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Wen-Lin Chiou [a], Woei-Ren Chiueh [a] & Stephen S.-T. Yau [b]

[a] Department of Mathematics, Fu-Jen University, Taipei, Taiwan

[b] Department of Mathematics, Statistics and Computer Science (M/C 249), University of Illinois at Chicago, Chicago, IL, USA

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Mitter conjecture for low dimensional estimation algebras in non-linear filtering†

Wen-Lin Chiou[a], Woei-Ren Chiueh[a] and Stephen S.-T. Yau[b]*

[a]*Department of Mathematics, Fu-Jen University, Taipei, Taiwan;* [b]*Department of Mathematics, Statistics and Computer Science (M/C 249), University of Illinois at Chicago, Chicago, IL, USA*

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It is well known that a systematical way to construct finite dimensional filter is to classify all finite dimensional estimation algebras. Mitter conjecture, which states that all functions in any finite dimensional estimation algebra are necessarily degree one polynomial, plays a crucial role in classifying all finite dimensional estimation algebras. The purpose of this paper is to prove the Mitter Conjecture for estimation algebra of dimension at most 5 with arbitrary state space dimension.

Keywords: Duncan-Mortensen-Zakai equation; finite-dimensional filter; finite-dimension estimation algebra; Mitter conjecture

1. Introduction

In the late seventies, a basic approach to non-linear filtering theory was independently proposed by Mitter (1979), (1983); Brockett (1980) and Brockett and Clark (1980). They suggested that the construction of the filter can be divided into two parts: (i) a universal filter which is the evolution equation describing the unnormalised conditional density, the Duncan–Mortensen–Zakai equation (cf. for example Zakai 1969) and (ii) a state-output map, which depends on the statistics being computed, where the state of the filter is the unnormalised conditional density. The reason for focusing on the Duncan–Mortensen–Zakai equation is that it is a linear equation and is a much simpler object than the non-linear conditional density equation and can be treated using geometric ideas. In 1983, Brockett formally proposed the problem of classifying all finite-dimensional estimation algebras in his lecture at the International Congress of Mathematicians. Recent work on estimation algebras have given us a deeper understanding of the Duncan–Mortensen–Zakai equation which was essential for progress in non-linear filtering as well as in stochastic control. Despite the usefulness of the Kalman–Bucy filter, however, it is not perfect. One of its weaknesses is that it is restricted to linear dynamical systems. Another weakness is that it needs a Gaussian assumption for the initial distribution. The advantage of the Brockett–Mitter approach of using the estimation algebra method to solve the Duncan–Mortensen–Zakai equation is the following. As long as the estimation algebra is finite dimensional, we will get a finite dimensional recursive filter and there is not a need to make any assumption on the initial distribution. Moreover, the approach applies well to non-linear dynamical systems. In fact, the finite dimensional filters constructed by the estimation algebra method are universal in the sense of Chaleyat–Maurel and Michel (1984). Wong (1987a,b) introduced a fundamental notion of Wong matrix which plays an important role in the classification of finite-dimensional estimation algebras, and gave a structure theorem of estimation algebra in case the drift term $f(x)$ is real analytic and its first, second and third derivatives of $f(x)$ are bounded functions. Nevertheless, the structure and classification of finite-dimensional estimation algebras were studied in detail only in the early 1990s by Tam, Wong and Yau (1990); Chiou and Yau (1994); Yau (1994); Chen and Yau (1996, 1997); Chen, Yau and Leung (1996, 1997); Chiou (1996); Dong, Tam, Wong and Yau (1997); Wu, Yau and

*Corresponding author. Email: yau@uic.edu
†Dedicated to Professor Han-Fu Chen on the occasion of his 70th Birthday.

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Hu (2002) and Yau and Hu (2008). In particular, Yau and Hu (2005) finished the final step in classification of finite-dimensional estimation algebras with maximal rank.

The success of the classification of finite dimensional estimation algebras of maximal rank was partly inspired by the Mitter Conjecture which was communicated to the third named author in 1990 by Mitter. The conjecture states that all the functions in any finite dimensional estimation algebra are necessarily degree one polynomials. An affirmative answer to this conjecture will give us a deeper understanding of the structure of finite dimensional estimation algebras without maximal rank. Rasoulian and Yau (1997) gave a general method to construct finite dimensional estimation algebras without maximal rank. But all their finite dimensional estimation algebras can be viewed as estimation algebras with maximal rank for certain filtering models. Recently Wu and Yau (2006) were able to classify all finite dimensional estimation algebras with state space dimension two. Their results are much deeper than the corresponding results of Chiou and Yau (1994) in the maximal rank case. Subsequently the Mitter Conjecture was proved in the case of state space dimension 2. The purpose of this paper is to prove the Mitter Conjecture for estimation algebra of dimension at most 5 with arbitrary state space dimension.

Theorem 1: *Let E be a finite dimensional estimation algebra associate to the filtering model* (1) *with arbitrary state space dimension. Then any function in E is a polynomial of degree at most one if* $\dim E \leq 5$.

One consequence of the classification of finite dimensional estimation algebras with maximal rank is the following. In order for an estimation algebra with maximal rank to be finite dimensional, the dynamical system has to be quite special, i.e. the drift term f in (1) must be of the form $f(x) = (\ell_1, \ldots, \ell_n) + \nabla\phi$, where $\ell_1, \ldots, \ell_n$ are degree one polynomials in $x_1, \ldots, x_n$ and ϕ is a C^∞ function. Therefore it is very desirable to classify finite dimensional estimation algebra without maximal rank. Hopefully we can substantially increase our knowledge in finite dimensional filters. In the course of classifying five dimensional estimation algebras, we discover a new class of finite dimensional filters of which the drift term $f(x)$ cannot be written as $(\ell_1, \ldots, \ell_n) + \nabla\phi$ as above. This is quite exciting because it indicates that there should be a lot of new classes of finite dimensional filters. Although the present paper is only an initial step towards understanding all finite dimensional estimation algebras, we believe that the technique developed in this paper (especially Lemma 1 which holds for arbitrary dimension) will be useful in the general situation. Historically, classification of finite dimensional estimation algebras with maximal rank was first done for state space dimension ≤ 4 before one can give a general proof. It is quite possible that the techniques developed in the proof of this paper and our previous paper (2006) together with the techniques developed in the classification of finite dimensional estimation algebras with maximal rank in Yau and Hu (2005) and Yau (2003) will solve the general problem of Mitter Conjecture and classification of finite dimensional estimation algebras.

In § 2, we recall the basic notations and some known theorems which will be used frequently in the proof of Theorem 1. In § 3, we shall prove a lemma which plays a key role in the proof of Theorem 1. This lemma may have independent interest in other branches of mathematics. In § 4, we prove Theorem 1. In § 5, we construct a new class of finite dimensional filters.

2. Basic notations and theorems

The filtering problem considered here is based on the following signal observation model:

$$\begin{cases} \mathrm{d}x(t) = f(x(t))\mathrm{d}t + g(x(t))\mathrm{d}v(t), & x(0) = x_0 \\ \mathrm{d}y(t) = h(x(t))\mathrm{d}t + \mathrm{d}w(t), & y(0) = 0 \end{cases} \tag{1}$$

in which x, v, y and w are respectively $\mathbb{R}^n$, $\mathbb{R}^n$, $\mathbb{R}^m$, and $\mathbb{R}^m$ valued processes, and v and w have components that are independent, standard Brownian processes. We further assume that f, h are C^∞ smooth, and that g is an orthogonal matrix.

Let $\rho(t, x)$ denote the conditional density of the state given the observation $\{y(s)\colon 0 \leq s \leq t\}$. It is well known that $\rho(t, x)$ is given by normalising a function $\sigma(t, x)$ which satisfies the Duncan–Mortensen–Zakai equation in Stratonovich form:

$$\mathrm{d}\sigma(t,x) = L_0\sigma(t,x)\mathrm{d}t + \sum_{i=1}^{m} L_i\sigma(t,x)\mathrm{d}y_i(t), \quad \sigma(0,x) = x_0, \tag{2}$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i} - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2$$

and for $i=1,\ldots,m$, L_i is the zero degree differential operator of multiplication by h_i. Here σ_0 is the probability density of the initial point x_0. Let

$$D_i=\frac{\partial}{\partial x_i}-f_i,\quad \eta=\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}+\sum_{i=1}^{n}f_i^2+\sum_{i=1}^{m}h_i^2. \tag{3}$$

Then L_0 can be rewritten as $L_0=\frac{1}{2}(\sum_{i=1}^{n}D_i^2-\eta)$.

Equation (2) is a stochastic partial differential equation. In real application, we are interested in constructing state estimators from observed sample paths with some property of robustness. M.H.A. Davis studied this problem and proposed some robust algorithms. In our case, his basic idea reduces to defining a new unnormalised density

$$u(t,x)=\exp\left(-\sum_{i=1}^{m}h_i(x)y_i(t)\right)\sigma(t,x).$$

It is easy to show that $u(t,x)$ satisfies the following time-varying partial differential equation

$$\begin{cases}\dfrac{\partial u}{\partial t}(t,x)=L_0u(t,x)+\displaystyle\sum_{i=1}^{m}y_i(t)[L_0,L_i]u(t,x)\\ \qquad +\dfrac{1}{2}\displaystyle\sum_{i,j=1}^{m}y_i(t)y_j(t)[[L_0,L_i],L_j]u(t,x)\\ u(0,x)=\sigma_0,\end{cases} \tag{4}$$

where $[\cdot,\cdot]$ is the Lie bracket.

