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HERMITE SPECTRAL METHOD WITH HYPERBOLIC CROSS APPROXIMATIONS TO HIGH-DIMENSIONAL PARABOLIC PDES*

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Dedicated to Professor Peter Caines on the occasion of his 68th birthday

Abstract. It is well known that the sparse grid algorithm has been widely accepted as an efficient tool to overcome the "curse of dimensionality" in some degree. In this note, we first give the error estimate of hyperbolic cross (HC) approximations with generalized Hermite functions. The exponential convergence in both regular and optimized HC approximations has been shown. Moreover, the error estimate of Hermite spectral method to high-dimensional linear parabolic PDEs with HC approximations has been investigated in the properly weighted Korobov spaces. The numerical result verifies the exponential convergence of this approach.

Key words. hyperbolic cross, Hermite spectral method, high-dimensional parabolic PDEs, convergence rate

AMS subject classifications. 65N35, 65N22, 35K10

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1. Introduction.

Our study is motivated by solving the conditional density function of the states of certain nonlinear filtering. The conditional density function satisfies a linear parabolic PDE, which comes from the robust Duncan–Mortensen–Zakai equation after some exponential transformation; see [18], [30]. We need to solve this equation in $\mathbb{R}^d$, since the states lived in the whole space, where d is the number of the states. Moreover, the real-time solution is expected in the filtering problems, so it is natural to adopt the spectral methods. Among the existing literature, the Hermite and Laguerre spectral methods are the commonly used approaches based on orthogonal polynomials in infinite interval, referring to [7], [29]. Although the Hermite spectral method (HSM) appears to be a natural choice, it is not commonly used as Chebyshev and Fourier spectral methods, due to its poor resolution (see [8]) and the lack of fast algorithm for the transformation (see [3]). However, it is shown in [2] that an appropriately chosen scaling factor could greatly improve the resolution. Some further investigations on the scaling factor can be found in [28] and also in Chapter 7 [24]. Moreover, recently a guideline of choosing the suitable scaling factors for Gaussian/super-Gaussian functions is described in [19], as well as the application of HSM to 1-dim forward Kolmogorov equation.

Nevertheless, the number of the states is generally greater than one. Taking the target tracking problem in 3-dimensions as an example, there are at least six states involved in this system (three for position, three for velocity). That is, we need to solve a linear parabolic PDE in $\mathbb{R}^6$. Naively, if we implement the spectral method

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with tensor product formulation and assume the first N modes need to be computed in each direction, then the total amount of the computation is N^6. Even if with moderately small N, it is still not within the reasonable computing capacity. This is the so-called "curse of dimensionality." An efficient tool to reduce this effect is the sparse grids approximations from Smolyak's algorithm [27], which is based on a hierarchy of one-dimensional quadrature. It has a potential to obtain higher rates of convergence than many existing methods, under certain regularity conditions. For example, the convergence rate of Monte Carlo simulations are $\mathcal{O}(N^{-\frac{1}{2}})$ with N sample points, while the sparse grids from [27] achieve $\mathcal{O}(N^{-r}(\log N)^{(d-1)(r+1)})$, under the condition that the function has bounded mix derivatives of order r. The studies of sparse grids start from the basis functions in the physical spaces: piecewise linear multiscale bases [5], wavelets [5], [22]. In the most recent decade, the hyperbolic cross (HC) approximation in the frequency space has also been investigated with various basis functions: Fourier series [10], [12], polynomial approximations generated from the Chebyshev–Gauss–Lobatto points [1], Jacobi polynomials [25].

Although the regular hyperbolic cross (RHC) approximation (2.23) reduces the effect of the "curse of dimensionality" in some degree, the convergence rate is still deteriorated slowly with the dimension increasing (noting the term $(\log N)^{(d-1)(r+1)}$ in the previous paragraph). To completely break the "curse of dimensionality", the optimized hyperbolic cross (OHC) approximation (2.38) is introduced in [12]. It has been shown in [17] that the convergence rate of the OHC approximation with $\gamma \in (0,1)$ (see the definition in (2.37)) with Fourier series is of $\mathcal{O}(N^{-r})$ in our notation, where the dimension enters the constant in front. The first purpose of this paper is to establish the error estimate for the HC approximations with the generalized Hermite functions in the weighted Korobov spaces $\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)$; see (2.25). In particular, we obtain the following results for the RHC/OHC approximation with the generalized Hermite functions.

