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REAL TIME SOLUTION OF THE NONLINEAR FILTERING PROBLEM WITHOUT MEMORY II*

SHING-TUNG YAU† AND STEPHEN S.-T. YAU‡

Dedicated to Professor Tyrone Duncan on the occasion of his 65th birthday

Abstract. It is well known that the nonlinear filtering problem has important applications in both military and commercial industries. The central problem of nonlinear filtering is to solve the Duncan–Mortensen–Zakai (DMZ) equation in real time and in a memoryless manner. The purpose of this paper is to show that, under very mild conditions (which essentially say that the growth of the observation $|h|$ is greater than the growth of the drift $|f|$), the DMZ equation admits a unique nonnegative weak solution u which can be approximated by a solution u_R of the DMZ equation on the ball B_R with $u_R|_{\partial B_R} = 0$. The error of this approximation is bounded by a function of R which tends to zero as R goes to infinity. The solution u_R can in turn be approximated efficiently by an algorithm depending only on solving the observation-independent Kolmogorov equation on B_R. In theory, our algorithm can solve basically all engineering problems in real time. Specifically, we show that the solution obtained from our algorithms converges to the solution of the DMZ equation in the L^1 sense. Equally important, we have a precise error estimate of this convergence, which is important in numerical computation.

Key words. nonlinear filtering, DMZ equation, conditional probability density, Kolmogorov equation

AMS subject classifications. 35K15, 60G35, 62M20, 65N15, 90E10, 93E11

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1. Introduction.

In 1961, Kalman and Bucy [Ka-Bu] first established the finite-dimensional filter for the linear filtering model with Gaussian initial distribution, which is highly influential in modern industry. Since then filtering theory has proved useful in science and engineering, for example, the navigational and guidance systems, radar tracking, sonar ranging, and satellite and airplane orbit determination. Despite its usefulness, however, the Kalman–Bucy filter is not perfect. Its main weakness is that it is restricted to the linear dynamical system with Gaussian initial distribution. Therefore there has been tremendous interest in solving the nonlinear filtering problem which involves the estimation of a stochastic process $x = \{x_t\}$ (called the signal or state process) that cannot be observed directly. Information containing x is obtained from observations of a related process $y = \{y_t\}$ (the observation process). The goal of nonlinear filtering is to determine the conditional density $\rho(t, x)$ of x_t given the observation history of $\{y_s : 0 \leq s \leq t\}$. In the late 1960s, Duncan [Du], Mortensen [Mo], and Zakai [Za] independently derived the Duncan–Mortensen–Zakai (DMZ) equation for the nonlinear filtering theory, which the conditional probability density $\rho(t, x)$ has to satisfy. The central problem of nonlinear filtering theory is to solve the DMZ equation in real time and in a memoryless way.

In 2000, we [Ya-Ya] proposed a novel algorithm to do just that. Under the as-

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†Department of Mathematics, Harvard University, Cambridge, MA 02138 (yau@math.harvard.edu). Research partially supported by the U.S. Army Research Office.

‡Institute of Mathematics, East China Normal University, Shanghai, China. Current address: Department of Mathematics, Statistics and Computer Science (MC 249), University of Illinois at Chicago, 851 South Morgan Street, Chicago, IL 60607-7045 (yau@uic.edu).

sumptions that the drift terms $f_i(x)$, $1 \le i \le n$, and their first and second derivatives, and the observation terms $h_i(x)$, $1 \le i \le m$, and their first derivatives, have linear growth, we showed that the solution obtained from our algorithms converges to the true solution of the DMZ equation. Although the above approach is quite successful, so far it cannot handle the famous cubic sensor in engineering in which $f(x) = 0$ and $h(x) = x^3$. It is well known that there is no finite-dimensional filter for the cubic sensor [Su].

The purpose of this paper is to show that under very mild conditions (A.2), (A.17), and (C.3) (which essentially say that the growth of $|h|$ is greater than the growth of $|f|$), the DMZ equation admits a unique nonnegative solution $u \in W_0^{1,1}((0,T) \times \mathbb{R}^n)$ which can be approximated by solutions u_R of the DMZ equation on the ball B_R with $u_R|_{\partial B_R} = 0$. The rate of convergence can be efficiently estimated in the L^1 norm. The solution u_R can in turn be approximated efficiently by an algorithm depending only on solving the time-independent Kolmogorov equation on B_R. Our algorithm can solve practically all engineering problems, including the cubic sensor problem in real time and in a memoryless fashion. Specifically we show that the solution obtained from our algorithms converges to the solution of the DMZ equation in the L^1 sense. Equally important, we have a precise error estimate of this convergence, which is important in numerical computation.

The filtering problem considered here is based on the signal observation model

$$\begin{cases} dx(t) = f(x(t))\,dt + dv(t), & x(0) = x_0, \\ dy(t) = h(x(t))\,dt + dw(t), & y(0) = 0, \end{cases} \tag{1.1}$$

in which x, v, y, and w are, respectively, $\mathbb{R}^n$-, $\mathbb{R}^n$-, $\mathbb{R}^m$-, and $\mathbb{R}^m$-valued processes and v and w have components that are independent, standard Brownian processes. We further assume that f and h are C^∞ smooth vector-valued. We shall refer to $x(t)$ as the state of the system at time t and $y(t)$ as the observation at time t.

Let $\rho(t,x)$ denote the conditional probability density of the state given the observation $\{y(s)\colon 0 \le s \le t\}$. It is well known that $\rho(t,x)$ is given by normalizing a function, $\sigma(t,x)$, which satisfies the following DMZ equation:

$$d\sigma(t,x) = L_0\sigma(t,x)\,dt + \sum_{i=1}^{n} L_i\sigma(t,x)dy_i(t), \quad \sigma(0,x) = \sigma_0, \tag{1.2}$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i} - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2, \tag{1.3}$$

and for $i = 1, \dots, m$, L_i is the zero degree differential operator of multiplication by h_i. (Here we have used the notation p_i to represent the ith component of the vector p.) σ_0 is the probability density of the initial point x_0.

Equation (1.2) is a stochastic partial differential equation. In real applications, we are interested in constructing robust state estimators from observed sample paths with some property of robustness. Davis [Da] studied this problem and proposed some robust algorithms. In our case, his basic idea reduces to defining a new unnormalized density

$$u(t,x) = \exp\left(-\sum_{i=1}^{m} h_i(x)y_i(t)\right)\sigma(t,x). \tag{1.4}$$

It is easy to show that $u(t,x)$ satisfies the time-varying partial differential equation

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = L_0 u(t,x) + \sum_{i=1}^m y_i(t)[L_0, L_i]u(t,x) \\ \qquad\qquad + \frac{1}{2}\sum_{i,j=1}^m y_i(t)y_j(t)[[L_0,L_i],L_j]u(t,x), \\ u(0,x) = \sigma_0, \end{cases} \tag{1.5}$$

where $[\cdot,\cdot]$ denotes the Lie bracket. It is shown in [Ya-Ya, p. 236] that the robust DMZ equation (1.5) is of the form

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + (-f(x) + \nabla K(t,x))\cdot \nabla u(t,x) \\ \qquad\qquad + \Big(-\operatorname{div} f(x) - \frac{1}{2}|h(x)|^2 + \frac{1}{2}\Delta K(t,x) \\ \qquad\qquad\qquad - f(x)\cdot\nabla K(t,x) + \frac{1}{2}|\nabla K(t,x)|^2\Big)u(t,x), \\ u(0,x) = \sigma_0(x), \end{cases} \tag{1.6}$$

where $K = \sum_{j=1}^m y_i(t)h_j(x)$, $f = (f_1,\dots,f_n)$, and $h = (h_1,\dots,h_m)$.

To simplify our presentation, we introduce the following condition.

Condition (C_1).

$$-\frac{1}{2}|h|^2 - \frac{1}{2}\Delta K - f\cdot\nabla K + \frac{1}{2}|\nabla K|^2 + |f - \nabla K| \le c_1 \quad \forall\ (t,x)\in[0,T]\times\mathbb{R}^n,$$

where c_1 is a constant possibly depending on T.

Our main theorems are as follows.

THEOREM A. *Consider the filtering model* (1.1). *For any $T > 0$, let u be a solution of the robust DMZ equation* (1.6) *in $[0,T]\times\mathbb{R}^n$. Assume Condition* (C_1) *is satisfied.*

Then

$$\sup_{0\le t\le T}\int_{\mathbb{R}^n} e^{\sqrt{1+|x|^2}}u(t,x) \le e^{(c_1+\frac{n+1}{2})T}\int_{\mathbb{R}^n} e^{\sqrt{1+|x|^2}}u(0,x). \tag{1.7}$$

In particular,

$$\sup_{0\le t\le T}\int_{|x|\ge R} u(t,x) \le e^{-\sqrt{1+R^2}}e^{(c_1+\frac{n+1}{2})T}\int_{\mathbb{R}^n} e^{\sqrt{1+|x|^2}}u(0,x). \tag{1.8}$$

Theorem A above says that one can choose a ball large enough to capture almost all the density. In fact by (1.8) we have a precise estimate of density lying outside this ball.

THEOREM B. *Consider the filtering model* (1.1). *For any $T > 0$, let u be a solution of the robust DMZ equation* (1.6) *in $[0,T]\times\mathbb{R}^n$. Assume the following:*

(1) *Condition* (C_1) *is satisfied.*

(2) $-\frac{1}{2}|h|^2 - \frac{1}{2}\Delta K - f(x)\cdot\nabla K(t,x) + \frac{1}{2}|\nabla K|^2 + 12 + 2n + 4|f-\nabla K| \le c_2$ *for all $(t,x)\in[0,T]\times\mathbb{R}^n$, where c_2 is a constant possibly depending on T.*

(3) $e^{-\sqrt{1+|x|^2}}[12 + 2n + 4|f-\nabla K|] \le c_3$ *for all $(t,x)\in[0,T]\times\mathbb{R}^n$.*

Let $R\ge 1$ and u_R be the solution of the following DMZ equation on the ball B_R:

$$\begin{cases} \frac{\partial u_R}{\partial t} = \frac{1}{2}\Delta u_R + (-f + \nabla K)\cdot\nabla u_R \\ \qquad\qquad + \Big(-\operatorname{div} f - \frac{1}{2}|h|^2 + \frac{1}{2}\Delta K - f\cdot\nabla K + \frac{1}{2}|\nabla K|^2\Big)u_R, \\ u_R(t,x) = 0 \qquad \text{for } (t,x)\in[0,T]\times\partial B_R, \\ u_R(0,x) = \sigma_0(x). \end{cases} \tag{1.9}$$

Let $v = u - u_R$. Then $v \geq 0$ for all $(t,x) \in [0,T] \times B_R$ and

$$\int_{B_R} \phi v(T,x) \leq \frac{e^{c_2T}-1}{c_2} c_3 e^{-R} e^{(c_1+\frac{n+1}{2})T} \int_{\mathbb{R}^n} e^{\sqrt{1+|x|^2}} u(0,x), \tag{1.10}$$

where $\phi(x) = e^{\frac{|x|^4}{R^3} - \frac{2|x|^2}{R}} - e^{-R}$. In particular

$$\int_{B_{\frac{R}{2}}} v(T,x) \leq \frac{2(e^{c_2T}-1)}{c_2} c_3 e^{-\frac{9}{16}R} e^{(c_1+\frac{n+1}{2})T} \int_{\mathbb{R}^n} e^{\sqrt{1+|x|^2}} u(0,x). \tag{1.11}$$

Theorem B above says that we can approximate u by u_R. The approximation is good if R is large enough. In fact we have a precise error estimate of this approximation by (1.11).

Theorem C. *Let Ω be a bounded domain in $\mathbb{R}^n$. Let $F\colon [0,T] \times \Omega \to \mathbb{R}^n$ be a family of vector fields C^∞ in x and Hölder continuous in t with exponent α and let $J\colon [0,T] \times \Omega \to \mathbb{R}$ be a C^∞ function in x and Hölder continuous in t with exponent α such that the following properties are satisfied:*

$$|\operatorname{div} F(t,x)| + 2|J(t,x)| + |F(t,x)| \leq c \quad \text{for } (t,x) \in [0,T] \times \Omega, \tag{1.12}$$

$$|F(t,x) - F(\bar{t},x)| + |\operatorname{div} F(t,x) - \operatorname{div} F(\bar{t},x)| + |J(t,x) - J(\bar{t},x)| \leq c_1|t-\bar{t}|^\alpha \quad \text{for } (t,x), (\bar{t},x) \in [0,T] \times \Omega. \tag{1.13}$$

Let $u(t,x)$ be the solution on $[0,T] \times \Omega$ of the equation

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + F(t,x) \cdot \nabla u(t,x) + J(t,x)u(t,x), \\ u(0,x) = \sigma_0(x), \\ u(t,x)\big|_{\partial\Omega} = 0. \end{cases} \tag{1.14}$$

For any $0 \leq \tau \leq T$, let $\mathcal{P}_k = \{0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_k = \tau\}$ be a partition of $[0,\tau]$, where $\tau_i = \frac{i\tau}{k}$. Let $u_i(t,x)$ be the solution on $[\tau_{i-1}, \tau_i] \times \Omega$ of the equation

$$\begin{cases} \frac{\partial u_i}{\partial t}(t,x) = \frac{1}{2}\Delta u_i(t,x) + F(\tau_{i-1},x) \cdot \nabla u_i(t,x) + J(\tau_{i-1},x)u_i(t,x), \\ u_i(\tau_{i-1},x) = u_{i-1}(\tau_{i-1},x), \\ u_i(t,x)\big|_{\partial\Omega} = 0. \end{cases} \tag{1.15}$$

Here we use the convention $u_0(t,x) = \sigma(x)$. Then the solution $u(t,x)$ of (1.14) can be computed by means of the solution $u_i(t,x)$ of (1.15). More specifically, $u(\tau,x) = \lim_{k\to\infty} u_k(\tau,x)$ in the L^1 sense on Ω and the following estimate holds:

$$\int_\Omega |u - u_k|(\tau_k,x) \leq \frac{2c_2}{\alpha+1} \frac{T^{\alpha+1}e^{cT}}{k^\alpha}, \tag{1.16}$$

where

$$c_2 = c_1 e^{cT} + c_1\sqrt{\text{Vol }(\Omega)}\, e^{c^2T} \sqrt{2c^2T \int_\Omega u^2(0,x) + \int_\Omega |\nabla u(0,x)|^2}. \tag{1.17}$$

The right-hand side of (1.16) goes to zero as $k \to \infty$.