Definition 1: The estimation algebra E of a filtering model (1) is defined to be the Lie algebra generated by $L_0,L_1,\ldots,L_m$ or $E=\langle L_0,L_1,\ldots,L_m\rangle_{\text{L.A.}}$ If $x_i\in E$ for every $1\le i\le n$, then E is called an estimation algebra of maximal rank. If E as a vector space over R is a finite dimensional vector space, then E is called a finite dimensional estimation algebra.

Definition 2: Wong matrix $\Omega=(\omega_{ij})$ is a $n\times n$ skew-symmetric matrix with $\omega_{ij}=[D_j,D_i]=\partial f_j/\partial x_i-\partial f_i/\partial x_j$.

Theorem 2 (Ocone 1981)**:** *Any function in a finite dimensional estimation algebra is a polynomial of degree at most two.*

The following theorem in Yau (1994) is very useful in the classification of finite dimension estimation algebras.

Theorem 3 (Yau 1994)**:** *Let $F(x_1,\ldots,x_n)$ be a polynomial on $\mathbb{R}^n$. Suppose there exists a polynomial path c: $\mathbb{R}\to\mathbb{R}^n$ such that $\lim_{t\to\infty}c(t)=\infty$ and $\lim_{t\to\infty}F(c(t))=-\infty$. Then there are no C^∞ function $f_1,\ldots,f_n$ on $\mathbb{R}^n$ such that*

$$\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}+\sum_{i=1}^{n}f_i^2=F.$$

Let U_k be the vector space of differential operators of order up to and including k. Assume that the coefficients of these differential operators are C^∞ functions. Wong (1987a) proved the following two theorems.

Theorem 4 (Wong 1987a)**:** *If $Y=\sum_{i=1}^{n}\gamma_iD_i \bmod U_0$ is an element in a finite dimensional estimation algebra, then γ_i are polynomials of $x_1,\ldots,x_n$ for all i.*

Theorem 5 (Wong 1987a)**:** *If $Y=\sum_{1\le i\le j\le n}\gamma_{ij}\times LD_iD_j \bmod U_1$ is an element in a finite dimensional estimation algebra, then γ_{ij} are polynomials of $x_1,\ldots,x_n$ for all i,j.*

The following structure theorem (Yau and Rasoulian 1999) for the Euler operator $E_\ell=x_1(\partial/\partial x_1)+x_2(\partial/\partial x_2)+\cdots+x_\ell(\partial/\partial x_\ell)$ will be used frequently.

Theorem 6 (Yau and Rasoulian 1999)**:** *Suppose $m\in\mathbb{Z}$ is a constant integer and ζ is a C^∞ function on $\mathbb{R}^n$ and $\ell\le n$ such that $E_\ell(\zeta)+m\zeta$ is a polynomial of degree k, k a positive integer, in $x_1,x_2,\ldots,x_\ell$ variables with coefficients in C^∞ functions of $x_{\ell+1},\ldots,x_n$ variables. If $k+m\ge 0$, ζ is a polynomial of degree k in $x_1,x_2,\ldots,x_\ell$ variables with coefficients in C^∞ functions of $x_{\ell+1},\ldots,x_n$. If $k+m<0$, ζ is a polynomial of degree at most $m'=-m$ in $x_1,x_2,\ldots,x_\ell$ variables with coefficients in C^∞ functions of $x_{\ell+1},\ldots,x_n$.*

Theorem 7 (Yau and Rasoulian 1996)**:** *Suppose that $h=\frac{1}{2}\sum_{i=1}^{\ell}x_i^2$ for some $1\le\ell\le n$ is in the finite dimensional estimation algebra E. Let $\alpha_i=\sum_{j=1}^{\ell}x_j\omega_{ij}$. Then*

(i) *for $1\le i,j\le\ell$, $E_\ell(\omega_{ij})+2(\omega_{ij})=(\partial\alpha_i/\partial x_j)-(\partial\alpha_j/\partial x_i)$ and ω_{ij} is a polynomial in $x_1,\ldots,x_n$*
(ii) *for $i\ge\ell+1$, $j\le\ell$, $E_\ell(\omega_{ij})+\omega_{ij}=(\partial\alpha_i/\partial x_j)-(\partial\alpha_j/\partial x_i)$ and ω_{ij} is a polynomial in $x_1,\ldots,x_n$.*
(iii) *for $\ell+1\le i,j\le\ell$, $E_\ell(\omega_{ij})=(\partial\alpha_i/\partial x_j)-(\partial\alpha_j/\partial x_i)$ and ω_{ij} is a polynomial in $x_1,\ldots,x_n$ plus a C^∞ function in $x_{\ell+1},\ldots,x_n$.*

3. Fundamental lemma

The following lemma plays an important role in proving the Mitter conjecture for low dimensional estimation algebras.

Lemma 1: *For any $1\le\ell\le n$, if γ_i, $i=1,\ldots,\ell$, are polynomials in $x_1,\ldots,x_\ell$ with coefficients in C^∞*

functions of $x_{\ell+1}, \dots, x_n$ *satisfying*

$$\frac{\partial \gamma_j}{\partial x_i} + \frac{\partial \gamma_i}{\partial x_j} = 0 \quad \text{for all } 1 \le i, j \le \ell, \tag{5}$$

then each γ_i *is necessary of the form*

$$\gamma_i = \sum_{1 \le j \le \ell} c_i^j(x_{\ell+1}, \dots, x_n) x_j + d_i(x_{\ell+1}, \dots, x_n) \tag{6}$$

where $c_i^j(x_{\ell+1}, \dots, x_n)$ *and* $d_i(x_{\ell+1}, \dots, x_n)$ *are* C^∞ *functions and* $c_i^j = -c_j^i$.

Proof: Taking $i = j$ in (5), we get

$$\frac{\partial \gamma_i}{\partial x_i} = 0 \quad \text{for } 1 \le i \le \ell. \tag{7}$$

Differentiating (5) with respect to x_i and using (7), we get

$$\frac{\partial^2 \gamma_j}{\partial x_i^2} = 0 \quad \text{for } 1 \le i, j \le \ell. \tag{8}$$

Since γ_j is a polynomial in $x_1, \dots, x_\ell$ with coefficients C^∞ functions of $x_{\ell+1}, \dots, x_n$, by (7) and (8), γ_j can be written as

$$\gamma_j = \sum_{s_i \in \{0,1\}} c_j^{(s_1, \dots, s_{j-1}, s_{j+1}, \dots, s_\ell)} x_1^{s_1} \cdots x_{j-1}^{s_{j-1}} x_{j+1}^{s_{j+1}} \cdots x_\ell^{s_\ell} + d_j, \tag{9}$$

where $c_j^{(s_1, \dots, s_{j-1}, s_{j+1}, \dots, s_\ell)}$ and d_j are C^∞ smooth functions of $x_{\ell+1}, \dots, x_n$.

From equations (9) and (5), it is easy to deduce the result of the lemma if $\ell = 2$. As for the case $\ell > 2$, we shall prove that the degree of γ_j is at most one in $x_1, \dots, x_\ell$ variables. Clearly the degree of γ_j is at most $\ell - 1$ in $x_1, \dots, x_\ell$ variables. We can write γ_j in the form

$$\gamma_j = c_j^{(1,\dots,1)} x_1 x_2 \cdots x_{j-1} x_{j+1} \cdots x_\ell + \text{lower degree in } x_1, \dots, x_\ell.$$

Putting the functions γ_j into (5), we get (assuming $i < j$)

$$c_j^{(1,\dots,1)} x_1 \cdots x_{i-1} x_{i+1} \cdots x_{j-1} x_{j+1} \cdots x_\ell + c_i^{(1,\dots,1)} x_1 \cdots x_{i-1} x_{i+1} \cdots x_{j-1} x_{j+1} \cdots x_\ell = 0. \tag{10}$$

Hence $c_j^{(1,\dots,1)} = -c_i^{(1,\dots,1)}$. Replacing i, j by i, k and k, j respectively, we get $c_k^{(1,\dots,1)} = -c_i^{(1,\dots,1)}$ and $c_k^{(1,\dots,1)} = -c_j^{(1,\dots,1)}$. It follows that $c_j^{(1,\dots,1)} = 0$ for all $1 \le j \le \ell$. We have shown that the degree of γ_j is at most $\ell - 2$ in $x_1, \dots, x_\ell$ variables. By the same method and decreasing induction, we can show easily that $c_j^{(s_1, \dots, s_{j-1}, s_{j+1}, \dots, s_\ell)}$ are zeroes except possibly the coefficients of the linear term in $x_1, \dots, x_\ell$. The lemma follows immediately from (5). □

4. Mitter Conjecture for estimation algebras of dimension at most 5

Yau and Rasoulian (1999) have classified all finite dimensional estimation algebras of dimension at most 4. As a corollary of their classification result, they have proved the Mitter Conjecture for estimation algebras of dimension at most 4. Therefore to prove Theorem 1, we only need to prove the Mitter Conjecture holds for estimation algebras of dimension 5. In what follows we shall assume that all the estimate algebras are of dimension 5. For the sake of convenience to our readers, we collect our notations together as follows:

$$L_0 = \frac{1}{2}\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^n f_i \frac{\partial}{\partial x_i} - \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^m h_i^2$$

$$D_i = \frac{\partial}{\partial x_i} - f_i, \quad \eta = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^n f_i^2 + \sum_{i=1}^m h_i^2$$

$$\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$$

$U_k =$ space of differential operations of order up to and inclding k

$$E_\ell = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + \cdots + x_\ell \frac{\partial}{\partial x_\ell}$$

$$\alpha_i = \sum_{j=1}^\ell x_j \omega_{ij}, \quad 1 \le i \le n$$

$$\beta_\ell = \sum_{i=1}^\ell \frac{\partial \alpha_i}{\partial x_i}, \quad \beta_n = \sum_{i=1}^n \frac{\partial \alpha_i}{\partial x_i}.$$

Recall that estimation algebra E of the filtering model (1) is a Lie algebra generated by $L_0, h_1, h_2, \dots, h_m$. By Ocone's theorem, $h_1, h_2, \dots, h_m$ are polynomials of degree at most two. It is easy to see that dim $E \le 2$ if and only if all h_is are constants. Observe

$$[L_0, x_1] = D_1, \quad [D_1, x_1] = 1,$$
$$\left[L_0, \sum_{j=2}^n a_j x_j + c\right] = \sum_{j=2}^n a_j D_j. \tag{11}$$

Proposition 1: If dim $E = 5$, *then*

(i) *E cannot contain two linear independent degree one polynomials*,
(ii) *E cannot contain a degree one polynomial and a degree two polynomial*,
(iii) *E cannot contain two linearly independent degree two polynomials*.

