2^2 + 2a_2' D_1 D_2 + c_1 \frac{\partial \eta}{\partial x_2} D_1 \\
&\quad + \left(-\omega_{12}^3 + \frac{1}{2}\omega_{12}\frac{\partial^2 \eta}{\partial x_1^2} + \frac{1}{2}\omega_{12}\frac{\partial^2 \eta}{\partial x_2^2} + \frac{1}{2}c_1 \frac{\partial \eta}{\partial x_1}\right) D_2, \quad \mathrm{mod}\ U_0, \\
Z_2 &= [L_0, Z_1] = c_1^2 D_1 D_2^2 + r_5 D_1^2 + \left(3a_2'\omega_{12} + \frac{1}{2}\omega_{12}\frac{\partial^3 \eta}{\partial x_2^3} + \frac{1}{2}c_1 \frac{\partial^2 \eta}{\partial x_1 \partial x_2}\right) D_2^2 \\
&\quad + \left(2r_4\omega_{12} - 5c_1\omega_{12}^2 + \frac{3}{2}c_1\frac{\partial^2 \eta}{\partial x_2^2} + c_1\frac{\partial^2 \eta}{\partial x_1^2} + \frac{1}{2}\omega_{12}\frac{\partial^3 \eta}{\partial x_1^3}\right) D_1 D_2, \quad \mathrm{mod}\ U_1, \\
Z_3 &= [L_0, Z_2] = r_6 D_1^2 D_2 + \left(r_7 + 2c_1\frac{\partial^3 \eta}{\partial x_2^3}\right) D_1 D_2^2 + \left(c_1^2\omega_{12} + \frac{1}{2}\omega_{12}\frac{\partial^4 \eta}{\partial x_2^4}\right) D_2^3 \\
&= r_6 D_1^2 D_2 + (r_7 + 2c_1 a_0''') D_1 D_2^2 + \left(c_1^2 + \frac{1}{2}a_0''''\right)\omega_{12} D_2^3, \quad \mathrm{mod}\ U_2, \\
Y_0 &= [Z_3, x_1] = 2r_6 D_1 D_2 + (r_7 + 2c_1 a_0''') D_2^2, \quad \mathrm{mod}\ U_1.
\end{aligned}
$$

From the proof of Lemma 4.13 (iii), $\frac{\partial}{\partial x_2}(r_7 + 2c_1 a_0''') = 0 \to a_0'''' = 0$. By Lemma 4.15, η is a degree 4 polynomial in x_1, x_2 with principal part $\eta^{(4)} = \frac{1}{4}c_1^2 x_1^4$. □

THEOREM 4.17. *If $c_1 \neq 0, c_2 = 0$, E is not finite-dimensional.*

Proof. From Lemma 4.15, (4.28), and (4.27),

$$H_1 = [D_1, H_0] = c_1 D_2 - \omega_{12}^2 + \frac{1}{2}\frac{\partial^2 \eta}{\partial x_1^2}$$

$$= c_1 D_2 + \frac{1}{2}c_1^2 x_1^2 + (3a_3 - 2c_1 c_0)x_1 + a_2 - c_0^2, \tag{4.44}$$

$$H_0 = [L_0, D_1] = \omega_{12} D_2 + \frac{1}{2}\frac{\partial \eta}{\partial x_1} + \frac{1}{2}c_2$$

$$= (c_1 x_1 + c_0)D_2 + \frac{1}{2}(a_1 + 2a_2 x_1 + 3a_3 x_1^2 + 4a_4 x_1^3)$$

$$= x_1 H_1 + d_0 H_1 + \left(\frac{1}{2}dx_1^2 + d_0 dx_1 + d_2 x_2 + d_3\right), \tag{4.45}$$

where

$$d = 3c_0 c_1 - 3a_3,\ d_0 = \frac{c_0}{c_1},\ d_2 = \frac{1}{2}a_1' - d_0 a_2', \tag{4.46}$$

and d_3 is a constant with unknown value. (Note that all the d's here are constants.)