THEOREM 1.1. *For any* $u \in \mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)$, $0 \le l < m$ *(and* $0 < \gamma \le \frac{l}{m}$*),*

$$\inf_{U_N \in X_N (or\ X_{N,\gamma})} ||u - U_N||_{\mathcal{K}^l_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d) (or\ \mathcal{W}^l_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d))} \le C N^{\frac{l-m}{2}} |u|_{\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)} \quad \forall\, 0 \le l < m,$$

where C is some constant depending on $\boldsymbol{\alpha}$, l, m, and d (or γ), X_N (or $X_{N,\gamma}$) is defined in (2.23) *(or* (2.38)*),* $\mathcal{W}^l_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)$ *and* $\mathcal{K}^l_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)$ *are the Sobolev-type spaces* (2.18) *and the weighted Korobov spaces* (2.25)*, respectively.*

We follow the error analysis developed in [25] to show Theorem 1.1. But it is necessary to point out that there is a gap in the proof of Theorem 2.3 [25]. We circumvent this by more delicate analysis.

We are also interested in the dimensional adaptive HC approximation. The following error estimate is obtained with respect to the dependence of dimensions.

THEOREM 1.2 (see THEOREM B.1). *For any* $u \in \mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)$*, for* $0 < l \le m$*, we have*

$$\inf_{U_N \in X_N} |u - U_N|_{W^l_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)} \lesssim |\boldsymbol{\alpha}|^{l-m}_\infty \left(N_1^{l-m} + N_2^{\frac{1-\gamma}{d-d_1-\gamma}(l-m)} \right)^{\frac{1}{2}} |u|_{\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)},$$

where X_N is defined in (B.2)*, γ is in the definition of OHC* (2.37)*, and N_1, d_1, N_2 are clarified in* (B.1).

To avoid the distraction of our main results, we leave the detailed proof of this theorem in Appendix B.

The second purpose of this paper is to study the application of the Galerkin-type HSM with the HC approximation to high-dimensional linear parabolic PDEs. The error estimates in appropriate weighted Korobov spaces are investigated under various conditions (cf. conditions (C_1)–(C_6) in section 3). There also exist rich literature of the applications of sparse grid algorithms to solve equations. It has already been successfully applied to problems from the integral equations [14], to interpolation and approximation [16], to the stochastic differential equations [23], [20], to high dimensional integration problems from physics and finance [9], and to the solutions to elliptic PDEs [31], [26]. As for the parabolic PDEs, they are treated with a wavelet-based sparse grid discretization in [21]. Besides the finite element approaches, they are also handled with finite differences on sparse grids [11] and finite volume schemes [15]. Griebel and Oeltz [13] proposed a space-time sparse grid technique, where the tensor product of one-dimensional multilevel basis in time and a proper multilevel basis in space has been employed. To the best of our knowledge, it is the first time in this paper that the Galerkin HSM with sparse grids algorithm is applied to parabolic PDEs, and the error estimates are obtained in the appropriate spaces.

THEOREM 1.3. *Assume that conditions (C_1)–(C_3) are satisfied, and the solution to* (3.1) $u \in L^\infty(0,T;\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)) \cap L^2(0,T;\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d))$*, for* $m > 1$*. Let* u_N *be the approximate solution obtained by HSM* (3.3)*, then*

$$||u - u_N||(t) \lesssim c^* N^{\frac{1-m}{2}},$$

where c^* *depends on* $\boldsymbol{\alpha}$*, the norms of* $L^2(0,T;\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d))$*, and* $L^\infty(0,T;\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d))$*.*

THEOREM 1.4. *Assume that conditions (C_3)–(C_6) are satisfied and the solution to* (3.1) $u \in L^2(0,T;\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d))$*, for some integer* $m > \max\{|\boldsymbol{\gamma}|_1, |\boldsymbol{\delta}|_1 + 1\}$ *($\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ are two parameters in condition (C_6)), and* u_N *is the approximate solution obtained by HSM* (3.3)*, then*

$$||u - u_N||(t) \lesssim c^\sharp N^{\frac{\max\{|\boldsymbol{\gamma}|_1, |\boldsymbol{\delta}|_1+1\}-m}{2}},$$

where $c^\sharp$ *depends on* $\boldsymbol{\alpha}$*,* T *and the norm of* $L^2(0,T;\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d))$*.*