In case (1.14) and (1.15) are DMZ equations, i.e., $F(t,x) = -f(x) + \nabla K$ and $J(t,x) = -\operatorname{div} f - \frac{1}{2}|h|^2 + \frac{1}{2}\Delta K - f \cdot \nabla K + \frac{1}{2}|\nabla K|^2$, by Proposition 2.1 below (which is similar to Proposition 3.1 of [Ya-Ya]), $u_i(\tau_i, x)$ can be computed by $\widetilde{u}_i(\tau_i, x)$, where $\widetilde{u}_i(t,x)$ for $\tau_{i-1} \le t \le \tau_i$ satisfies the Kolmogorov equation

(1.18)

$$\begin{cases} \frac{\partial \widetilde{u}_i}{\partial t}(t,x) = \frac{1}{2}\Delta \widetilde{u}_i(t,x) - \sum_{j=1}^n f_j(x) \frac{\partial \widetilde{u}_i}{\partial x_j}(t,x) - \left(\operatorname{div} f(x) + \frac{1}{2}\sum_{j=1}^m h_j^2(x)\right) \widetilde{u}_i(t,x), \\ \widetilde{u}_i(\tau_{i-1}, x) = \exp\left(\sum_{j=1}^m \left(y_j(\tau_{i-1}) - y_j(\tau_{i-2})\right) h_j(x)\right) \widetilde{u}_{i-1}(\tau_{i-1}, x). \end{cases}$$

In fact

$$u_i(\tau_i, x) = \exp\left(-\sum_{j=1}^m y_j(\tau_{i-1}) h_j(x)\right) \widetilde{u}_i(\tau_i, x). \tag{1.19}$$

Therefore theoretically to solve the DMZ equation in a real time manner, we only need to compute the following Kolmogorov equation off-line:

$$\begin{cases} \frac{\partial \widetilde{u}}{\partial t}(t,x) = \frac{1}{2}\Delta \widetilde{u}(t,x) - \sum_{j=1}^n f_j(x) \frac{\partial \widetilde{u}}{\partial x_j}(t,x) - \left(\operatorname{div} f(x) + \frac{1}{2}\sum_{j=1}^m h_j^2(x)\right)\widetilde{u}(t,x), \\ \widetilde{u}(0,x) = \phi_i(x), \end{cases}$$

where $\{\phi_i(x)\}$ is an orthonormal base in $L^2(\mathbb{R}^n)$. The only real time computation here is to express arbitrary initial condition $\phi(x)$ as the linear combination of $\phi_i(x)$. But this can be done by means of parallel computation.

The idea of solving the Kolmogorov equation "off-line" for the elements of an orthogonal basis has a substantial history; see, for example, [L-M-R] and the references therein. In the Lototsky–Mikulevicius–Rozovskii [L-M-R] approach, the authors used the Cameron–Martin expansion for the solution of the DMZ equation. Unfortunately, to determine the coefficients of the expansion, they need to consider a system of Kolmogorov-type equations which is a recursive system. The advantage of our method is that we need to deal with only one Kolmogorov equation.

THEOREM D. *Let u_R be the solution of* (1.9)*, the DMZ equation on B_R. Assume the following:*

(1) *$f(x)$ and $h(x)$ have at most polynomial growth.*

(2) *For any $0 \le t \le T$, there exist positive integer m and positive constants c' and c'' independent of R such that the following two inequalities hold on $\mathbb{R}^n$:*

  (a) $\frac{m^2}{2}|x|^{2m-2} - \frac{m}{2}(m+n-2)|x|^{m-2} - m|x|^{m-2}x \cdot (f - \nabla K) - \frac{\Delta K}{2} - \frac{1}{2}|h|^2 - f \cdot \nabla K + \frac{1}{2}|\nabla K|^2 \ge -c'$.

  (b) $|\frac{m^2|x|^{2m-2}}{2} - \frac{m(m+n-2)}{2}|x|^{m-2} - m|x|^{m-2}(f - \nabla K)\cdot x| \le \frac{m(m+1)}{2}|x|^{2m-2} + c''$.

(3) $-\frac{1}{2}|h|^2 - \frac{1}{2}\Delta K - \sum_{j=1}^n f_j \frac{\partial K}{\partial x_j} + \frac{1}{2}|\nabla K|^2 \le c_1$ *for all* $(t,x) \in [0,T] \times \mathbb{R}^n$*, where c_1 is a constant possibly depending on T.*

Then for any $R_0 < R$,

$$\begin{aligned} &\int_{B_{R_0}} (e^{-|x|^m} - e^{-R_0^m}) u_R(T,x) \\ &\quad \ge e^{-c'T} \int_{B_{R_0}} (e^{-|x|^m} - e^{-R_0^m}) \sigma_0(x) \\ &\qquad + \frac{e^{-R_0^m}}{c'} \left(\frac{m(m+1)}{2} R_0^{2m-2} + c''\right) (1 - e^{c'T}) \int_{B_R} \sigma_0(x). \end{aligned}$$

In particular, the solution u of the robust DMZ equation on $\mathbb{R}^n$ has the estimate

$$\int_{\mathbb{R}^n} e^{-|x|^m} u(T,x) \geq e^{-c'T} \int_{\mathbb{R}^n} e^{-|x|^m} \sigma_0(x).$$

In practical nonlinear filtering computation, it is important to know how much density remains within the given ball. Theorem D provides such a lower estimate. In particular, the solution u of the DMZ equation in $\mathbb{R}^n$ obtained by taking $\lim_{R\to\infty} u_R$, where u_R is the solution of the DMZ equation in the ball B_R, is a nontrivial solution.

In the appendix, we give a priori estimation of derivatives of the solution of the DMZ equation up to second order. As a consequence we prove the existence of a weak solution of the DMZ equation. The uniqueness of the weak solution is shown in Appendix C.

Existence and uniqueness of solutions to the robust DMZ equation (1.6) have been treated by many well-known authors, including Pardoux [Pa1], [Pa2], Chaleyat-Maurel, Michel, and Pardoux [C-M-P], Rozovskii [Ro], Bensoussan [Be], Fleming and Mitter [Fl-Mi], Sussmann [Su], Michel [Mi], and Baras, Blankenship, and Hopkins [B-B-H]. They all obtained important estimates on the DMZ equation under special conditions. For example, Fleming and Mitter [Fl-Mi] treated the case where f and ∇f are bounded, while Michel [Mi] analyzed regularity properties of solutions to DMZ equations with bounded f and h. Pardoux's earlier paper [Pa1] treated the case f, h bounded using arguments based on coercivity. It also contains many other interesting ideas. Pardoux [Pa2] has also treated nonlinear filtering problems with unbounded coefficients (f, h have linear growth). Starting with methods somewhat like those used by [Pa3], Baras, Blankenship, and Hopkins also obtained important results on existence, uniqueness, and asymptotic behavior of solutions to a class of DMZ equations with unbounded coefficients. However, they focused on only one spatial dimension and their result cannot cover the linear case. The Sobolev space setup of Appendices B and C in this paper is quite standard in partial differential equations and has been used by many people; see, for example, [Pa1].

The splitting up method has been used extensively by many authors. This technique is like the Trotter product formula from semigroup theory. Hopkins and Wong [Ho-Wo] used the Trotter product formula to study nonlinear filtering. The approximation method proposed for the DMZ equation, that of operator splitting, has a history going back to Bensoussan, Glowinski, and Rascanu [B-G-R1], [B-G-R2]. More recent articles on operator splitting methods in nonlinear filtering are [Gy-Kr], [Na], [It], [It-Ro]. Rates of convergence and "true" numerical schemes are developed in [Fl-Le], [It], and [It-Ro]. As pointed out by Bensoussan, Glowinski, and Rascanu [B-G-R1, section 4.3, p. 1431] the method bears the serious limitation that h must be bounded. The numerics of the Kushner–Stratonovitch equations were studied by many people. Two highly competitive classes of methods are "particle methods" (see, for example, [D-J-P] and [Cr-Ly]), in which particles move according to the signal dynamics and are weighted, killed, or duplicated according to their likelihood, and "discrete state" approximations (see, for example, [Ku] and [Pa-Ph]). These methods work nicely under the assumption that h is bounded (cf. [D-J-P, p. 348]).

2. Some basic results. In this section, we recall some results from our previous paper. The following proposition plays a fundamental role in our real time solution to the robust DMZ equation (1.6) in a memoryless manner.

PROPOSITION 2.1. *$\widetilde{u}(t,x)$ satisfies the Kolmogorov equation*

$$\frac{\partial \widetilde{u}}{\partial t}(t,x) = \frac{1}{2}\Delta \widetilde{u}(t,x) - f(x)\cdot \nabla \widetilde{u}(t,x) - \left(\operatorname{div} f(x) + \frac{1}{2}\sum_{i=1}^{m} h_i^2(x)\right)\widetilde{u}(t,x) \tag{2.1}$$

for $\tau_{\ell-1} \le t \le \tau_\ell$ if and only if

$$u(t,x) = e^{-\sum_{i=1}^{m} y_i(\tau_{\ell-1})h_i(x)}\widetilde{u}(t,x) \tag{2.2}$$

satisfies the robust DMZ equation with observation being frozen at $y(\tau_{\ell-1})$:

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x) = {} & \frac{1}{2}\Delta u(t,x) + (-f(x) + \nabla K(\tau_{\ell-1},x))\cdot \nabla u(t,x)\\ & + \bigg(-\operatorname{div} f(x) - \frac{1}{2}|h(x)|^2 + \frac{1}{2}\Delta K(\tau_{\ell-1},x)\\ & \quad - f(x)\cdot \nabla K(\tau_{\ell-1},x) + \frac{1}{2}|\nabla K(\tau_{\ell-1},x)|^2\bigg) u(t,x).\end{aligned} \tag{2.3}$$

Proof. Proposition 2.1 is the left-hand version of Proposition 3.1 in [Ya-Ya]. The proof is a straightforward computation. ∎

We remark that (2.3) is obtained from the robust DMZ equation by freezing the observation term $y(t)$ to $y(\tau_{\ell-1})$. We shall show that the solution of (2.3) approximates the solution of the robust DMZ equation very well in the L^1 sense.

Suppose that $u(t,x)$ is the solution of the robust DMZ equation and we want to compute $u(\tau,x)$. Let $\mathcal{P}_k = \{0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_k = \tau\}$ be a partition of $[0,\tau]$, where $\tau_i = \frac{i\tau}{k}$. Let $u_i(t,x)$ be a solution of the following partial differential equation for $\tau_{i-1} \le \tau \le \tau_i$:

$$\begin{cases} \frac{\partial u_i}{\partial t}(t,x) = \frac{1}{2}\Delta u_i(t,x) + (-f(x) + \nabla K(\tau_{i-1},x))\cdot \nabla u_i(t,x) \\ \qquad + \Big(-\operatorname{div} f(x) - \frac{1}{2}|h(x)|^2 + \frac{1}{2}\Delta K(\tau_{i-1},x) \\ \qquad\quad - f(x)\cdot\nabla K(\tau_{i-1},x) + \frac{1}{2}|\nabla K(\tau_{i-1},x)|^2\Big) u_i(t,x), \\ u_i(\tau_{i-1},) = u_{i-1}(\tau_{i-1},x). \end{cases} \tag{2.4}$$

In section 4 below we shall show that $u(\tau,x) = \lim_{k\to\infty} u_k(\tau_k,x)$ in the L^1 sense. By Proposition 2.1, $u_1(\tau_1,x)$ can be computed by $\widetilde{u}_1(\tau_1,x)$, where $\widetilde{u}_1(t,x)$ for $0 \le t \le \tau_1$ satisfies (2.1) with initial condition

$$\widetilde{u}_1(0,x) = \sigma_0(x). \tag{2.5}$$

In fact

$$u_1(\tau_1,x) = \widetilde{u}_1(\tau_1,x). \tag{2.6}$$

In general Proposition 2.1 tells us that for $i \ge 2$, $u_i(\tau_i,x)$ can be computed by $\widetilde{u}_i(\tau_i,x)$, where $\widetilde{u}_i(t,x)$ for $\tau_{i-1} \le t \le \tau_i$ satisfies (2.1) with initial condition

$$\widetilde{u}_i(\tau_{i-1},x) = \exp\left[\sum_{j=1}^{m}(y_j(\tau_{i-1}) - y_j(\tau_{i-2}))h_j(x)\right]\widetilde{u}_{i-1}(\tau_{i-1},x), \tag{2.7}$$

where the last initial condition comes from

$$\begin{aligned}\widetilde{u}_i(\tau_{i-1},x) &= u_i(\tau_{i-1},x)\exp\left(\sum_{j=1}^m y_j(\tau_{i-1})h_j(x)\right)\\ &= u_{i-1}(\tau_{i-1},x)\exp\left(\sum_{j=1}^m y_j(\tau_{i-1})h_j(x)\right)\\ &= \exp\left(-\sum_{j=1}^m y_j(\tau_{i-2})h_j(x)\right)\widetilde{u}_{i-1}(\tau_{i-1},x)\exp\left(\sum_{j=1}^m y_j(\tau_{i-1})h_j(x)\right)\\ &= \exp\left[\sum_{j=1}^m (y_j(\tau_{i-1})-y_j(\tau_{i-2})h_j(x)\right]\widetilde{u}_{i-1}(\tau_{i-1},x).\end{aligned}$$

In fact,

$$u_i(\tau_i,x) = \exp\left(-\sum_{j=1}^m y_j(\tau_{i-1})h_j(x)\right)\widetilde{u}_i(\tau_i,x). \tag{2.8}$$

3. Reduction of the problem to the bounded domain case. In this section, we shall prove that in order to solve the robust DMZ equation (1.6) in $\mathbb{R}^n$, it suffices to solve the same equation in a bounded ball B_R with radius R. The important points here are that we know how large the R needs to be and that a precise error estimate is given. These are the essential ingredients for a successful implementation of nonlinear filters.

Proof of Theorem A. Let ϕ be a C^∞ function on $\mathbb{R}^n$ and $B_R = \{x \in \mathbb{R}^n : |x| \le R\}$.