From (4.44), D_2 can be represented in terms of H_1, x_1, x_2, i.e.,

$$D_2 = \frac{1}{c_1}\left(H_1 - \left(\frac{1}{2}c_1^2 x_1^2 + (3a_3 - 2c_1 c_0)x_1 + a_2 - c_0^2\right)\right). \tag{4.47}$$

Thus, any element in E can be represented in terms of D_1, H_1, x_1, x_2. The following relations hold:

$$[D_1, x_1] = 1, \qquad [D_1, x_2] = 0, \tag{4.48}$$

$$[H_1, x_1] = 0, \qquad [H_1, x_2] = c_1, \tag{4.49}$$

$$[D_1, H_1] = 3a_3 - 3c_1 c_0 = -d. \tag{4.50}$$

Definition. Let A be a differential operator of order k in E. Then A has a unique representation:

$$A = \sum_{i=0}^{k}\sum_{j=0}^{i} h_{ij} D_1^{i-j} H_1^j, \tag{4.51}$$

where h_{ij}'s are polynomials in x_1, x_2. (It is worth mentioning that in the scope of this theorem, η is a polynomial by Lemma 4.16. In general, it is not clear whether h_{ij} is a polynomial or not.) The *degree* of A is defined to be $deg\ A = \max_i \max_j \{deg\ h_{ij} + 2i\}$. In other words, the degree of the differential operator A is the degree of the polynomial (of x_1, x_2, x_3, x_4 variables) obtained by replacing D_1, H_1 by x_3^2, x_4^2 in the form (4.51) of A.

For example, in (4.45), H_0 is a sum of three terms. The first, $x_1 H_1$, is of degree 3; the second, $d_0 H_1$, is of degree 2; the order 0 term $\frac{1}{2}dx_1^2 + d_0 dx_1 + d_2 x_2 + d_3$ is of degree 2. Hence, $deg\ H_0 = 3$. Similarly, $deg\ D_2 = 2$ by (4.47).

Since $D_1 H_1 = H_1 D_1 - d$, $H_1 h = hH_1 + c_1 \frac{\partial h}{\partial x_2}$, and $D_1 h = hD_1 + \frac{\partial h}{\partial x_1}$, it is easy to show by induction that deg is commutative. Moreover, for any $X, Y \in E$,

$$deg\ X + Y \leq \max\{deg\ X, deg\ Y\}, \quad deg\ XY = deg\ YX, \tag{4.52}$$

$$deg\ XY \leq deg\ X + deg\ Y.$$

Let $Y_l = Ad^l_{L_0} D_1$. The following will show that $deg\ Y_l = l + 2$ and

$$Y_{2l} = D_1 H_1^l + Z_{2l}, \qquad Y_{2l+1} = x_1 H_1^{l+1} + Z_{2l+1}, \tag{4.53}$$

where Z_k is a differential operator with $deg\ Z_k \le k + 1$:

- $l = 0$, $Y_0 = D_1$, $deg\ Y_0 = 2$, $Z_0 = 0$.
- $l = 1$, $Y_1 = [L_0, D_1] = H_0 = x_1 H_1 + Z_1$, where $Z_1 = d_0 H_1 + (\frac{1}{2} dx_1^2 + d_0 dx_1 + d_2 x_2 + d_3)$ by (4.45). $deg\ Y_1 = 3$, $deg\ Z_1 \le 2$.

Suppose $Y_{2l-1} = x_1 H_1^l + Z_{2l-1}$ with $deg\ Z_{2l-1} \le 2l$.

$$\begin{aligned} Y_{2l} &= [L_0, Y_{2l-1}] = [L_0, x_1 H_1^l + Z_{2l-1}] \\ &= [L_0, x_1] H_1^l + x_1 [L_0, H_1^l] + [L_0, Z_{2l-1}] = D_1 H_1^l + Z_{2l}, \end{aligned}$$

where $Z_{2l} = x_1 [L_0, H_1^l] + [L_0, Z_{2l-1}]$. By Lemma 4.18,

$$\begin{aligned} &deg\ Z_{2l} \le \max\{1 + deg\ [L_0, H_1^l], deg\ [L_0, Z_{2l-1}]\} \le 2l + 1. \\ &Y_{2l+1} = [L_0, Y_{2l}] = [L_0, D_1 H_1^l + Z_{2l}] = [L_0, D_1] H_1^l + D_1 [L_0, H_1^l] + [L_0, Z_{2l}] \\ &\qquad = (x_1 H_1 + Z_1) H_1^l + D_1 [L_0, H_1^l] + [L_0, Z_{2l}] = x_1 H_1^{l+1} + Z_{2l+1}, \end{aligned}$$

where $Z_{2l+1} = Z_1 H_1^l + D_1 [L_0, H_1^l] + [L_0, Z_{2l}]$ with

$$deg\ Z_{2l+1} \le \max\{deg\ Z_1 H_1^l, 2 + deg\ [L_0, H_1^l], deg\ [L_0, Z_{2l}]\} = 2l + 2.$$

