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The problem of classification of finite-dimensional estimation algebras was formally proposed by Brockett in his lecture at International Congress of Mathematicians in 1983. Due to the difficulty of the problem, in the early 1990s Brockett suggested that one should understand the low-dimensional estimation algebras first. In this article, we extend Yau and his coauthors' work of the Mitter conjecture for low-dimensional estimation algebras in nonlinear filtering problem. And, we apply the results to give classification of estimation algebras of dimension six.

Keywords: estimation algebra; nonlinear filter; Wei–Norman approach; Yau filter

1. Introduction

Kalman and Kalman–Bucy published two historically important mathematical papers in ASME's *Journal of Basic Engineering* in 1960 and 1961, respectively. Kalman–Bucy filter has been used in many areas such as navigational and guidance systems, radar tracking, solar mapping and satellite orbit determination, see Cipra (1996). Despite the usefulness of Kalman–Bucy filter, it is not perfect. One of its weaknesses is that it is restricted to linear dynamical systems. Another weakness is the Gaussian assumption of initial value.

In the late 1970s, Brockett and Clark (1980), Brockett (1981) and Mitter (1979) proposed the idea of using estimation algebras to construct a finite-dimensional nonlinear filter. The advantage of the Brockett–Mitter approach of using estimation algebra method to solve the Duncan–Mortensen–Zakai equation is the following. As long as the estimation algebra is finite-dimensional, we will get a finite-dimensional recursive filter and there is no need to make any assumption on the initial distribution. Moreover, the approach applies well to nonlinear dynamical systems. Wong (1987a) introduced a fundamental notion of Wong matrix which plays an important role in the classification of finite-dimensional estimation algebras. Wong (1987b) gave a structure theorem of estimation algebra in case the drift term $f(x)$ is real analytic and its first, second and third derivatives of $f(x)$ are bounded functions. Under these assumptions, Wong also proved that all finite-dimensional estimation algebras are solvable and observation $h(x)$ is polynomial of degree one. During 1991–1997, Yau et al., in a series of papers (Chiou and Yau 1994; Yau 1994; Chen, Leung, and Yau 1996a, 1997a; Chen and Yau 1996b, 1997b; Rasoulian and Yau 1997; Yau and Yau 1997), introduced new concepts and classified new finite-dimensional estimation algebras. The self-contained proof of the classification of finite-dimensional estimation algebras with maximal rank can be found in Yau (2003).

Despite the success of the classification of finite-dimensional estimation algebras with maximal rank, the problem of classification of non-maximal rank finite-dimensional estimation algebras is still wide open except for the case of state-space dimension two which was finished by Wu and Yau (2006) and some construction of non-maximal rank finite-dimensional algebras by Rasoulian and Yau (1997). Due to the difficulty of the problem, Brockett suggested that one should understand the low-dimensional estimation algebras first. Rasoulian and Yau (1999) have classified estimation algebras of dimension at most four. Chiou, Chiueh, and Yau (2006) have classified estimation algebras of dimension five. In this article, we give a structure theorem for estimation algebras of dimension six. Section 2 offers the filter model and some basic theorems which we need to prove our main theorem. Mitter conjectured that functions in finite-dimensional estimation algebras are polynomials of

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degree at most one. In Section 3, we prove the Mitter conjecture when $n=6$. In Section 4, we give the structure of six-dimensional estimation algebra and two examples. In the last section, we give the conclusion.

2. Filtering model and basic theorems

The filtering problem considered here is based on following signal observation model:

$$\begin{cases} \mathrm{d}x(t) = f(x(t))\mathrm{d}t + g(x(t))\mathrm{d}v(t), & x(0) = x_0, \\ \mathrm{d}y(t) = h(x(t))\mathrm{d}t + \mathrm{d}w(t), & y(0) = 0, \end{cases} \tag{2.1}$$

where x, v, y and w are, respectively, $\mathbb{R}^n$, $\mathbb{R}^p$, $\mathbb{R}^m$ and $\mathbb{R}^m$ valued processes, and v and w have components that are independent, standard Brownian processes. We further assume $n=p$, f, h are $\mathbb{C}^\infty$ smooth and g is an orthogonal matrix.

Let $\rho(t,x)$ denote the conditional density of the state given the observation $\{y(s)\colon 0 \leqslant s \leqslant t\}$. It is well known that $\rho(t,x)$ is given by normalising a function $\sigma(t,x)$ which satisfies the Duncan–Mortensen–Zakai equation:

$$\begin{cases} \mathrm{d}\sigma(t,x) = L_0\sigma(t,x) + \sum_{i=1}^{m} L_i\sigma(t,x)\mathrm{d}y_i(t), \\ \sigma(0,x) = \sigma_0, \end{cases} \tag{2.2}$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i} - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2,$$

and for $i=1,\dots,m$, L_i is the zero degree differential operator of multiplication by h_i. Here σ_0 is the probability density of the initial point x_0. Let

$$D_i = \frac{\partial}{\partial x_i} - f_i,$$
$$\eta = \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{m} h_i^2. \tag{2.3}$$

Then L_0 can be written as $L_0 = \frac{1}{2}(\sum_{i=1}^{n} D_i^2 - \eta)$.

Definition 2.1: The estimation algebra E of a filtering model (2.1) is defined to be the Lie algebra generated by L_0, $L_1,\dots,L_m$ or $E = \langle L_0, L_1,\dots,L_m\rangle_{\mathrm{LA}}$. If $x_i \in E$ for every $1 \leqslant i \leqslant n$, then E is called an estimation algebra of maximal rank.

Definition 2.2: Wong matrix $\Omega = (\omega_{ij})$ is an $n\times n$ skew symmetric matrix with $\omega_{ij} = (\partial f_j/\partial x_i) - (\partial f_i/\partial x_j)$.

Theorem 2.3 (Ocone 1981)**:** *Any function in a finite-dimensional estimation algebra is polynomial of degree at most two.*

The following theorem in Yau (1994) is useful in the classification of finite-dimensional estimation algebras.

Theorem 2.4 (Yau 1994)**:** *Let $F(x_1,\dots,x_n)$ be a polynomial on $\mathbb{R}^n$. Suppose there exists a polynomial path c: $\mathbb{R}\to\mathbb{R}^n$ such that $\lim_{t\to\infty} c(t) = \infty$ and $\lim_{t\to\infty} F(c(t)) = -\infty$. Then there are no $\mathbb{C}^\infty$ functions $f_1,\dots,f_n$ on $\mathbb{R}^n$ such that*

$$\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = F.$$

Let U_k be the vector space of differential operators of order up to and including k. Assume that the coefficients of these differential operators are $\mathbb{C}^\infty$ functions. Wong (1987a) proved the following theorem.

Theorem 2.5 (Wong 1987a)**:** *If $Y = \sum_{i=1}^{n}\gamma_i D_i \bmod U_0$ is an element in a finite-dimensional estimation algebra, then γ_i are polynomials of $x_1,\dots,x_n$ for all i.*

Theorem 2.6 (Wong 1987a)**:** *If $Y = \sum_{1\leqslant i\leqslant j\leqslant n}\gamma_{ij}\times D_iD_j \bmod U_1$ is an element in a finite-dimensional estimation algebra, then γ_{ij} are polynomials of $x_1,\dots,x_n$ for all i, j.*

The following structure theorem (Rasoulian and Yau 1999) for Euler operator $E_l = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + \cdots + x_l\frac{\partial}{\partial x_l}$ will be used frequently.

Theorem 2.7 (Rasoulian and Yau 1999)**:** *Suppose $m\in Z$ is a constant integer and ζ is a $\mathbb{C}^\infty$ function on $\mathbb{R}^n$ and $l\leqslant n$ such that $E_l(\zeta)+m\zeta$ is a polynomial of degree k, k a positive integer, in $x_1, x_2,\dots,x_l$ variables with coefficient in $\mathbb{C}^\infty$ functions of $x_{l+1},\dots,x_n$ variables. If $k+m\geqslant 0$, ζ is a polynomial of degree k in x_1, $x_2,\dots,x_l$ variables with coefficients in $\mathbb{C}^\infty$ functions of $x_{l+1},\dots,x_n$. If $k+m<0$, ζ is a polynomial of degree at most $m'=-m$ in $x_{l+1},\dots,x_n$ variables with coefficient in $\mathbb{C}^\infty$ functions of $x_{l+1},\dots,x_n$.*

Theorem 2.8 (Rasoulian and Yau 1996)**:** *Suppose that $h = \frac{1}{2}\sum_{i=1}^{l} x_i^2$ for some $1\leqslant l\leqslant n$ is in the finite-dimensional estimation algebra E. Let $\alpha_i = \sum_{j=1}^{l} x_j\omega_{ij}$. Then*

- (i) *for $1\leqslant i, j\leqslant l$, $E_l(\omega_{ij}) + 2(\omega_{ij}) = \frac{\partial\alpha_i}{\partial x_j} - \frac{\partial\alpha_j}{\partial x_i}$ and ω_{ij} is a polynomial in $x_1, x_2,\dots,x_n$,*
- (ii) *for $i\geqslant l+1$, $j\leqslant l$, $E_l(\omega_{ij}) + \omega_{ij} = \frac{\partial\alpha_i}{\partial x_j} - \frac{\partial\alpha_j}{\partial x_i}$ and ω_{ij} is a polynomial in $x_1, x_2,\dots,x_n$ and*
- (iii) *for $l+1\leqslant i, j\leqslant l$, $E_l(\omega_{ij}) = \frac{\partial\alpha_i}{\partial x_j} - \frac{\partial\alpha_j}{\partial x_i}$ and ω_{ij} is a polynomial in $x_1, x_2,\dots,x_n$ plus a $\mathbb{C}^\infty$ function in $x_{l+1},\dots,x_n$.*

3. Six-dimensional estimation algebras

Rasoulian and Yau (1999) have classified all finite-dimensional estimation algebras of dimension at most four. As a corollary of their classification result, they have proved the Mitter conjecture for estimation algebras of dimension at most four. Chiou et al. (2006), have classified all finite-dimensional estimation algebras of dimension five, based on the result (Chiou et al. 2006) about Mitter conjecture estimation algebras of dimension five. Therefore, we need to prove the Mitter conjecture to classify all finite-dimensional estimation algebras of dimension six.

The following lemma plays an important role in proving the Mitter conjecture for low-dimensional estimation algebras.

Lemma 3.1 (Chiou et al. 2006)**:** *For any* $1 \leqslant l \leqslant n$, *if* γ_i, $i = 1, \ldots, l$, *are polynomials in* $x_1, \ldots, x_l$ *with coefficients in* $\mathbb{C}^\infty$ *functions of* $x_{l+1}, \ldots, x_n$ *satisfying*

$$\frac{\partial \gamma_j}{\partial x_i} + \frac{\partial \gamma_i}{\partial x_j} = 0 \quad \text{for all } 1 \leqslant i, j \leqslant l,$$

then each γ_i *is necessary of the form*

$$\gamma_i = \sum_{1 \leqslant j \leqslant l} c_i^j (x_{l+1}, \ldots, x_n) x_j + d_i(x_{l+1}, \ldots, x_n),$$

where $c_i^j (x_{l+1}, \ldots, x_n)$ *and* $d_i(x_{l+1}, \ldots, x_n)$ *are* $\mathbb{C}^\infty$ *functions and* $c_i^j = -c_j^i$.