Proof: (i) Suppose there are two linear independent polynomials of degree one in E. Without loss of generality, we may assume that these are of the form x_1

and $\sum_{j=2}^n a_j x_j + c$. In view of (11), E contains six linearly independent elements L_0, x_1, D_1, 1, $\sum_{j=2}^n a_j x_j$ and $\sum_{j=2}^n a_j D_j$ and (i) is proven.

Let

$$h = \sum_{i,j=1}^n a_{ij} x_i x_j + \sum_{j=1}^n b_j x_j + c_1,$$
$$\widetilde{h} = \sum_{i,j=1}^n d_{ij} x_i x_j + \sum_{j=1}^n e_j x_j + c_2$$

be two linearly independent degree two polynomials. Observe

$$Y_1 = [L_0, h] = \sum_{i,j=1}^n a_{ij}(x_i D_j + x_j D_i) + \sum_{j=1}^n b_j D_j \mod U_0 \tag{12}$$

$$\widetilde{Y}_1 = [L_0, \widetilde{h}] = \sum_{i,j=1}^n d_{ij}(x_i D_j + x_j D_i) + \sum_{j=1}^n e_j D_j \mod U_0 \tag{13}$$

$$Y_2 = [L_0, Y_1] = \sum_{i,j=1}^n a_{ij} D_i D_j \mod U_1 \tag{14}$$

where U_i is the space of differential operators with order at most i.

(ii) If E contains a degree one polynomial and a degree two polynomial, we may assume that x_1, and degree two polynomial $h = \sum_{i,j=1}^n a_{ij} x_i x_j + \sum_{j=1}^n b_j x_j + c_1$, where $a_{ij} \neq 0$ for some i, j, are in E. From (11) and (12), we see that E contains six linear independent elements L_0, x_1, D_1,1, h and Y_1.

(iii) Suppose E contains two linear independent degree two polynomials, say h and $\widetilde{h}$. Denote $h^{(i)}$ be the degree i monomial part of h. If $h^{(2)} = c\widetilde{h}^{(2)}$ for some constant c, then E contains two degree one polynomials $h_1{}^{(1)} + c_1$ and $\widetilde{h}_1{}^{(1)} + c_2$. In view of the proof of (i) and (ii) above, dim E is at least 6. On the other hand, if $h^{(2)} \neq c\widetilde{h}^{(2)}$ for any constant c, then we may assume that $\Sigma a_{ij} x_i x_j \neq \sum_{i=1}^n x_i^2$. From (12), (13) and (14), we see that E contains the six linearly independent elements $L_0, h, \widetilde{h}, Y_1, \widetilde{Y}_1$ and Y_2. □

Theorem 8: *Let E be the estimation algebra of a filtering model* (5) *with state space dimension n. Suppose* dim $E = 5$. *Then*

(i) *n is at least two,*
(ii) *if E contains a degree two polynomial, then E contains a polynomial of the form $h_1(x) = \sum_{i=1}^{\ell} x_i^2$ for some $1 \leq \ell \leq n$.*

Proof: (i) It follows from Theorem A of Chiou and Yau (1994, p. 303) which states that there is no finite dimensional estimation algebra of dimension ≥ 5 when the state space dimension is one.

(ii) If E contains a degree two polynomial, say h_1, then by using the affine transformation $\widetilde{x} = Ax + b$, where A is orthogonal, we may assume h_1 is of the form $\sum_{i=1}^{\ell} a_i x_i^2 + \sum_{i=\ell+1}^n a_i x_i + a_0$, where $a_1, \ldots, a_\ell$ are non-zero real numbers, and $\ell \leq n$. (If $\ell = n$, the second summation vanishes.) Let $X_0 = h_1$, and define X_i for $i \geq 1$ recursively by $X_i = [[L_0, X_{i-1}], X_0]$. Since $L_0 = \frac{1}{2}\left(\sum_{i=1}^n D_i - \eta\right)$, it is easy to see that

$$X_1 = 4\sum_{i=1}^{\ell} c_i^2 x_i^2 + \sum_{i=\ell+1}^n c_i^\ell$$

and for $j > 1$

$$X_j = 4^j \sum_{i=1}^{\ell} c_i^{j+1} x_i^2.$$

By the invertibility of the Vandermonde matrix, it follows after some relabelling, if necessary, that $\sum_{i=1}^{\ell} x_i^2$ is in E. □

In view of the above discussion, in order to prove the Mitter Conjecture for estimation algebras of dimension five, we only need to consider two possible cases: case 1, $h_1(x) = \frac{1}{2}\sum_{i=1}^{\ell} x_i^2$, $1 \leq \ell < n$ and case 2, $h_1(x) = \frac{1}{2}\sum_{i=1}^n x_i^2$. In both cases, $h_j = c_j h_1$ for some constants c_j, $j = 2, \ldots, m$. We may also assume that $n \geq 2$. Notice that $\sum_{j=1}^m h_j^2 = c^2 h_1^2$ where $c^2 = 1 + c_2^2 + \cdots + c_m^2$.

Proof of Theorem 1:

Case 1: $h_1(x) = \frac{1}{2}\sum_{i=1}^{\ell} x_i^2$, $\ell \leq n-1$. Observe that

$$Y_1 := [L_0, h_1] = \sum_{i=1}^{\ell} x_i D_i + \frac{\ell}{2}$$
$$Y_2 := [L_0, Y_1] = \sum_{i=1}^{\ell} D_i^2 - \sum_{i=1}^n \alpha_i D_i - \frac{1}{2}\beta_n + \frac{1}{2}E_\ell(\eta),$$

where $\alpha_i = \sum_{j=1}^{\ell} x_j \omega_{ij}$ and $\beta_n = \sum_{i=1}^n (\partial\alpha_i/\partial x_i) = \sum_{i=1}^n \sum_{j=1}^{\ell} x_j(\partial\omega_{ij}/\partial x_i)$

$$Z_{21} := [Y_2, Y_1] = 2\sum_{i=1}^{\ell} D_i^2 + \sum_{i=1}^n E_\ell(\alpha_i) D_i - 3\sum_{i=1}^{\ell} \alpha_i D_i - \beta_\ell + \sum_{i=1}^n \alpha_i^2 + \frac{1}{2}E_\ell(\beta_n) - \frac{1}{2}E_\ell(E_\ell(\eta)),$$

where $\beta_\ell = \sum_{i=1}^{\ell}(\partial\alpha_i/\partial x_i) = \sum_{i=1}^{\ell}\sum_{j=1}^{\ell} x_j(\partial\omega_{ij}/\partial x_i)$ and $\sum_{i=1}^{n}\alpha_i^2 = \sum_{i=1}^{n}\sum_{j,k=1}^{\ell} x_j x_k \omega_{ij}\omega_{ik}$,

$$\Delta := Z_{21} - 2Y_2 = \sum_{i=1}^{\ell}(E_\ell(\alpha_i) - \alpha_i)D_i + \sum_{i=\ell+1}^{n}(E_\ell(\alpha_i) + 2\alpha_i)D_i + \sum_{i=1}^{n}\alpha_i^2 - \beta_\ell + \frac{1}{2}(E_\ell(\beta_n) + 2\beta_n) - \frac{1}{2}E_\ell(E_\ell(\eta) + 2\eta).$$

By Theorem 4 and Theorem 6, α_i, $1 \le i \le n$, are polynomials in $x_1, \dots, x_\ell$ with coefficients in C^∞ functions of $x_{\ell+1}, \dots, x_n$. Clearly L_0, h_1, Y_1 and Y_2 are linearly independent since $\ell < n$. Suppose that Δ is a linear combination of L_0, h_1, Y_1, Y_2. Since L_0 and Y_2 are differential operators of order 2 while Δ is a differential operator of order 1, there exist $\lambda, \gamma \in \mathbb{R}$ such that

$$\Delta = \lambda Y_1 + \gamma h_1 = \lambda\left(\sum_{i=1}^{\ell} x_i D_i + \frac{\ell}{2}\right) + \gamma\left(\frac{1}{2}\sum_{i=1}^{\ell} x_i^2\right).$$

But this implies $E_\ell\,(\alpha_i) + 2\alpha_i = 0$ for $\ell+1 \le i \le n$ and $E_\ell(\alpha_i) - \alpha_i = \lambda x_i$, $1 \le i \le \ell$. We are going to prove that $\lambda = 0$.