Hence, (4.53) holds by induction. $deg\ Y_l = l + 2 \to \infty$. E is not finite-dimensional. □

LEMMA 4.18.
(i) *Let* $A_l = [L_0, H_1^l]$, $B_l = [L_0, D_1^l]$. *Then* $deg\ A_l \le 2l$ *and* $deg\ B_l \le 2l + 1$.
(ii) *Let* $C = [L_0, h D_1^{i_1} H_1^{i_2}]$. *Then* $deg\ C \le deg\ h + 2(i_1 + i_2) + 1$.
(iii) $deg\ [L_0, A] \le deg\ A + 1$.

Proof. (i) $l = 1$,

$$\begin{aligned} A_1 = [L_0, H_1] &= \left[L_0, c_1 D_2 + \frac{1}{2} c_1^2 x_1^2 + (3a_3 - 2c_1 c_0) x_1 + a_2 - c_0^2\right] \\ &= -dD_1 + d_1 H_1 + (d_1 d + c_1 d_2) x_1 + d_4, \end{aligned}$$

where d, d_2 are in (4.46), and $d_1 = \frac{a_2'}{c_1}$ and $d_4 = \frac{1}{2} c_1 a_0' - \frac{1}{c_1} a_2 a_2' + d_1 c_0^2$ are constants. (Note: Actually one extra condition has been obtained on a_0 that $a_0'' = 2(\frac{a_2'}{c_1})^2 = 2d_1^2$ since d_4 is in E and independent of x_2.) Thus, $deg\ A_1 = 2$. By (4.45), $deg\ B_1 = deg\ H_0 = 3$.

$$\begin{aligned} A_{l+1} &= [L_0, H_1^{l+1}] = [L_0, H_1^l] H_1 + H_1^l [L_0, H_1] \\ &= A_l H_1 + H_1^l A_1, \\ B_{l+1} &= [L_0, D_1^{l+1}] = [L_0, D_1^l] D_1 + D_1^l [L_0, D_1] \\ &= B_l D_1 + D_1^l B_1. \end{aligned}$$

By induction and (4.52), (i) holds since $deg\ A_{l+1} \le \max\{deg\ A_l + deg\ H_1, deg\ H_1^l + deg\ A_1\} \le 2(l+1)$, $deg\ B_{l+1} \le \max\{deg\ B_l + deg\ D_1, deg\ D_1^l + deg\ B_1\} \le 2(l+1) + 1$.

(ii)

$$\begin{aligned} C &= [L_0, h D_1^{i_1} H_1^{i_2}] \\ &= [L_0, h] D_1^{i_1} H_1^{i_2} + h [L_0, D_1^{i_1}] H_1^{i_2} + h D_1^{i_1} [L_0, H_1^{i_2}]. \\ &= \sum_{j=1}^{2} \frac{\partial h}{\partial x_j} D_j D_1^{i_1} H_1^{i_2} + \frac{1}{2} \sum_{j=1}^{2} \frac{\partial^2 h}{\partial x_j^2} D_1^{i_1} H_1^{i_2} + h B_{i_1} H_1^{i_2} + h D_1^{i_1} A_{i_2}. \end{aligned}$$

Since $deg\ D_2 = 2$ by (4.47),

$$\begin{aligned}
&deg\ C \\
&\le \max\left\{deg\ \frac{\partial h}{\partial x_1} D_1^{i_1+1} H_1^{i_2}, deg\ \frac{\partial h}{\partial x_2} D_2 D_1^{i_1} H_1^{i_2}, deg\ \frac{\partial^2 h}{\partial x_1^2} D_1^{i_1} H_1^{i_2},\right. \\
&\qquad \left. deg\ \frac{\partial^2 h}{\partial x_2^2} D_1^{i_1} H_1^{i_2}, deg\ hB_{i_1} H_1^{i_2}, deg\ hD_1^{i_1} A_{i_2}\right\} \\
&\le \max\{deg\ h - 1 + 2(i_1 + i_2 + 1), deg\ h - 1 + 2(i_1 + i_2 + 1), deg\ h - 2 + 2i_1 + 2i_2, \\
&\qquad deg\ h - 2 + 2i_1 + 2i_2, deg\ h + 2i_1 + 1 + 2i_2, deg\ h + 2i_1 + 2i_2\} \\
&= deg\ h + 2(i_1 + i_2) + 1.
\end{aligned}$$