Let u_R be the solution of (1.9), the DMZ equation on the ball B_R:

$$\begin{aligned}\frac{d}{dt}\int_{B_R} e^\phi u_R &= \frac{1}{2}\int_{B_R} e^\phi \Delta u_R + \int_{B_R} e^\phi(-f+\nabla K)\cdot\nabla u_R\\ &\quad + \int_{B_R}\left(-\operatorname{div} f - \frac{1}{2}|h|^2 + \frac{1}{2}\Delta K - f\cdot\nabla K + \frac{1}{2}|\nabla K|^2\right)e^\phi u_R\\ &= -\frac{1}{2}\int_{B_R} e^\phi\nabla\phi\cdot\nabla u_R + \frac{1}{2}\int_{\partial B_R} e^\phi\frac{\partial u_R}{\partial\nu} - \int_{B_R}\operatorname{div}[e^\phi(-f+\nabla K)]u_R\\ &\quad + \int_{\partial B_R} e^\phi u_R(-f+\nabla K)\cdot\nu\\ &\quad + \int_{B_R} e^\phi u_R\left(-\operatorname{div} f - \frac{1}{2}|h|^2 + \frac{1}{2}\Delta K - f\cdot\nabla K + \frac{1}{2}|\nabla K|^2\right)\\ &= \frac{1}{2}\int_{B_R}\operatorname{div}[e^\phi\nabla\phi]u_R - \frac{1}{2}\int_{\partial B_R} u_R e^\phi\nabla\phi\cdot\nu + \frac{1}{2}\int_{\partial B_R} e^\phi\frac{\partial u_R}{\partial\nu}\\ &\quad - \int_{B_R} e^\phi\nabla\phi\cdot(-f+\nabla K)u_R - \int_{B_R} e^\phi(-\operatorname{div} f + \Delta K)u_R\end{aligned}$$

$$
\begin{aligned}
&+\int_{\partial B_R} u_R e^{\phi}(-f+\nabla K)\cdot\nu \\
&+\int_{B_R} e^{\phi}u_R\left(-\operatorname{div} f-\frac{1}{2}|h|^2+\frac{1}{2}\Delta K-f\cdot\nabla K+\frac{1}{2}|\nabla K|^2\right)\\
=&\ \frac{1}{2}\int_{B_R} e^{\phi}u_R(\Delta\phi+|\nabla\phi|^2)+\int_{B_R} e^{\phi}u_R\nabla\phi\cdot(f-\nabla K)\\
&+\int_{B_R} e^{\phi}u_R\left(-\frac{1}{2}|h|^2-\frac{1}{2}\Delta K-f\cdot\nabla K+\frac{1}{2}|\nabla K|^2\right)\\
&-\frac{1}{2}\int_{\partial B_R} e^{\phi}u_R\nabla\phi\cdot\nu+\frac{1}{2}\int_{\partial B_R} e^{\phi}\frac{\partial u_R}{\partial\nu}+\int_{\partial B_R} e^{\phi}u_R(-f+\nabla K)\cdot\nu,
\end{aligned}
$$

where ν is the unit outward normal of ∂B_R. Choose $\phi=\sqrt{1+|x|^2}$. Then $\phi_i=\frac{x_i}{\sqrt{1+|x|^2}}$, $\phi_{ii}=\frac{1}{\sqrt{1+|x|^2}}-\frac{x_i^2}{(1+|x|^2)^{3/2}}$. Recall that $u|_{\partial B_R}=0$ and $\frac{\partial u_R}{\partial\nu}|_{\partial B_R}\le 0$. It follows that

$$
\begin{aligned}
\frac{d}{dt}\int_{B_R} e^{\phi}u_R \le&\ \int_{B_R} e^{\phi}u_R\Big[-\frac{1}{2}|h|^2-\frac{1}{2}\Delta K-f\cdot\nabla K+\frac{1}{2}|\nabla K|^2\\
&\qquad+\frac{1}{2}\Delta\phi+\frac{1}{2}|\nabla\phi|^2+\nabla\phi\cdot(f-\nabla K)\Big]\\
=&\ \int_{B_R} e^{\phi}u_R\Big[-\frac{1}{2}|h|^2-\frac{1}{2}\Delta K-f\cdot\nabla K+\frac{1}{2}|\nabla K|^2+\frac{n}{2\sqrt{1+|x|^2}}\\
&\qquad-\frac{|x|^2}{2(1+|x|^2)^{3/2}}+\frac{1}{2}\frac{|x|^2}{1+|x|^2}+\frac{x}{\sqrt{1+|x|^2}}(f-\nabla K)\Big]\\
\le&\ \int_{B_R} e^{\phi}u_R\left[-\frac{1}{2}|h|^2-\frac{1}{2}\Delta K-f\cdot\nabla K+\frac{1}{2}|\nabla K|^2+\frac{n+1}{2}+|f-\nabla K|\right]\\
\le&\ \left(c_1+\frac{n+1}{2}\right)\int_{B_R} e^{\phi}u_R.
\end{aligned}
$$

Hence

$$
\int_{B_R} e^{\phi}u_R(t,x)\le e^{(c_1+\frac{n+1}{2})t}\int_{B_R} e^{\phi}u_R(0,x)\qquad\forall\ t\in[0,T].
$$

Let R go to infinity. We have

$$
\int_{\mathbb{R}^n} e^{\phi}u(t,x)\le e^{(c_1+\frac{n+1}{2})t}\int_{\mathbb{R}^n} e^{\phi}u(0,x)\qquad\forall\ t\in[0,T],
$$

which implies

$$
\sup_{0\le t\le T}\int_{\mathbb{R}^n} e^{\phi}u(t,x)\le e^{(c_1+\frac{n+1}{2})T}\int_{\mathbb{R}^n} e^{\phi}u(0,x).
$$

In particular

$$
\begin{aligned}
e^{\sqrt{1+R^2}}\sup_{0\le t\le T}\int_{|x|\ge R} u(t,x)\le \sup_{0\le t\le T}\int_{|x|\ge R} e^{\phi}u(t,x)\le \sup_{0\le t\le T}\int_{\mathbb{R}^n} e^{\phi}u(t,x)\\
\le e^{(c_1+\frac{n+1}{2})T}\int_{\mathbb{R}^n} e^{\phi}u(0,x).
\end{aligned}
$$

This implies

$$\sup_{0\le t\le T}\int_{|x|\ge R} u(t,x) \le e^{-\sqrt{1+R^2}} e^{(c_1+\frac{n+1}{2})T} \int_{\mathbb{R}^n} e^{\sqrt{1+|x|^2}} u(0,x). \qquad \square$$

Theorem A says that we can choose R large enough so that $\sup_{0\le t\le T}\int_{|x|\ge R} u(t,x)$ is arbitrarily small. For numerical calculation, we can restrict the DMZ equation to the ball B_R. In fact we can prove Theorem B, which states that u_R is a good approximation of u when R is large.

Proof of Theorem B. By the maximum principle (cf. Theorem 1, p. 34 in Friedman's book [Fr]), we have $v \ge 0$ for $(t,x) \in [0,T]\times B_R$ since $v|_{\partial B_R} \ge 0$ for $0 \le t \le T$:

$$\begin{aligned}
\frac{d}{dt}\int_{B_R}\psi v = &\int_{B_R}\psi\frac{dv}{dt}\\
&= \frac{1}{2}\int_{B_R}\psi\Delta v + \int_{B_R}\psi(-f+\nabla K)\cdot\nabla v\\
&\quad + \int_{B_R}\left(-\operatorname{div} f - \frac{1}{2}|h|^2 + \frac{1}{2}\Delta K - f\cdot\nabla K + \frac{1}{2}|\nabla K|^2\right)\psi v\\
&= -\frac{1}{2}\int_{B_R}\nabla\psi\cdot\nabla v + \frac{1}{2}\int_{\partial B_R}\psi\frac{\partial v}{\partial \nu} - \int_{B_R}\operatorname{div}[\psi(-f+\nabla K)]v\\
&\quad + \int_{\partial B_R}\psi v(-f+\nabla K)\cdot\nu\\
&\quad + \int_{B_R}\left(-\operatorname{div} f - \frac{1}{2}|h|^2 + \frac{1}{2}\Delta K - f\cdot\nabla K + \frac{1}{2}|\nabla K|^2\right)\psi v\\
&= \frac{1}{2}\int_{B_R}(\Delta\psi)v - \frac{1}{2}\int_{\partial B_R} v\frac{\partial\psi}{\partial\nu} + \frac{1}{2}\int_{\partial B_R}\psi\frac{\partial v}{\partial\nu} - \int_{B_R}\nabla\psi\cdot(-f+\nabla K)v\\
&\quad - \int_{B_R}\psi(-\operatorname{div} f + \Delta K)v + \int_{\partial B_R}\psi v(-f+\nabla K)\cdot\nu\\
&\quad + \int_{B_R}\left(-\operatorname{div} f - \frac{1}{2}|h|^2 + \frac{1}{2}\Delta K - f\cdot\nabla K + \frac{1}{2}|\nabla K|^2\right)\psi v.
\end{aligned}$$

Let ϕ be a radial symmetric function such that $\phi|_{\partial B_R} = R$, $\nabla\phi|_{\partial B_R} = 0$, and ϕ is increasing with $|x|$. Let

$$\psi = e^{-\phi(x)} - e^{-R}.$$

Then $\psi|_{\partial B_R} = 0$ and $\nabla\psi|_{\partial B_R} = 0$. Hence

$$\begin{aligned}
\frac{d}{dt}\int_{B_R}\psi v &= \frac{1}{2}\int_{B_R}(\Delta\psi)v - \int_{B_R}\nabla\psi\cdot(-f+\nabla K)v\\
&\quad + \int_{B_R}\left(-\frac{1}{2}|h|^2 - \frac{1}{2}\Delta K - f\cdot\nabla K + \frac{1}{2}|\nabla K|^2\right)\psi v\\
&= \frac{1}{2}\int_{B_R} ve^{-\phi}(-\Delta\phi + |\nabla\phi|^2) - \int_{B_R} e^{-\phi}v[\nabla\phi\cdot(f-\nabla K)]
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{B_R} \left(-\frac{1}{2}|h|^2 - \frac{1}{2}\Delta K - f \cdot \nabla K + \frac{1}{2}|\nabla K|^2 \right) \psi v \\
&= \int_{B_R} \psi v \left[-\frac{1}{2}\Delta\phi + \frac{1}{2}|\nabla\phi|^2 - \nabla\phi \cdot (f - \nabla K) \right. \\
&\qquad\qquad \left. -\frac{1}{2}|h|^2 - \frac{1}{2}\Delta K - f \cdot \nabla K + \frac{1}{2}|\nabla K|^2 \right] \\
&+ e^{-R} \int_{B_R} \left[\frac{1}{2}(-\Delta\phi + |\nabla\phi|^2) - \nabla\phi \cdot (f - \nabla K) \right] v \\
&\le \sup_{B_R} \left[-\frac{1}{2}\Delta\phi + \frac{1}{2}|\nabla\phi|^2 - \nabla\phi \cdot (f - \nabla K) \right. \\
&\qquad\qquad \left. -\frac{1}{2}|h|^2 - \frac{1}{2}\Delta K - f \cdot \nabla K + \frac{1}{2}|\nabla K|^2 \right] \cdot \int_{B_R} \psi v \\
&+ e^{-R} \sup_{B_R} \left\{ e^{-\sqrt{1+|x|^2}} \left[\frac{1}{2}(-\Delta\phi + |\nabla\phi|^2) - \nabla\phi \cdot (f - \nabla K) \right] \right\} \\
&\qquad \times \int_{B_R} e^{\sqrt{1+|x|^2}} v.
\end{aligned}
$$

Observe that $0 \le v \le u$ for $(t, x) \in [0, T] \times B + R$. Let

$$
\chi(x) = 1 - (1 - x)^2 \quad \text{and} \quad \phi(x) = R\,\chi\left(\frac{|x|^2}{R^2}\right).
$$

Then

$$
\chi'(x) = 2(1 - x), \quad \chi''(x) = -2, \quad \chi(1) = 1, \quad \chi'(1) = 0,
$$

$$
\nabla\phi(x) = \frac{2x}{R}\chi'\left(\frac{|x|^2}{R^2}\right), \quad \Delta\phi = \frac{4|x|^2}{R^3}\chi''\left(\frac{|x|^2}{R^2}\right) + \frac{2n}{R}\chi'\left(\frac{|x|^2}{R^2}\right),
$$

$$
\begin{aligned}
&\sup_{B_R}\left[-\frac{1}{2}\Delta\phi + \frac{1}{2}|\nabla\phi|^2 - \nabla\phi \cdot (f - \nabla K) - \frac{1}{2}|h|^2 - \frac{1}{2}\Delta K - f \cdot \nabla K + \frac{1}{2}|\nabla K|^2 \right] \\
&\quad = \sup_{B_R}\left\{ -\frac{2|x|^2}{R^3}\chi''\left(\frac{|x|^2}{R^2}\right) - \frac{n}{R}\chi'\left(\frac{|x|^2}{R^2}\right) + 2\frac{|x|^2}{R^2}\left[\chi'\left(\frac{|x|^2}{R^2}\right)\right]^2 \right. \\
&\qquad\qquad \left. - \frac{2}{R}\chi'\left(\frac{|x|^2}{R^2}\right)[x \cdot f - x \cdot \nabla K] - \frac{1}{2}|h|^2 - \frac{1}{2}\Delta K - f \cdot \nabla K + \frac{1}{2}|\nabla K|^2 \right\} \\
&\quad \le \sup_{B_R}\left[\frac{4|x|^2}{R^3} + \frac{2n}{R} + \frac{8|x|^2}{R} + \frac{4}{R}|x||f - \nabla K| - \frac{1}{2}|h|^2 - \frac{1}{2}\Delta K - f \cdot \nabla K + \frac{1}{2}|\nabla K|^2 \right] \\
&\quad \le 12 + 2n + 4|x||f - \nabla K| - \frac{1}{2}|h|^2 - \frac{1}{2}\Delta K - f \cdot \nabla K + \frac{1}{2}|\nabla K|^2 \\
&\quad \le c_2.
\end{aligned}
$$

Similarly

$$\sup_{B_R}\left\{e^{-\sqrt{1+|x|^2}}\left[\frac{1}{2}(-\Delta\phi+|\nabla\phi|^2)-\nabla\phi\cdot(f-\nabla K)\right]\right\}$$
$$\le \sup_{B_R}\left\{e^{-\sqrt{1+|x|^2}}[12+2n+4|x||f-\nabla K|]\right\}\le c_3.$$

In view of Theorem A, we have

$$\begin{aligned}\frac{d}{dt}\int_{B_R}\psi v &\le c_2\int_{B_R}\psi v+e^{-R}c_3\int_{B_R}e^{\sqrt{1+|x|^2}}u\\ &\le c_2\int_{B_R}\psi v+e^{-R}c_3e^{(c_1+\frac{n+1}{2}T)}\int_{\mathbb{R}^n}e^{\sqrt{1+|x|^2}}u(0,x),\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\left[e^{-c_2t}\int_{B_R}\psi v\right] &= e^{-c_2t}\frac{d}{dt}\int_{B_R}\psi v-c_2e^{-c_2t}\int_{B_R}\psi v\\ &\le c_3e^{-R}e^{-c_2t}e^{c_1+\frac{n+1}{2}T}\int_{\mathbb{R}^n}e^{\sqrt{1+|x|^2}}u(0,x).\end{aligned}$$

Recall that $v(0,x)=0$ on B_R. Therefore we have

$$e^{-c_2T}\int_{B_R}\psi v(T,x)\le\frac{e^{-c_2T}-1}{-c_2}c_3e^{-R}e^{(c_1+\frac{n+1}{2})T}\int_{\mathbb{R}^n}e^{\sqrt{1+|x|^2}}u(0,x),$$

which implies

$$\int_{B_R}\psi v(T,x)\le\frac{e^{c_2T}-1}{c_2}c_3e^{-R}e^{(c_1+\frac{n+1}{2})T}\int_{\mathbb{R}^n}e^{\sqrt{1+|x|^2}}u(0,x).$$

Notice that $\psi(x)=e^{\frac{|x|^4}{R^3}-\frac{2|x|^2}{R}}-e^{-R}$:

$$\begin{aligned}\int_{B_R}\psi v(T,x) &\ge \int_{B_{\frac{R}{2}}}[e^{\frac{|x|^4}{R^3}-\frac{2|x|^2}{R}}-e^{-R}]v(T,x)\\ &\ge (e^{-\frac{7}{16}R}-e^{-R})\int_{B_{\frac{R}{2}}}v(T,x)\\ &= e^{-\frac{7}{16}R}(1-e^{-\frac{9}{16}R})\int_{B_{\frac{R}{2}}}v(T,x)\\ &\ge \frac{1}{2}e^{-\frac{7}{16}R}\int_{B_{\frac{R}{2}}}v(T,x).\end{aligned}$$

Therefore

$$\int_{B_{\frac{R}{2}}}v(T,x)\le\frac{2(e^{c_2T}-1)}{c_2}c_3e^{-\frac{9}{16}R}e^{(c_1+\frac{n+1}{2})T}\int_{\mathbb{R}^n}e^{\sqrt{1+|x|^2}}u(0,x).\qquad\square$$

Theorem B above says that if we replace u by u_R, then the error is small when R goes to infinity. In fact (1.11) gives the precise error estimate.