$$E_\ell(\alpha_1) - \alpha_1 = \lambda x_1 \Rightarrow \sum_{i=1}^{\ell} x_i \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{\ell} x_j\omega_{1j}\right) - \sum_{j=1}^{\ell} x_j\omega_{1j} = \lambda x_1 \Rightarrow \sum_{j=1}^{\ell} x_j E_\ell(\omega_{1j}) = \lambda x_1.$$

The above equality is impossible (because $E_\ell(\omega_{1j})$ is a C^∞ function, simply put $x_1 = x_3 = \cdots = x_\ell = 0$) unless $\lambda = 0$, which implies $E_\ell(\alpha_i) - \alpha_i = 0$, $1 \le i \le \ell$. Next by the argument of Yau and Rasoulian (1999, p. 2315), we have $\beta_n = \beta_l$ and $\epsilon(\beta_n) = 0$. Thus we have proved that $\Delta = \sum_{i=1}^{n}\alpha_i^2 - \frac{1}{2}[E_\ell(E_\ell(\eta) + 2\eta)]$, where $\sum_{i=1}^{n}\alpha_i^2$ is a polynomial of degree two in $x_1, \dots, x_\ell$ with coefficients in C^∞ functions of $x_{\ell+1}, \dots, x_n$. By Theorem 6, η is a polynomial of degree two in $x_1, \dots, x_\ell$ with coefficients in C^∞ functions of $x_{\ell+1}, \dots, x_n$. Now we have by (3) that

$$\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = -\sum_{j=1}^{m} h_j^2 + \eta = -c^2\left(\sum_{i=1}^{\ell} x_i^2\right)^2 + \eta. \tag{15}$$

However, there is no C^∞ solution $f = (f_1, \dots, f_n)$ for Equation (15) by Theorem 3, a contradiction. We have proved that the five elements L_0, h_1, Y_1, Y_2 and Δ are linearly independent and they form a basis of E as vector space over $\mathbb{R}$.

Consider

$$Y_3 := [L_0, Y_2] = 2\sum_{i=1}^{n}\sum_{j=1}^{\ell}\omega_{ji}D_jD_i - \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial\alpha_j}{\partial x_i}D_iD_j \mod U_1$$
$$= -\sum_{1\le i,j\le\ell}\left(\frac{\partial\alpha_j}{\partial x_i} + \frac{\partial\alpha_i}{\partial x_j}\right)D_iD_j - \sum_{\ell+1\le i,j\le n}\left(\frac{\partial\alpha_j}{\partial x_i} + \frac{\partial\alpha_i}{\partial x_j}\right)D_iD_j + \sum_{i=\ell+1}^{n}\sum_{j=1}^{\ell}\left[2\omega_{ji} - \left(\frac{\partial\alpha_j}{\partial x_i} + \frac{\partial\alpha_i}{\partial x_j}\right)\right]D_iD_j \mod U_1.$$

The last equality came from the fact that $D_iD_j = D_jD_i + \omega_{ji}$. Suppose there exist $\lambda_1, \dots, \lambda_5$ in $\mathbb{R}$ such that

$$Y_3 = \lambda_1 L_0 + \lambda_2 h_1 + \lambda_3 Y_1 + \lambda_4 Y_2 + \lambda_5\Delta.$$

Then we have

$$-\sum_{1\le i\le j\le\ell}\left(\frac{\partial\alpha_j}{\partial x_i} + \frac{\partial\alpha_i}{\partial x_j}\right)D_iD_j - \sum_{\ell+1\le i,j\le n}\left(\frac{\partial\alpha_j}{\partial x_i} + \frac{\partial\alpha_i}{\partial x_j}\right)D_iD_j + \sum_{i=\ell+1}^{n}\sum_{j=1}^{\ell}\left[2\omega_{ji} - \left(\frac{\partial\alpha_j}{\partial x_i} + \frac{\partial\alpha_i}{\partial x_j}\right)\right]D_iD_j$$
$$= \lambda_1\sum_{i=1}^{n} D_i^2 + \lambda_4\sum_{i=1}^{\ell} D_i^2. \tag{16}$$

From (16), we get

$$\frac{\partial\alpha_j}{\partial x_i} + \frac{\partial\alpha_i}{\partial x_j} = 0, \quad \text{for } i \ne j \quad \text{and} \quad 1 \le i,j \le \ell \tag{17}$$

$$\frac{\partial\alpha_i}{\partial x_i} = \frac{\lambda}{2} = \begin{cases} \dfrac{\lambda_1 + \lambda_4}{2} & \text{if } 1 \le i \le \ell \\ \dfrac{\lambda_1}{2} & \text{if } \ell+1 \le i \le n. \end{cases} \tag{18}$$

Recall that $\alpha_i = \sum_{j=1}^{\ell} x_j\omega_{ij}$ and $\omega_{ii} = 0$ for all i. It is easy to see that (18) implies

$$\frac{\lambda}{2} = \frac{\partial}{\partial x_i}\left(\sum_{k=1}^{\ell} x_k\omega_{ik}\right) = \sum_{k=1}^{\ell} x_k\frac{\partial\omega_{ik}}{\partial x_i}. \tag{19}$$

Putting $x_1 = x_2 = \cdots = x_\ell = 0$ in (19), we get $\lambda = 0$. Recall that $\beta_j = \sum_{i=1}^{j}(\partial\alpha_i/\partial x_i)$. Since $(\partial\alpha_i/\partial x_i) = 0$ for $1 \le i \le n$, we get $\beta_n = 0 = \beta_\ell$. In view of (17), by Lemma 1, we have

$$\alpha_i = \sum_{\substack{1\le j\le\ell \\ j\ne i}} c_i^j(x_{\ell+1}, \dots, x_n)x_j + d_i(x_{\ell+1}, \dots, x_n), \quad 1 \le i \le \ell. \tag{20}$$

Now we are ready to simplify Δ. Putting (20) in the expression of Δ and using the fact that

$\beta_n = 0 = \beta_\ell$, and $E_\ell(\alpha_i) - \alpha_i = -d_i(x_{\ell+1}, \ldots, x_n)$ for $1 \le i \le \ell$, we get

$$\Delta = -\sum_{i=1}^{\ell} d_i(x_{\ell+1}, \ldots, x_n) D_i + \sum_{i=\ell+1}^{n} (E_\ell(\alpha_i) + 2\alpha_i) D_i + \sum_{i=1}^{n} \alpha_i^2 - \frac{1}{2} E_\ell(E_\ell(\eta) + 2\eta)$$

$$[\Delta, h_1] = -\sum_{i=1}^{\ell} d_i(x_{\ell+1}, \ldots, x_n) x_i.$$

Since dim $E=5$, $\sum_{i=1}^{\ell} d_i(x_{\ell+1}, \ldots, x_n) x_i$ must be a constant multiple of $\sum_{i=1}^{\ell} x_i^2$. This implies $d_i(x_{\ell+1}, \ldots, x_n) = 0$ for $i = 1, \ldots, \ell$. So for any fixed $\ell \ge 1$,

$$\Delta = \sum_{i=\ell+1}^{n} (E_\ell(\alpha_i) + 2\alpha_i) D_i + \sum_{i=1}^{n} \alpha_i^2 - \frac{1}{2} E_\ell[E_\ell(\eta) + 2\eta]$$

$$\begin{aligned}[Y_2, \Delta] &= 2\sum_{j=1}^{\ell} \sum_{i=\ell+1}^{n} \frac{\partial}{\partial x_j} (E_\ell(\alpha_i) + 2\alpha_i) D_j D_i \mod U_1 \\ &= \widetilde{\lambda}_1 L_0 + \widetilde{\lambda}_2 h_1 + \widetilde{\lambda}_3 Y_1 + \widetilde{\lambda}_4 Y_2 + \widetilde{\lambda}_5 \Delta \\ &\quad \text{for some } \widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_5 \in \mathbb{R},\end{aligned}$$

because $\{L_0, h_1, Y_1, Y_2, \Delta\}$ forms a basis of E and $[Y_2, \Delta] \in E$. It follows that

$$2\sum_{j=1}^{\ell} \sum_{i=\ell+1}^{n} \frac{\partial}{\partial x_j} (E_\ell(\alpha_i) + 2\alpha_i) D_j D_i = \widetilde{\lambda}_1 \left(\sum_{i=1}^{n} D_i^2 \right) + \widetilde{\lambda}_4 \left(\sum_{i=1}^{\ell} D_i^2 \right).$$

Therefore

$$\frac{\partial}{\partial x_j} [E_\ell(\alpha_i) + 2\alpha_i] = 0 \quad \text{for } 1 \le j \le \ell, \ \ell + 1 \le i \le n. \tag{21}$$

Observe that $E_\ell(\alpha_i) + 2\alpha_i$, $\ell + 1 \le i \le n$, are polynomials of $x_1, \ldots, x_n$ by Theorem 4. (4) implies that $E_\ell(\alpha_i) + 2\alpha_i$, $\ell + 1 \le i \le n$, are polynomials of $x_{\ell+1}, \ldots, x_n$. In view of Theorem 6, α_i, $\ell + 1 \le i \le n$, are polynomials of $x_{\ell+1}, \ldots, x_n$.