This paper is organized as following. The error analysis of the HC approximations with generalized Hermite functions is in section 2. Section 3 is devoted to the error estimate of HSM with HC approximation applying to linear parabolic PDE in suitable spaces under certain conditions. Finally, in section 4, the numerical experiment has been included to verify the exponential convergence of the HSM with the HC approximation to PDE. In the appendices, the error analysis of the full grid approximation and the dimensional adaptive HC approximation with generalized Hermite function are illustrated in detail.

2. Hyperbolic cross approximation with generalized Hermite functions.

2.1. Notations. Let us first clarify the notations to be used throughout this paper.

- Let $\mathbb{R}$ (resp., $\mathbb{N}$) denote all the real numbers (resp., natural numbers), and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
- For any $d \in \mathbb{N}$, we use boldface lowercase letters to denote d-dimensional multi-indices and vectors, e.g., $\boldsymbol{k} = (k_1, k_2, \ldots, k_d) \in \mathbb{N}_0^d$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{R}^d$.

- Let $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{N}^d$, and let $\boldsymbol{e}_i = (0, \dots, 1, \dots, 0)$ be the ith unit vector in $\mathbb{R}^d$. For any scalar $s \in \mathbb{R}$, we define the componentwise operations

$$\boldsymbol{\alpha} \pm \boldsymbol{k} = (\alpha_1 \pm k_1, \dots, \alpha_d \pm k_d), \quad \boldsymbol{\alpha} \pm s := \boldsymbol{\alpha} \pm s\mathbf{1} = (\alpha_1 \pm s, \dots, \alpha_d \pm s),$$
$$\frac{1}{\boldsymbol{\alpha}} = \left(\frac{1}{\alpha_1}, \dots, \frac{1}{\alpha_d}\right), \quad \boldsymbol{\alpha}^{\boldsymbol{k}} = \alpha_1^{k_1} \cdots \alpha_d^{k_d},$$

  and

$$\boldsymbol{\alpha} \geq \boldsymbol{k} \Leftrightarrow \alpha_j \geq k_j \quad \forall 1 \leq j \leq d; \quad \boldsymbol{\alpha} \geq s \Leftrightarrow \alpha_j \geq s \quad \forall 1 \leq j \leq d.$$

- The frequently used norms are denoted as

$$|\boldsymbol{k}|_1 = \sum_{j=1}^{d} k_j; \quad |\boldsymbol{k}|_\infty = \max_{1 \leq j \leq d} k_j; \quad |\boldsymbol{k}|_{\text{mix}} = \prod_{j=1}^{d} \bar{k}_j,$$

  where $\bar{k}_j = \max\{1, k_j\}$.
- Given a multivariate function $u(\boldsymbol{x})$, we denote, the $\boldsymbol{k}$th mixed partial derivative by

$$\boldsymbol{\partial}_{\boldsymbol{x}}^{\boldsymbol{k}} u = \frac{\partial^{|\boldsymbol{k}|_1} u}{\partial x_1^{k_1} \cdots \partial x_d^{k_d}} = \partial_{x_1}^{k_1} \cdots \partial_{x_d}^{k_d} u.$$

  In particular, we denote $\boldsymbol{\partial}_{\boldsymbol{x}}^{s} u = \boldsymbol{\partial}_{\boldsymbol{x}}^{s\mathbf{1}} u = \boldsymbol{\partial}_{\boldsymbol{x}}^{(s,s,\dots,s)} u$.
- Let $L^2(\mathbb{R}^d)$ be the Lebesgue space in $\mathbb{R}^d$, equipped with the norm $\|\cdot\| = \left(\int_{\mathbb{R}^d} |\cdot|^2 d\boldsymbol{x}\right)^{\frac{1}{2}}$ and the scalar product $\langle \cdot, \cdot \rangle$.
- We follow the convention in the asymptotic analysis, $a \sim b$ means that there exist some constants $C_1, C_2 > 0$ such that $C_1 a \leq b \leq C_2 a$; $a \lesssim b$ means that there exists some constant $C_3 > 0$ such that $a \leq C_3 b$; $N \gg 1$ means that N is sufficiently large.
- We denote C as some generic positive constant, which may vary from line to line.