4. L^1-convergence. In this section, we shall show that our algorithm described in section 2 will yield an L^1-convergence for bounded domains, i.e., $u(\tau, x) = \lim_{k\to\infty} u_k(\tau_k, x)$ in the L^1 sense for bounded domain. We first begin with the following technical lemma.

LEMMA 4.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ and let $v\colon [0,T]\times\overline{\Omega}\to\mathbb{R}$ be a C^1 function. Assume that $v(t,x) = 0$ for $(t,x)\in[0,T]\times\partial\Omega$. Let $\Omega_t^+ = \{x\in\Omega\colon v(t,x)\geq 0\}$. Then*

$$\frac{d}{dt}\int_{\Omega_t^+} v(t,x) = \int_{\Omega_t^+}\frac{dv}{dt}(t,x) \quad \textit{for almost all } t\in[0,t].$$

Proof.

$$\begin{aligned}
\frac{d}{dt}\int_{\Omega_t^+} v(t,x) &= \lim_{\Delta t\to 0}\frac{\int_{\Omega^+_{t+\Delta t}} v(t+\Delta t,x) - \int_{\Omega_t^+} v(t,x)}{\Delta t}\\
&= \lim_{\Delta t\to 0}\int_{\Omega^+_{t+\Delta t}}\frac{v(t+\Delta t,x)-v(t,x)}{\Delta t} + \lim_{\Delta t\to 0}\frac{\int_{\Omega^+_{t+\Delta t}} v(t,x) - \int_{\Omega_t^+} v(t,x)}{\Delta t}\\
&= \int_{\Omega_t^+}\frac{dv}{dt}(t,x) + \lim_{\Delta t\to 0}\frac{\int_{\Omega^+_{t+\Delta t}-\Omega_t^+} v(t,x) - \int_{\Omega_t^+-\Omega^+_{t+\Delta t}} v(t,x)}{\Delta t}\\
&= \int_{\Omega_t^+}\frac{dv}{dt}(t,x) + \lim_{\Delta t\to 0}\frac{v(t,\xi_1)\mathrm{Vol}\,(\Omega^+_{t+\Delta t}-\Omega_t^+)}{\Delta t}\\
&\qquad - \lim_{\Delta t\to 0}\frac{v(t,\xi_2)\mathrm{Vol}\,(\Omega_t^+-\Omega^+_{t+\Delta t})}{\Delta t},
\end{aligned} \tag{4.1}$$

where $\xi_1\in\Omega^+_{t+\Delta t}-\Omega_t^+$ and $\xi_2\in\Omega_t^+-\Omega^+_{t+\Delta t}$. Clearly we have $\lim_{\Delta t\to 0} v(t,\xi_1) = 0 = \lim_{\Delta t\to 0} v(t,\xi_2)$. Therefore it remains to prove that

$$\lim_{\Delta t\to 0}\frac{\mathrm{Vol}\,(\Omega^+_{t+\Delta t}-\Omega_t^+)}{\Delta t} \quad\text{and}\quad \lim_{\Delta t\to 0}\frac{\mathrm{Vol}\,(\Omega_t^+-\Omega^+_{t+\Delta t})}{\Delta t}$$

are bounded for almost all t.

Let $w\colon A\to\mathbb{R}$ be a Lipschitz function, where $A\subset\mathbb{R}^n$ is measurable. The coarea formula for Euclidean space, an important tool of geometric measure theory (cf. [Fe], [Ma]), reads as follows:

$$\int_A h(x)|\nabla w(x)|\,dx = \int_{\mathbb{R}}\int_{w^{-1}(y)} h(x)\mathcal{H}^{n-1}_{|\cdot|}(x)\,dy, \tag{4.2}$$

where $\mathcal{H}^{n-1}_{|\cdot|}$ denotes the Hausdorff measure with respect to the Euclidean distance and $h\colon A\to[-\infty,\infty]$ is a measurable function.

Let $A = \overline{\Omega_t^+-\Omega^+_{t+\Delta t}} = \{x\in\Omega_t^+\colon v(t+\Delta t,x)\leq 0\}$. Let L be the Lipschitz constant such that

$$|v(t,x)-v(t+\Delta t,x)|\leq L\Delta t \quad \forall\ x\in\overline{\Omega}. \tag{4.3}$$

Since $v(t,x)\geq 0$ for $x\in A$, we have

$$v(t+\Delta t,x)\geq v(t,x)-L\Delta t\geq -L\Delta t \quad \text{for } x\in A. \tag{4.4}$$

Let $h(x) = \frac{1}{|\nabla v(t+\Delta t,x)|}$ and $w(x) = v(t+\Delta t, x)$ in the coarea formula (4.2). We have

$$\begin{aligned} \text{Vol }(\Omega_t^+ - \Omega_{t+\Delta t}^+) &= \text{Vol }(A) \\ &= \int_{-L\Delta t}^0 \int_{\{x\in A:\, v(t+\Delta t,x)=y\}} \frac{1}{|\nabla v(t+\Delta t, x)|} \mathcal{H}_{|\cdot|}^{n-1}(x)\, dy. \end{aligned} \tag{4.5}$$

Consider the map $\Phi\colon [0,T]\times\overline{\Omega} \to [0,T]\times\mathbb{R}$ given by $\Phi(t,x) = (t, v(t,x))$. By Sard's theorem, the set of critical values of Φ has Lebesgue measure zero. Therefore for almost all t, almost all Δt, and almost all y, $\nabla v(t+\Delta t, x) \neq 0$ for all $x \in \{x\colon v(t+\Delta t, x) = y\}$. It follows that $\lim_{\Delta t\to 0} \frac{\text{Vol }(\Omega_t^+ - \Omega_{t+\Delta t}^+)}{\Delta t}$ is bounded for almost all t. Similarly one can prove that $\lim_{\Delta t\to 0}\frac{\text{Vol }(\Omega_{t+\Delta t}^+ - \Omega_t^+)}{\Delta t}$ is bounded for almost all t. ∎

Remark 4.2. Lemma 4.1 is not true for all $t\in[0,T]$. As we shall see from the following example, this is because $\lim_{\Delta t\to 0}\frac{\text{vol }(\Omega_t^+ - \Omega_{t+\Delta t}^+)}{\Delta t}$ is not necessarily bounded for all t.

Example 4.3. Let $0 < a < b < c < d < \infty$ and $\overline{\Omega} = [a,d]$. Let $v(t,x) = (x-a)(x-b-\sqrt{t})(x-c-\sqrt{t})(x-d)$ be defined on $[0,T]\times\overline{\Omega}$. Then $\Omega_t^+ = [b+\sqrt{t}, c+\sqrt{t}]$ and $\Omega_{t+\Delta t} = [b+\sqrt{t+\Delta t},\, c+\sqrt{t+\Delta t}]$, and

$$\lim_{\Delta t\to 0}\frac{\text{length }(\Omega_t^+ - \Omega_{t+\Delta t}^+)}{\Delta t} = \lim_{\Delta t\to 0}\frac{\sqrt{t+\Delta t}-\sqrt{t}}{\Delta t} = \frac{1}{2\sqrt{t}},$$

which is finite except at $t = 0$.

Proof of Theorem C. Observe that (4.14) and (4.15) imply

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) - \frac{\partial u_i}{\partial t}(t,x) &= \frac{1}{2}\Delta(u-u_i)(t,x) + F(\tau_{i-1},x)\cdot\nabla(u-u_i)(t,x) \\ &\quad + (F(t,x) - F(\tau_{i-1},x))\cdot\nabla u(t,x) + J(\tau_{i-1},x)(u-u_i)(t,x) \\ &\quad + (J(t,x) - J(\tau_{i-1},x))u(t,x). \end{aligned} \tag{4.6}$$

Let $\Omega_t^+ = \{x\in\Omega\colon u(t,x) - u_i(t,x) \geq 0\}$.

$$\begin{aligned} \frac{d}{dt}\int_{\Omega_t^+}(u-u_i)(t,x) &= \frac{1}{2}\int_{\Omega_t^+}\Delta(u-u_i)(t,x) + \int_{\Omega_t^+}F(\tau_{i-1},x)\cdot\nabla(u-u_i)(t,x) \\ &\quad + \int_{\Omega_t^+}(F(t,x) - F(\tau_{i-1},x))\cdot\nabla u(t,x) \\ &\quad + \int_{\Omega_t^+}J(\tau_{i-1},x)(u-u_i)(t,x) \\ &\quad + \int_{\Omega_t^+}(J(t,x) - J(\tau_{i-1},x))u(t,x) \\ &= \frac{1}{2}\int_{\partial\Omega_t^+}\frac{\partial(u-u_i)}{\partial\nu}(t,x) - \int_{\Omega_t^+}\operatorname{div}F(\tau_{i-1},x)(u-u_i)(t,x) \\ &\quad + \int_{\Omega_t^+}(F(t,x) - F(\tau_{i-1},x))\cdot\nabla u(t,x) \end{aligned}$$

$$
\begin{aligned}
&+\int_{\Omega_t^+} J(\tau_{i-1},x)(u-u_i)(t,x)\\
&+\int_{\Omega_t^+} (J(t,x)-J(\tau_{i-1},x))u(t,x)\\
&\le c\int_{\Omega_t^+}(u-u_i)(t,x)+c_1(t-\tau_{i-1})^{\alpha}\int_{\Omega_t^+}u(t,x)\\
&+c_1(t-\tau_{i-1})^{\alpha}\int_{\Omega_t^+}|\nabla u(t,x)|. \qquad (4.7)
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\frac{d}{dt}\int_{\Omega}u(t,x)&=\frac{1}{2}\int_{\Omega}\Delta u+\int_{\Omega}F(t,x)\cdot\nabla u(t,x)+\int_{\Omega}J(t,x)u(t,x)\\
&=\frac{1}{2}\int_{\partial\Omega}\frac{\partial u}{\partial \nu}-\int_{\Omega}\operatorname{div}F(t,x)u(t,x)+\int_{\Omega}J(t,x)u(t,x)\\
&\le c\int_{\Omega}u(t,x). \qquad (4.8)
\end{aligned}
$$

This implies, for $0\le t\le T$,

$$
\int_{\Omega}u(t,x)\le e^{cT}\int_{\Omega}u(0,x). \qquad (4.9)
$$

In order to estimate $\int_{\Omega}|\nabla u(t,x)|^2$, we need to estimate the L^2 norm of $u(t,x)$.

$$
\begin{aligned}
\frac{d}{dt}\int_{\Omega}u^2(t,x)&=2\int_{\Omega}u(t,x)\frac{\partial u}{\partial t}(t,x)\\
&=\int_{\Omega}u(t,x)\Delta u(t,x)+2\int_{\Omega}u(t,x)F(t,x)\cdot\nabla u(t,x)+2\int_{\Omega}J(t,x)u^2(t,x). \qquad (4.10)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\int_{\Omega}u(t,x)F(t,x)\cdot\nabla u(t,x)&=-\int_{\Omega}u(t,x)\operatorname{div}[u(t,x)F(t,x)]+\int_{\partial\Omega}u^2(t,x)F(t,x)\cdot\nu\\
&=-\int_{\Omega}u(t,x)\nabla u(t,x)\cdot F(t,x)-\int_{\Omega}u^2(t,x)\operatorname{div}F(t,x).
\end{aligned}
$$

This implies

$$
\int_{\Omega}u(t,x)F(t,x)\nabla u(t,x)=-\frac{1}{2}\int_{\Omega}u^2(t,x)\operatorname{div}F(t,x). \qquad (4.11)
$$

Putting (4.11) into (4.10), we get

$$
\begin{aligned}
\frac{d}{dt}\int_{\Omega}u^2(t,x)=&-\int_{\Omega}|\nabla u(t,x)|^2+\int_{\partial\Omega}u(t,x)\frac{\partial u}{\partial\nu}(t,x)-\int_{\Omega}u^2(t,x)\operatorname{div}F(t,x)\\
&+2\int_{\Omega}J(t,x)u^2(t,x)\\
\le&\; c\int_{\Omega}u^2(t,x). \qquad (4.12)
\end{aligned}
$$

This implies

$$\int_\Omega u^2(t,x) \le e^{ct}\int_\Omega u^2(0,x) \le e^{cT}\int_\Omega u^2(0,x). \tag{4.13}$$

Now we are ready to estimate $\int_\Omega |\nabla u(t,x)|^2$.

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |\nabla u|^2(t,x) =& \int_\Omega 2\nabla\frac{\partial u}{\partial t}(t,x)\cdot\nabla u(t,x)\\
=& -2\int_\Omega \frac{\partial u}{\partial t}(t,x)\Delta u(t,x) + 2\int_{\partial\Omega}\frac{\partial u}{\partial t}(t,x)\frac{\partial u}{\partial \nu}(t,x)\\
=& -\int_\Omega (\Delta u(t,x))^2 - 2\int_\Omega F(t,x)\cdot\nabla u(t,x)\Delta u(t,x)\\
& -2\int_\Omega J(t,x)u(t,x)\Delta u(t,x)\\
\le& -\int_\Omega(\Delta u(t,x))^2 + 2\int_\Omega |F(t,x)|^2|\nabla u(t,x)|^2 + \frac{1}{2}\int_\Omega(\Delta u(t,x))^2\\
& +2\int_\Omega J^2(t,x)u^2(t,x) + \frac{1}{2}\int_\Omega (\Delta u(t,x))^2\\
\le& \, 2c^2\int_\Omega |\nabla u(t,x)|^2 + 2c^2\int_\Omega u^2(t,x)\\
\le& \, 2c^2\int_\Omega |\nabla u(t,x)|^2 + 2c^2e^{ct}\int_\Omega u^2(0,x). \qquad (4.14)
\end{aligned}$$

This implies

$$\begin{aligned}
\frac{d}{dt}\left[e^{-2c^2t}\int_\Omega |\nabla u(t,x)|^2\right] &= e^{-2c^2t}\left[\frac{d}{dt}\int_\Omega|\nabla u(t,x)|^2 - 2c^2\int_\Omega |\nabla u(t,x)|^2\right]\\
&\le 2c^2e^{-(2c^2-c)t}\int_\Omega u^2(0,x) \le 2c^2\int_\Omega u^2(0,x). \qquad (4.15)
\end{aligned}$$

Hence

$$e^{-2c^2t}\int_\Omega|\nabla u(t,x)|^2 - \int_\Omega |\nabla u(0,x)|^2 \le 2c^2t\int_\Omega u^2(0,x)$$

and

$$\begin{aligned}
\int_\Omega |\nabla u(t,x)|^2 &\le 2c^2te^{2c^2t}\int_\Omega u^2(0,x) + e^{2c^2t}\int_\Omega|\nabla u(0,x)|^2\\
&\le 2c^2Te^{2c^2T}\int_\Omega u^2(0,x) + e^{2c^2T}\int_\Omega |\nabla u(0,x)|^2. \qquad (4.16)
\end{aligned}$$