It follows that $\alpha_i, \ell + 1 \le i \le n$, must be zero because $\alpha_i = \sum_{j=1}^{\ell} x_j \omega_{ji}$ where ω_{ji} are C^∞ smooth. Now Δ is of the form

$$\begin{aligned}\Delta &= \sum_{i=1}^{\ell} \alpha_i^2 - \frac{1}{2} E_\ell[E_\ell(\eta) + 2\eta] \\ &= \sum_{i=1}^{\ell} \left(\sum_{j=1, j\ne i}^{\ell} c_i^j(x_{\ell+1}, \ldots, x_n) x_j \right)^2 - \frac{1}{2} E_\ell[E_\ell(\eta) + 2\eta] \\ &= e \sum_{i=1}^{\ell} x_i^2 \text{ for some constant } e.\end{aligned}$$

The last equality comes from the fact that dim $E = 5$. By Theorem 6, η is a polynomial of degree at most two in $x_1, \ldots, x_\ell$. We get a contradiction again since there is no C^∞ smooth solution for the equation (15) in view of Theorem 3. We have shown that five dimensional estimation algebra cannot contain an element of the form $\frac{1}{2}\sum_{i=1}^{\ell} x_i^2$ with $1 \le \ell < n$.

Case 2: $h_1(x) = \frac{1}{2}\sum_{i=1}^{n} x_i^2$. In this case, ω_{ij}, $1 \le i, j \le n$, are polynomials in $x_1, \ldots, x_n$ by Theorem 7. Observe that

$$Y_1 := [L_0, h_1] = \sum_{i=1}^{n} x_i D_i + \frac{n}{2}, \ [Y_1, h_1] = \sum_{i=1}^{n} x_i^2 = 2h_1,$$

$$Y_2 := 2L_0 - [L_0, Y_1] = \sum_{i=1}^{n} \alpha_i D_i + g_1,$$

where $\alpha_i = \sum_{j=1}^{n} x_j \omega_{ij}$, $\beta_n = \sum_{i=1}^{n} (\partial\alpha_i / \partial x_i)$ and $g_1 = \frac{1}{2}\beta_n - \frac{1}{2}[E_n(\eta) + 2\eta]$.

Before we proceed, we study the structure of α_i and Y_2. By Theorem 4, $\alpha_i, 1 \le i \le n$, are polynomials. From the definition of α_i, we know that α_i has no constant term. Next if all α_i are zero which implies $\beta_n = 0$, then there exists a constant e such that $Y_2 = -\frac{1}{2}[E_n(\eta) + 2\eta] = eh_1$ in view of Proposition 1. By Theorem 6, η is a polynomial of degree two. However, we obtain a contradiction since there is no C^∞ solution for equation (15) by Theorem 3. Therefore, at least one of α_i is non-zero polynomial of degree ≥ 1. Furthermore, if there exist constant C_1 and C_2 such that $Y_2 = C_1 h_1 + C_2 Y_1$, then we have

$$C_2 x_i = \alpha_i = \sum_{j=1}^{n} x_j \omega_{ij}, \quad 1 \le i \le n.$$

Since $\omega_{ii} = 0$, the above equation implies $\alpha_i = 0$ for $1 \le i \le n$. This leads to a contradiction as before. We have shown that the four elements L_0, h_1, Y_1 and Y_2 are linearly independent and $(\alpha_1, \ldots, \alpha_n) \ne C(x_1, \ldots, x_n)$ for any constant C.

Observe that

$$[Y_2, h_1] = \sum_{i=1}^{n} \alpha_i x_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} x_j \omega_{ij} \right) x_i = 0$$

$$\Delta := [Y_1, Y_2] = \sum_{i=1}^{n} [E_n(\alpha_i) - \alpha_i] D_i + \sum_{i=1}^{n} \alpha_i^2 + E_n\left[\frac{1}{2}\beta_n - \frac{1}{2} E_n(\eta) - \eta \right].$$

We claim that Δ is a linear combination of L_0, h_1, Y_1 and Y_2. Suppose on the contrary that L_0, h_1, Y_1, Y_2

and Δ are linearly independent. Then the operator

$$\begin{aligned} Y_3 &:= [L_0, Y_2] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial\alpha_j}{\partial x_i}D_iD_j - \alpha_j\omega_{ij}D_i + \frac{1}{2}\frac{\partial^2\alpha_j}{\partial x_i^2}D_j - \frac{1}{2}\alpha_j\frac{\partial\omega_{ij}}{\partial x_i}\right) \\ &\quad + \sum_{i=1}^{n}\left[\frac{\partial g_1}{\partial x_i}D_i + \frac{1}{2}\frac{\partial^2 g_1}{\partial x_i^2}\right] + \frac{1}{2}\sum_{i=1}^{n}\alpha_i\frac{\partial\eta}{\partial x_i} \\ &= \sum_{1\le i\le j\le n}\left(\frac{\partial\alpha_j}{\partial x_i} + \frac{\partial\alpha_i}{\partial x_j}\right)D_iD_j + \cdots \end{aligned}$$

must be a linear combination of L_0, h_1, Y_1, Y_2 and Δ. We write $Y_3 = \lambda_1 L_0 + \lambda_2 h_1 + \lambda_3 Y_1 + \lambda_4 Y_2 + \lambda_5\Delta$. From this we get

$$\frac{\partial\alpha_j}{\partial x_i} + \frac{\partial\alpha_i}{\partial x_j} = 0, \quad 1 \le i,j \le n,\ i \ne j \tag{22}$$

$$\frac{\partial\alpha_i}{\partial x_i} = \frac{\lambda_1}{2}, \quad 1 \le i \le n. \tag{23}$$

By the similar argument as in case 1, we get $\Delta = \sum_{i=1}^{n}(\sum_{j=1, j\ne i}^{n} c_i^j x_j)^2 - \frac{1}{2}E_n[E_n(\eta) + 2\eta] = eh_1$ for some constant e. The last equality comes from the fact that dim $E = 5$. Same argument as case 1 using Theorem 3 will provide a contradiction. Therefore we can find constants λ_1,λ_2,λ_3 and λ_4 such that

$$\Delta = \lambda_1 L_0 + \lambda_2 h_1 + \lambda_3 Y_1 + \lambda_4 Y_2.$$

Then

$$\begin{aligned} &\sum_{i=1}^{n}[E_n(\alpha_i) - \alpha_i]D_i + \sum_{i=1}^{n}\alpha_i^2 + E_n(g_1) \\ &\quad = \frac{\lambda_1}{2}\left(\sum_{i=1}^{n}D_i^2 - \eta\right) + \frac{\lambda_2}{2}\sum_{i=1}^{n}x_i^2 \\ &\qquad + \lambda_3\left(\sum_{i=1}^{n}x_iD_i + \frac{n}{2}\right) + \lambda_4\left(\sum_{i=1}^{n}\alpha_iD_i + g_1\right) \end{aligned}$$

$$\Rightarrow \lambda_1 = 0 \tag{24}$$

$$E_n(\alpha_i) = \lambda_3 x_i + (\lambda_4 + 1)\alpha_i, \quad 1 \le i \le n, \tag{25}$$

$$E_n(g_1) - \lambda_4 g_1 = -\sum_{i=1}^{n}\alpha_i^2 + \frac{\lambda_2}{2}\sum_{i=1}^{n}x_i^2 + \frac{n\lambda_3}{2}. \tag{26}$$

Let $p_i = \deg(\alpha_i)$. Recall that $(\alpha_1, \ldots, \alpha_n) \ne C(x_1, \ldots, x_n)$ for any constant C and $p_j \ne 0$ for some j. For any polynomial ψ, we shall denote the homogeneous part of degree s of ψ by $\psi^{(s)}$. Equation (25) implies

$$\sum_{s=1}^{p_i} s\alpha_i^{(s)} = E_n(\alpha_i) = \lambda_3 x_i + (\lambda_4 + 1)\sum_{s=1}^{n}\alpha_i^{(s)}.$$

Therefore $\alpha_i^{(1)} = \lambda_3 x_i + (\lambda_4 + 1)\alpha_i^{(1)}$, and $p_i = \lambda_4 + 1$ if $p_i > 1$. Without loss of generality, we shall assume $\deg(\alpha_1) = \deg(\alpha_2) = \cdots = \deg(\alpha_\nu) = \lambda_4 + 1 := p$, $\deg(\alpha_{\nu+1}) = \cdots = \deg(\alpha_\mu) = 1$, $\alpha_{\mu+1} = \cdots = \alpha_n = 0$ for some $\mu \ge \nu$ (if $\nu = 0$, then $\mu > 0$) and

$$\begin{cases} \alpha_i = \alpha_i^{(p)} + \dfrac{\lambda_3}{1-p}x_i & \text{for } 1 \le i \le \nu \quad (27) \\ \alpha_j = \alpha_j^{(1)} & \text{for } \nu + 1 \le j \le \mu. \quad (28) \end{cases}$$

We now claim that α_i, $1 \le i \le \nu$, are of the form

$$\alpha_i = \sum a^i_{j_1\ldots j_p} x_{j_1}\ldots x_{j_p}, \quad 1 \le i \le \nu, \quad \nu > 0 \tag{29}$$

where the summation is over distinct indices $j_1 \ne i$, $j_2 \ne i, \ldots, j_p \ne i$ and $a^i_{j_1\ldots j_p}$ are real constants. To see this, we first observe that L_0, h_1, Y_1, Y_2 and Y_3 must be linearly independent. This is because we have previously proved that Y_3 is not a linear combination of L_0, h_1, Y_1, Y_2 and Δ. We next consider the operator

$$\begin{aligned} Y_4 &:= [L_0, Y_3] \\ &= \sum_{i,j,k=1}^{n}\frac{\partial^2\alpha_j}{\partial x_k\partial x_i}D_kD_iD_j \\ &\quad + \sum_{i,j,k=1}^{n}\frac{\partial\alpha_j}{\partial x_i}(\omega_{jk}D_iD_k + \omega_{ik}D_kD_j) \mod U_1. \end{aligned}$$