Observe

$$[L_0, x_1] = D_1, \quad [D_1, x_1] = 1,$$
$$\left[L_0, \sum_{j=2}^n a_j x_j + c\right] = \sum_{j=2}^n a_j D_j. \tag{3.1}$$

Proposition 3.2: *If* $\dim E = 6$, *then*

(i) *E cannot contain a degree one polynomial and a degree two polynomial and*
(ii) *E cannot contain two linearly independent degree two polynomials.*

Proof: Let

$$h = \sum_{i,j=1}^n a_{ij} x_i x_j + \sum_{j=1}^n b_j x_j + c_1,$$
$$\widetilde{h} = \sum_{i,j=1}^n d_{ij} x_i x_j + \sum_{j=1}^n e_j x_j + c_2$$

be two linearly independent degree two polynomials. Observe

$$Y_1 = [L_0, h] = \sum_{i,j=1}^n a_{ij}(x_i D_j + x_j D_i) + \sum_{j=1}^n b_j D_j \mod U_0, \tag{3.2}$$

$$\widetilde{Y}_1 = [L_0, \widetilde{h}] = \sum_{i,j=1}^n d_{ij}(x_i D_j + x_j D_i) + \sum_{j=1}^n e_j D_j \mod U_0, \tag{3.3}$$

$$Y_2 = [L_0, Y_1] = \sum_{i,j=1}^n a_{ij} D_i D_j \mod U_1, \tag{3.4}$$

$$\widetilde{Y}_2 = [L_0, \widetilde{Y}_1] = \sum_{i,j=1}^n d_{ij} D_i D_j \mod U_1, \tag{3.5}$$

where U_i is the space of differential operators with order at most i.

(i) If E contains a degree one polynomial and degree two polynomial, we may assume that x_1, and degree two polynomial $h = \sum_{i,j=1}^n a_{ij} x_i x_j + \sum_{j=1}^n b_j x_j + c_1$, where $a_{ij} \neq 0$ for some i, j, are in E. From (3.1) and (3.2), we see that E contains six linearly independent elements $L_0, x_1, D_1, 1, h$ and Y_1. Notice that

$$[D_1, h] = \sum_{i=1}^n (a_{1i} + a_{i1}) x_i + b_1$$

need to be a linear combination of x_1, and 1, which implied that $a_{1i} + a_{i1} = 0$, $i \neq 1$. Now again, we consider $[L_0, D_1]$, which must be linear combination of x_1, 1, h and Y_1:

$$\begin{aligned}[L_0, D_1] &= \sum_{j=2}^n \omega_{1j} D_j + \frac{1}{2} \sum_{j=2}^n \frac{\partial \omega_{1j}}{\partial x_j} + \frac{1}{2} \frac{\partial \eta}{\partial x_1} \\ &= \lambda_1 x_1 + \lambda_2 + \lambda_3 h + \lambda_4 Y_1. \end{aligned} \tag{3.6}$$

If $a_{ij} + a_{ji} \neq 0$ for some $i \neq j$, then from (3.4), we see that E contains seven linearly independent elements L_0, x_1, D_1, 1, h, Y_1 and Y_2.

If $a_{ij} + a_{ji} = 0$ for all $i \neq j$, then $h = \sum_{i=1}^n 2a_{ii} x_i^2 + \sum_{j=1}^n b_j x_j + c_1$. We have $\lambda_4(2a_{11}x_1 + b_1) = 0$ (D_1 term) from (3.6). If $\lambda_4 \neq 0$, then we know $a_{11} = 0$. Again, E contains seven linearly independent elements L_0, x_1, D_1, 1, h, Y_1 and Y_2. If $\lambda_4 = 0$, then $\omega_{1i} = 0$ for $1 \leqslant i \leqslant n$ from (3.6). Then we have

$$\frac{\partial \eta}{\partial x_1} = \lambda_1 x_1 + \lambda_2 + \lambda_3 h$$
$$\Rightarrow \eta = \lambda_1 x_1^2 + \frac{2}{3} \lambda_3 x_1^3 + 2 x_1 f_1(x_2, \ldots, x_n),$$

where f_1 is a function of $x_2, \ldots, x_n$ only. Thus, $\sum_{i=1}^n \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^n f_i^2 = \eta - h^2 - x_1^2$. Then by Theorem 2.4, there is no $\mathbb{C}^\infty$ solution $f = (f_1, \ldots, f_n)$ for Equation (2.3), a contradiction.

(ii) Suppose E contains two linearly independent degree two polynomials, h and $\widetilde{h}$. Let $h^{(i)}$ be the degree i monomial part of h. If $h^{(2)} = c\widetilde{h}^{(2)}$ for some constant c, then E contains a degree one polynomial $h^{(1)} + c_1$ and a degree two polynomial $h^{(2)} + c_2$. In view of the proof of (i), we know it is impossible. One the other hand, if $h^{(2)} \neq \mathrm{c}h^{(2)}$ for any constant c, then we may assume that $\sum a_{ij} x_i x_j \neq \sum_{i=1}^n x_i^2$. From (3.2)–(3.5), we see that E contains the seven linearly independent elements $L_0, h, \widetilde{h}, Y_1, \widetilde{Y}_1, Y_2$ and $\widetilde{Y}_2$. □

The following result is provided by Chiou et al. (2006) with $\dim E = 5$. However, without any change of the proof, we will find that the result holds for $\dim E = 6$ too.

Theorem 3.3: *Let E be the estimation algebra of a filtering model* (2.1) *with state-space dimension n. Suppose* $\dim E = 6$. *Then*

(i) *n is at least two and*
(ii) *if E contains a degree two polynomial, then E contains a polynomial of the form* $h_1 = \sum_{i=1}^{l} x_i^2$ *for some* $1 \leqslant l \leqslant n$.

In view of the above discussion, in order to prove the Mitter conjecture of estimation algebras of dimension six, we only need to consider two possible cases: case 1, $h_1(x) = \frac{1}{2}\sum_{i=1}^{l} x_i^2, 1 \leqslant l < n$ and case 2, $h_1(x) = \frac{1}{2}\sum_{i=1}^{n} x_i^2$. In both cases, $h_j = c_j h_1$ for some constants c_j, $j = 2, \dots, m$. We may assume that $n \geqslant 2$. Notice that $\sum_{j=1}^{m} h_j^2 = c^2 h_1^2$ where $c^2 = 1 + c_2^2 + \cdots + c_m^2$.

Theorem 3.4: *Let E be a finite-dimensional estimation algebra associated with filtering model* (2.1) *with arbitrary state-space dimension. Then any function in E is a polynomial of degree at most one if* $\dim E = 6$.

Proof: Denote

$$E_l = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + \cdots + x_l \frac{\partial}{\partial x_l}.$$

Case 1: $h_1(x) = \frac{1}{2}\sum_{i=1}^{l} x_i^2$, $l \leqslant n-1$. Observe that

$$Y_1 := [L_0, h_1] = \sum_{i=1}^{l} x_i D_i + \frac{l}{2},$$

$$Y_2 := [L_0, Y_1] = \sum_{i=1}^{l} D_i^2 - \sum_{i=1}^{n} \alpha_i D_i - \frac{1}{2}\beta_n + \frac{1}{2}E_l(\eta),$$

where $\alpha_i = \sum_{j=1}^{l} x_j \omega_{ij}$, and $\beta_n = \sum_{i=1}^{n} \frac{\partial \alpha_i}{\partial x_i} = \sum_{i=1}^{n}\sum_{j=1}^{l} x_j \frac{\partial \omega_{ij}}{\partial x_i}$.

$$Z_{21} := [Y_2, Y_1] = 2\sum_{i=1}^{l} D_i^2 + \sum_{i=1}^{n} E_l(\alpha_i) D_i - 3\sum_{i=1}^{l} \alpha_i D_i - \beta_l + \sum_{i=1}^{n} \alpha_i^2 + \frac{1}{2}E_l(\beta_n) - \frac{1}{2}E_l(E_l(\eta)),$$

where $\beta_l = \sum_{i=1}^{n} \frac{\partial \alpha_i}{\partial x_i} = \sum_{i=1}^{l}\sum_{j=1}^{l} x_j \frac{\partial \omega_{ij}}{\partial x_i}$ and $\sum_{i=1}^{n} \alpha_i^2 = \sum_{i=1}^{n}\sum_{j,k=1}^{l} x_j x_k \omega_{ij}\omega_{ik}$,

$$\Delta := Z_{21} - 2Y_2 = \sum_{i=1}^{l}(E_l(\alpha_i) - \alpha_i) D_i + \sum_{i=l+1}^{n}(E_l(\alpha_i) + 2\alpha_i) D_i + \sum_{i=1}^{n} \alpha_i^2 - \beta_l + \frac{1}{2}(E_l(\beta_n) + 2\beta_n) - \frac{1}{2}E_l(E_l(\eta) + 2\eta)).$$

Based on the proof of the main theorem from Chiou, Chiueh, and Yau (2008), L_0, h_1, Y_1, Y_2 and Δ are linearly independent. We have the following multiplication table:

	L_0	h_1	Y_1	Y_2	Δ
L_0	0	$Y_1 = [L_0, h_1]$	$Y_2 = [L_0, Y_1]$	$[L_0, Y_2]$	$[L_0, \Delta]$
h_1	$-[L_0, h_1]$	0	$[h_1, Y_1]$	$[h_1, Y_2]$	$[h_1, \Delta]$
Y_1	$-[L_0, Y_1]$	$-[h_1, Y_1]$	0	$\Delta - 2Y_2 = [Y_1, Y_2]$	$[Y_1, \Delta]$
Y_2	$-[L_0, Y_2]$	$-[h_1, Y_2]$	$-[Y_1, Y_2]$	0	$[Y_2, \Delta]$
Δ	$-[L_0, \Delta]$	$-[h_1, \Delta]$	$-[Y_1, \Delta]$	$-[Y_2, \Delta]$	0

Since $\dim E = 6$, then there is one and only one operator, which is linearly independent of L_0, h_1, Y_1, Y_2 and Δ, among the upper triangle operators in the above table. Notice that $[h_1, Y_1]$, $[h_1, \Delta] \in U_0$ and $[h_1, Y_2] - Y_1 \in U_0$, so $[h_1, Y_1]$, $[h_1, \Delta]$ and $[h_1, Y_2] - Y_1$ must be a constant multiple of h_1. Thus the operator, which is linearly independent of L_0, h_1, Y_1, Y_2 and Δ, must be among $[L_0, Y_2]$, $[L_0, \Delta]$, $[Y_1, \Delta]$ and $[Y_2, \Delta]$. Let us assume $Y_3 = [L_0, Y_2]$ is linearly independent of L_0, h_1, Y_1, Y_2 and Δ. Then we have,

$$\begin{aligned} Y_3 := [L_0, Y_2] &= 2\sum_{i=1}^{n}\sum_{j=1}^{l} \omega_{ji} D_j D_i - \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial \alpha_j}{\partial x_i} D_i D_j \mod U_1 \\ &= -\sum_{1 \leqslant i \leqslant j \leqslant l} \left(\frac{\partial \alpha_j}{\partial x_i} + \frac{\partial \alpha_i}{\partial x_j}\right) D_i D_j - \sum_{l+1 \leqslant i \leqslant j \leqslant n} \left(\frac{\partial \alpha_j}{\partial x_i} + \frac{\partial \alpha_i}{\partial x_j}\right) D_i D_j \\ &\quad + \sum_{i=l+1}^{n}\sum_{j=1}^{l}\left[2\omega_{ji} - \left(\frac{\partial \alpha_j}{\partial x_i} + \frac{\partial \alpha_i}{\partial x_j}\right)\right] D_i D_j \mod U_1. \end{aligned}$$