2.2. Generalized Hermite functions and its properties. Recall that the univariate physical Hermite polynomials $H_n(x)$ are given by $H_n(x) = (-1)^n e^{x^2} \partial_x^n e^{-x^2}$, $n \geq 0$. Two well-known and useful facts of Hermite polynomials are the mutualy orthogonality with respect to the weight $w(x) = e^{-x^2}$ and the three-term recurrence, i.e.,

$$H_0 \equiv 1, \quad H_1(x) = 2x, \quad \text{and} \quad H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x). \tag{2.1}$$

It is studied in [28] that the scaling and translating factors are crucial to the resolution of Hermite functions. And the necessity of the translating factor is discussed in [19]. Let us define the generalized Hermite functions as

$$\mathcal{H}_n^{\alpha,\beta}(x) = \left(\frac{\alpha}{2^n n! \sqrt{\pi}}\right)^{\frac{1}{2}} H_n(\alpha(x-\beta)) e^{-\frac{1}{2}\alpha^2 (x-\beta)^2}, \tag{2.2}$$

for $n \geq 0$, where $\alpha > 0$ is the scaling factor, and $\beta \in \mathbb{R}$ is the translating factor. The

Hermite functions readily derive the following properties for (2.2):

- The $\{\mathcal{H}_n^{\alpha,\beta}\}_{n\in\mathbb{N}_0}$ forms an orthonormal basis of $L^2(\mathbb{R})$, i.e.,

$$\int_{\mathbb{R}} \mathcal{H}_n^{\alpha,\beta}(x)\mathcal{H}_m^{\alpha,\beta}(x)dx = \delta_{nm}, \tag{2.3}$$

  where δ_{nm} is the Kronecker function.
- $\mathcal{H}_n^{\alpha,\beta}(x)$ is the nth eigenfunction of the following Strum–Liouville problem:

$$e^{\frac{1}{2}\alpha^2(x-\beta)^2}\partial_x(e^{-\alpha^2(x-\beta)^2}\partial_x(e^{\frac{1}{2}\alpha^2(x-\beta)^2}u(x))) + \lambda_n u(x) = 0, \tag{2.4}$$

  with the corresponding eigenvalue $\lambda_n = 2\alpha^2 n$.
- By convention, $\mathcal{H}_n^{\alpha,\beta} \equiv 0$, for $n < 0$. For $n \geq 0$, the three-term recurrence is inherited from the Hermite polynomials:

$$2\alpha^2(x-\beta)\mathcal{H}_n^{\alpha,\beta}(x) = \sqrt{\lambda_n}\mathcal{H}_{n-1}^{\alpha,\beta}(x) + \sqrt{\lambda_{n+1}}\mathcal{H}_{n+1}^{\alpha,\beta}(x). \tag{2.5}$$

- The derivative of $\mathcal{H}_n^{\alpha,\beta}(x)$ is explicitly expressed, namely

$$\partial_x\mathcal{H}_n^{\alpha,\beta}(x) = \frac{1}{2}\sqrt{\lambda_n}\mathcal{H}_{n-1}^{\alpha,\beta}(x) - \frac{1}{2}\sqrt{\lambda_{n+1}}\mathcal{H}_{n+1}^{\alpha,\beta}(x). \tag{2.6}$$

- Let $\mathcal{D}_x = \partial_x + \alpha^2(x-\beta)$. Then

$$\mathcal{D}_x^k\mathcal{H}_n^{\alpha,\beta}(x) = \sqrt{\mu_{n,k}}\mathcal{H}_{n-k}^{\alpha,\beta}(x) \quad \forall\, n \geq k \geq 1, \tag{2.7}$$

  where

$$\mu_{n,k} = \prod_{j=0}^{k-1}\lambda_{n-j} = \frac{2^k\alpha^{2k}n!}{(n-k)!} \quad \forall\, n \geq k \geq 1. \tag{2.8}$$