(iii) Let $A = \sum_{i=0}^{k} \sum_{j=0}^{i} h_{ij} D_1^{i-j} H_1^j$. Then by (ii)

$$\begin{aligned}
deg\ [L_0, A] &= deg\ \left[L_0, \sum_{i=0}^{k} \sum_{j=0}^{i} h_{ij} D_1^{i-j} H_1^j\right] \\
&\le \max_{i,j} deg\ [L_0, h_{ij} D_1^{i-j} H_1^j] \le \max_{i,j} deg\ h_{ij} + 2i + 1 \\
&= deg\ A + 1. \qquad \square
\end{aligned}$$

By putting Theorems 4.12, 4.14, and 4.17 together, we obtain the following.

THEOREM 4.19. *If E has linear rank 1 and is finite-dimensional, then the Ω-matrix has constant entries.*

4.3. Classification results. First assume E has linear rank 1. By Theorem 4.9, h_i's must be degree at most 1 polynomials in x_1. By Theorem 4.19, ω_{12} is a constant.

(i) If $\omega_{12} = 0$, $[L_0, D_1] = \frac{1}{2}\frac{\partial \eta}{\partial x_1} \in E$. Thus, η must be a degree 2 polynomial in x_1 plus a C^∞-function in x_2. $E = \{L_0, x_1, D_1, 1\}$. For example, $f_1 = x_1$, $f_2 = \sin x_2$, $h_1 = x_1$, $\eta = 2x_1^2 + 1 + \cos x_2 + \sin^2 x_2$.

(ii) If $\omega_{12} \neq 0$,

$$A_1 := [L_0, D_1] = \omega_{12} D_2 + \frac{1}{2}\frac{\partial \eta}{\partial x_1} \in E, \tag{4.54}$$

$$A_2 := [D_1, A_1] = -\omega_{12}^2 + \frac{1}{2}\frac{\partial^2 \eta}{\partial x_1^2} \in E, \tag{4.55}$$

$$\begin{aligned}
A_3 := [L_0, A_1] = &\left(\frac{1}{2}\frac{\partial^2 \eta}{\partial x_1^2} - \omega_{12}^2\right) D_1 + \frac{1}{2}\frac{\partial^2 \eta}{\partial x_1 \partial x_2} D_2 \\
&+ \frac{1}{2}\omega_{12}\frac{\partial \eta}{\partial x_2} + \frac{1}{4}\frac{\partial^3 \eta}{\partial x_1^3} + \frac{1}{4}\frac{\partial^3 \eta}{\partial x_1 \partial x_2^2} \in E.
\end{aligned} \tag{4.56}$$

By (4.55), $\eta = d_0 x_1^3 + d_1 x_1^2 + e_2(x_2) x_1 + e_3(x_2)$. By Theorem 3.21, $d_0 = 0$. By (4.56) and $D_1 \in E$,

$$\frac{1}{2}\frac{\partial^2 \eta}{\partial x_1 \partial x_2} D_2 + \frac{1}{2}\omega_{12}\frac{\partial \eta}{\partial x_2} + \frac{1}{4}\frac{\partial^3 \eta}{\partial x_1 \partial x_2^2} \in E. \tag{4.57}$$

By Theorem 4.1, $\frac{\partial^2 \eta}{\partial x_1 \partial x_2} = e_2'(x_2)$ is a polynomial of degree at most 1. Thus, e_2 is a polynomial of degree at most 2 in x_2. By substituting e_2 and e_3 into η and removing

D_1, x_1, and 1 from (4.54) and (4.57), one has

$$\bar{A}_1 = \omega_{12} D_2 + \frac{1}{2} e_2 \in E, \tag{4.58}$$

$$\bar{A}_3 = \frac{1}{2} e_2' D_2 + \frac{1}{2} \omega_{12} (e_2' x_1 + e_3') \in E. \tag{4.59}$$