It follows that

$$\begin{aligned}
\int_\Omega |\nabla u(t,x)| &\le \sqrt{\text{Vol }(\Omega)}\left[\int_\Omega |\nabla u(t,x)|^2\right]^{\frac{1}{2}}\\
&\le \sqrt{\text{Vol }(\Omega)}e^{c^2T}\sqrt{2c^2T\int_\Omega u^2(0,x) + \int_\Omega |\nabla u(0,x)|^2}. \qquad (4.17)
\end{aligned}$$

Putting (4.9), (4.17) into (4.7), we get

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t^+}(u-u_i)(t,x) &\le c\int_{\Omega_t^+}(u-u_i)(t,x)+c_1(t-\tau_{i-1})^\alpha\int_\Omega u(t,x)\\ &\quad + c_1(t-\tau_{i-1})^\alpha\int_\Omega |\nabla u(t,x)|\\ &\le c\int_{\Omega_t^t}(u-u_i)(t,x)+c_1(t-\tau_{i-1})^\alpha e^{cT}\int_\Omega u(0,x)\\ &\quad + c_1(t-\tau_{i-1})^\alpha\sqrt{\text{Vol }(\Omega)}e^{c^2T}\\ &\quad \sqrt{2c^2T\int_\Omega u^2(0,x)+\int_\Omega|\nabla u(0,x)|^2}\\ &= c\int_{\Omega_t^+}(u-u_i)(t,x)+c_2(t-\tau_{i-1})^\alpha, \end{aligned} \tag{4.18}$$

where

$$c_2 = c_1 e^{cT}\int_\Omega u(0,x)+c_1\sqrt{\text{Vol }(\Omega)}e^{c^2T}\sqrt{2c^2T\int_\Omega u^2(0,x)+\int_\Omega|\nabla u(0,x)|^2}, \tag{4.19}$$

$$\begin{aligned}&\frac{d}{dt}\left[e^{-c(t-\tau_{i-1})}\int_{\Omega_t^+}(u-u_i)(t,x)\right]\\ &\quad = e^{-c(t-\tau_{i-1})}\left[\frac{d}{dt}\int_{\Omega_t^+}(u-u_i)(t,x)-c\int_{\Omega_t^+}(u-u_i)(t,x)\right]\\ &\quad \le c_2(t-\tau_{i-1})^\alpha e^{-c(t-\tau_{i-1})}.\end{aligned}$$

This implies

$$\begin{aligned}&e^{-c(t-\tau_{i-1})}\int_{\Omega_t^+}(u-u_i)(t,x)-\int_{\Omega^+_{\tau_{i-1}}}(u-u_i)(\tau_{i-1},x)\\ &\quad \le c_2\int_{\tau_{i-1}}^t (s-\tau_{i-1})^\alpha e^{-c(s-\tau_{i-1})}\le c_2\frac{(t-\tau_{i-1})^{\alpha+1}}{\alpha+1}.\end{aligned} \tag{4.20}$$

Hence

$$\begin{aligned}\int_{\Omega_t^+}(u-u_i)(t,x) &\le e^{c(t-\tau_{i-1})}\int_{\Omega^+_{\tau_{i-1}}}(u-u_{i-1})(\tau_{i-1},x)\\ &\quad + c_2\frac{(t-\tau_{i-1})^{\alpha+1}}{\alpha+1}e^{c(t-\tau_{i-1})}.\end{aligned} \tag{4.21}$$

Similarly one can prove that

$$\begin{aligned}\int_{\Omega_t^-}(u-u_i)(t,x) &\le e^{c(t-\tau_{i-1})}\int_{\Omega^-_{\tau_{i-1}}}(u_{i-1}-u)(\tau_{i-1},x)\\ &\quad + c_2\frac{(t-\tau_{i-1})^{\alpha+1}}{\alpha+1}e^{c(t-\tau_{i-1})}.\end{aligned} \tag{4.22}$$

Consequently, we have

$$\begin{aligned}\int_\Omega |u-u_i|(t,x) &= \int_{\Omega_t^+}(u-u_i)(t,x)+\int_{\Omega_t^-}(u_i-u)(t,x)\\ &\le e^{c(t-\tau_{i-1})}\left[\int_\Omega |u-u_{i-1}|(\tau_{i-1},x)+2c_2\frac{t-\tau_{i-1})^{\alpha+1}}{\alpha+1}\right].\end{aligned}\tag{4.23}$$

By applying (4.23) inductively, we have the following estimate:

$$\begin{aligned}&\int_\Omega |u-u_k|(\tau_k,x)\\ &\quad\le e^{c(\tau_k-\tau_{k-1})}\left[\int_\Omega |u-u_{k-1}|(\tau_{k-1},x)+2c_2\frac{(\tau_k-\tau_{k-1})^{\alpha+1}}{\alpha+1}\right]\\ &\quad\le e^{c(\tau_k-\tau_{k-1})}e^{c(\tau_{k-1}-\tau_{k-2})}\left[\int_\Omega |u-u_{k-2}|(\tau_{k-2},x)+2c_2\frac{(\tau_{k-1}-\tau_{k-2})^{\alpha+1}}{\alpha+1}\right]\\ &\qquad+e^{c(\tau_k-\tau_{k-1})}2c_2\frac{(\tau_k-\tau_{k-1})^{\alpha+1}}{\alpha+1}\\ &\quad= e^{c(\tau_k-\tau_{k-2})}\int_\Omega |u-u_{k-2}|(\tau_{k-2},x)+\frac{2c_2}{\alpha+1}[(\tau_{k-1}-\tau_{k-2})^{\alpha+1}e^{c(\tau_k-\tau_{k-2})}\\ &\qquad+(\tau_k-\tau_{k-1})^{\alpha+1}e^{c(\tau_k-\tau_{k-1})}]\\ &\quad\le e^{c(\tau_k-\tau_{k-i})}\int_\Omega |u-u_{k-i}|(\tau_{k-i},x)\\ &\qquad+\frac{2c_2}{\alpha+1}[(\tau_k-\tau_{k-1})^{\alpha+1}e^{c(\tau_k-\tau_{k-1})}+(\tau_{k-1}-\tau_{k-2})^{\alpha+1}e^{c(\tau_k-\tau_{k-2})}\\ &\qquad+\cdots+(\tau_{k-i+1}-\tau_{k-i})^{\alpha+1}e^{c(\tau_k-\tau_{k-i})}]\\ &\quad\le e^{cT}\int_\Omega |u-u_0|(0,x)+\frac{2c_2}{\alpha+1}[(\tau_k-\tau_{k-1})^{\alpha+1}e^{c(\tau_k-\tau_{k-1})}\\ &\qquad+(\tau_{k-1}-\tau_{k-2})^{\alpha+1}e^{c(\tau_k-\tau_{k-2})}+\cdots+(\tau_1-\tau_0)^{\alpha+1}e^{c(\tau_k-\tau_0)}]\\ &\quad=\frac{2c_2}{\alpha+1}\frac{T^{\alpha+1}}{k^{\alpha+1}}[e^{c\frac{T}{k}}+e^{c\frac{2T}{k}}+\cdots+e^{c\frac{kT}{k}}]\\ &\quad\le\frac{2c_2}{\alpha+1}\frac{T^{\alpha+1}e^{cT}}{k^\alpha},\end{aligned}\tag{4.24}$$

which goes to zero as $k\to\infty$. ☐

5. Lower estimate of density function. In practical nonlinear filtering computation, it is important to know how much density remains within a given ball. In this section, we shall provide such a lower estimate. In particular, the solution u of the DMZ equation in $\mathbb{R}^n$ obtained by taking $\lim_{R\to\infty} u_R$, where u_R is the solution of the DMZ equation in the ball B_R, is a nontrivial solution.

Proof of Theorem D. Let $\phi = e^{-\rho(x)} - e^{-\rho(R_0)}$, where ρ is an increasing function of $|x|$. Observe that $\phi\ge 0$ for $x\in B_{R_0}$, $\phi=0$ on ∂B_{R_0}, and $\frac{\partial\phi}{\partial\nu}|_{\partial B_{R_0}}\le 0$, where ν is the outward normal of ∂B_{R_0}.

$$\begin{aligned}\frac{d}{dt}\int_{B_{R_0}} \phi u_R &= \frac{1}{2}\int_{B_{R_0}} \phi\Delta u_R + \int_{B_{R_0}} \phi(-f+\nabla K)\cdot\nabla u_R \\ &\quad + \int_{B_{R_0}} \left(-\operatorname{div} f - \frac{1}{2}|h|^2 + \frac{\Delta K}{2} - f\cdot\nabla K + \frac{1}{2}|\nabla K|^2\right)\phi u_R \\ &= \frac{1}{2}\int_{B_{R_0}} u_R\Delta\phi - \frac{1}{2}\int_{\partial B_{R_0}} \frac{\partial\phi}{\partial\nu}u_R + \frac{1}{2}\int_{\partial B_{R_0}} \phi\frac{\partial u_R}{\partial\nu} \\ &\quad - \int_{B_{R_0}} u_R \operatorname{div}[\phi(-f+\nabla K)] + \int_{\partial B_{R_0}} \phi u_R(-f+\nabla K)\cdot\nu \\ &\quad + \int_{B_{R_0}} \left(-\operatorname{div} f - \frac{1}{2}|h|^2 + \frac{\Delta K}{2} - f\cdot\nabla K + \frac{1}{2}|\nabla K|^2\right)\phi u_R \\ &\geq \frac{1}{2}\int_{B_{R_0}} u_R\Delta\phi + \int_{B_{R_0}} u_R\nabla\phi\cdot(f-\nabla K) \\ &\quad + \int_{B_{R_0}} \left(-\frac{\Delta K}{2} - \frac{1}{2}|h|^2 - f\cdot\nabla K + \frac{1}{2}|\nabla k|^2\right)\phi u_R.\end{aligned}$$

Notice that $\nabla\phi = -e^{-\rho(x)}\nabla\rho$ and $\Delta\phi = e^{-\rho(x)}(|\nabla\rho|^2 - \Delta\rho)$. Hence

$$\begin{aligned}\frac{d}{dt}\int_{B_{R_0}} \phi u_R \geq \int_{B_{R_0}} u_R e^{-\rho(x)}\left[-\frac{\Delta\rho}{2} + \frac{|\nabla\rho|^2}{2} - \nabla\rho\cdot(f-\nabla K)\right] \\ + \int_{B_{R_0}} \left(-\frac{\Delta K}{2} - \frac{1}{2}|h|^2 - f\cdot\nabla K + \frac{1}{2}|\nabla K|^2\right)\phi u_R.\end{aligned}$$

Let $r = |x|$. We have

$$\nabla\rho = \frac{\rho'(\gamma)}{r}x \quad \text{and} \quad \Delta\rho = \rho''(r) + \rho'(r)\frac{n-1}{r}.$$

Hence

$$\begin{aligned}\frac{d}{dt}\int_{B_{R_0}} \phi u_R &\geq \int_{B_{R_0}} u_R e^{-\rho(R_0)}\left[-\frac{\Delta\rho}{2} + \frac{|\nabla\rho|^2}{2} - \nabla\rho\cdot(f-\nabla K)\right] \\ &\quad + \int_{B_{R_0}} \left[-\frac{\Delta\rho}{2} + \frac{|\nabla\rho|^2}{2} - \nabla\rho\cdot(f-\nabla K)\right. \\ &\qquad\qquad \left. - \frac{\Delta K}{2} - \frac{1}{2}|h|^2 - f\cdot\nabla K + \frac{1}{2}|\nabla K|^2\right]\phi u_R \\ &= e^{-\rho(R_0)}\int_{B_{R_0}} \left[\frac{\rho'^2}{2} - \frac{\rho''}{2} - \frac{n-1}{2r}\rho' - \rho'(f-\nabla K)\cdot\frac{x}{r}\right]u_R \\ &\quad + \int_{B_{R_0}} \left[-\frac{\Delta\rho}{2} + \frac{|\nabla\rho|^2}{2} - \nabla\rho\cdot(f-\nabla K)\right. \\ &\qquad\qquad \left. - \frac{\Delta K}{2} - \frac{1}{2}|h|^2 - f\cdot\nabla K + \frac{1}{2}|\nabla K|^2\right]\phi u_R. \end{aligned} \tag{5.1}$$

We want to choose ρ such that

$$e^{-\rho(R_0)}\left|\int_{B_{R_0}}\left[\frac{\rho^{12}}{2}-\frac{\rho''}{2}-\frac{n-1}{2r}\rho'-\rho'(f-\nabla K)\cdot\frac{x}{r}\right]u_R\right|\leq\epsilon(R_0),$$

where $\epsilon(R_0)$ is small and will be determined later. Then (5.1) implies

$$\frac{d}{dt}\int_{B_{R_0}}\phi u_R\geq -\epsilon(R_0)+\int_{B_{R_0}}\left[-\frac{\Delta\rho}{2}+\frac{|\nabla\rho|^2}{2}-\nabla\rho\cdot(f-\nabla K)\right.$$
$$\left.-\frac{\Delta K}{2}-\frac{1}{2}|h|^2-f\cdot\nabla K+\frac{1}{2}|\nabla K|^2\right]\phi u_R. \tag{5.2}$$

We now take $\rho=|x|^m$. Then

$$\Delta\rho=\rho''(r)+\rho'(r)\frac{n-1}{r}=m(m+n-2)r^{m-2},$$
$$|\nabla\rho|^2=(\rho'(r))^2=m^2r^{2m-2}.$$

Since f and h are of polynomial growth, we can choose a positive integer m large enough such that

$$\begin{aligned}&-\frac{\Delta\rho}{2}+\frac{|\nabla\rho|^2}{2}-\nabla\rho\cdot(f-\nabla K)-\frac{\Delta K}{2}-\frac{1}{2}|h|^2-f\cdot\nabla K+\frac{1}{2}|\nabla K|^2\\&\quad=-\frac{1}{2}m(m+n-2)r^{m-2}+\frac{1}{2}m^2r^{2m-2}\\&\qquad-\frac{\rho'(r)}{r}x\cdot(f-\nabla K)-\frac{\Delta K}{2}-\frac{1}{2}|h|^2-f\cdot\nabla K+\frac{1}{2}|\nabla K|^2\\&\quad\leq -c'\qquad\text{on }\mathbb{R}^n,\end{aligned}$$

where c' is a positive constant independent of R and R_0. Hence

$$\frac{d}{dt}\int_{B_{R_0}}\phi u_R\geq -\epsilon(R_0)-c'\int_{B_{R_0}}\phi u_R$$
$$\Rightarrow\frac{d}{dt}\left[e^{c't}\int_{B_{R_0}}\phi u_R\right]\geq-\epsilon(R_0)e^{c't}$$
$$\Rightarrow e^{c'T}\int_{B_{R_0}}\phi u_R(T,x)-\int_{B_{R_0}}\phi u_R(0,x)\geq\frac{\epsilon(R_0)}{c'}(1-e^{c'T}). \tag{5.3}$$