Consider

$$\begin{aligned} Y_4 &:= [L_0, Y_3] \\ &= \sum_{k=1}^{n}\sum_{1 \leqslant i \leqslant j \leqslant l} \frac{\partial}{\partial x_k}\left(\frac{\partial \alpha_j}{\partial x_i} + \frac{\partial \alpha_i}{\partial x_j}\right) D_k D_i D_j \\ &\quad - \sum_{k=1}^{n}\sum_{l+1 \leqslant i \leqslant j \leqslant n} \frac{\partial}{\partial x_k}\left(\frac{\partial \alpha_j}{\partial x_i} + \frac{\partial \alpha_i}{\partial x_j}\right) D_k D_i D_j \\ &\quad + \sum_{k=1}^{n}\sum_{i=l+1}^{n}\sum_{j=1}^{l} \frac{\partial}{\partial x_k} \times \left[2\omega_{ji} - \left(\frac{\partial \alpha_j}{\partial x_i} + \frac{\partial \alpha_i}{\partial x_j}\right)\right] D_k D_i D_j \mod U_2. \end{aligned} \tag{3.7}$$

Suppose there exist $\lambda_1, \ldots, \lambda_6$ in $\mathbb{R}$ such that

$$Y_4 = \lambda_1 L_0 + \lambda_2 h_1 + \lambda_3 Y_1 + \lambda_4 Y_2 + \lambda_5 \Delta + \lambda_6 Y_3. \tag{3.8}$$

Then we get:

$$\alpha_k = \sum_{s=1}^{l} d_s^k x_s x_k + c_k x_k + e_k,$$
where d_s^k, c_k are functions independent of $x_1, \ldots, x_l$,
$d_k^k = 0$,
and e_k is independent of x_k, for $1 \leqslant k \leqslant l$,
(3.9)

$$\alpha_k = \sum_{s=l+1}^{n} d_s^k x_s x_k + c_k x_k + e_k,$$
where d_s^k, c_k are functions independent of $x_{l+1}, \ldots, x_n$,
$d_k^k = 0$,
and e_k is independent of x_k, for $1+1 \leqslant k \leqslant n$,
(3.10)

$$\frac{\partial}{\partial x_k}\left[2\omega_{ji} - \left(\frac{\partial \alpha_j}{\partial x_i} + \frac{\partial \alpha_i}{\partial x_j}\right)\right] = 0 \tag{3.11}$$
for $1 \leqslant j \leqslant l, l+1 \leqslant i \leqslant n, 1 \leqslant k \leqslant n$.

The analysis is similar to case 2 in Chiou et al. (2008). Notice $\alpha_k = \sum_{s=1}^{l} d_s^k x_s x_k + c_k x_k + e_k = \sum_{i=1}^{l} x_i \omega_{ki}$, for $1 \leqslant k \leqslant l$, from (3.9), we have $\sum_{s=1}^{l} d_s^k x_s x_k + c_k x_k = 0$ (simply put $x_k = 0$), which implies that $d_s^k = c_k = 0$, $1 \leqslant s,\ k \leqslant l$. Similarly, $d_s^k = c_k = 0,\ l+1 \leqslant s,\ k \leqslant n$ from (3.10).

Then $E_l(\alpha_i) - \alpha_i = E_l(e_i) - e_i$, and $E_l(e_i) - e_i$ is independent of x_i for $1 \leqslant i \leqslant l$. We get

$$\Delta = \sum_{i=1}^{l}(E_l(e_i) - e_i)D_i + \sum_{i=l+1}^{n}(E_l(\alpha_i) + 2\alpha_i)D_i + \sum_{i=1}^{n}\alpha_i^2 - \beta_l + \frac{1}{2}(E_l(\beta_n) + 2\beta_n) - \frac{1}{2}E_l(E_l(\eta) + 2\eta)),$$

$$[\Delta, h_1] = \sum_{i=1}^{l}(E_l(e_i) - e_i)x_i.$$

Since $\dim E = 6$, $\sum_{i=1}^{l}(E_l(e_i) - e_i)x_i$ must be a constant multiple of $\sum_{i=1}^{l} x_i^2$. This implies $E_l(e_i) - e_i = 0$ for $i = 1, \ldots, l$ (put $x_j = 0, j \neq i$). So for any fixed $l \geqslant 1$

$$\Delta = \sum_{i=l+1}^{n}(E_l(\alpha_i) + 2\alpha_i)D_i + \sum_{i=1}^{n}\alpha_i^2 - \beta_l + \frac{1}{2}(E_l(\beta_n) + 2\beta_n) - \frac{1}{2}E_l(E_l(\eta) + 2\eta)). \tag{3.12}$$

From (3.11), we have

$$\left[2\omega_{ji} - \left(\frac{\partial \alpha_j}{\partial x_i} + \frac{\partial \alpha_i}{\partial x_j}\right)\right] = \text{constant } \lambda_{ij}, \quad \text{for } 1 \leqslant j \leqslant l, l+1 \leqslant i \leqslant n. \tag{3.13}$$

From (3.12)

$$\begin{aligned}[Y_2, \Delta] &= 2\sum_{j=1}^{l}\sum_{i=l+1}^{n}\frac{\partial}{\partial x_j}(E_l(\alpha_i) + 2\alpha_i)D_j D_i \mod U_1 \\ &= \widetilde{\lambda}_1 L_0 + \widetilde{\lambda}_2 h_1 + \widetilde{\lambda}_3 Y_1 + \widetilde{\lambda}_4 Y_2 + \widetilde{\lambda}_5 \Delta + \widetilde{\lambda}_6 Y_3, \\ &\text{for some } \widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_6 \in \mathbb{R}.\end{aligned} \tag{3.14}$$

It follows that (by (3.12))

$$\frac{\partial}{\partial x_j}(E_l(\alpha_i) + 2\alpha_i) = \text{constant } \widetilde{\lambda}_{ij}, \quad \text{for } 1 \leqslant j \leqslant l, l+1 \leqslant i \leqslant n. \tag{3.15}$$

Observe that $E_l(\alpha_i) + 2\alpha_i$, $l+1 \leqslant i \leqslant n$ are polynomials of $x_1, \ldots, x_n$ by Theorem 2.5. Equation (3.15) implies that $E_l(\alpha_i) + 2\alpha_i$, $l+1 \leqslant i \leqslant n$ are polynomials of degree one in $x_1, x_2, \ldots, x_l$ with coefficients in $\mathbb{C}^\infty$ functions of $x_{l+1}, \ldots, x_n$. In view of Theorem 2.7, α_i, $l+1 \leqslant i \leqslant n$ are polynomials of degree one in $x_1, x_2, \ldots, x_l$ with coefficients in $\mathbb{C}^\infty$ functions of $x_{l+1}, \ldots, x_n$,

$$\alpha_i = \sum_{j=1}^{l} c_i^j(x_{l+1}, \ldots, x_n)x_j + d_i(x_{l+1}, \ldots, x_n), \quad \text{for } l+1 \leqslant i \leqslant n.$$

It follows that $d_i(x_{l+1}, \ldots, x_n) = 0$, $c_i^j(x_{l+1}, \ldots, x_n) = \omega_{ij}$ because $\alpha_i = \sum_{j=1}^{l} x_j \omega_{ij}$ where ω_{ij} are $\mathbb{C}^\infty$ smooth.

Then from (3.13), we have

$$\begin{aligned}&2\omega_{ji} - \left(\omega_{ij} + \sum_{k=1}^{l} x_k\left(\frac{\partial \omega_{jk}}{\partial x_i} + \frac{\partial \omega_{ik}}{\partial x_j}\right)\right) = \lambda_{ij}, \\ &1 \leqslant j \leqslant l, l+1 \leqslant i \leqslant n \\ &\Rightarrow 3\omega_{ji} - \sum_{k=1}^{l} x_k \frac{\partial \omega_{jk}}{\partial x_i} = \lambda_{ij} \\ &(\omega_{ik} \text{ is independent of } x_1, \ldots, x_l) \\ &\Rightarrow 3\omega_{ji} = \lambda_{ij}, \ \sum_{k=1}^{l} x_k \frac{\partial \omega_{jk}}{\partial x_i} = 0 \\ &(\text{by setting } x_1 = \cdots = x_l = 0).\end{aligned}$$

Then $\alpha_i = \frac{1}{3}\sum_{j=1}^{l} \lambda_{ji} x_j$. We have $\alpha_i = \widetilde{\lambda}_{ij}$ from (3.15), which implies that $\widetilde{\lambda}_{ij} = \lambda_{ji} = 0$, further, $\alpha_i = 0$, $l+1 \leqslant i \leqslant n$.

From $E_l(e_i) - e_i = 0$, $1 \leqslant i \leqslant l$, by Theorem 2.7, we know e_i is a polynomial of degree at most one in $x_1, \ldots, x_l$ with coefficients in $\mathbb{C}^\infty$ functions of

$x_{l+1}, \ldots, x_n$. Thus, α_i is a polynomial of degree at most one in $x_1, \ldots, x_l$ with coefficients in $\mathbb{C}^\infty$ functions of $x_{l+1}, \ldots, x_n$. Now Δ is of the form

$$\Delta = \sum_{i=1}^{l} \alpha_i^2 - \beta_l + \frac{1}{2}(E_l(\beta_n) + 2\beta_n) - \frac{1}{2}E_l(E_l(\eta) + 2\eta)$$
$$= e\sum_{i=1}^{l} x_i^2, \quad \text{for some constant } e.$$

The last equality comes from the fact that $\dim E = 6$. Notice that $\beta_l = \sum_{i=1}^{l} \frac{\partial \alpha_i}{\partial x_i}$ and $\beta_n = \sum_{i=1}^{n} \frac{\partial \alpha_i}{\partial x_i}$, β_l and β_n are polynomials of degree at most one by (3.9), (3.10). By Theorem 2.7, η is a polynomial of degree at most two in $x_1, \ldots, x_l$. We get a contradiction since there is no $\mathbb{C}^\infty$ smooth solution for $\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = \sum_{j=1}^{m} h_j^2 + \eta = -c^2(\sum_{i=1}^{l} x_i^2)^2 + \eta$ in view of Theorem 2.4 . We have shown that six-dimensional estimation algebra cannot contain an element of the form $\frac{1}{2}\sum_{i=1}^{l} x_i^2$ with $1 \leqslant l < n$ if L_0, h_1, Y_1, Y_2, Δ and Y_3 are linearly independent.