- The orthogonality of $\{\mathcal{D}_x^k\mathcal{H}_n^{\alpha,\beta}(x)\}_{n\in\mathbb{N}_0}$ holds, i.e.,

$$\int_{\mathbb{R}} \mathcal{D}_x^k\mathcal{H}_n^{\alpha,\beta}(x)\mathcal{D}_x^k\mathcal{H}_m^{\alpha,\beta}(x)dx = \mu_{n,k}\delta_{nm}. \tag{2.9}$$

For notational convenience, we extend $\mu_{n,k}$ in (2.8) for all $n, k \in \mathbb{N}_0$:

$$\mu_{n,k} = \begin{cases} 1 & \text{if } n \geq k, k = 0, \\ 0 & \text{if } k > n \geq 0. \end{cases} \tag{2.10}$$

Now we define the d-dimensional tensorial generalized Hermite functions as

$$\boldsymbol{\mathcal{H}}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{x}) = \prod_{j=1}^{d}\mathcal{H}_{n_j}^{\alpha_j,\beta_j}(x_j),$$

for $\boldsymbol{\alpha} > 0$, $\boldsymbol{\beta} \in \mathbb{R}^d$, and $\boldsymbol{x} \in \mathbb{R}^d$. It verifies readily that the properties (2.7)–(2.9) can be extended correspondingly to multivariate generalized Hermite functions. Let $\boldsymbol{\mathcal{D}}_{\boldsymbol{x}}^{\boldsymbol{k}} = \mathcal{D}_{x_1}^{k_1}\cdots\mathcal{D}_{x_d}^{k_d}$, then

$$\boldsymbol{\mathcal{D}}_{\boldsymbol{x}}^{\boldsymbol{k}}\boldsymbol{\mathcal{H}}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} = \sqrt{\boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{k}}}\boldsymbol{\mathcal{H}}_{\boldsymbol{n}-\boldsymbol{k}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}, \tag{2.11}$$

and

$$\int_{\mathbb{R}^d} \boldsymbol{\mathcal{D}}_{\boldsymbol{x}}^{\boldsymbol{k}} \boldsymbol{\mathcal{H}}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{x}) \boldsymbol{\mathcal{D}}_{\boldsymbol{x}}^{\boldsymbol{k}} \boldsymbol{\mathcal{H}}_{\boldsymbol{m}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{x}) d\boldsymbol{x} = \boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{k}} \boldsymbol{\delta}_{\boldsymbol{nm}}, \tag{2.12}$$

for $\boldsymbol{\alpha} > 0$, $\boldsymbol{\beta} \in \mathbb{R}^d$, where

$$\boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{k}} = \prod_{j=1}^{d} \mu_{n_j,k_j} \quad \text{and} \quad \boldsymbol{\delta}_{\boldsymbol{nm}} = \prod_{j=1}^{d} \delta_{n_j m_j}. \tag{2.13}$$

Here, $\mu_{\cdot,\cdot}$ is defined in (2.8) and (2.10), and $\boldsymbol{\delta}_{\boldsymbol{nm}}$ is the tensorial Kronecker function.

The generalized Hermite functions $\{\boldsymbol{\mathcal{H}}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{x})\}_{\boldsymbol{n}\in\mathbb{N}_0^d}$ form an orthonormal basis of $L^2(\mathbb{R}^d)$. That is, any function $u \in L^2(\mathbb{R}^d)$ can be written in the form

$$u(\boldsymbol{x}) = \sum_{\boldsymbol{n}\geq 0} \hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} \mathcal{H}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{x}), \quad \text{with} \quad \hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} = \int_{\mathbb{R}^d} u(\boldsymbol{x}) \mathcal{H}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{x}) d\boldsymbol{x}. \tag{2.14}$$

Hence, we have $\boldsymbol{\mathcal{D}}_{\boldsymbol{x}}^{\boldsymbol{k}} u(\boldsymbol{x}) = \sum_{\boldsymbol{n}\geq\boldsymbol{k}} \hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} \boldsymbol{\mathcal{D}}_{\boldsymbol{x}}^{\boldsymbol{k}} \mathcal{H}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{x})$. Furthermore,

$$||\boldsymbol{\mathcal{D}}_{\boldsymbol{x}}^{\boldsymbol{k}} u||^2 = \sum_{\boldsymbol{n}\geq\boldsymbol{k}} \boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{k}} |\hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}|^2 \overset{(2.10)}{=} \sum_{\boldsymbol{n}\in\mathbb{N}_0^d} \boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{k}} |\hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}|^2. \tag{2.15}$$