We are now ready to estimate $\epsilon(R_0)$. Observe that $\frac{\partial u_R}{\partial\nu}\leq 0$ on ∂B_R, where ν is the outward normal of ∂B_R.

$$\begin{aligned}\frac{d}{dt}\int_{B_R}u_R&=\frac{1}{2}\int_{B_R}\Delta u_R+\int_{B_R}(-f+\nabla K)\cdot\nabla u_R\\&\quad+\int_{B_R}\left(-\operatorname{div}f-\frac{1}{2}|h|^2+\frac{1}{2}\Delta K-f\cdot\nabla K+\frac{1}{2}|\nabla K|^2\right)u_R\\&=\frac{1}{2}\int_{\partial B_R}\frac{\partial u_R}{\partial\nu}-\int_{B_R}u_R\operatorname{div}(-f+\nabla K)+\int_{\partial B_R}u_R(-f+\nabla K)\cdot\nu\end{aligned}$$

$$+ \int_{B_R} \left(-\operatorname{div} f - \frac{1}{2}|h|^2 + \frac{\Delta K}{2} - f \cdot \nabla K + \frac{1}{2}|\nabla K|^2 \right) u_R$$

$$\leq \int_{B_R} \left(-\frac{1}{2}|h|^2 - \frac{\Delta K}{2} - f \cdot \nabla K + \frac{1}{2}|\nabla K|^2 \right) u_R \leq c_1 \int_{B_R} u_R.$$

Hence

$$\int_{B_R} u_R(t,x) \leq e^{c_1 t} \int_{B_R} u_R(0,x), \qquad 0 \leq t \leq T. \tag{5.4}$$

In order to estimate $\epsilon(R_0)$, we need to determine the upper bound of

$$e^{-\rho(R_0)} \left| \int_{B_{R_0}} \left[\frac{\rho'^2}{2} - \frac{\rho''}{2} - \frac{n-1}{2r}\rho' - \rho'(f - \nabla K) \cdot \frac{x}{r} \right] u_R. \right.$$

Recall that $\rho = r^m$. Then for m large enough,

$$\left| \frac{\rho'^2}{2} - \frac{\rho''}{2} - \frac{n-1}{2r}\rho' - \rho'(f - \nabla K) \cdot \frac{x}{r} \right|$$

$$= \left| \frac{m^2 r^{2m-2}}{2} - \frac{m(m+n-2)}{2} r^{m-2} - m r^{m-2}(f - \nabla K) \cdot x \right|$$

$$\leq \frac{m(m+1)}{2} r^{2m-2} + c'',$$

where c'' is independent of R and R_0. Therefore

$$e^{-\rho(R_0)} \left| \int_{B_{R_0}} \left[\frac{\rho'^2}{2} - \frac{\rho''}{2} - \frac{n-1}{2r}\rho' - \rho'(f - \nabla K) \cdot \frac{x}{r} \right] u_R(t,x) \right|$$

$$\leq e^{-R_0^m} \left(\frac{m(m+1)}{2} R_0^{2m-2} + c'' \right) \int_{B_{R_0}} u_R(t,x)$$

by (5.4)

$$\leq e^{c'T - R_0^m} \left(\frac{m(m+1)}{2} R_0^{2m-2} + c'' \right) \int_{B_R} u_R(0,x),$$

and we can set

$$\epsilon(R_0) = e^{c'T - R_0^m} \left(\frac{m(m+1)}{2} R_0^{2m-2} + c'' \right) \int_{B_R} u_R(0,x).$$

In view of (5.3), we have

$$e^{c'T} \int_{B_{R_0}} \phi u_R(T,x) - \int_{B_{R_0}} \phi u_R(0,x)$$

$$\geq \frac{e^{c'T - R_0^m}}{c'} \left(\frac{m(m+1)}{2} R_0^{2m-2} + c'' \right) (1 - e^{c'T}) \int_{B_R} u_R(0,x),$$

which implies

$$\int_{B_{R_0}} \phi u_R(T,x) \geq e^{-c'T} \int_{B_{R_0}} \phi u_R(0,x)$$

$$+ \frac{e^{-R_0^m}}{c'} \left(\frac{m(m+1)}{2} R_0^{2m-2} + c'' \right) (1 - e^{c'T}) \int_{B_R} u_R(0,x). \tag{5.5}$$

Observe that the second term on the right-hand side of (5.5) tends to zero as $R_0 \to \infty$. Therefore we have

$$\int_{\mathbb{R}^n} e^{-|x|^m} u(T,x) \geq e^{-c'T} \int_{\mathbb{R}^n} e^{-|x|^m} u(0,x). \qquad \square$$

Appendix A. A priori estimation of derivatives up to second order. In this section, we shall give a priori estimation of zero, first, and second derivatives of the solution of the robust DMZ equation on $[0,T] \times B_R$.

THEOREM A.1. *Consider the robust DMZ equation* (1.9) *on* $[0,T] \times B_R$, *where* $B_R = \{x \in \mathbb{R}^n \colon |x| \leq R\}$ *is a ball of radius R. Let* $C_1 = \max_{0 \leq t \leq T} \left[\sum_{i=1}^m |y_i(t)|^2 \right]^{\frac{1}{2}}$ *be the smallest constant such that*

$$|\nabla K(t,x)| \leq C_1 |\nabla h(x)| \quad \text{for } (t,x) \in [0,T] \times B_R, \tag{A.1}$$

where $|\nabla h|^2 = \sum_{i=1}^m |\nabla h_i(x)|^2$.

Suppose that there exists a constant $C > 0$ *such that for any* $r \geq 0$

$$\min_{|x|=r} \frac{|h|^2 + \operatorname{div} f + C}{\sqrt{|f|^2 + |h|^2 + \operatorname{div} f + C} + |f|} - C_1 \max_{|x|=r} |\nabla h| \geq 0. \tag{A.2}$$

Let $g(x)$ *be a positive radial symmetric function on* $\mathbb{R}^n$ *(i.e.,* $g = g(r)$*, where* $r = |x| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$*) such that*

$$|g'(r)| \leq \min_{|x|=r} \frac{|h|^2 + \operatorname{div} f + C}{\sqrt{|f|^2 + |h|^2 + \operatorname{div} f + C} + |f|} - C_1 \max_{|x|=r} |\nabla h|. \tag{A.3}$$

Then, for $0 \leq t \leq T$,

$$\int_{B_R} e^{2g} u_R^2(t,x) \leq e^{ct} \int_{B_R} e^{2g} \sigma^2(x). \tag{A.4}$$

Proof. Let ρ be any smooth function on $\mathbb{R} \times \mathbb{R}^n$. Then

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}\int_{B_R} \rho^2 u_R^2\right) &= \int_{B_R} \rho\rho_t u_R^2 + \int_{B_R} \rho^2 u_R \frac{\partial u_R}{\partial t} \\
&= \int_{B_R} \rho\rho_t u_R^2 + \int_{B_R} \frac{1}{2}\rho^2 u_R \Delta u_R - \int_{B_R} \rho^2 u_R (f - \nabla K)\cdot \nabla u_R \\
&\quad - \int_{B_R} \rho^2 u_R^2 \left[\frac{1}{2}|h|^2 + \operatorname{div} f - \frac{1}{2}\Delta K + f\cdot\nabla K - \frac{1}{2}|\nabla K|^2\right] \\
&= \int_{B_R} \rho\rho_t u_R^2 - \int_{B_R} \rho u_R \nabla\rho\cdot\nabla u_R - \frac{1}{2}\int_{B_R} \rho^2 |\nabla u_R|^2 \\
&\quad + \int_{B_R} \rho u_R^2 \nabla\rho\cdot(f - \nabla K) + \frac{1}{2}\int_{B_R} \rho^2 (\operatorname{div} f - \nabla K) u_R^2 \\
&\quad - \int_{B_R} \rho^2 u_R^2 \left[\frac{1}{2}|h|^2 + \operatorname{div} f - \frac{1}{2}\Delta K + f\cdot\nabla K - \frac{1}{2}|\nabla K|^2\right] \\
&\leq \int_{B_R} \rho\rho_t u_R^2 + \frac{1}{2}\int_{B_R} |\nabla\rho|^2 u_R^2 - \int_{B_R} \rho^2 u_R^2 \Bigg[\frac{1}{2}|h|^2 + \frac{1}{2}\operatorname{div} f \\
&\quad + f\cdot\nabla K - \frac{1}{2}|\nabla K|^2 - \sum_{i=1}^n \nabla(\log\rho)\cdot(f - \nabla K)\Bigg].
\end{aligned} \tag{A.5}$$

If we set $\rho = e^g$, then we get

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\int_{B_R}\rho^2 u_R^2\right] \\
&\quad\le \int_{B_R}\left[g_t + \frac{|\nabla g|^2}{2} - \frac{1}{2}|h|^2 - \frac{1}{2}\operatorname{div} f - f\cdot\nabla K\right.\\
&\qquad\qquad \left. + \frac{1}{2}|\nabla K|^2 + \nabla g\cdot(f-\nabla K)\right]\rho^2 u_R^2\\
&\quad= \int_{B_R}\left[g_t + \frac{|\nabla g|^2}{2} + \frac{1}{2}|f-\nabla K|^2 - \frac{1}{2}|f|^2\right.\\
&\qquad\qquad \left. - \frac{1}{2}|h|^2 - \frac{1}{2}\operatorname{div} f + \nabla g\cdot(f-\nabla K)\right] g^2 u_R^2\\
&\quad= \int_{B_R}\left[g_t + \frac{1}{2}|\nabla g + f - \nabla K|^2 - \frac{1}{2}|f|^2 - \frac{1}{2}|h|^2 - \frac{1}{2}\operatorname{div} f\right]\rho^2 u_R^2. \qquad \text{(A.6)}
\end{aligned}$$

We shall choose g to be independent of t and a constant $C > 0$ so that

$$|\nabla g + f - \nabla K| \le \sqrt{|f|^2 + |h|^2 + \operatorname{div} f + C}. \qquad \text{(A.7)}$$

Notice that (A.6) and (A.7) imply

$$\frac{d}{dt}\left[\frac{1}{2}\int_{B_R}\rho^2 u_R^2\right] \le C\int_{B_R}\frac{1}{2}\rho^2 u_R^2. \qquad \text{(A.8)}$$

Inequality (A.7) can be achieved if we have

$$\begin{aligned}
|\nabla g| &\le \sqrt{|f|^2 + |h|^2 + \operatorname{div} f + C} - |f| - |\nabla K|\\
&= \frac{|h|^2 + \operatorname{div} f + C}{\sqrt{|f|^2 + |h|^2 + \operatorname{div} f + C} + |f|} - |\nabla K|. \qquad \text{(A.9)}
\end{aligned}$$

Notice that, for $0 \le t \le T$,

$$\begin{aligned}
|\nabla K(t,x)| = \left|\sum_{i=1}^m y_i(t)\nabla h_i\right| &\le \sum_{i=1}^m |y_i(t)||\nabla h_i(x)|\\
&\le \left(\sum_{i=1}^m |y_i(t)|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^m |\nabla h_i(x)|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Let $C_1 = \max_{0\le t\le T}\left(\sum_{i=1}^m |y_i(t)|^2\right)^{\frac{1}{2}}$. Then we have

$$|\nabla K(t,x)| \le C_1|\nabla h(x)|. \qquad \text{(A.10)}$$

Now we shall choose g to be radial symmetric so that

$$|g'(r)| = |\nabla g| \le \min_{|x|=r}\frac{|h|^2 + \operatorname{div} f + C}{\sqrt{|f|^2 + |h|^2 + \operatorname{div} f + C} + |f|} - C_1\max_{|x|=r}|\nabla h|. \qquad \text{(A.11)}$$

Inequality (A.9) implies

$$\int_{B_R} e^{2g} u_R^2(t,x) \le e^{ct} \int_{B_R} e^{2g}\sigma(x). \qquad \square \tag{A.12}$$

Remark A.2. Notice that from (A.11), if h grows fast, then we can allow g to grow fast.

We next give a priori estimation of the first and second derivatives of the solution of the robust DMZ equation on $[0,T]\times B_R$. We first observe that for estimation of second derivatives it is sufficient to estimate the Laplacian of the solution.

LEMMA A.3. *Let ρ be a smooth function with compact support in B_R. Let u_R be the solution of* (1.9). *Then*

$$\int_{B_R} \sum_{i,j=1}^n \rho^2 (u_R)_{ji}^2 \le 4\int_{B_R} \rho^2(\Delta u_R)^2 + 6\sup|\nabla\rho|^2 \int_{B_R} |\nabla u_R|^2. \tag{A.13}$$

Proof.

$$\begin{aligned}\int_{B_R}\rho^2(\Delta u_R)^2 &= -\int_{B_R} 2\rho(\nabla\rho\cdot\nabla u_R)\Delta u_R - \int_{B_R}\rho^2\nabla(\Delta u_R)\cdot\nabla u_R\\ &= -\int_{B_R} 2\rho\Delta u_R(\nabla u_R\cdot\nabla\rho) - \int_{B_R}\rho^2\sum_{i,j=1}^n (u_R)_{jji}(u_R)_i\\ &= -\int_{B_R} 2\rho\Delta u_R(\nabla u_R\cdot\nabla\rho)\\ &\quad + \sum_{i,j=1}^n\int_{B_R} 2\rho\rho_j(u_R)_{ji}u_i + \sum_{i,j=1}^n\int_{B_R}\rho^2(u_R)_{ji}^2. \end{aligned}\tag{A.14}$$

By the Schwartz inequality, we have

$$\begin{aligned}\int_{B_R} 2\rho\Delta u_R(\nabla u_R\cdot\nabla\rho) &\le \int_{B_R} 2\rho\Delta u_R|\nabla u_R|\,|\nabla\rho|\\ &\le \int_{B_R}\rho^2(\Delta u_R)^2 + \int_{B_R}|\nabla u_R|^2|\nabla\rho|^2\end{aligned}\tag{A.15}$$

$$\sum_{i,j=1}^n\int_{B_R} 2\rho\rho_j(u_R)_{ji}(u_R)_i \le \sum_{i,j=1}^n\int_{B_R}\left[\frac{1}{2}\rho^2(u_R)_{ji}^2 + 2\rho_j^2(u_R)_i^2\right]. \tag{A.16}$$

Inequalities (A.14), (A.15), and (A.16) imply

$$\begin{aligned}\int_{B_R}\rho^2(\Delta u_R)^2 \ge& -\int_{B_R}\rho^2(\Delta u_R)^2 - \int_{B_R}|\nabla u_R|^2|\nabla\rho|^2\\ &-\frac{1}{2}\int_{B_R}\sum_{i,j=1}^n\rho^2(u_R)_{ji}^2 - 2\sum_{i,j=1}^n\int_{B_R}\rho_j^2u_i^2 + \sum_{i,j=1}^n\int_{B_R}\rho^2(u_{ji})^2\\ =& -\int_{B_R}\rho^2(\Delta u_R)^2 - \int_{B_R}|\nabla u_R|^2|\nabla\rho|^2 - 2\int_{B_R}|\nabla\rho|^2|\nabla u_R|^2\\ &+\frac{1}{2}\int_{B_R}\sum_{i,j=1}^n\rho^2(u_R)_{ji}^2,\end{aligned}$$

which is equivalent to

$$\frac{1}{2}\int_{B_R}\sum_{i,j=1}^{n}\rho^2(u_R)_{ji}^2 \le 2\int_{B_R}\rho^2(\Delta u_R)^2 + 3\int_{B_R}|\nabla\rho|^2|\nabla u_R|^2.$$

Hence

$$\int_{B_R}\sum_{i,j=1}^{n}\rho^2(u_R)_{ji}^2 \le 4\int_{B_R}\rho^2(\Delta u_R)^2 + 6\sup|\nabla\rho|^2\int_{B_R}|\nabla u_R|^2. \qquad \square$$

Now we are ready to give a priori estimation of the first and second derivatives of the solution of the robust DMZ equation on $[0,T]\times B_R$.