Since Y_4 has to be a linear combination of the five elements L_0, h_1, Y_1, Y_2 and Y_3, and notice that $D_iD_j = D_jD_i + \omega_{ji}$, we have

$$Y_4 = \sum_{i,j=1}^{n}\left[\sum_{k=1}^{n}\left(\frac{\partial\alpha_k}{\partial x_i}\omega_{kj} + \frac{\partial\alpha_j}{\partial x_k}\omega_{ki}\right)\right]D_iD_j \mod U_1, \tag{30}$$

$$\frac{\partial^2\alpha_i}{\partial x_i^2} = 0, \quad 1 \le i \le n, \quad (D_i^3 \text{ terms}) \tag{31}$$

$$\frac{\partial^2\alpha_j}{\partial x_i^2} + 2\frac{\partial^2\alpha_i}{\partial x_i\partial x_j} = 0, \quad 1 \le i,j \le n, \quad (D_i^2D_j \text{ terms}) \tag{32}$$

$$\frac{\partial^2\alpha_k}{\partial x_i\partial x_j} + \frac{\partial^2\alpha_j}{\partial x_i\partial x_k} + \frac{\partial^2\alpha_i}{\partial x_j\partial x_k} = 0, \quad 1 \le i,j,k \le n. \quad (D_iD_jD_h \text{ terms}) \tag{33}$$

Differentiating (33) with respect to x_k, we get

$$\frac{\partial^3\alpha_k}{\partial x_i\partial x_j\partial x_k} + \frac{\partial^3\alpha_j}{\partial x_i\partial x_k^2} + \frac{\partial^3\alpha_i}{\partial x_j\partial x_k^2} = 0.$$

By (32), we have

$$0=\frac{\partial^3\alpha_k}{\partial x_i\partial x_j\partial x_k}+\frac{\partial}{\partial x_i}\left(-2\frac{\partial^2\alpha_k}{\partial x_k\partial x_j}\right)+\frac{\partial}{\partial x_j}\left(-2\frac{\partial^2\alpha_k}{\partial x_k\partial x_i}\right)=\frac{3\partial^3\alpha_k}{\partial x_i\partial x_j\partial x_k},$$

which implies

$$\frac{\partial\alpha_k}{\partial x_k}=\sum_{s=1}^{n}d_s^k x_s+c_k \tag{34}$$

where d_s^k, c_k are constants and $d_k^k=0$ by (31). It follows that

$$\alpha_k=\sum_{s=1}^{n}d_s^k x_s x_k+c_k x_k+e_k,\quad 1\le k\le n, \tag{35}$$

where e_k is independent of x_k variable. (32) and (35) imply

$$0=\frac{\partial^2\alpha_j}{\partial x_i^2}+2\frac{\partial}{\partial x_j}\left(\frac{\partial\alpha_i}{\partial x_i}\right)=\frac{\partial^2 e_j}{\partial x_i^2}+2d_j^i$$

$$\Rightarrow\frac{\partial^2 e_j}{\partial x_i^2}=-2d_j^i. \tag{36}$$

(a) For the case $p>2$, in view of (27), (28), (35), (36) and $\alpha_i^{(0)}=0$, we have

$$\begin{aligned}
c_i&=\frac{\lambda_3}{1-p}:=c, && \text{for } 1\le i\le\nu,\\
d_j^i&=0, && \text{for } 1\le i\le n,\ 1\le j\le n,\\
\frac{\partial^2 e_j}{\partial x_i^2}&=0, && \text{for } 1\le i\le n,\ 1\le j\le n.
\end{aligned}$$

Hence (35) is of the form

$$\alpha_i=\sum a^i_{j_1\ldots j_p}x_{j_1}\ldots x_{j_p}+cx_i\quad\text{for some constants } a^i_{j_1\ldots j_p},\ 1\le i\le\nu,$$

where summation is over distinct indices $j_1\ne i,\ldots,j_p\ne i$. Recall that $\sum_{i=1}^n x_i\alpha_i=0$. It follows that $c\sum_{i=1}^{\nu}x_i^2+\sum_{i=\nu+1}^n x_i\alpha_i^{(1)}=0$ and hence $c=0$. We have established (29) for $p>2$.

(b) For the case $p=2$, from (27), (35), (36) and the fact that e_j is independent of x_j, we have

$$\alpha_j=\sum_{s=1}^{n}d_s^j x_s x_j-\sum_{s=1}^{n}d_j^s x_s^2+\sum_{1\le s,r\le n}c_{sr}^j x_s x_r+\alpha_j^{(1)},\quad 1\le j\le n, \tag{37}$$

for some constants c and c_{sr}^j with $c_{jr}^j=c_{rj}^j=c_{rr}^j=0$, $\forall 1\le r\le n$, $1\le j\le\nu$; $d_s^j=d_j^s=0=c_{sr}^j$, $\forall$ $j\ge\nu+1$, $1\le s,r\le n$; and $\alpha_j^{(1)}=cx_j$ for $1\le j\le\nu$. In view of $\alpha_j=\sum_{s=1}^n x_s(d_s^j x_j-d_j^s x_s+\sum_{r=1}^n c_{sr}^j x_r)+cx_j=\sum_{s=1}^n x_s\omega_{js}$, we have

$$\begin{aligned}
&x_1(d_1^j x_j-d_j^1 x_1)+\cdots+x_n(d_n^j x_j-d_j^n x_n)\\
&\quad+c_{12}^j x_1x_2+c_{13}^j x_1x_3+\cdots+c_{1(j-1)}^j x_1x_{j-1}+0\\
&\quad+c_{1(j+1)}^j x_1x_{j+1}+\cdots+c_{1n}^j x_1x_n+c_{21}^j x_2x_1\\
&\quad+c_{23}^j x_2x_3+\cdots+c_{2(j-1)}^j x_2x_{j-1}+0+c_{2(j+1)}^j x_2x_{j+1}\\
&\quad+\cdots+c_{2n}^j x_2x_n+\cdots+c_{n1}^j x_nx_1+c_{n2}^j x_nx_2+\cdots\\
&\quad+c_{n(j-1)}^j x_nx_{j-1}+0+c_{n(j+1)}^j x_jx_{j+1}+\cdots\\
&\quad+c_{n(n-1)}^j x_nx_{n-1}+cx_j=x_1\omega_{j1}+x_2\omega_{j2}+\cdots+x_n\omega_{jn}.
\end{aligned}$$

Recall that ω_{ij}, $1\le i,j\le n$, are polynomials. $\omega_{js}^{(1)}$, homogeneous polynomial of degree 1 part of ω_{js}, must be of the form

$$\omega_{js}^{(1)}=d_s^j x_j-d_j^s x_s+\sum_{r=1}^{n}d_r^{js}x_r,\quad 1\le j\le n,\ 1\le s\le n, \tag{38}$$

for some constants d_r^{js} with $d_j^{js}=d_s^{js}=0$.

Next we shall prove that all d_s^j are zeros for $1\le j\le\nu$, $1\le s\le n$. Compute the coefficient $\sum_{k=1}^n((\partial\alpha_k/\partial x_i)+(\partial\alpha_i/\partial x_k))\omega_{ki}$ of the term D_i^2, $(1\le i\le n)$ in equation (30), which is a polynomial by Theorem 5. Denote this polynomial by q_i. We have

$$\begin{aligned}
q_i&=\sum_{\substack{k=1\\k\ne i}}\left[-d_i^k x_k-d_k^i x_i+\sum_{r=1}^{n}(c_{ir}^k+c_{ri}^k+c_{kr}^i+c_{rk}^i)x_k\right]\\
&\quad\cdot\left[d_i^k x_k-d_k^i x_i+\sum_{r=1}^{n}d_r^{ki}x_r+(\omega_{ki}-\omega_{ki}^{(1)})\right]\\
&=\sum_{\substack{k=1\\k\ne i}}\Big[-(d_i^k)^2x_k^2-d_i^k d_k^i x_k x_i\\
&\quad+\sum_{r=1}^{n}(c_{ir}^k+c_{ri}^k+c_{kr}^i+c_{rk}^i)d_i^k x_k x_r+d_i^k d_k^i x_k x_i+(d_k^i)^2x_i^2\\
&\quad-\sum_{r=1}^{n}(c_{ir}^k+c_{ri}^k+c_{kr}^i+c_{rk}^i)d_k^i x_i x_r-\sum_{r=1}^{n}d_r^{ki}d_i^k x_r x_k\\
&\quad-\sum_{r=1}^{n}d_r^{ki}d_k^i x_r x_i-d_i^k x_k(\omega_{ki}-\omega_{ki}^{(1)})\\
&\quad+\sum_{r=1}^{n}\sum_{m=1}^{n}(c_{ir}^k+c_{ri}^k+c_{kr}^i+c_{rk}^i)d_m^{ki}x_r x_m-d_k^i x_i(\omega_{ki}-\omega_{ki}^{(1)})\\
&\quad+\sum_{r=1}^{n}(c_{ir}^k+c_{ri}^k+c_{kr}^i+c_{rk}^i)x_r(\omega_{ki}-\omega_{ki}^{(1)})\Big].
\end{aligned}$$

Observe that the coefficients $c_{ii}^k, c_{ki}^i, c_{ik}^i$ and d_i^{ki} are all zeros. Therefore there is no x_i^2 terms in the four expressions: $\sum_{r=1}^n(c_{ir}^k+c_{ri}^k+c_{kr}^i+c_{rk}^i)d_k^i x_i x_r$,