If e_2 is a degree 2 polynomial in x_2, then $[\bar{A}_1, \bar{A}_3] - \frac{1}{2} e_2'' \bar{A}_1 = \frac{1}{2} \omega_{12}^2 e_3'' - \frac{1}{4}(e_2 e_2'' + e_2' e_2') + \frac{1}{2} \omega_{12}^2 e_2'' x_1 \in E$. Since the degree 2 term of $e_2 e_2'' + e_2' e_2'$ will never be zero, $e_3(x_2)$ must be a degree 4 polynomial. Now, consider

$$\begin{aligned} B_1 = [L_0, \bar{A}_3] &= \frac{1}{2} e_2'' D_2^2 + \frac{1}{2} \omega_{12} (e_2'' x_1 + e_3'') D_2 + \frac{1}{4} e_2' (e_2' x_1 + e_3') + \frac{1}{4} \omega_{12} e_3'' \in E, \\ B_2 = [L_0, B_1] &= \frac{1}{2} e_2'' [L_0, D_2^2] + \left[L_0, \frac{1}{2} \omega_{12} (e_2'' x_1 + e_3'') D_2 \right] \\ &= \frac{1}{2} \omega_{12} e_3''' D_2^2 - \frac{1}{2} \omega_{12} e_2'' D_2 D_1 \in E, \qquad \text{mod } U_1. \end{aligned}$$

From the discussion of the proof of Lemma 4.13 (iii), e_3''' must be a constant. Contradiction! Hence, e_2 must be a degree 1 polynomial.

Consider $\omega_{12} \bar{A}_3 - \frac{1}{2} e_2' \bar{A}_1 = \frac{1}{2} \omega_{12}^2 e_2' x_1 + \frac{1}{2} \omega_{12}^2 e_3' - \frac{1}{4} e_2' e_2 \in E \Longrightarrow \frac{1}{2} \omega_{12}^2 e_3' - \frac{1}{4} e_2' e_2$ is independent of x_2 and e_3 must be a degree 2 polynomial. Hence, η is a degree 2 polynomial in x_1 and x_2. E is of dimension 5 and $E = \{L_0, x_1, D_1, D_2 + c x_2, 1\}$.

For example, $f_1 = 5x_1 - 3x_2$, $f_2 = 4x_2$, $h_1 = x_1$. Then $\omega_{12} = 3$ and $\eta = 26x_1^2 - 30 x_1 x_2 + 25 x_2^2 + 9$. It is easy to show that $E = \{L_0, x_1, D_1, D_2 - 5x_2, 1\}$.

If E has linear rank 0, h_i's must be constants, and $E = \{L_0\}$ or $E = \{L_0, 1\}$.

If E has linear rank 2, E is of maximal rank. The Ω-matrix must have constant entries and $E = \{L_0, x_1, x_2, D_1, D_2, 1\}$ by Theorem 2.7. In summary, we have the following.

THEOREM 4.20. *Let $n = 2$. If E is finite-dimensional, then*

(1) *if h_i's are constants, $E = \{L_0\}$ or $E = \{L_0, 1\}$;*

(2) *otherwise, Ω-matrix has constant entries. h_i's must be affine in x_1 and x_2. E has dimension of either 4, 5, or 6.*

Moreover, from the above discussion, it is easy to see that if E is finite-dimensional, it has only elements with order less than or equal to 2. Thus, the Levine conjecture holds for the finite-dimensional estimation algebras with state dimension 2.

REFERENCES

[1] V. E. BENES, *Exact finite dimensional filters for certain diffusions with nonlinear drift*, Stochastics, 5 (1981), pp. 65–92.

[2] R. W. BROCKETT, *Classification and equivalence in estimation theory*, in Proceedings of the 18th Annual IEEE Conference on Decision and Control, Ft. Lauderdale, FL, 1979, pp. 172–175.

[3] R. W. BROCKETT, *Nonlinear systems and nonlinear estimation theory*, in Stochastic Systems: The Mathematics of Filtering and Identification and Applications, M. Hazewinkel and J. S. Willems, eds., Reidel, Dordrecht, 1981, pp. 479–504.

[4] R. W. BROCKETT, *Nonlinear control theory and differential geometry*, in Proceedings of the International Congress of Mathematics, Warsaw, 1983, pp. 1357–1368.

[5] R. W. BROCKETT AND J. M. C. CLARK, *The geometry of the conditional density functions*, in Analysis and Optimization of Stochastic Systems, O. L. R. Jacobs et. al., eds., Academic Press, New York, 1980, pp. 299–309.

[6] R. S. BUCY AND J. PAGES, *A prior error bounds for the cubic sensor problem*, IEEE Trans. Automat. Control, 24 (1979), pp. 948–953.