2.3. Multivariate orthogonal projection and approximations. In this section, we aim to arrive at some typical error estimate of the form

$$\inf_{U_N \in X_N} ||u - U_N||_l \lesssim N^{-c(l,r)} ||u||_r,$$

where $c(l,r)$ is some positive constant depending on l and r, $||\cdot||_l$ is the norm of some functional space, l indicates the regularity of the function in some sense, and X_N is an approximation space. In this paper, X_N is defined as

$$X_N^{\boldsymbol{\alpha},\boldsymbol{\beta}} = \text{span}\{\boldsymbol{\mathcal{H}}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} : \ \boldsymbol{n} \in \boldsymbol{\Omega}_N\}, \tag{2.16}$$

where $\boldsymbol{\Omega}_N \subset \mathbb{N}_0^d$ is some index set. With different choices of $\boldsymbol{\Omega}_N$, it yields full grid, RHC, OHC, etc..

Let us denote the orthogonal projection operator $P_N^{\boldsymbol{\alpha},\boldsymbol{\beta}} : L^2(\mathbb{R}^d) \to X_N^{\boldsymbol{\alpha},\boldsymbol{\beta}}$, i.e., for any $u \in L^2(\mathbb{R}^d)$,

$$\langle (u - P_N^{\boldsymbol{\alpha},\boldsymbol{\beta}} u), v \rangle = 0 \quad \forall\, v \in X_N^{\boldsymbol{\alpha},\boldsymbol{\beta}},$$

or, equivalently,

$$P_N^{\boldsymbol{\alpha},\boldsymbol{\beta}} u(\boldsymbol{x}) = \sum_{\boldsymbol{n}\in\boldsymbol{\Omega}_N} \hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} \boldsymbol{\mathcal{H}}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{x}). \tag{2.17}$$

We shall estimate how close the projected function $P_N^{\boldsymbol{\alpha},\boldsymbol{\beta}} u$ is to u, with respect to various index sets $\boldsymbol{\Omega}_N$ and norms.

2.3.1. Approximations on the full grid. The index set $\boldsymbol{\Omega}_N$ corresponding to the d-dimensional full tensor grid is

$$\boldsymbol{\Omega}_N = \{\boldsymbol{n} \in \mathbb{N}_0^d : \ |\boldsymbol{n}|_\infty \le N\}.$$

And $X_N^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is defined in (2.16). Let us define the Sobolev-type space as

$$\mathcal{W}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d) = \{u : \ \boldsymbol{\mathcal{D}}^{\boldsymbol{k}}_{\boldsymbol{x}} u \in L^2(\mathbb{R}^d), 0 \le |\boldsymbol{k}|_1 \le m\} \quad \forall\, m \in \mathbb{N}_0, \tag{2.18}$$

equipped with the norm and seminorm

$$||u||_{\mathcal{W}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)} = \left(\sum_{0 \le |\boldsymbol{k}|_1 \le m} \left|\left| \boldsymbol{\mathcal{D}}^{\boldsymbol{k}}_{\boldsymbol{x}} u \right|\right|^2 \right)^{\frac{1}{2}}, \tag{2.19}$$

$$|u|_{\mathcal{W}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)} = \left(\sum_{j=1}^d \left|\left| \mathcal{D}^m_{x_j} u \right|\right|^2 \right)^{\frac{1}{2}}. \tag{2.20}$$

It is clear that $\mathcal{W}^0_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d) = L^2(\mathbb{R}^d)$, and

$$|u|^2_{\mathcal{W}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)} \overset{(2.15)}{=} \sum_{j=1}^d \sum_{\boldsymbol{n} \in \mathbb{N}_0^d} \mu_{n_j,m} \left| \hat{u}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\boldsymbol{n}} \right|^2. \tag{2.21}$$