THEOREM A.4. *Consider the robust DMZ equation* (1.9) *on* $[0,T]\times B_R$*, where* $B_R=\{x\in\mathbb{R}^n\colon |x|<R\}$ *is a ball of radius R. Assume that*

$$\text{(A.17)}\qquad \sqrt{\frac{1}{2}|h|^2+\operatorname{div} f-\frac{1}{2}\Delta K+f\cdot\nabla K-\frac{1}{2}|\nabla K|^2+\frac{C}{2}}-|f|-|\nabla K|\ge 0,$$

where C is the constant in Theorem A.1. *Choose a nonnegative function* $\widetilde{g}$ *so that*

$$\text{(A.18)}\qquad |\nabla\widetilde{g}|\le\sqrt{\frac{1}{2}|h|^2+\operatorname{div} f-\frac{1}{2}\Delta K+f\cdot\nabla K-\frac{1}{2}|\nabla K|^2+\frac{C}{2}}-|f|-|\nabla K|$$

and

$$\text{(A.19)}\qquad e^{2\widetilde{g}}\left|\nabla\left(\frac{1}{2}|h|^2+\operatorname{div} f-\frac{1}{2}\Delta K+f\cdot\nabla K-\frac{1}{2}|\nabla K|^2\right)\right|^2\le e^{2g},$$

where g is chosen as in Theorem A.1. *Then*

$$\text{(A.20)}\qquad \int_{B_R}e^{2\widetilde{g}}|\nabla u_R|^2(T,x)+\frac{1}{2}\int_0^T\int_{B_R}e^{2\widetilde{g}}(\Delta u_R)^2(t,x) \le \int_{B_R}e^{2\widetilde{g}}|\nabla u_R|^2(0,x)+T\int_{B_R}e^{2g}\sigma^2(x).$$

Proof. Recall that $\frac{\partial u_R}{\partial t}\big|_{\partial B_R}=0$. We have

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\int_{B_R}e^{2\widetilde{g}}|\nabla u_R|^2\right]\\
&\quad=\int_{B_R}e^{2\widetilde{g}}\nabla u_R\cdot\nabla\frac{\partial u_R}{\partial t}\\
&\quad=\int_{B_R}e^{2\widetilde{g}}(-2\nabla u_R\cdot\nabla g-\Delta u_R)\frac{\partial u_R}{\partial t}\\
&\quad=\int_{B_R}e^{2\widetilde{g}}(-2\nabla u_R\cdot\nabla\widetilde{g}-\Delta u_R)\Bigg[\frac{1}{2}\Delta u_R-(f-\nabla K)\cdot\nabla u_R\\
&\qquad\quad-\left(\frac{1}{2}|h|^2+\operatorname{div} f-\frac{1}{2}\Delta K+f\cdot\nabla K-\frac{1}{2}|\nabla K|^2\right)u_R\Bigg]
\end{aligned}$$

$$= \int_{B_R} e^{2\widetilde{g}}\Big\{-\frac{1}{2}(\Delta u_R)^2 + \Big[-\nabla u_R\cdot\nabla\widetilde{g} + (f-\nabla K)\cdot\nabla u_R\Big]\Delta u_R$$
$$+ 2(\nabla u_R\cdot\nabla\widetilde{g})(f-\nabla K)\cdot\nabla u_R$$
$$+ u_R\Delta u_R\Big(-\frac{1}{2}|h|^2 + \operatorname{div} f - \frac{1}{2}\Delta K + f\cdot\nabla K - \frac{1}{2}|\nabla K|^2\Big)$$
$$+ 2u_R\nabla u_R\cdot\nabla\widetilde{g}\Big(\frac{1}{2}|h|^2 + \operatorname{div} f - \frac{1}{2}\Delta K + f\cdot\nabla K - \frac{1}{2}|\nabla K|^2\Big)\Big\}.$$

This implies

$$\frac{d}{dt}\Big[\frac{1}{2}\int_{B_R} e^{2\widetilde{g}}|\nabla u_R|^2\Big] + \frac{1}{4}\int_{B_R} e^{2\widetilde{g}}(\Delta u_R)^2$$
$$= \int_{B_R} e^{2\widetilde{g}}\Big\{-\frac{1}{4}\Big[(\Delta u_R)^2 - 4\Big(-\nabla u_R\cdot\nabla\widetilde{g} + (f-\nabla K)\cdot\nabla u_R\Big)\Delta u_R\Big]$$
$$+ 2(\nabla u_R\cdot\nabla g)(f-\nabla K)\cdot\nabla u_R$$
$$+ u_R\Delta u_R\Big(-\frac{1}{2}|h|^2 + \operatorname{div} f - \frac{1}{2}\Delta K + f\cdot\nabla K - \frac{1}{2}|\nabla K|^2\Big)$$
$$+ 2u_R\nabla u_R\cdot\nabla\widetilde{g}\Big(\frac{1}{2}|h|^2 + \operatorname{div} f - \frac{1}{2}\Delta K + f\cdot\nabla K - \frac{1}{2}|\nabla K|^2\Big)\Big\}$$
$$= \int_{B_R} e^{2\widetilde{g}}\Big\{-\frac{1}{4}\Big[\Delta u_R + 2\nabla u_R\cdot\nabla\widetilde{g} - 2(f-\nabla K)\cdot\nabla u_R\Big]^2 + \frac{1}{4}[2\nabla u_R\cdot\nabla\widetilde{g}$$
$$- 2(f-\nabla K)\cdot\nabla u_R]^2 + 2(\nabla u_R\cdot\nabla\widetilde{g})(f-\nabla K)\cdot\nabla u_R$$
$$+ u_R\Delta u_R\Big(-\frac{1}{2}|h|^2 + \operatorname{div} f - \frac{1}{2}\Delta K + f\cdot\nabla K - \frac{1}{2}|\nabla K|^2\Big)$$
$$+ 2u_R(\nabla u_R\cdot\widetilde{g})\Big(\frac{1}{2}|h|^2 + \operatorname{div} f - \frac{1}{2}\Delta K + f\cdot\nabla K - \frac{1}{2}|\nabla K|^2\Big)\Big\}. \tag{A.21}$$

Notice that

$$\frac{1}{4}\Big[2\nabla u_R\cdot\nabla\widetilde{g} - 2(f-\nabla K)\cdot\nabla u_R\Big]^2 + 2(\nabla u_R\cdot\nabla\widetilde{g})\Big[(f-\nabla K)\cdot\nabla u_R\Big]$$
$$= \Big[\nabla u_R\cdot\nabla\widetilde{g} - (f-\nabla K)\cdot\nabla u_R\Big]^2 + 2(\nabla u_R\cdot\nabla\widetilde{g})\Big[(f-\nabla K)\cdot\nabla u_R\Big]$$
$$= \big[\nabla u_R\cdot(\nabla\widetilde{g} + f - \nabla K)\big]^2 \tag{A.22}$$

and

$$\int_{B_R} e^{2\widetilde{g}} u_R\Delta u_R\Big(\frac{1}{2}|h|^2 + \operatorname{div} f - \frac{1}{2}\Delta K + f\cdot\nabla K - \frac{1}{2}|\nabla K|^2\Big)$$
$$= -\int_{B_R} e^{2\widetilde{g}}\Big[2u_R\nabla u_R\cdot\nabla\widetilde{g}\Big(\frac{1}{2}|h|^2 + \operatorname{div} f - \frac{1}{2}\Delta K + f\cdot\nabla K - \frac{1}{2}|\nabla K|^2\Big)\Big]$$

$$- \int_{B_R} e^{2\widetilde{g}} |\nabla u_R|^2 \left(\frac{1}{2}|h|^2 + \operatorname{div} f - \frac{1}{2}\Delta K + f \cdot \nabla K - \frac{1}{2}|\nabla K|^2 \right)$$

$$- \int_{B_R} e^{2\widetilde{g}} u_R \nabla u_R \cdot \nabla \left(\frac{1}{2}|h|^2 + \operatorname{div} f - \frac{1}{2}\Delta K + f \cdot \nabla K - \frac{1}{2}|\nabla K|^2 \right). \tag{A.23}$$

Putting (A.22), (A.23) in (A.21), we get

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\int_{B_R} e^{2\widetilde{g}} |\nabla u_R|^2 \right] + \frac{1}{4}\int_{B_R} e^{2\widetilde{g}} (\Delta u_R)^2 \\ &\quad \le \int_{B_R} e^{2\widetilde{g}} |u_R|^2 |\nabla \widetilde{g} + f - \nabla K|^2 \\ &\quad\quad - \int_{B_R} e^{2\widetilde{g}} |\nabla u_R|^2 \left(\frac{1}{2}|h|^2 + \operatorname{div} f - \frac{1}{2}\Delta K + f \cdot \nabla K - \frac{1}{2}|\nabla K|^2 \right) \\ &\quad\quad - \int_{B_R} e^{2\widetilde{g}} u_R \nabla u_R \cdot \nabla \left(\frac{1}{2}|h|^2 + \operatorname{div} f - \frac{1}{2}\Delta K + f \cdot \nabla K - \frac{1}{2}|\nabla K|^2 \right). \end{aligned} \tag{A.24}$$

As before, we look for $\widetilde{g}$ so that

$$|\nabla \widetilde{g} + f - \nabla K| \le \left(\frac{1}{2}|h|^2 + \operatorname{div} f - \frac{1}{2}\Delta K + f \cdot \nabla K - \frac{1}{2}|\nabla K|^2 + \frac{C}{2} \right)^{\frac{1}{2}}.$$

Hence it suffices to set $\widetilde{g}$ so that

$$|\nabla \widetilde{g}| \le \left(\frac{1}{2}|h|^2 + \operatorname{div} f - \frac{1}{2}\Delta K + f \cdot \nabla K - \frac{1}{2}|\nabla K|^2 + \frac{C}{2} \right)^{\frac{1}{2}} - |f| - |\nabla K|.$$

For such $\widetilde{g}$, we have

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\int_{B_R} e^{2\widetilde{g}} |\nabla u_R|^2 \right] + \frac{1}{4}\int_{B_R} e^{2\widetilde{g}} (\Delta u_R)^2 \\ &\quad \le C\left[\frac{1}{2}\int_{B_R} e^{2\widetilde{g}} |\nabla u_R|^2 \right] + \frac{1}{2}\int_{B_R} e^{2\widetilde{g}} |\nabla u_R|^2 \\ &\quad\quad + \frac{1}{2}\int_{B_R} e^{2\widetilde{g}} u_R^2 \left| \nabla \left(\frac{1}{2}|h|^2 + \operatorname{div} f - \frac{1}{2}\Delta K + f \cdot \nabla K - \frac{1}{2}|\nabla K|^2 \right) \right|^2. \end{aligned} \tag{A.25}$$

We can choose $\widetilde{g}$ so that

$$e^{2\widetilde{g}} \left| \nabla \left(\frac{1}{2}|h|^2 + \operatorname{div} f - \frac{1}{2}\Delta K + f \cdot \nabla K - \frac{1}{2}|\nabla K|^2 \right) \right|^2 \le e^{2g}. \tag{A.26}$$

Inequalities (A.25) and (A.26) imply

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\int_{B_R} e^{2\widetilde{g}} |\nabla u_R|^2 \right] + \frac{1}{4}\int_{B_R} e^{2\widetilde{g}} (\Delta u_R)^2 \\ &\quad \le (c+1)\int_{B_R} \frac{1}{2} e^{2\widetilde{g}} |\nabla u_R|^2 + \frac{1}{2} e^{ct} \int_{B_R} e^{2g} \sigma^2(x). \end{aligned}$$

Hence

$$\frac{d}{dt}\left[e^{-(c+1)t}\int_{B_R}\frac{1}{2}e^{2\widetilde{g}}|\nabla u_R|^2\right]$$
$$= e^{-(c+1)t}\left[\frac{d}{dt}\int_{B_R}\frac{1}{2}e^{2\widetilde{g}}|\nabla u_R|^2 - (c+1)\int_{B_R}\frac{1}{2}e^{2\widetilde{g}}|\nabla u_R|^2\right]$$
$$\leq \frac{1}{2}e^{-t}\int_{B_R}e^{2g}\sigma^2(x) - \frac{1}{4}e^{-(c+1)t}\int_{B_R}e^{2\widetilde{g}}(\Delta u_R)^2,$$

which implies

$$e^{-(c+1)T}\int_{B_R}\frac{1}{2}e^{2\widetilde{g}}|\nabla u_R|^2(T,x) - \int_{B_R}\frac{1}{2}e^{2\widetilde{g}}|\nabla u_R|^2(0,x)$$
$$\leq \frac{1}{2}T\int_{B_R}e^{2g}\sigma^2(x) - \frac{1}{4}\int_0^T e^{-(c+1)t}\int_{B_R}e^{2\widetilde{g}}(\Delta u_R)^2(t,x).$$

It follows that

$$e^{-(c+1)T}\int_{B_R}\frac{1}{2}e^{2\widetilde{g}}|\nabla u_R|^2(T,x) + \frac{1}{4}e^{-(c+1)T}\int_0^T\int_{B_R}e^{2\widetilde{g}}(\Delta u_R)^2(t,x)$$
$$\leq \int_{B_R}\frac{1}{2}e^{2\widetilde{g}}|\nabla u_R|^2(0,x) + \frac{1}{2}T\int_{B_R}e^{2g}\sigma^2(x),$$

and (A.20) follows immediately. ◻

Appendix B. Existence of a weak solution for the DMZ equation. Let $Q = (0,T)\times\mathbb{R}^n$ and let $L^2(Q)$ be the space of functions that are square integrable over Q. The scalar product of two elements v_1, v_2 of $L^2(Q)$ is defined by the equation

$$(v_1, v_2) = \iint_Q v_1 v_2\, dx\, dt.$$

The class of C^∞ functions in $\overline{Q}$ with compact supports in Q will be denoted by $C_0^\infty(Q)$.

DEFINITION B.1. *A locally L^2-integrable function is called a generalized derivative of a locally L^2-integrable function $v(t,x)$ in Q with respect to x if for each $\Phi(t,x) \in C_0^\infty(Q)$ the equation*

$$\iint_Q\left(v\frac{\partial\Phi}{\partial x_k} + w\Phi\right)dx\,dt = 0 \tag{B.1}$$

holds. In this case, we write $w = \frac{\partial v}{\partial x_k}$. The generalized derivative with respect to t and generalized derivatives of higher order are defined similarly (see [So]*).*

Remark B.2. If the sequence of functions $v_m(t,x)$ weakly tends to $v(t,x)$ in the space $L^2(Q)$ as $m\to\infty$ and the norms of $\frac{\partial v_m}{\partial x_k}$ in $L^2(Q)$ are uniformly bounded with respect to m, then $v(t,x)$ has a generalized derivative $\frac{\partial v}{\partial x_k}\in L^2(Q)$ and $\frac{\partial v_m}{\partial x_k}$ weakly tends to $\frac{\partial v}{\partial x_k}$ [So].