[7] M. Chaleyat-Maurel and D. Michel, *Des resultats de non existence de filtre de dimension finie*, Stochastics, 13 (1984), pp. 83–102.
[8] W.-L. Chiou and S. S.-T. Yau, *Finite-dimensional filters with nonlinear drift* II: *Brockett's problem on classification of finite-dimensional estimation algebras*, SIAM J. Control Optim., 32 (1994), pp. 297–310.
[9] R.-T. Dong, L.-F. Tam, W. S. Wong, and S. S.-T. Yau, *Structure and classification theorems of finite-dimensional exact estimation algebras*, SIAM J. Control Optim., 29 (1991), pp. 866–877.
[10] M. Hazewinkel, *Lectures on linear and nonlinear filtering*, in Analysis and Estimation of Stochastic Mechanical Systems, W. Schiehlen and W. Wedig, eds., Springer-Verlag, New York, 1988, pp. 103–135.
[11] M. Hazewinkel, S. I. Marcus, and H. J. Sussmann, *Nonexistence of finite dimensional filters for conditional statistics of the cubic sensor problem*, Systems Control Lett., 3 (1983), pp. 331–340.
[12] J. Levine, *Finite dimensional realizations of stochastic P.D.E.'s and application to filtering*, Stochastics, 37 (1991), pp. 75–103.
[13] R. S. Liptser and A. N. Shiryaev, *Statistics of Random Processes* I, Springer-Verlag, New York, 1977.
[14] S. I. Marcus, *Algebraic and geometric methods in nonlinear filtering*, SIAM J. Control Optim., 22 (1984), pp. 817–844.
[15] S. K. Mitter, *On the analogy between mathematical problems of nonlinear filtering and quantum physics*, Ricerche Automat., 10 (1979), pp. 163–216.
[16] D. Ocone, *Finite dimensional estimation algebras in nonlinear filtering*, in Stochastic Systems: The Mathematics of Filtering and Identification and Applications, M. Hazewinkel and J. S. Willems, eds., Reidel, Dordrecht, 1981, pp. 629–636.
[17] A. Rasoulian and S. S.-T. Yau, *Finite dimensional filters with nonlinear drift* IX: *Construction of finite dimensional estimation algebras of nonmaximal rank*, Systems Control Lett., 30 (1997), pp. 109–118.
[18] L.-F. Tam, W. S. Wong, and S. S.-T. Yau, *On a necessary and sufficient condition for finite dimensionality of estimation algebras*, SIAM J. Control Optim., 28 (1990), pp. 173–185.
[19] J. Wei and E. Norman, *On the global representation of the solutions of linear differential equations as a product of exponentials*, Proc. Amer. Math. Sci., 15 (1964), pp. 327–334.
[20] W. S. Wong, *New classes of finite dimensional nonlinear filters*, Systems Control Lett., 3 (1983), pp. 155–164.
[21] W. S. Wong, *On a new class of finite dimensional estimation algebras*, Systems Control Lett., 9 (1987), pp. 79–83.
[22] W. S. Wong, *Theorems on the structure of finite dimensional estimation algebras*, Systems Control Lett., 9 (1987), pp. 117–124.
[23] W. S. Wong and S. S.-T. Yau, *The estimation algebra of nonlinear filtering systems*, in Mathematical Control Theory, Special Volume Dedicated to the 60th Birthday of Brockett, J. Baillieul and J. C. Willems eds., Springer-Verlag, New York, 1998, pp. 33–65.
[24] S. S.-T. Yau, *Finite-dimensional filters with nonlinear drift* I: *A class of filters including both Kalman-Bucy filters and Benes filters*, J. Math. Systems Estim. Control, 4 (1994), pp. 181–203.
[25] S. S.-T. Yau, *Brockett's problem on nonlinear filtering theory*, in Lectures on Systems, Control, and Information, AMS/IP Stud. Adv. Math. 17, AMS, Providence, RI, 2000, pp. 177–212.
[26] S. S.-T. Yau and W. L. Chiou, *Recent results on classification of finite dimensional estimation algebras: Dimension of state space* 2, in Proceedings of the 30th Annual Conference on Decision and Control, Brighton, England, 1991, pp. 2758–2760.
[27] S. S.-T. Yau and A. Rasoulian, *Classification of four-dimensional estimation algebras*, IEEE Trans. Automat. Control, 44 (1999), pp. 2312–2318.