If $Y_3' := [Y_1, \Delta]$ is linearly independent of L_0, h_1, Y_1, Y_2 and Δ, then we can still use a similar proof since we are focusing on the coefficients of $D_iD_jD_k$ terms and $[Y_1, \Delta]$, $[Y_2, \Delta] \in U_2$. We only need to change Equation (3.8) to $Y_4 = \lambda_1 L_0 + \lambda_2 h_1 + \lambda_3 Y_1 + \lambda_4 Y_2 + \lambda_5 \Delta + \lambda_6 Y_3'$ and Equation (3.14) to

$$[Y_2, \Delta] = 2\sum_{j=1}^{l}\sum_{i=l+1}^{n} \frac{\partial}{\partial x_j}(E_l(\alpha_i) + 2\alpha_i)D_jD_i \mod U_1$$
$$= \tilde{\lambda}_1 L_0 + \tilde{\lambda}_2 h_1 + \tilde{\lambda}_3 Y_1 + \tilde{\lambda}_4 Y_2 + \tilde{\lambda}_5 \Delta + \tilde{\lambda}_6 Y_3',$$
$$\text{for some } \tilde{\lambda}_1, \ldots, \tilde{\lambda}_6 \in \mathbb{R}.$$

If $Y_3' := [L_0, \Delta]$ is linearly independent of L_0, h_1, Y_1, Y_2 and Δ, we need not change Equation (3.8). We only need to change Equation (3.14) to

$$[Y_2, \Delta] = -\sum_{1\leqslant i\leqslant j\leqslant l}\left(\frac{\partial \alpha_j}{\partial x_i} + \frac{\partial \alpha_i}{\partial x_j}\right)D_iD_j$$
$$- \sum_{l+1\leqslant i\leqslant j\leqslant n}\left(\frac{\partial \alpha_j}{\partial x_i} + \frac{\partial \alpha_i}{\partial x_j}\right)D_iD_j$$
$$+ \sum_{i=l+1}^{n}\sum_{j=1}^{l}\left[2\omega_{ji} - \left(\frac{\partial \alpha_j}{\partial x_i} + \frac{\partial \alpha_i}{\partial x_j}\right)\right]D_iD_j \mod U_1$$
$$= \tilde{\lambda}_1 L_0 + \tilde{\lambda}_2 h_1 + \tilde{\lambda}_3 Y_1 + \tilde{\lambda}_4 Y_2 + \tilde{\lambda}_5 \Delta + \tilde{\lambda}_6 Y_3',$$
$$\text{for some } \tilde{\lambda}_1, \ldots, \tilde{\lambda}_6 \in \mathbb{R}.$$

If $\tilde{\lambda}_6 \neq 2$, then we could still use a similar proof. If $\tilde{\lambda}_6 = 2$, there does not exist a $\tilde{\lambda}_1 \in \mathbb{R}$ such that (3.14) holds. (Assume $\alpha_i \neq 0$, $1 \leqslant i \leqslant l$. If $\alpha_i = 0$, $1 \leqslant i \leqslant l$, we could directly obtain the contradiction in view of Theorem 2.4.)

If $Y_3' := [Y_2, \Delta]$ is linearly independent of L_0, h_1, Y_1, Y_2 and Δ, we need not change Equation (3.8). We only need to change Equation (3.14) to

$$[L_0, Y_2] = -\sum_{1\leqslant i\leqslant j\leqslant l}\left(\frac{\partial \alpha_j}{\partial x_i} + \frac{\partial \alpha_i}{\partial x_j}\right)D_iD_j$$
$$- \sum_{l+1\leqslant i\leqslant j\leqslant n}\left(\frac{\partial \alpha_j}{\partial x_i} + \frac{\partial \alpha_i}{\partial x_j}\right)D_iD_j$$
$$+ \sum_{i=l+1}^{n}\sum_{j=1}^{l}\left[2\omega_{ji} - \left(\frac{\partial \alpha_j}{\partial x_i} + \frac{\partial \alpha_i}{\partial x_j}\right)\right]D_iD_j \mod U_1$$
$$= \tilde{\lambda}_1 L_0 + \tilde{\lambda}_2 h_1 + \tilde{\lambda}_3 Y_1 + \tilde{\lambda}_4 Y_2 + \tilde{\lambda}_5 \Delta + \tilde{\lambda}_6 Y_3',$$
$$\text{for some } \tilde{\lambda}_1, \ldots, \tilde{\lambda}_6 \in \mathbb{R}.$$

If $\tilde{\lambda}_6 \neq 1/2$, then we could still use a similar proof. If $\tilde{\lambda}_6 = 1/2$, there does not exist a $\tilde{\lambda}_1 \in \mathbb{R}$ such that (3.14) holds. (Assume $\alpha_i \neq 0$, $1 \leqslant i \leqslant l$. If $\alpha_i = 0$, $1 \leqslant i \leqslant l$, we could directly obtain the contradiction in view of Theorem 2.4.)

Case 2: $h_1(x) = \frac{1}{2}\sum_{i=1}^{n} x_i^2$. In this case, ω_{ij}, $1 \leqslant i, j \leqslant n$ are polynomial in $x_1, \ldots, x_n$ by Theorem 2.8. Observe that

$$Y_1 := [L_0, h_1] = \sum_{i=1}^{n} x_iD_i + \frac{n}{2}, \quad [Y_1, h_1] = \sum_{i=1}^{n} x_i^2 = 2h_1,$$

$$Y_2 := 2L_0 - [L_0, Y_1] = \sum_{i=1}^{n} \alpha_iD_i + g_1,$$

$$\Delta := [Y_1, Y_2] = \sum_{i=1}^{n}[E_n(\alpha_i) - \alpha_i]D_i + \sum_{i=1}^{n}\alpha_i^2$$
$$+ E_n\left[\frac{1}{2}\beta_n - \frac{1}{2}E_n(\eta) - \eta\right],$$

$$Y_3 := [L_0, Y_2] = \sum_{1\leqslant i\leqslant j\leqslant n}\left(\frac{\partial \alpha_j}{\partial x_i} + \frac{\partial \alpha_i}{\partial x_j}\right)D_iD_j \mod U_1,$$

$$[L_0, Y_3] = \sum_{i,j,k=1}^{n} \frac{\partial^2 \alpha_j}{\partial x_k \partial x_i} D_kD_iD_j$$
$$+ \sum_{i,j,k=1}^{n} \frac{\partial \alpha_j}{\partial x_i}(\omega_{jk}D_iD_k + \omega_{ik}D_kD_j) \mod U_1,$$

where $\alpha_i = \sum_{j=1}^{n} x_j\omega_{ij}$, $\beta_n = \sum_{j=1}^{n} \frac{\partial \alpha_i}{\partial x_i}$ and $g_1 = \frac{1}{2}\beta_n - \frac{1}{2}[E_n(\eta) + 2\eta]$. Based on the analysis from Chiou et al. (2008), L_0, h_1, Y_1, Y_2 and Y_3 are linearly independent. If L_0, h_1, Y_1, Y_2, Y_3 and Δ are linearly dependent, then $[L_0, Y_3]$ is a linear combination of L_0, h_1, Y_1, Y_2, Y_3 and Δ. Using the analysis from Chiou et al. (2008), case 2 of their main theorem, we will obtain a contradiction. So Δ is a linear combination of L_0, h_1, Y_1, Y_2, Y_3: $\Delta = \lambda_1 L_0 + \lambda_2 h_1 + \lambda_3 Y_1 + \lambda_4 Y_2 + \lambda_5 Y_3$. If $\lambda_5 \neq 0$, then we could use the same argument in Chiou et al. (2008) to discuss the coefficient of D_iD_j terms to obtain a contradiction.

So $\Delta = \lambda_1 L_0 + \lambda_2 h_1 + \lambda_3 Y_1 + \lambda_4 Y_2$, and

$$\begin{aligned} &\lambda_1 = 0, \\ &E_n(\alpha_i) = \lambda_3 x_i + (\lambda_4+1)\alpha_i, \quad 1 \leqslant i \leqslant n, \end{aligned} \tag{3.16}$$

$$E_n(g_1) - \lambda_4 g_1 = -\sum_{i=1}^n \alpha_i^2 + \frac{\lambda_2}{2}\sum_{i=1}^n x_i^2 + \frac{n\lambda_3}{2}, \tag{3.17}$$

from (3.16), we have $\sum_{j=1}^n x_j(E_n(\omega_{1j}) - \lambda_4\omega_{1j}) = \lambda_3 x_1$, which implies $\lambda_3 = 0$. Let $p_i = \deg(\alpha_i)$. Chiou et al. (2008) show that $(\alpha_1, \dots, \alpha_n) \neq C(x_1, \dots, x_n)$ for any constant C and $p_i \neq 0$. For any polynomial ψ, we shall denote the homogeneous part of degree s of ψ by $\psi^{(s)}$. Equation (3.16) implies

$$\sum_{s=1}^{p_i} s\alpha_i^{(s)} = E_n(\alpha_i) = (\lambda_4+1)\sum_{s=1}^{p_i}\alpha_i^{(s)}.$$

Therefore $\alpha_i = \alpha_i^{(p_i)}$, where $p_i = \lambda_4 + 1$ if $p_i > 1$. Without loss of generality, we shall assume $\deg\alpha_1 = \deg\alpha_2 = \cdots = \deg\alpha_\mu = \lambda_4 + 1 := p$, $\deg\alpha_{\mu+1} = \cdots = \deg\alpha_n = 0$ for some $0 < \mu \leqslant n$.

We now claim that

$$\alpha_i = \sum a^i_{j_1 j_p} x_{j_1} x_{j_p}, \quad 1 \leqslant i \leqslant \mu, \tag{3.18}$$

where the summation is over distinct indices $j_1 \neq i$, $j_2 \neq i, \dots, j_p \neq i$ and $a^i_{j_1 j_p}$ are real constants.

For L_0, h_1, Y_1, Y_2 and Y_3, we have the following multiplication table:

	L_0	h_1	Y_1	Y_2	Y_3
L_0	0	$Y_1 = [L_0, h_1]$	$2L_0 - Y_2 = [L_0, Y_1]$	$Y_3 = [L_0, Y_2]$	$[L_0, Y_3]$
h_1	$-[L_0, h_1]$	0	$[h_1, Y_1]$	$[h_1, Y_2]$	$[h_1, Y_3]$
Y_1	$-[L_0, Y_1]$	$-[h_1, Y_1]$	0	$[Y_1, Y_2]$	$[Y_1, Y_3]$
Y_2	$-[L_0, Y_2]$	$-[h_1, Y_2]$	$-[Y_1, Y_2]$	0	$[Y_2, Y_3]$
Y_3	$-[L_0, Y_3]$	$-[h_1, Y_3]$	$-[Y_1, Y_3]$	$-[Y_2, Y_3]$	0

Since $\dim E = 6$, then there is one and only one operator, which is linearly independent of L_0, h_1, Y_1, Y_2 and Y_3, among the upper triangle operators in the above table. Notice $[h_1, Y_1]$, $[h_1, Y_2] \in U_0$, so $[h_1, Y_1]$, $[h_1, Y_2]$ must be a constant multiple of h_1. We already know $\Delta = [Y_1, Y_2]$ is a linear combination of L_0, h_1, Y_1, Y_2 and Y_3. Thus the operator, which is linearly independent of L_0, h_1, Y_1, Y_2 and Y_3, among $[L_0, Y_3] \in U_3$, $[Y_1, Y_3]$, $[Y_2, Y_3] \in U_2$, $[h_1, Y_3] \in U_1$. Let us assume $Y_4 := [L_0, Y_3]$ (if $Y_4 := [Y_1, Y_3]$, $[Y_2, Y_3]$, or $[h_1, Y_3]$, the proof is similar and easier since they do not contain $D_i D_j D_k$ terms).