THEOREM 2.1. *Given $u \in \mathcal{W}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)$, we have for any $0 \le l \le m$,*

$$\left| P_N^{\boldsymbol{\alpha},\boldsymbol{\beta}} u - u \right|_{\mathcal{W}^l_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)} \lesssim |\boldsymbol{\alpha}|_\infty^{l-m} N^{\frac{l-m}{2}} |u|_{\mathcal{W}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)}, \tag{2.22}$$

for $N \gg 1$. Furthermore,

$$\left|\left| P_N^{\boldsymbol{\alpha},\boldsymbol{\beta}} u - u \right|\right|_{\mathcal{W}^l_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)} \lesssim C_{\boldsymbol{\alpha},l,m} N^{\frac{l-m}{2}} |u|_{\mathcal{W}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)},$$

where $C_{\boldsymbol{\alpha},l,m}$ is some constant depending on $\boldsymbol{\alpha}$, l, and m.

Since the proof of this theorem is similar to that in [25], and to avoid the distraction of our main results, we put the proof in Appendix A. It is clear that the convergence rate deteriorates rapidly with respect to the cardinality of the full grid. That is,

$$\left|\left| P_N^{\boldsymbol{\alpha},\boldsymbol{\beta}} u - u \right|\right|_{\mathcal{W}^l_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)} \lesssim C_{\boldsymbol{\alpha},l,m} M^{\frac{l-m}{2d}} |u|_{\mathcal{W}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)},$$

where $M = \mathrm{card}(\boldsymbol{\Omega}_N) = (N+1)^d$.

2.3.2. RHC approximation. As we mentioned in the introduction, the HC approximation is an efficient tool to overcome the "curse of dimensionality" in some degree. The index set of RHC approximation is $\boldsymbol{\Omega}_N = \{\boldsymbol{n} \in \mathbb{N}_0^d : \ |\boldsymbol{n}|_{\mathrm{mix}} \le N\}$. It is known that the cardinality of $\boldsymbol{\Omega}_N$ is $\mathcal{O}(N(\ln N)^{d-1})$ [12]. Correspondingly, the finite dimensional subspace $X_N^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is

$$X_N^{\boldsymbol{\alpha},\boldsymbol{\beta}} = \mathrm{span}\{\boldsymbol{\mathcal{H}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\boldsymbol{n}} : \ |\boldsymbol{n}|_{\mathrm{mix}} \le N\}. \tag{2.23}$$

Let the orthogonal projection operator $P_N^{\boldsymbol{\alpha},\boldsymbol{\beta}} : L^2(\mathbb{R}^d) \to X_N^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ be defined before. Denote the $\boldsymbol{k}$-complement of $\boldsymbol{\Omega}_N$ by

$$\boldsymbol{\Omega}_{N,\boldsymbol{k}}^c := \{\boldsymbol{n} \in \mathbb{N}_0^d : \ |\boldsymbol{n}|_{\text{mix}} > N \text{ and } \boldsymbol{n} \geq \boldsymbol{k}\} \quad \forall \boldsymbol{k} \in \mathbb{N}_0^d. \tag{2.24}$$

We define the Koborov-type space as

$$\mathcal{K}_{\boldsymbol{\alpha},\boldsymbol{\beta}}^r(\mathbb{R}^d) = \{u : \ \mathcal{D}_{\boldsymbol{x}}^{\boldsymbol{k}} u \in L^2(\mathbb{R}^d), \ 0 \leq |\boldsymbol{k}|_\infty \leq r\} \quad \forall m \in \mathbb{N}_0^d, \tag{2.25}$$

equipped with the norm and seminorm

$$||u||_{\mathcal{K}_{\boldsymbol{\alpha},\boldsymbol{\beta}}^r(\mathbb{R}^d)} = \left(\sum_{0 \leq |\boldsymbol{k}|_\infty \leq r} \left|\left| \mathcal{D}_{\boldsymbol{x}}^{\boldsymbol{k}} u \right|\right|^2 \right)^{\frac{1}{2}}, \tag{2.26}$$

$$|u|_{\mathcal{K}_{\boldsymbol{\alpha},\boldsymbol{\beta}}^r(\mathbb{R}^d)} = \left(\sum_{|\boldsymbol{k}|_\infty = r} \left|\left| \mathcal{D}_{\boldsymbol{x}}^{\boldsymbol{k}} u \right|\right|^2 \right)^{\frac{1}{2}}. \tag{2.27}$$

Remark 2.1. It is easy to see from the definitions that $\mathcal{K}_{\boldsymbol{\alpha},\boldsymbol{\beta}}^0(\mathbb{R}^d) = L^2(\mathbb{R}^d)$ and $\mathcal{W}_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{dl}(\mathbb{R}^d) \subset \mathcal{K}_{\boldsymbol{\alpha},\boldsymbol{\beta}}^l(\mathbb{R}^d) \subset \mathcal{W}_{\boldsymbol{\alpha},\boldsymbol{\beta}}^l(\mathbb{R}^d)$.