DEFINITION B.3. *We denote by $W^1(\mathbb{R}^n)$ the space of functions $\phi(x)$ such that $\phi(x) \in L^2(\mathbb{R}^n)$ and $\frac{\partial \phi}{\partial x_i} \in L^2(\mathbb{R}^n)$ for $i = 1, \ldots, n$, with the scalar product*

$$(\phi_1, \phi_2)_1 := \int_{\mathbb{R}^n} \phi_1(x)\phi_2(x)\,dx + \int_{\mathbb{R}^n} \sum_{i=1}^{n} \frac{\partial \phi_1}{\partial x_i}\frac{\partial \phi_2}{\partial x_i}\,dx. \tag{B.2}$$

We shall denote by $W^{1,1}(Q)$ the space of functions $v(t,x)$ for which $v(t,x) \in L^2(Q)$, $\frac{\partial v(t,x)}{\partial x_i} \in L^2(Q)$ $(i = 1, \ldots, n)$, and $\frac{\partial v(t,x)}{\partial t} \in L^2(Q)$, with the scalar product

$$(v_1, v_2)_{1,1} := \iint_Q v_1(t,x)v_2(t,x)\,dt\,dx + \iint_Q \left(\sum_{i=1}^{n} \frac{\partial v_1}{\partial x_i}\frac{\partial v_2}{\partial x_i} + \frac{\partial v_1}{\partial t}\frac{\partial v_2}{\partial t} \right) dx\,dt. \tag{B.3}$$

It is known [So] *that $W^1(\mathbb{R}^n)$ and $W^{1,1}(\mathbb{R}^n)$ are complete. The norms in $L^2(Q)$, $W^1(\mathbb{R}^n)$, and $W^{1,1}(Q)$ will be written $\|v\|_0$, $\|v\|_1$, and $\|v\|_{1,1}$, respectively.*

Remark B.4. It follows from the embedding theorems of Sobolev that a function of $W^{1,1}(Q)$ can be modified on a set of measure zero in such a way that it is L^2-integrable on the section of the cylinder Q by any n-dimensional plane or n-dimensional C^1 surface. In particular, such a function is L^2-integrable on the section of Q by any plane $t =$ constant. Moreover, the values of $v(t,x) \in W^{1,1}(Q)$ on sufficiently close n-dimensional planes will differ in mean by as little as we please [So]. In particular, if $v(t,x) \in W^{1,1}(Q)$ and $v(x,0) = \phi(x)$, then $\int_Q [v(t,x) - \phi(x)]^2\,dx \longrightarrow 0$ as $t \longrightarrow 0$.

DEFINITION B.5. *The subspace of $W^1(\mathbb{R}^n)$ consisting of functions that have compact supports in $\mathbb{R}^n$ is written $W_0^1(\mathbb{R}^n)$, and the subspace of $W^{1,1}(Q)$ consisting of functions $v(t,x)$ which have compact supports in $\mathbb{R}^n$ for any t is written $W_0^{1,1}(Q)$.*

DEFINITION B.6. *The function $u(t,x)$ in $W_0^{1,1}(Q)$ is called a weak solution of the initial value problem*

$$\begin{cases} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij}(t,x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} B_i(t,x)\frac{\partial u}{\partial x_i} + C(t,x)u = \frac{\partial u}{\partial t}, \\ u(0,x) = \phi(x) \end{cases} \tag{B.4}$$

if for any function $\Phi(t,x) \in W_0^{1,1}(Q)$ the following relation is valid:

$$\iint_Q \left[\sum_{i,j=1}^{n} A_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \Phi}{\partial x_i} - \left(\sum_{i=1}^{n} B_i \frac{\partial u}{\partial x_i} + Cu - \frac{\partial u}{\partial t} \right)\Phi \right] dx\,dt = 0 \tag{B.5}$$

and $u(0,x) = \phi(x)$.

We now recall some facts concerning convergence in Hilbert spaces.

Remark B.7. A sequence $\{u_m\}$, in a Hilbert space H with scalar product $(\cdot,\cdot)$, is said to be *weakly convergent* (to u) if the sequence $\{(u_m, f)\}$ is convergent (to (u,f)) for any $f \in H$. A weakly convergent sequence is bounded. From any bounded sequence $\{u_m\}$ in H one can extract a weakly convergent subsequence. If $\{u_m\}$ is weakly convergent to u, then there exists a subsequence $\{u_{m'}\}$ whose arithmetic means converge to u in the H norm (see [Fr, p. 273]).

THEOREM B.8. *Under the hypothesis of Theorem* A.4 *the robust DMZ equation* (1.6) *on $[0,T] \times \mathbb{R}^n$ with initial condition $\sigma_0(x) \in W_0^1(\mathbb{R}^n)$ has a weak solution.*

Proof. Let $\{R_k\}$ be a sequence of positive number such that $\lim_{k\to\infty} R_k = \infty$. Let $u_k(x)$ be the solution of the robust DMZ equation (1.9) on $[0,T] \times B_{R_k}$, where

$B_{R_k} = \{x \in \mathbb{R}^n : |x| \le R_k\}$ is a ball of radius R_k. Let

$$u_k(t,x) \begin{cases} u_{R_k}(t,x) & \text{if } x \in B_{R_k}, \\ 0 & \text{if } x \notin B_{R_k}, \end{cases} \qquad \sigma_k(x) \begin{cases} \sigma_0(x) & \text{if } x \in B_{R_k}, \\ 0 & \text{if } x \notin B_{R_k}. \end{cases}$$

In view of Theorems A.1 and A.4, the sequence $\{u_k\}$ is a bounded set in $W_0^{1,1}(Q)$. By Remark B.7, there exists a subsequence $\{u_{k'}\}$ which is weakly convergent to u. Moreover, $u(t,x)$ has generalized derivative $\frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i^2} \in L^2(Q)$, and $\frac{\partial u_{k'}}{\partial x_i}, \frac{\partial^2 u_{k'}}{\partial x_i^2}$ weakly tend to $\frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i^2}$, respectively. Now we claim that the weak derivative $\frac{\partial u}{\partial t}$ exists and is equal to the right-hand side of (B.3). To see this, let $\Phi(t,x) \in W_0^{1,1}(Q)$. Then

$$\begin{aligned} \iint \bigg[\frac{1}{2}\Delta u - (f(x) - \nabla K) \cdot \nabla u - \bigg(\frac{1}{2}|h|^2 + \operatorname{div} f - \frac{1}{2}\Delta K \\ + f \cdot \nabla K - \frac{1}{2}|\nabla K|^2 \bigg) u \bigg] \Phi(t,x)\, dx\, dt \\ = \lim_{k' \to \infty} \iint \bigg[\frac{1}{2}\Delta u_{k'} - (f(x) - \nabla K) \cdot u_{k'} \\ - \bigg(\frac{1}{2}|h|^2 + \operatorname{div} f - \frac{1}{2}\Delta K \\ + f \cdot \nabla K - \frac{1}{2}|\nabla K|^2 \bigg) u_{k'} \bigg] \Phi(t,x)\, dx\, dt \\ = \lim_{k' \to \infty} \iint \frac{\partial u_{k'}}{\partial t} \Phi(t,x)\, dx\, dt \\ = -\lim_{k' \to \infty} \iint u_{k'} \frac{\partial \Phi}{\partial t}(t,x)\, dx\, dt \\ = -\iint u \frac{\partial \Phi}{\partial t}(t,x)\, dx\, dt. \end{aligned}$$

Clearly $u(0,x) = \lim_{k' \to \infty} u_{k'}(0,x) = \lim_{k' \to \infty} \sigma_{k'}(x) = \sigma_0(x)$. ◻

Appendix C. Uniqueness of a weak solution for the DMZ equation. We are now ready to establish the uniqueness of a weak solution for the DMZ equation. We shall follow the notation in previous sections.

THEOREM C.1. *Let $Q = (0,T) \times \mathbb{R}^n$. Assume that for some $c > 0$,*

$$\sup_{0 \le t \le T} \int_{\mathbb{R}^n} e^{cr} u^2(t,x)\, dx < \infty, \tag{C.1}$$

$$\int_0^T \int_{\mathbb{R}^n} e^{cr} |\nabla u(t,x)|^2\, dx\, dt < \infty, \tag{C.2}$$

where $r = \sqrt{x_1^2 + \cdots + x_n^2}$. Suppose that there exists a finite number α such that

$$\left| \frac{c}{2}\nabla r + f - \nabla K \right|^2 - 2\left(\frac{1}{2}|h|^2 + \operatorname{div} f - \frac{1}{2}\Delta K + f \cdot \nabla K - \frac{1}{2}|\nabla K|^2 \right) \le \alpha. \tag{C.3}$$

Then the weak solution $u(t,x)$ of the robust DMZ *equation on Q is unique.*

Proof. We only need to prove that $u(t,x)=0$ on Q if $u(0,x)=0$. By iteration, we may assume that $\alpha T<1$. Let $\Phi\in C_0^\infty(\mathbb{R}^n)$. According to the definition of weak solution (B.5), we have

$$\begin{aligned}\iint_Q \frac{\partial u}{\partial t}\Phi\,dt\,dx = &-\frac{1}{2}\int_0^T\int_{\mathbb{R}^n}\nabla u\cdot\nabla\Phi\,dx\,dt-\int_0^T\int_{\mathbb{R}^n}(f-\nabla K)\cdot\nabla u\Phi\,dx\,dt\\ &-\int_0^T\int_{\mathbb{R}^n}\left[\frac{1}{2}|h|^2+\operatorname{div} f-\frac{1}{2}\Delta K+f\cdot\nabla K-\frac{1}{2}|\nabla K|^2\right]u\Phi\,dx\,dt.\end{aligned}\tag{C.4}$$

Replacing Φ by Φe^{cr}, we have

$$\begin{aligned}\int_{\mathbb{R}^n}u(T,x)\Phi e^{cr} = &-\frac{1}{2}\int_0^T\int_{\mathbb{R}^n}\nabla u\cdot(e^{cr}\nabla\Phi)-\frac{c}{2}\int_0^T\int_{\mathbb{R}^n}\Phi e^{cr}\nabla r\cdot\nabla u\\ &+\int_0^T\int_{\mathbb{R}^n}e^{cr}\Phi(-f+\nabla K)\cdot\nabla u\\ &-\int_0^T\int_{\mathbb{R}^n}\left[\frac{1}{2}|h|^2+\operatorname{div} f-\frac{1}{2}\Delta K+f\cdot\nabla K-\frac{1}{2}|\nabla K|^2\right]\Phi u e^{cr}\\ &+\int_0^T\int_{\mathbb{R}^n}u\frac{\partial\Phi}{\partial t}e^{cr}.\end{aligned}\tag{C.5}$$

Approximating u by Φ in the $W^{1,1}(Q)$ norm, we get

$$\begin{aligned}\int_{\mathbb{R}^n}u^2(T,x)e^{cr} = &-\frac{1}{2}\int_0^T\int_{\mathbb{R}^n}e^{cr}|\nabla u|^2-\frac{c}{2}\int_0^T\int_{\mathbb{R}^n}ue^{cr}\nabla r\cdot\nabla u\\ &+\int_0^T\int_{\mathbb{R}^n}e^{cr}u(-f+\nabla K)\cdot\nabla u\\ &-\int_0^T\int_{\mathbb{R}^n}\left[\frac{1}{2}|h|^2+\operatorname{div} f-\frac{1}{2}\Delta K+f\cdot\nabla K-\frac{1}{2}|\nabla K|^2\right]u^2e^{cr}\\ &+\int_0^T\int_{\mathbb{R}^n}u\frac{\partial u}{\partial t}e^{cr}\\ = &-\frac{1}{2}\int_0^T\int_{\mathbb{R}^n}e^{cr}\Big\{|\nabla u|^2+\Big[cu\nabla r-2u(-f+\nabla K)\Big]\cdot\nabla u\Big\}\\ &-\int_0^T\int_{\mathbb{R}^n}\left[\frac{1}{2}|h|^2+\operatorname{div} f-\frac{1}{2}\Delta K+f\cdot\nabla K-\frac{1}{2}|\nabla K|^2\right]u^2e^{cr}\\ &+\int_0^T\int_{\mathbb{R}^n}\frac{1}{2}e^{cr}u\Delta u-\int_0^T\int_{\mathbb{R}^n}e^{cr}u(f-\nabla K)\cdot\nabla u\\ &-\int_0^T\int_{\mathbb{R}^n}e^{cr}u^2\left[\frac{1}{2}|h|^2+\operatorname{div} f-\frac{1}{2}\Delta K+f\cdot\nabla K-\frac{1}{2}|\nabla K|^2\right]\\ = &-\int_0^T\int_{\mathbb{R}^n}e^{cr}\Big\{|\nabla u|^2+\Big[cu\nabla r-2u(-f+\nabla K)\Big]\cdot\nabla u\Big\}\end{aligned}$$

$$
\begin{aligned}
&- 2\int_0^T \int_{\mathbb{R}^n} \left[\frac{1}{2}|h|^2 + \operatorname{div} f - \frac{1}{2}\Delta K + f\cdot\nabla K - \frac{1}{2}|\nabla K|^2\right] u^2 e^{cr} \\
&= -\int_0^T \int_{\mathbb{R}^n} e^{cr}\left|\nabla u - \frac{cu}{2}\nabla r - uf + u\nabla K\right|^2 \\
&\quad + \int_0^T \int_{\mathbb{R}^n} e^{cr}u^2\left\{\left|\frac{c}{2}\nabla r + f - \nabla K\right|^2 \right. \\
&\qquad\qquad \left. - \left(|h|^2 + 2\ \operatorname{div} f - \Delta K + 2f\cdot\nabla K - |\nabla K|^2\right)\right\} \\
&\le \alpha \int_0^T \int_{\mathbb{R}^n} e^{cr}u^2(t,x). \qquad \text{(C.6)}
\end{aligned}
$$

By the mean value theorem, there exists $T_1 \in (0,T)$ such that

$$\int_0^T \int_{\mathbb{R}^n} e^{cr}u^2(t,x) = T\int_{\mathbb{R}^n} e^{cr}u^2(T_1,x).$$

In view of (C.5), we have

$$\int_{\mathbb{R}^n} u^2(T,x)e^{cr} \le \alpha T \int_{\mathbb{R}^n} u^2(T_1,x)e^{cr}. \tag{C.7}$$

By applying (C.5) successfully, there exists $T_m \in (0,T)$ such that

$$\int_{\mathbb{R}^n} u^2(T,x)e^{cr} \le (\alpha T)^m \int_{\mathbb{R}^n} u^2(T_m,x)e^{cr}. \tag{C.8}$$

As $\alpha T < 1$, we conclude that $u = 0$. □

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