We next consider

$$\begin{aligned} Y_5 &:= [L_0, Y_4] \\ &= \sum_{i,j,k,l=1}^n \frac{\partial^3\alpha_j}{\partial x_k \partial x_i \partial x_l} D_l D_k D_i D_j \\ &\quad + \sum_{i,j,k,l=1}^n \left(\frac{\partial^2\alpha_l}{\partial x_i \partial x_k}\omega_{lj} + \frac{\partial^2\alpha_j}{\partial x_l \partial x_k}\omega_{li}\right) D_k D_i D_j \mod U_2. \end{aligned}$$

Then Y_5 has to be a linear combination of L_0, h_1, Y_1, Y_2, Y_3 and Y_4, we have

$$Y_5 = \sum_{i,j,k=1}^n \left[\sum_{l=1}^n \left(\frac{\partial^2\alpha_l}{\partial x_i \partial x_k}\omega_{lj} + \frac{\partial^2\alpha_j}{\partial x_l \partial x_k}\omega_{li}\right)\right] D_k D_i D_j \mod U_2, \tag{3.19}$$

$$\frac{\partial^3\alpha_i}{\partial x_i^3} = 0, \quad 1 \leqslant i \leqslant n \ (D_i^4 \text{ terms}), \tag{3.20}$$

$$\frac{\partial^3\alpha_i}{\partial x_j^3} + 3\frac{\partial^3\alpha_j}{\partial x_i \partial x_j^2} = 0, \quad 1 \leqslant i, j \leqslant n \ (D_i^3 D_j \text{ terms}), \tag{3.21}$$

$$\frac{\partial^3\alpha_i}{\partial x_j \partial x_k^2} + \frac{\partial^3\alpha_j}{\partial x_i \partial x_k^2} + 2\frac{\partial^3\alpha_k}{\partial x_i \partial x_j \partial x_k} = 0, \quad 1 \leqslant i, j, k \leqslant n \ (D_i^2 D_j D_k \text{ terms}), \tag{3.22}$$

$$\frac{\partial^3\alpha_i}{\partial x_j \partial x_k \partial x_l} + \frac{\partial^3\alpha_j}{\partial x_k \partial x_l \partial x_i} + \frac{\partial^3\alpha_k}{\partial x_l \partial x_i \partial x_j} + \frac{\partial^3\alpha_l}{\partial x_k \partial x_i \partial x_j} = 0, \quad 1 \leqslant i, j, k, l \leqslant n \ (D_i D_j D_k D_l \text{ terms}). \tag{3.23}$$

Differentiating (3.23) with respect to x_k, together with (3.22), we get

$$\frac{\partial^4\alpha_k}{\partial x_i \partial x_j \partial x_k \partial x_l} = 0,$$

which implies

$$\frac{\partial\alpha_k}{\partial x_k} = \sum_{s,r=1}^n d^k_{sr} x_s x_r + \sum_{s=1}^n d^k_s x_s + c_k, \tag{3.24}$$

where d^k_{sr}, d^k_s, c_k are constants and $d^k_{kk} = 0$. It follows that

$$\alpha_k = \sum_{s,r=1}^n d^k_{sr} x_s x_r x_k + \sum_{s=1}^n d^k_s x_s x_k + c_k x_k + e_k,$$

where e_k is independent of x_k variable. Equations (3.20) and (3.23) imply

$$\frac{\partial^3 e_i}{\partial x_j^3} = -3(d^j_{ij} + d^j_{ji}). \tag{3.25}$$

(a) For the case $p > 3$, in view of (3.24), (3.25), we have

$$\begin{aligned} &c_k = 0, \quad \text{for } 1 \leqslant k \leqslant \mu, \\ &d^k_i, d^k_{ij} = 0, \quad \text{for } 1 \leqslant i, j, k \leqslant n, \\ &\frac{\partial^3 e_i}{\partial x_j^3} = 0, \quad \text{for } 1 \leqslant i, j \leqslant n. \end{aligned}$$

From (3.22) we have $\frac{\partial}{\partial x_j}(\frac{\partial^2 \alpha_i}{\partial x_k^2}) + \frac{\partial}{\partial x_i}(\frac{\partial^2 \alpha_j}{\partial x_k^2}) = 0$. From Lemma 3.1, we have

$$\frac{\partial^2 e_j}{\partial x_k^2} = \sum_{s=1}^{n} c_s^j x_s + d_j, \quad \text{for all } 1 \leqslant j, k \leqslant n,$$

where c_s^j, d_j are constants and $c_s^j = c_j^s$. Notice deg $(\alpha_j) > 3$, we have $c_s^j = 0$, $d_j = 0$. Therefore α_j does not contain x_i^k terms, $k > 1$. We have established (3.18) for $p > 3$.

(b) For the case $p = 3$, from (3.24) and (3.25) and the fact that e_j is independent of x_j, we have

$$\alpha_j = \sum_{r,s=1}^{n} d_{rs}^j x_r x_s x_j - \frac{1}{2}\sum_{s=1}^{n}(d_{sj}^s + d_{js}^s)x_s^3 + \sum_{r,s=1}^{n} b_{rs}^j x_r x_s^2 + \sum_{r,s,k=1}^{n} c_{rsk}^j x_r x_s x_k, \tag{3.26}$$

for some constants $d_{rs}^j, b_{rs}^j, c_{rsk}^j$ with $d_{jj}^j = 0$, $b_{rj}^j = b_{jr}^j = b_{jj}^j = 0$, $c_{jsk}^j = c_{rjk}^j = c_{rsj}^j = c_{rrk}^j = c_{rsr}^j = c_{rss}^j = 0$ $\forall 1 \leqslant r, s, k \leqslant n$, $1 \leqslant j \leqslant \mu$, $d_{rs}^j = d_{js}^s = d_{sj}^s = b_{rs}^j = c_{rsk}^j = 0$ $\forall 1 \leqslant r, s, k \leqslant n$, $\mu + 1 \leqslant j \leqslant n$. In view of

$$\alpha_j = \sum_{s=1}^{n} x_s \left(\sum_{r=1}^{n} d_{rs}^j x_r x_j - \frac{1}{2}(d_{sj}^s + d_{js}^s)x_s^2 + \sum_{r=1}^{n} b_{rs}^j x_r x_s + \sum_{r,k=1}^{n} c_{rsk}^j x_r x_k \right) = \sum_{s=1}^{n} x_s \omega_{js}$$

recall that ω_{ij}, $1 \leqslant i, j \leqslant n$ are polynomials. Then $\omega_{js}^{(2)}$, homogeneous polynomial of degree two part of ω_{js} must be of the form

$$\omega_{js}^{(2)} = \sum_{r=1}^{n} d_{rs}^j x_r x_j - \frac{1}{2}(d_{sj}^s + d_{js}^s)x_s^2 + \sum_{r=1}^{n} b_{rs}^j x_r x_s + \sum_{r,k=1}^{n} c_{rk}^{js} x_r x_k, \quad 1 \leqslant j, s \leqslant n, \tag{3.27}$$

for some constant c_{rk}^{js} with $c_{jk}^{js} = c_{rj}^{js} = c_{sk}^{js} = c_{rs}^{js} = 0$ Next, we show that $d_{js}^j + d_{sj}^j = d_{rr}^j = b_{rs}^j = 0$. Compute the coefficient $\sum_{l=1}^{n}(\frac{\partial^2 \alpha_l}{\partial x_i^2} + \frac{\partial^2 \alpha_i}{\partial x_l \partial x_i})\omega_{li}$ of the term D_i^3, $1 \leqslant i \leqslant n$ in Equation (3.19), which is a polynomial. Denote this polynomial by u_i, we have

$$u_i = \sum_{\substack{l=1\\ l\neq i}} \left[2d_{ii}^l x_l - 3(d_{il}^l + d_{li}^l)x_i + \sum_{r=1}^{n} 2b_{ri}^l x_r + \sum_{r=1}^{n}(d_{rl}^i + d_{lr}^i)x_r \right] \cdot \left[\sum_{p=1}^{n} d_{pi}^l x_p x_l - \frac{1}{2}(d_{il}^l + d_{li}^l)x_i^2 + \sum_{p=1}^{n} b_{pi}^l x_p x_i + \sum_{p,q=1}^{n} c_{pq}^{li} x_p x_q + (\omega_{ji} - \omega_{li}^{(2)}) \right]. \tag{3.28}$$

Since Y_5 is a linear combination of L_0, h_1, Y_1, Y_2, Y_3 and Y_4, and notice that the coefficient $\frac{\partial^2 \alpha_i}{\partial x_i^2}$ of D_i^3 term in Y_4 is $2\sum_{s=1}^{n} d_{is}^i x_s + \text{constant}$, which is a degree one polynomial, we have that the coefficients $\frac{3}{2}(d_{il}^l + d_{li}^l)^2$ of x_i^3, $(d_{ii}^l)^2$ of $x_i x_l^2$, and $2(b_{ri}^l)^2$ of $x_r^2 x_i$ are necessary zero. It follows that $d_{il}^l + d_{li}^l = d_{ii}^l = b_{ri}^l = 0$ for $i \neq l$. Therefore $\alpha_j = \sum_{r,s=1}^{n} d_{rs}^j x_r x_s x_j + \sum_{r,s,k=1}^{n} c_{rsk}^j x_r x_s x_k$. Therefore, α_j, $1 \leqslant j \leqslant \mu$ are of the form (3.18) with $p = 3$.

(c) For the case $p = 2$, from (3.24), (3.25) and fact that e_j is independent of x_j, we have

$$\alpha_j = \sum_{s=1}^{n} d_s^j x_s x_j - \sum_{s=1}^{n} b_s^j x_s^2 + \sum_{1 \leqslant s, r \leqslant n} c_{rs}^j x_r x_s, \quad 1 \leqslant j \leqslant n. \tag{3.29}$$

With the similarity to case $p = 3$, we will have $b_s^j = 0$. Therefore α_j, $1 \leqslant j \leqslant \mu$ are of the form (3.18) with $p = 2$.