$\sum_{r=1}^{n} d_r^{ki} d_k^i x_r x_i$, $\sum_{r,m=1}^{n}(c_{ir}^k + c_{ri}^k + c_{kr}^i + c_{rk}^i) d_m^{ki} x_r x_m$ and $\sum_{r=1}^{n}(c_{ir}^k + c_{ri}^k + c_{kr}^i + c_{rk}^i) x_r(\omega_{ki} - \omega_{ki}^{(1)})$. It follows that

$$q_i = \sum_{\substack{k=1\\k\neq 1}}^{n} \Big[(d_k^i)^2 x_i^2 - d_k^i x_i(\omega_{ki} - \omega_{ki}^{(1)}) + \text{terms contain no } x_i^2\Big].$$

Since ω_{ki}, $1 \le k \le n$, are polynomials, $\sum_{\substack{k=1\\k\neq 1}}^{n} d_k^i x_i(\omega_{ki} - \omega_{ki}^{(1)})$ does not contain x_i^2 term. Next, since Y_4 is a linear combination of L_0, h_1, Y_1, Y_2 and Y_3, and notice that the coefficient $2(\partial\alpha_i/\partial x_i)$ of D_i^2 term in Y_3 is $2\sum_{s=1}^{n} d_s^i x_s +$ constant, we have that the coefficient $\sum_{k=1}^{n}(d_k^i)^2$ of x_i^2 in q_i is necessary zero. It follows that $d_k^i = 0$ for $i \neq k$. Therefore $\alpha_j = \sum_{r,s=1}^{n} c_{sr}^j x_s x_r + \alpha_j^{(1)}$, where $\alpha_j^{(1)} = c x_j$ for $1 \le j \le \nu$. Using the fact that $\sum_{i=1}^{n} x_i \alpha_i = 0$, we get $c = 0$ again. Therefore α_j, $1 \le j \le \nu$, are of the form (29) with $p = 2$.

We have shown equation (29) holds and $d_s^k = 0 \ \forall\, k, s$. Hence (34) implies

$$\frac{\partial\alpha_k}{\partial x_k} = c_k, \quad 1 \le k \le n. \tag{39}$$

(26) becomes

$$E_n(g_1) - (p-1)g_1 = -\sum_{i=1}^{\nu}\Bigg(\sum_{\substack{j_1\neq i,\ldots,j_p\neq i\\ j_1,\ldots,j_p \text{ distinct}}} a^i_{j_1\ldots j_p} x_{j_1}\ldots x_{j_p}\Bigg)^2 - \sum_{i=\nu+1}^{\mu}\Bigg(\sum_{j=1}^{n} a_{ij}x_j\Bigg)^2 + \lambda_2\Bigg(\frac{1}{2}\sum_{i=1}^{n} x_i^2\Bigg) + \frac{\lambda_3 n}{2} \quad \text{if } \nu > 0 \tag{40}$$

$$E_n(g_1) = -\sum_{i=1}^{\mu}\Bigg(\sum_{j=1}^{n} b_{ij}x_j\Bigg)^2 + \lambda_2\Bigg(\frac{1}{2}\sum_{i=1}^{n} x_i^2\Bigg) + \frac{\lambda_3 n}{2} \quad \text{if } \nu = 0 \tag{41}$$

where a_{ij} and b_{ij} are constants. By Theorem 6, g_1 is a polynomial of degree $2p$ (if $\nu > 0$) or 2 (if $\nu = 0$). By expressing g_1 as a polynomial in general form and by comparing the coefficients in both sides in equation (39) (noticing that $j_1 \neq i, \ldots, j_p \neq i$ are distinct indices), we get that g_1 is a polynomial of degree 2 in x_r for some x_r variable. Therefore g_1 can be written in the form $u_2(x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n)x_r^2 + u_1(x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n)x_r + u_0(x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n)$, where u_i are polynomials of $x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n$. Recall $g_1 = \beta_n - \frac{1}{2}(E_n(\eta) + 2\eta)$, where $\beta_n = \sum_{i=1}^{n}(\partial\alpha_i/\partial x_i) =$ constant by (39). By Theorem 6, η is a polynomial of degree $2p$ in $x_1, \ldots, x_n$. Therefore we can write η in the form $\theta_2(x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n)x_r^2 + \theta_1(x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n)x_r + \theta_0(x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n)$, where θ_i are polynomials of $x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n$. It follows that

$$\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = \eta - \sum_{j=1}^{m} h_j^2 = \eta - c^2\Bigg(\sum_{i=1}^{n} x_i^2\Bigg)^2 = -c^2 x_r^4 + \text{ lower degree in } x_r.$$

By Theorem 3, we get a contradiction by taking a polynomial path $x_r = t$, $x_i = 0$ for $i \neq r$. We have shown that a five dimensional estimation algebra cannot contain an element of the form $\frac{1}{2}\sum_{i=1}^{n} x_i^2$. This finishes the proof of Theorem 1. □

5. New class of finite dimensional filters

It is well known that if the estimation algebra is of finite dimensional, then the finite dimensional filter exists (cf. Yau (1994)). One of the consequences of the classification of finite dimensional estimation algebras with maximal rank is the following. In order for an estimation algebra with maximal rank to be finite dimensional, the dynamical system has to be quite special, i.e. the drift term f must be of the form

$$f(x) = (\ell_1, \ldots, \ell_n) + \nabla\phi$$

where $\ell_1, \ldots, \ell_n$ are degree one polynomials in $x_1, \ldots, x_n$ and ϕ is a C^∞ function. The significance of proving the Mitter Conjecture and hence the classification of 5-dimensional estimation algebras (cf. Chiou et al. 2006) is that it provides a new class of finite dimensional estimation algebras and hence finite dimensional filters which are totally unknown.

Example 1: Consider the filtering model

$$\begin{cases} dx(t) = f(x(t))dt + g(x(t))dv(t) \\ dy(t) = h(x(t))dt + dw(t), \end{cases}$$

where $f_1 = ax_1$, $f_2 = bx_1x_3$, $f_3 = -bx_1x_2$, $f_i = g_i(x_4, \ldots, x_n)$, $4 \le i \le n$, $h(x) = x_1$; a, b are non-zero constants and g_i are C^∞ functions in $x_4, \ldots, x_n$ variables. Then

$$\omega_{12} = \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} = bx_3$$
$$\omega_{13} = \frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3} = -bx_2$$
$$\omega_{23} = \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} = -2bx_1$$
$$\omega_{1j} = \frac{\partial f_j}{\partial x_1} - \frac{\partial f_1}{\partial x_j} = 0, \quad 4 \le i \le n,$$

$$\sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} = a^2x_1^2 + b^2x_1^2x_3^2 + b^2x_1^2x_2^2$$

Table 1. Multiplication Table.

	1	x_1	D_1	Y_1	L_0
1	0	0	0	0	0
x_1	0	0	-1	0	$-D_1$
D_1	0	1	0	a^2+1	$-Y_1$
Y_1	0	0	$-(a^2+1)$	0	$-(a^2+1)D_1$
L_0	0	D_1	Y_1	$(a^2+1)D_1$	0

$$
\begin{aligned}
&+\sum_{i=4}^{n} g_i^2(x_4,\ldots,x_n)\\
&+a+\sum_{i=4}^{n}\frac{\partial g_i}{\partial x_i}(x_4,\ldots,x_n).
\end{aligned}
$$

It is easy to check that the estimation algebra E is 5-dimensional with basis $\{1, x_1, D_1, Y_1 = bx_3D_2 - bx_2D_3 + (a^2+1+b^2x_2^2+b^2x_3^2)x_1, L_0\}$. In fact the Lie algebra E has the Lie bracket multiplication show in Table 1.

The following theorem gives an explicit construction of new finite dimensional filters.

Theorem 9: *Consider the filtering model as in the above example. Then the robust Duncan–Mortensen–Zakai equation* (14) *has a solution for all* $t \geq 0$ *of the form* $u(t,x) = e^{T(t)}e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0$ *where* $r(t)$, $s_1(t)$, $s_2(t)$ *and* $T(t)$ *satisfy the following differential equations*

$$\frac{\mathrm{d}r}{\mathrm{d}t}(t) = 0 \tag{42}$$

$$\frac{\mathrm{d}s_1}{\mathrm{d}t}(t) = s_2(t) \tag{43}$$

$$\frac{\mathrm{d}s_2}{\mathrm{d}t}(t) = r(t) + (a^2+1)s_1(t) + y(t) \tag{44}$$

$$
\begin{aligned}
\frac{\mathrm{d}T}{\mathrm{d}t} = &-\frac{r^2(t)}{2} + \frac{a^2+1}{2}s_2^2(t) - (a^2+1)r(t)s_1(t)\\
&-\frac{(a^2+1)^2}{2}s_1^2(t) - (a^2+1)s_2(t)\frac{\mathrm{d}s_1}{\mathrm{d}t}(t)\\
&+r(t)\frac{\mathrm{d}s_2}{\mathrm{d}t}(t) + \frac{1}{2}y^2(t).
\end{aligned}
\tag{45}
$$