We have shown Equation (3.18), hence

$$\frac{\partial \alpha_k}{\partial x_k} = 0, \ 1 \leqslant k \leqslant n. \tag{3.30}$$

Equation (3.17) becomes

$$E_n(g_1) - (p-1)g_1 = -\sum_{i=1}^{\mu} \left(\sum_{\substack{j_1 \neq i, \ldots, j_p \neq i \\ j_1, \ldots, j_p \text{ distinct}}} a_{j_1 \ldots j_p}^i x_{j_1} \ldots x_{j_p} \right)^2 + \lambda_2 \left(\frac{1}{2}\sum_{i=1}^{n} x_i^n \right) + \frac{\lambda_3 n}{2}, \quad \text{if } \mu > 0, \tag{3.31}$$

$$E_n(g_1) = \lambda_2 \left(\frac{1}{2}\sum_{i=1}^{n} x_i^n \right) + \frac{\lambda_3 n}{2}, \quad \text{if } \mu = 0. \tag{3.32}$$

By Theorem 2.7, g_1 is a polynomial of degree $2p$ (if $\mu > 0$) or 2 (if $\mu = 0$). By expressing g_1 as a polynomial on general form and by comparing the coefficient on both sides of (3.31) and (3.32), we get that g_1 is a polynomial of degree two in x_r for some x_r variable. Therefore, g_1 can be written in the form

$$u_2(x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n)x_r^2 + u_1(x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n)x_r + u_0(x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n),$$

where u_i are polynomials of $x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n$. Recall that $g_1 = \beta_n - \frac{1}{2}(E_n(\eta) + 2\eta)$, where $\beta_n = \sum_{i=1}^{n} \frac{\partial \alpha_i}{\partial x_i} = 0$ by (3.30). By Theorem 2.7, η is a polynomial of degree $2p$, $p \geqslant P1$ in $x_1, \ldots, x_n$. Therefore, we can write η in the form

$$\theta_2(x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n)x_r^2 + \theta_1(x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n)x_r + \theta_0(x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n),$$

where u_i are polynomials of $x_1, \dots, x_{r-1}, x_{r+1}, \dots, x_n$. It follows that

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = \sum_{j=1}^{m} h_j^2 + \eta = -c^2 \left(\sum_{i=1}^{l} x_i^2 \right)^2 + \eta$$

$$= -c^2 x_r^4 + \text{lower degree in } x_r.$$

By Theorem 2.4, we get a contradiction by taking a polynomial path $x_r = t$, $x_i = 0$ for $i \neq r$. Thus we have finished the proof Mitter conjecture for $\dim E = 6$. □

4. Structure theorem of six-dimensional estimation algebras

In this section, we shall prove the structure of six-dimensional estimation algebra. Since $\dim E = 6$, we only need to consider the following two cases by Theorem 3.4. Otherwise, if E contains more than two degree one polynomial, we will find $\dim E > 6$.

Case 1: E does not contain two degree one polynomials. By Proposition 3.2 and Theorem 3.4, we have $m = 1$ (hence $\sum_{i=1}^{m} h_i^2 = h_1^2$) and we may assume that $h_1 = x_1$. Observe that

$$[L_0, x_1] = D_1, \quad [D_1, x_1] = 1, \tag{4.1}$$

we have $\{1, x_1, D_x, L_0\} \subseteq E$. Consider

$$Y_1 = [L_0, D_1] = \sum_{i=1}^{n} \omega_{1i} D_i + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial \omega_{1i}}{\partial x_i} + \frac{1}{2} \frac{\partial \eta}{\partial x_1}. \tag{4.2}$$

From Chiou et al. (2006), we know that $\omega_{1i} \neq 0$ for some $2 \leqslant i \leqslant n$. And 1, x, D_1, L_0 and Y_1 are linearly independent.

$$\begin{aligned} Y_2 &:= [L_0, Y_1] \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \omega_{1j}}{\partial x_i} D_i D_j + \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_{1j} \omega_{ji} D_i \\ &\quad + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 \omega_{1j}}{\partial x_i^2} D_j + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_{1j} \frac{\partial \omega_{ji}}{\partial x_i} \\ &\quad + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 \omega_{1j}}{\partial x_i \partial x_j} D_i + \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^3 \omega_{1j}}{\partial x_i^2 \partial x_j} \\ &\quad + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 \eta}{\partial x_i \partial x_1} D_i + \frac{1}{4} \sum_{i=1}^{n} \frac{\partial^3 \eta}{\partial x_1 \partial x_i^2} + \frac{1}{2} \sum_{i=1}^{n} \omega_{1i} \frac{\partial \eta}{\partial x_i}. \end{aligned} \tag{4.3}$$

If $\frac{\partial \omega_{1j}}{\partial x_i} + \frac{\partial \omega_{1i}}{\partial x_j} = 0$, $\sum_{j=1}^{n} \omega_{1j} \omega_{ji} + \frac{1}{2} \frac{\partial^2 \eta}{\partial x_i \partial x_1} = 0$, in view of (4.2), Proposition 3.2 and Theorem 3.4, E is a five-dimensional estimation algebra. Consider

$$Y_3 := [L_0, Y_2] = \sum_{i,j,k=1}^{n} \frac{\partial^2 \omega_{1j}}{\partial x_i \partial x_k} D_k D_i D_j \mod U_2. \tag{4.4}$$

Since Y_3 has to be a linear combination of 1, x_1, D_1, L_0, Y_1 and Y_2, we can write

$$Y_3 = 2\lambda L_0 + C_0 x_1 + C_1 Y_1 + C_2 + C_3 D_1 + C_4 Y_2. \tag{4.5}$$

From the above equation, we have

$$\frac{\partial^2 \omega_{1i}}{\partial x_i^2} = 0, \quad 1 \leqslant i \leqslant n \ (D_i^3 \text{ terms}), \tag{4.6}$$

$$\frac{\partial^2 \omega_{1j}}{\partial x_i^2} + 2 \frac{\partial^2 \omega_{1i}}{\partial x_i \partial x_j} = 0, \quad 1 \leqslant i, j \leqslant n \ (D_i^2 D_j \text{ terms}), \tag{4.7}$$

$$\frac{\partial^2 \omega_{1k}}{\partial x_i \partial x_j} + \frac{\partial^2 \omega_{1j}}{\partial x_i \partial x_k} + \frac{\partial^2 \omega_{1i}}{\partial x_j \partial x_k} = 0, \quad 1 \leqslant i, j, k \leqslant (D_i D_j D_k \text{ terms}). \tag{4.8}$$

It follows

$$\omega_{1i} = \sum_{s=1}^{n} d_s^i x_s x_i + c_i x_i + e_i, \quad 1 \leqslant i \leqslant n, \tag{4.9}$$

$$\frac{\partial^2 e_j}{\partial x_i^2} = -2 d_j^i, \tag{4.10}$$

where d_s^i, c_i are constants and $d_i^i = 0$, e_i are independent of x_i.

It follows that

$$\begin{aligned} Y_2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \omega_{1j}}{\partial x_i} D_i D_j + \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_{1j} \omega_{ji} D_i \\ &\quad + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 \omega_{1j}}{\partial x_i^2} D_j + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_{ij} \frac{\partial \omega_{ji}}{\partial x_i} \\ &\quad + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 \omega_{1j}}{\partial x_i \partial x_j} D_i + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 \eta}{\partial x_i \partial x_1} D_i \\ &\quad + \frac{1}{4} \sum_{i=1}^{n} \frac{\partial^3 \eta}{\partial x_1 \partial x_i^2} + \frac{1}{2} \sum_{i=1}^{n} \omega_{1j} \frac{\partial \eta}{\partial x_j}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} Y_3 &= \sum_{i,j,k=1}^{n} \frac{\partial^2 \omega_{1j}}{\partial x_i \partial x_k} D_k D_i D_j + \sum_{i,j,k=1}^{n} \frac{\partial \omega_{1j}}{\partial x_i} \omega_{jk} D_i D_k \\ &\quad + \sum_{i,j,k=1}^{n} \frac{\partial \omega_{1j}}{\partial x_i} \omega_{ik} D_k D_j + \sum_{i,j,k=1}^{n} \frac{\partial \omega_{1j}}{\partial x_i} \frac{\partial \omega_{jk}}{\partial x_i} D_k \\ &\quad + \frac{1}{2} \sum_{i,j,k=1}^{n} \frac{\partial \omega_{1j}}{\partial x_i} \frac{\partial \omega_{jk}}{\partial x_k} D_i + \frac{1}{2} \sum_{i,j,k=1}^{n} \frac{\partial \omega_{1j}}{\partial x_i} \frac{\partial \omega_{ik}}{\partial x_k} D_j \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{i,j,k=1}^{n}\frac{\partial\omega_{1j}}{\partial x_i}\frac{\partial^2\omega_{jk}}{\partial x_i\partial x_k}+\sum_{i,j,k=1}^{n}\frac{\partial(\omega_{1j}\omega_{ji})}{\partial x_k}D_kD_i\\
&-\sum_{i,j,k=1}^{n}\omega_{1j}\omega_{ji}\omega_{ki}D_k+\frac{1}{2}\sum_{i,j,k=1}^{n}\frac{\partial^2(\omega_{1j}\omega_{ji})}{\partial x_k^2}D_i\\
&-\frac{1}{2}\sum_{i,j,k=1}^{n}\omega_{1j}\omega_{ji}\frac{\partial\omega_{ki}}{\partial x_k}-\frac{1}{2}\sum_{i,j,k=1}^{n}\frac{\partial^2\omega_{1j}}{\partial x_i^2}\omega_{kj}D_k\\
&-\frac{1}{4}\sum_{i,j,k=1}^{n}\frac{\partial^2\omega_{1j}}{\partial x_i^2}\frac{\partial\omega_{kj}}{\partial x_k}-\frac{1}{2}\sum_{i,j,k=1}^{n}\frac{\partial^2\omega_{1j}}{\partial x_i\partial x_j}\omega_{ki}D_k\\
&-\frac{1}{4}\sum_{i,j,k=1}^{n}\frac{\partial^2\omega_{1j}}{\partial x_i\partial x_j}\frac{\partial\omega_{ki}}{\partial x_k}+\frac{1}{2}\sum_{i,k=1}^{n}\frac{\partial^3\eta}{\partial x_k\partial x_i\partial x_1}D_kD_i\\
&-\frac{1}{2}\sum_{i,k=1}^{n}\frac{\partial^2\eta}{\partial x_i\partial x_1}\omega_{ki}D_k+\frac{1}{4}\sum_{i,k=1}^{n}\frac{\partial^4\eta}{\partial x_k^2\partial x_i\partial x_1}D_i\\
&-\frac{1}{4}\sum_{i,k=1}^{n}\frac{\partial^2\eta}{\partial x_i\partial x_1}\frac{\partial\omega_{ki}}{\partial x_k}+\frac{1}{2}\sum_{i,j,k=1}^{n}\frac{\partial}{\partial x_k}\left(\omega_{1j}\frac{\partial\omega_{ji}}{\partial x_i}\right)D_k\\
&+\frac{1}{4}\sum_{i,j,k=1}^{n}\frac{\partial^2}{\partial x_k^2}\left(\omega_{1j}\frac{\partial\omega_{ji}}{\partial x_i}\right)+\frac{1}{4}\sum_{i,k=1}^{n}\frac{\partial^4\eta}{\partial x_k\partial x_1\partial x_i^2}D_k\\
&+\frac{1}{8}\sum_{i,k=1}^{n}\frac{\partial^5\eta}{\partial x_k^2\partial x_1\partial x_i^2}+\frac{1}{2}\sum_{j,k=1}^{n}\frac{\partial}{\partial x_k}\left(\omega_{1j}\frac{\partial\eta}{\partial x_j}\right)D_k\\
&+\frac{1}{4}\sum_{j,k=1}^{n}\frac{\partial^2}{\partial x_k^2}\left(\omega_{1j}\frac{\partial\eta}{\partial x_j}\right)+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial\omega_{1j}}{\partial x_i}\frac{\partial\eta}{\partial x_i}D_j\\
&+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial\omega_{1j}}{\partial x_i}\frac{\partial\eta}{\partial x_j}D_i+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial\omega_{1j}}{\partial x_i}\frac{\partial^2\eta}{\partial x_i\partial x_j}\\
&+\frac{1}{2}\sum_{i,j=1}^{n}\omega_{1j}\omega_{ji}\frac{\partial\eta}{\partial x_i}+\frac{1}{4}\sum_{i,j=1}^{n}\frac{\partial^2\omega_{1j}}{\partial x_i^2}\frac{\partial\eta}{\partial x_j}\\
&+\frac{1}{4}\sum_{i,j=1}^{n}\frac{\partial^2\omega_{1j}}{\partial x_i\partial x_j}\frac{\partial\eta}{\partial x_j}+\frac{1}{4}\sum_{i=1}^{n}\frac{\partial^2\eta}{\partial x_i\partial x_1}\frac{\partial\eta}{\partial x_i}.
\end{aligned}\tag{4.12}
$$