Proof: Since L_0 is uniformly elliptic, for any $t>0$, $e^{tL_0}\sigma_0$ is C^∞. By differentiating $u(t,x)$, we have

$$
\begin{aligned}
\frac{\partial u}{\partial t}(t,x) = {}& e^{T(t)}e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}L_0e^{tL_0}\sigma_0\\
&+\frac{\mathrm{d}s_1}{\mathrm{d}t}(t)e^{T(t)}e^{r(t)x_1}e^{s_2(t)D_1}Y_1e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&+\frac{\mathrm{d}s_2}{\mathrm{d}t}(t)e^{T(t)}e^{r(t)x_1}D_1e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&+\frac{\mathrm{d}r}{\mathrm{d}t}(t)e^{T(t)}x_1e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&+\frac{\mathrm{d}T}{\mathrm{d}t}(t)e^{T(t)}e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0
\end{aligned}
$$

$$
\begin{aligned}
&= e^{T(t)}e^{r(t)x_1}e^{s_2(t)D_1}\left\{L_0 + s_1(t)[Y_1,L_0] + \frac{s_1^2(t)}{2}[Y_1,[Y_1,L_0]]\right.\\
&\quad\left.+\frac{s_1^3(t)}{3!}[Y_1,[Y_1,[Y_1,L_0]]]+\cdots\right\}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&\quad+\frac{\mathrm{d}s_1}{\mathrm{d}t}(t)e^{T(t)}e^{r(t)x_1}\left\{Y_1 + s_2(t)[D_1,Y_1]\right.\\
&\quad\left.+\frac{s_2^2(t)}{2}[D_1,[D_1,Y_1]]+\cdots\right\}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&\quad+\frac{\mathrm{d}s_2}{\mathrm{d}t}(t)e^{T(t)}\{D_1 + [r(t)x_1, D_1]+\cdots\}\\
&\quad\times e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_1}\sigma_0 + \frac{\mathrm{d}r}{\mathrm{d}t}(t)x_1u(t,x)\\
&\quad+\frac{dT}{\mathrm{d}t}(t)u(t,x)\\
&= e^{T(t)}e^{r(t)x_1}e^{s_2(t)D_1}\left\{L_0 - (a^2+1)s_1(t)D_1\right.\\
&\quad\left.+\frac{(a^2+1)^2}{2}s_1^2(t)\right\}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&\quad+\frac{\mathrm{d}s_1}{\mathrm{d}t}(t)e^{T(t)}e^{r(t)x_1}\{Y_1 + (a^2+1)s_2(t)\}\\
&\quad\times e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0 + \frac{\mathrm{d}s_2}{\mathrm{d}t}(t)e^{T(t)}\\
&\quad\times\{D_1 - r(t)\}e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&\quad+\left(\frac{\mathrm{d}r}{\mathrm{d}t}(t)x_1 + \frac{dT}{\mathrm{d}t}(t)\right)u(t,x)\\
&= e^{T(t)}e^{r(t)x_1}\left\{L_0 + s_2(t)[D_1,L_0]\right.\\
&\quad\left.+\frac{s_1^2(t)}{2}[D_1,[D_1,L_0]]+\cdots\right\}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&\quad-(a^2+1)s_1(t)e^{T(t)}e^{r(t)x_1}D_1e^{s_2(t)D_1}e^{s_2(t)Y_1}e^{tL_0}\sigma_0\\
&\quad+\frac{(a^2+1)^2}{2}s_1^2(t)u(t,x) + \frac{\mathrm{d}s_1}{\mathrm{d}t}(t)e^{T(t)}\\
&\quad\times\left\{Y_1 + r(t)[x_1,Y_1] + \frac{r^2(t)}{2}[x_1,[x_1,Y_1]]+\cdots\right\}\\
&\quad\times e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0 + (a^2+1)s_2(t)\frac{\mathrm{d}s_1}{\mathrm{d}t}(t)u(t,x)\\
&\quad+\frac{\mathrm{d}s_2}{\mathrm{d}t}(t)D_1u(t,x) - r(t)\frac{\mathrm{d}s_2}{\mathrm{d}t}(t)u(t,x)\\
&\quad+\left(\frac{\mathrm{d}r}{\mathrm{d}t}(t)x_1 + \frac{dT}{\mathrm{d}t}(t)\right)u(t,x)\\
&= e^{T(t)}e^{r(t)x_1}\left\{L_0 + s_2(t)(-Y_1) - \frac{a^2+1}{2}s_2^2(t)\right\}\\
&\quad\times e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&\quad-(a^2+1)s_1(t)e^{T(t)}e^{r(t)x_1}D_1e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(a^2+1)^2}{2}s_1^2(t)u(t,x)+\frac{\mathrm{d}s_1}{\mathrm{d}t}(t)\\
&\times e^{T(t)}Y_1e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&+(a^2+1)s_2(t)\frac{\mathrm{d}s_1}{\mathrm{d}t}(t)u(t,x)+\frac{\mathrm{d}s_2}{\mathrm{d}t}(t)D_1u(t,x)\\
&+\frac{\mathrm{d}r}{\mathrm{d}t}(t)x_1u(t,x)+\left(-r(t)\frac{\mathrm{d}s_2}{\mathrm{d}t}(t)+\frac{dT}{\mathrm{d}t}(t)\right)u(t,x)
\end{aligned}
$$

$$
\begin{aligned}
&=e^{T(t)}\left\{L_0+r(t)[x_1,L_0]+\frac{r^2(t)}{2}[x_1,[x_1,L_0]]+\cdots\right\}\\
&\quad\times e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&\quad-s_2(t)e^{T(t)}\{Y_1+r(t)[x_1,Y_1]+\cdots\}\\
&\quad\times e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&\quad-\frac{(a^2+1)}{2}s_2^2(t)u(t,x)-(a^2+1)s_1(t)e^{T(t)}\\
&\quad\times\{D_1+r(t)[x_1,D_1]+\cdots\}e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&\quad+\frac{(a^2+1)^2}{2}s_1^2(t)u(t,x)+\frac{\mathrm{d}s_1}{\mathrm{d}t}(t)Y_1u(t,x)\\
&\quad+(a^2+1)s_2(t)\frac{\mathrm{d}s_1}{\mathrm{d}t}(t)u(t,x)\\
&\quad+\frac{\mathrm{d}s_2}{\mathrm{d}t}(t)D_1u(t,x)+\frac{\mathrm{d}r}{\mathrm{d}t}(t)x_1u(t,x)\\
&\quad+\left(-r(t)\frac{\mathrm{d}s_2}{\mathrm{d}t}(t)+\frac{dT}{\mathrm{d}t}(t)\right)u(t,x)\\
&=L_0u(t,x)-r(t)D_1u(t,x)+\frac{r^2(t)}{2}u(t,x)-s_2(t)Y_1u(t,x)\\
&\quad-\frac{a^2+1}{2}s_2^2(t)u(t,x)-(a^2+1)s_2(t)D_1u(t,x)\\
&\quad+(a^2+1)r(t)s_1(t)u(t,x)+\frac{(a^2+1)^2}{2}s_1^2(t)u(t,x)\\
&\quad+\frac{\mathrm{d}s_1}{\mathrm{d}t}(t)Y_1u(t,x)+(a^2+1)s_2(t)\frac{\mathrm{d}s_1}{\mathrm{d}t}(t)u(t,x)\\
&\quad+\frac{\mathrm{d}s_2}{\mathrm{d}t}D_1u(t,x)+\frac{\mathrm{d}r}{\mathrm{d}t}(t)x_1u(t,x)\\
&\quad+\left(-r(t)\frac{\mathrm{d}s_2}{\mathrm{d}t}(t)+\frac{dT}{\mathrm{d}t}(t)\right)u(t,x)
\end{aligned}
$$

$$
\begin{aligned}
&=L_0u(t,x)+\left(\frac{\mathrm{d}s_1}{\mathrm{d}t}(t)-s_2(t)\right)Y_1u(t,x)\\
&\quad+\left[\frac{\mathrm{d}s_2}{\mathrm{d}t}(t)-r(t)-(a^2+1)s_1(t)\right]D_1u(t,x)\\
&\quad+\frac{\mathrm{d}r}{\mathrm{d}t}(t)x_1u(t,x)+\left[\frac{r^2}{2}(t)-\frac{a^2+1}{2}s_2^2(t)\right.\\
&\quad+(a^2+1)r(t)s_1(t)+\frac{(a^2+1)^2}{2}s_1^2(t)+(a^2+1)s_2(t)\\
&\quad\times\frac{\mathrm{d}s_1}{\mathrm{d}t}(t)-\frac{\mathrm{d}s_2}{\mathrm{d}t}r(t)+\left.\frac{dT}{\mathrm{d}t}\right]u(t,x). \qquad (46)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\frac{\partial u}{\partial t}(t,x)&=L_0u(t,x)+y(t)[L_0,x_1]u(t,x)\\
&\quad+\tfrac{1}{2}y^2(t)[[L_0,x_1],x_1]u(t,x)\\
&=L_0u(t,x)+y(t)D_1u(t,x)+\tfrac{1}{2}y^2(t)u(t,x). \qquad (47)
\end{aligned}
$$

By comparing (46) and (47), it is clear that $u(t,x)$ is a solution to robust DMZ equation (4) if (42), (43), (44) and (45) are satisfied. It is also clear that (42), (43), (44) and (45) have solutions for all t. □

Remark 1: Our new filters constructed in Theorem 9 are of real applied significance for the following reasons:

(a) Observe that (42), (43), (44) and (45) are independent of the initial distribution of x_0. Therefore one can implement this filter in hardware for real application. These are the so called universal filters.
(b) It is interesting to observe that the state space dimension of our filters is arbitrarily large while the dimension of our filters is only four. The real time computation of (42), (43) and (44) are a trivial matter because these are linear equations. Once we find out what $r(t)$, $s_1(t)$ and $s_2(t)$, we simply put them in (45). We only need to do simple integration to find out T. Therefore our filters are of real applied significance.
(c) Our filters are defined for all time t as one can see directly from (42), (43), (44) and (45).

In the future, we plan to prove theorems on structure of estimation algebras of dimension 6 by using Lemma 1 developed in this paper. We would like to find out what kind of new six dimensional filters can be constructed explicitly. Hopefully with the new knowledge of estimation algebras of dimension 6, we shall be able to understand all finite dimensional estimation algebras.

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