By comparing the coefficient of D_1^2 term, D_i^2, $2\leqslant i\leqslant n$ terms and D_iD_j, $1\leqslant i\neq j\leqslant n$ terms from (4.5) and (4.12), we have

$$
-\frac{3}{2}\sum_{j=1}^{n}\frac{\partial\omega_{1j}^2}{\partial x_1}+\frac{1}{2}\frac{\partial^3\eta}{\partial x_1^3}=\lambda \tag{4.13}
$$

$$
\begin{aligned}
&\sum_{j=1}^{n}\frac{\partial\omega_{1j}}{\partial x_i}\omega_{ji}+\sum_{j=1}^{n}\frac{\partial\omega_{1i}}{\partial x_j}\omega_{ji}+\sum_{j=1}^{n}\frac{\partial(\omega_{1j}\omega_{ji})}{\partial x_i}+\frac{1}{2}\frac{\partial^3\eta}{\partial x_i^2\partial x_1}\\
&=\lambda+C_4\frac{\partial\omega_{1i}}{\partial x_i},\quad 2\leqslant i\leqslant n
\end{aligned}\tag{4.14}
$$

$$
\begin{aligned}
&\sum_{j=1}^{n}\left(\frac{\partial\omega_{1j}}{\partial x_i}\omega_{jk}+\frac{\partial\omega_{1j}}{\partial x_k}\omega_{ji}\right)+\sum_{j=1}^{n}\left(\frac{\partial\omega_{1k}}{\partial x_j}\omega_{ji}+\frac{\partial\omega_{1i}}{\partial x_j}\omega_{jk}\right)\\
&+\sum_{j=1}^{n}\left(\frac{\partial\omega_{1j}\omega_{jk}}{\partial x_i}+\frac{\partial\omega_{1j}\omega_{ji}}{\partial x_k}\right)+\frac{\partial^3\eta}{\partial x_i\partial x_k\partial x_1}\\
&=C_4\left(\frac{\partial\omega_{1i}}{\partial x_k}+\frac{\partial\omega_{1k}}{\partial x_i}\right),\quad 1\leqslant i<k\leqslant n.
\end{aligned}\tag{4.15}
$$

By comparing the coefficient of D_1 term, D_k, $2\leqslant k\leqslant n$ terms from (4.5) and (4.12), we have

$$
\begin{aligned}
&-\sum_{i,j=1}^{n}\left(\frac{\partial\omega_{1j}}{\partial x_i}\right)^2+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial\omega_{1j}}{\partial x_1}\frac{\partial\omega_{ji}}{\partial x_i}-\sum_{i,j=1}^{n}\omega_{1j}\omega_{ji}\omega_{1i}\\
&-\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2(\omega_{1j}^2)}{\partial x_i^2}-\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2\omega_{1j}}{\partial x_i^2}\omega_{1j}-\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2\omega_{1j}}{\partial x_i\partial x_j}\omega_{1i}\\
&-\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2\eta}{\partial x_i\partial x_1}\omega_{1i}+\frac{1}{4}\sum_{i=1}^{n}\frac{\partial^4\eta}{\partial x_1^3\partial x_i}+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_1}\left(\omega_{1j}\frac{\partial\omega_{ji}}{\partial x_i}\right)\\
&+\frac{1}{4}\sum_{i=1}^{n}\frac{\partial^4\eta}{\partial x_1^2\partial x_i^2}+\frac{1}{2}\sum_{i=1}^{n}\frac{\partial}{\partial x_1}\left(\omega_{1i}\frac{\partial\eta}{\partial x_i}\right)+\frac{1}{2}\sum_{i=1}^{n}\frac{\partial\omega_{1i}}{\partial x_1}\frac{\partial\eta}{\partial x_i}\\
&=C_3+C_4\left(-\sum_{i=1}^{n}(\omega_{1i})^2+\frac{1}{2}\sum_{j=1}^{n}\frac{\partial^2\omega_{1j}}{\partial x_i\partial x_j}+\frac{1}{2}\frac{\partial^2\eta}{\partial x_1^2}\right).
\end{aligned}\tag{4.16}
$$

$$
\begin{aligned}
&\sum_{i,j=1}^{n}\frac{\partial\omega_{1j}}{\partial x_i}\frac{\partial\omega_{jk}}{\partial x_i}+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial\omega_{1j}}{\partial x_k}\frac{\partial\omega_{ji}}{\partial x_i}+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial\omega_{1k}}{\partial x_i}\frac{\partial\omega_{ij}}{\partial x_j}\\
&-\sum_{i,j=1}^{n}\omega_{1j}\omega_{ji}\omega_{ki}+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2(\omega_{1j}\omega_{jk})}{\partial x_i^2}-\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2\omega_{1j}}{\partial x_i^2}\omega_{kj}\\
&-\frac{1}{4}\sum_{i,j=1}^{n}\frac{\partial^2\omega_{1j}}{\partial x_i\partial x_j}\omega_{ki}-\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2\eta}{\partial x_i\partial x_1}\omega_{ki}+\frac{1}{4}\sum_{i=1}^{n}\frac{\partial^4\eta}{\partial x_k\partial x_i^2\partial x_1}\\
&+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_k}\left(\omega_{1j}\frac{\partial\omega_{ji}}{\partial x_i}\right)+\frac{1}{4}\sum_{i=1}^{n}\frac{\partial^4\eta}{\partial x_1\partial x_k\partial x_i^2}\\
&+\frac{1}{2}\sum_{i=1}^{n}\frac{\partial}{\partial x_k}\left(\omega_{1i}\frac{\partial\eta}{\partial x_i}\right)+\frac{1}{2}\sum_{i=1}^{n}\frac{\partial\omega_{1k}}{\partial x_i}\frac{\partial\eta}{\partial x_i}+\frac{1}{2}\sum_{i=1}^{n}\frac{\partial\omega_{1i}}{\partial x_k}\frac{\partial\eta}{\partial x_i}\\
&=C_1\omega_{1k}+C_4\left(\sum_{i=1}^{n}\omega_{1i}\omega_{ik}+\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2\omega_{1k}}{\partial x_i^2}+\frac{1}{2}\sum_{j=1}^{n}\frac{\partial^2\omega_{1j}}{\partial x_k\partial x_j}\right.\\
&\left.+\frac{1}{2}\frac{\partial^2\eta}{\partial x_1\partial x_k}\right),\quad 2\leqslant k\leqslant n.
\end{aligned}\tag{4.17}
$$

By comparing the zero-order differential operators on both sides of (4.5), we get

$$
\begin{aligned}
&\frac{1}{2}\sum_{i,j,k=1}^{n}\frac{\partial\omega_{1j}}{\partial x_i}\frac{\partial^2\omega_{jk}}{\partial x_i\partial x_k}-\frac{1}{2}\sum_{i,j,k=1}^{n}\omega_{1j}\omega_{ji}\frac{\partial\omega_{ki}}{\partial x_k}\\
&-\frac{1}{4}\sum_{i,j,k=1}^{n}\frac{\partial^2\omega_{1j}}{\partial x_i^2}\frac{\partial\omega_{kj}}{\partial x_k}-\frac{1}{4}\sum_{i,j,k=1}^{n}\frac{\partial^2\omega_{1j}}{\partial x_i\partial x_j}\frac{\partial\omega_{ki}}{\partial x_k}\\
&-\frac{1}{4}\sum_{i,k=1}^{n}\frac{\partial^2\eta}{\partial x_i\partial x_1}\frac{\partial\omega_{ki}}{\partial x_k}+\frac{1}{4}\sum_{i,j,k=1}^{n}\frac{\partial^2}{\partial x_k^2}\left(\omega_{1j}\frac{\partial\omega_{ji}}{\partial x_i}\right)\\
&+\frac{1}{8}\sum_{i,k=1}^{n}\frac{\partial^5\eta}{\partial x_k^2\partial x_1\partial x_i^2}+\frac{1}{4}\sum_{j,k=1}^{n}\frac{\partial^2}{\partial x_k^2}\left(\omega_{1j}\frac{\partial\eta}{\partial x_j}\right)\\
&+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial\omega_{1j}}{\partial x_i}\frac{\partial^2\eta}{\partial x_i\partial x_j}+\frac{1}{2}\sum_{i,j=1}^{n}\omega_{1j}\omega_{ji}\frac{\partial\eta}{\partial x_i}+\frac{1}{4}\sum_{i,j=1}^{n}\frac{\partial^2\omega_{1j}}{\partial x_i^2}\frac{\partial\eta}{\partial x_j}
\end{aligned}
$$

$$+\frac{1}{4}\sum_{i,j=1}^{n}\frac{\partial^2\omega_{1j}}{\partial x_i\partial x_j}\frac{\partial\eta}{\partial x_j}+\frac{1}{4}\sum_{i,j=1}^{n}\frac{\partial^2\eta}{\partial x_i\partial x_1}\frac{\partial\eta}{\partial x_i}$$
$$+\sum_{j=1}^{n}\sum_{1\leqslant i<k\leqslant n}\frac{\partial\omega_{1j}}{\partial x_i}\omega_{jk}\omega_{ki}+\sum_{i=1}^{n}\sum_{1\leqslant k<j\leqslant n}\frac{\partial\omega_{1j}}{\partial x_i}\omega_{ik}\omega_{kj}$$
$$+\sum_{j=1}^{n}\sum_{1\leqslant k<i\leqslant n}\frac{\partial(\omega_{1j}\omega_{ji})}{\partial x_k}\omega_{ik}+\frac{1}{2}\sum_{1\leqslant k<i\leqslant n}\frac{\partial^3\eta}{\partial x_k\partial x_i\partial x_1}\omega_{ik}$$
$$=-\lambda\eta+C_0x_1+C_1\left(\frac{1}{2}\sum_{i=1}^{n}\frac{\partial\omega_{1i}}{\partial x_i}+\frac{1}{2}\frac{\partial\eta}{\partial x_1}\right)+C_2+C_4$$
$$\cdot\left(\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\omega_{1j}\frac{\partial\omega_{ji}}{\partial x_i}+\frac{1}{4}\sum_{i=1}^{n}\frac{\partial^3\eta}{\partial x_1\partial x_i^2}\right.$$
$$\left.+\frac{1}{2}\sum_{i=1}^{n}\omega_{1i}\frac{\partial\eta}{\partial x_i}+\sum_{1\leqslant i<j\leqslant n}\frac{\partial\omega_{1j}}{\partial x_i}\omega_{ji}\right). \quad (4.18)$$

Case 2: E contains two degree one polynomials. Then $E=\{L_0,x_1,D_1,1,\sum_{j=2}^n a_jx_j,\sum_{j=2}^n a_jD_j\}$. Consider $Y_1:=[L_0,D_1]$ and $\widetilde{Y}_1:=[L_0,\sum_{j=2}^n a_jD_j]$, which must be linear combination of the six elements:

$$Y_1:=[L_0,D_1]=\sum_{j=2}^{n}\omega_{1j}D_j+\frac{1}{2}\sum_{j=2}^{n}\frac{\partial\omega_{1j}}{\partial x_j}+\frac{1}{2}\frac{\partial\eta}{\partial x_1}$$
$$=\lambda L_0+C_0x_1+C_1D_1+C_2$$
$$+C_3\sum_{j=2}^{n}a_jx_j+C_4\sum_{j=2}^{n}a_jD_j, \quad (4.19)$$

$$\widetilde{Y}_1:=\left[L_0,\sum_{j=2}^{n}a_jD_j\right]=\sum_{i=1}^{n}\sum_{j=2}^{n}\omega_{ji}D_i+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=2}^{n}\frac{\partial\omega_{ji}}{\partial x_i}$$
$$+\frac{1}{2}\sum_{j=2}^{n}\frac{\partial\eta}{\partial x_j}$$
$$=\widetilde{\lambda}L_0+\widetilde{C}_0x_1+\widetilde{C}_1D_1+\widetilde{C}_2+\widetilde{C}_3\sum_{j=2}^{n}a_jx_j$$
$$+\widetilde{C}_4\sum_{j=2}^{n}a_jD_j. \quad (4.20)$$

Then we have $\lambda=\widetilde{\lambda}=C_1=0$, $\widetilde{C}_1=\sum_{j=2}^n\omega_{j1}$, $\omega_{1j}=C_4a_j$, $2\leqslant j\leqslant n$, $\sum_{j=2}^n\omega_{ji}=\widetilde{C}_4a_i$, $2\leqslant i\leqslant n$. Then we have $\frac{1}{2}\sum_{j=2}^n\frac{\partial\omega_{1j}}{\partial x_j}=0$, $\frac{1}{2}\sum_{i=1}^n\sum_{j=2}^n\frac{\partial\omega_{ji}}{\partial x_i}=0$. So $\frac{1}{2}\frac{\partial\eta}{\partial x_1}$ is a linear combination of x_1, 1 and $\sum_{j=2}^n a_jx_j$:

$$\frac{1}{2}\frac{\partial\eta}{\partial x_1}=C_0x_1+C_2\sum_{j=2}^{n}a_jx_j+C_3. \quad (4.21)$$

And $\frac{1}{2}\sum_{j=2}^n\frac{\partial\eta}{\partial x_j}$ is also a linear combination of x_1, 1 and $\sum_{j=2}^n a_jx_j$:

$$\frac{1}{2}\sum_{j=2}^{n}\frac{\partial\eta}{\partial x_j}=\widetilde{C}_0x_1+\widetilde{C}_2\sum_{j=2}^{n}a_jx_j+\widetilde{C}_3. \quad (4.22)$$

Summarising what we have shown above, we have the following structure theorem for six-dimensional estimation algebras.

Theorem 4.1: *Suppose that the state-space of the filtering model* (2.1) *is of dimension at least two. Then the six-dimensional estimation algebra is isomorphic to a Lie algebra generated by*

(i) L_0 *and* x_1, *with a basis given by*

$$1,x_1,D_1=\frac{\partial}{\partial x_1}-f_1,\quad Y_1=[L_0,D_1],\quad Y_2=[L_0,Y_1]$$

and L_0, *and*:

(i.1) $\omega_{1i}\neq 0$ *for some* $2\leqslant i\leqslant n$ *and each* ω_{1i} *is of the form*

$$\omega_{1i}=\sum_{s=1}^{n}d_s^ix_sx_i+c_ix_i+e_i,\quad 1\leqslant i\leqslant n, \quad (4.23)$$

$$\frac{\partial^2e_j}{\partial x_i^2}=-2d_j^i, \quad (4.24)$$

where d_s^i, c_i *are constants and* $d_i^i=0$, e_i *are independent of* x_i.

(i.2) *There exist constants* λ, C_0, C_1, C_2, C_3, C_4 *such that Equations* (4.17)–(4.22) *are satisfied.*

Or,

(ii) L_0, x_1 *and* $\sum_{j=2}^n a_jx_j$, *with a basis given by* $L_0,x_1,D_1,1,\sum_{j=2}^n a_jx_j,\sum_{j=2}^n a_jD_j$, *and*:

(ii.1) $\omega_{1j}=C_4a_j, 2\leqslant j\leqslant n$, $\sum_{j=2}^n\omega_{ji}=\widetilde{C}_4a_i$, $2\leqslant i\leqslant n$, $\frac{1}{2}\sum_{j=2}^n\frac{\partial\omega_{1j}}{\partial x_j}=0$, *where* C_4 *and* $\widetilde{C}_4$ *are constants. And* $\frac{1}{2}\sum_{i=1}^n\sum_{j=2}^n\frac{\partial\omega_{ji}}{\partial x_i}=0$.

(ii.2) *There exist constants* $C_0, C_2, C_3, \widetilde{C}_0, \widetilde{C}_2$ *and* $\widetilde{C}_3$ *such that*

$$\frac{1}{2}\frac{\partial\eta}{\partial x_1}=C_0x_1+C_2\sum_{j=2}^{n}a_jx_j+C_3, \quad (4.25)$$

$$\frac{1}{2}\sum_{j=2}^{n}\frac{\partial\eta}{\partial x_j}=\widetilde{C}_0x_1+\widetilde{C}_2\sum_{j=2}^{n}a_jx_j+\widetilde{C}_3. \quad (4.26)$$

Example 1: Consider the filtering model

$$\begin{cases}\mathrm{d}x(t)=f(x(t))\mathrm{d}t+g(x(t))\mathrm{d}v(t),\\ \mathrm{d}y(t)=h(x(t))\mathrm{d}t+\mathrm{d}w(t),\end{cases}$$

where

$$f_1=a-\frac{3}{\sqrt{2}}x_1,$$
$$f_2=b+x_1+2x_3,$$

$$f_3 = c - x_2,$$
$$f_i = g_i(x_4, \ldots, x_n), \quad 4 \leqslant i \leqslant n,$$
$$h(x) = x_1,$$

and a, b, c are constants. Then

$$\omega_{12} = 1, \quad \omega_{13} = 0, \quad \omega_{23} = -3, \quad \omega_{1j} = 0, \quad 4 \leqslant i \leqslant n,$$
$$\sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} = \left(a - \frac{3}{\sqrt{2}} x_1\right)^2 + (b + x_1 + 2x_3)^2 + (c - x_2)^2 - \frac{3}{\sqrt{2}}.$$

It is easy to check that $\lambda = 0$, $C_0 = -\frac{33}{2}$, $C_1 = -\frac{7}{2}$, $C_2 = C_3 = C_4 = 0$. The estimation algebra E is six-dimensional with basis $\{1, x_1, D_1, Y_1 = D_2 - \frac{\sqrt{3}}{2} a + b + \frac{13}{2} x_1 + 2x_3,\ Y_2 = -D_3 - c + x_2, L_0\}$.

Example 2: Consider the filtering model

$$\begin{cases} \mathrm{d}x(t) = f(x(t))\mathrm{d}t + g(x(t))\mathrm{d}v(t), \\ \mathrm{d}y(t)\ h(x(t))\mathrm{d}t + \mathrm{d}w(t), \end{cases}$$

where

$$f_1 = x_1,$$
$$f_2 = ax_2 - dx_2^2 + bx_3 + 2dx_2x_3 - \frac{1}{2}(1 + 2d)x_3^2 + cx_4 + x_3x_4 - \frac{1}{2}x_4^2,$$
$$f_3 = bx_2 + dx_2^2 + (a + c - e)x_3 - (1 + 2d)x_2x_3 + \frac{1}{2}(1 + 2d - 2k)x_3^2 + ex_4 + x_2x_4 + 2kx_3x_4 + \left(\frac{1}{2} + 2d - 4d^2 - k\right)x_4^2,$$
$$f_4 = cx_2 + ex_3 + x_2x_3 + kx_3^2 + (a + b - e)x_4 - x_2x_4 + (1 + 4d - 8d^2 - 2k)x_3x_4 + (1 + k)x_4^2,$$
$$h_1(x) = x_1, \quad h_2(x) = x_2 + x_3 + x_4,$$

and a, b, c, d, e, k are constants. Then

$$\omega_{ij} = 0, \quad 1 \leqslant i \leqslant 4, 1 \leqslant i \leqslant 4,$$
$$\frac{1}{2}\frac{\partial \eta}{\partial x_1} = 2x_1,$$
$$\frac{1}{2}\sum_{j=2}^{n} \frac{\partial \eta}{\partial x_j} = (a + b + c)^2(x_2 + x_3 + x_4).$$

It is easy to check that $\lambda = C_1 = C_2 = C_3 = C_4 = \widetilde{\lambda} = \widetilde{C}_0 = \widetilde{C}_1 = \widetilde{C}_2 = \widetilde{C}_4 = 0, C_0 = 2, \quad \widetilde{C}_3 = (a + b + c)^2$. The estimation algebra E is six-dimensional with basis $\{1,\ x_1,\ D_1,\ x_2 + x_3 + x_4,\ D_2 + D_3 + D_4,\ L_0\}$.

5. Conclusions

Despite the success of the classification of finite-dimensional estimation algebras with maximal rank, the problem of classification of non-maximal rank finite-dimensional estimation algebras is still open except for the case of state-space dimension two which is finished by Wu and Yau (2006). Due the difficulty of the problem, Brockett suggested that one should understand the low-dimensional estimation algebras. Rasoulian and Yau (1999) and Chiou et al. (2006), have classified estimation algebras at most five. In this article, we have given a structure theorem for estimation algebras of dimension six. Note that there are two structures for estimation algebras. We have also provided two families of six-dimensional estimation algebras. In the first example, the filtering model is the same as those filtering models of Rasoulian and Yau. In the second example, the filtering model is of Yau-type, filtering model with nonlinear observations and drift terms equal to gradient vector field plus affine vector field. In the future work, we shall investigate whether there are more new classes of finite-dimensional estimation algebras, and new method to discuss the structure for low-dimensional estimation algebras.

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