THEOREM 2.2. *Given $u \in \mathcal{K}_{\boldsymbol{\alpha},\boldsymbol{\beta}}^m(\mathbb{R}^d)$, for $0 \leq \boldsymbol{l} \leq m$, we have*

$$\left|\left| \mathcal{D}_{\boldsymbol{x}}^{\boldsymbol{l}} \left(P_N^{\boldsymbol{\alpha},\boldsymbol{\beta}} u - u \right) \right|\right| \leq C_{\boldsymbol{\alpha},\boldsymbol{l},m,d} N^{\frac{|\boldsymbol{l}|_\infty - m}{2}} |u|_{\mathcal{K}_{\boldsymbol{\alpha},\boldsymbol{\beta}}^m(\mathbb{R}^d)},$$

where $C_{\boldsymbol{\alpha},\boldsymbol{l},m,d}$ is some constant depending on $\boldsymbol{\alpha}$, $\boldsymbol{l}$, m, and d, for $N \gg 1$ (more precisely, at least $N > m^d$). In particular, if $\boldsymbol{\alpha} = \boldsymbol{1}$, then

$$C_{\boldsymbol{1},\boldsymbol{l},m,d} = 2^{|\boldsymbol{l}|_\infty - m} m^{(2d-1)m - |\boldsymbol{l}|_1 - (d-1)|\boldsymbol{l}|_\infty}.$$

Proof. From (2.17), (2.15), we have

$$\begin{aligned}
\left|\left| \mathcal{D}_{\boldsymbol{x}}^{\boldsymbol{l}} (P_N^{\boldsymbol{\alpha},\boldsymbol{\beta}} u - u) \right|\right|^2 &= \sum_{\boldsymbol{n} \in \boldsymbol{\Omega}_N^c} \boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{l}} \left| \hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} \right|^2 \\
&= \sum_{\boldsymbol{n} \in \boldsymbol{\Omega}_{N,m}^c} \boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{l}} \left| \hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} \right|^2 + \sum_{\boldsymbol{n} \in \boldsymbol{\Omega}_{N,\boldsymbol{l}}^c \setminus \boldsymbol{\Omega}_{N,m}^c} \boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{l}} \left| \hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} \right|^2 \\
&:= II_1 + II_2.
\end{aligned}$$

For II_1:

$$II_1 \leq \max_{\boldsymbol{n} \in \boldsymbol{\Omega}_{N,m}^c} \left\{ \frac{\boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{l}}}{\boldsymbol{\mu}_{\boldsymbol{n},m}} \right\} \sum_{\boldsymbol{n} \in \boldsymbol{\Omega}_{N,m}^c} \boldsymbol{\mu}_{\boldsymbol{n},m} \left| \hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} \right|^2.$$

With the facts that

$$\begin{aligned}
\frac{\boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{l}}}{\boldsymbol{\mu}_{\boldsymbol{n},m}} &= 2^{|\boldsymbol{l}|_1 - dm} \prod_{j=1}^d \alpha_j^{2(l_j - m)} \prod_{j=1}^d \frac{1}{(n_j - l_j) \cdots (n_j - m + 1)} \\
&= 2^{|\boldsymbol{l}|_1 - dm} \prod_{j=1}^d \alpha_j^{2(l_j - m)} \prod_{j=1}^d n_j^{l_j - m} \prod_{j=1}^d \left(1 - \frac{l_j}{n_j}\right)^{-1} \cdots \left(1 - \frac{m-1}{n_j}\right)^{-1} \\
&\overset{(2.24)}{\leq} 2^{|\boldsymbol{l}|_1 - dm} \prod_{j=1}^d \alpha_j^{2(l_j - m)} N^{|\boldsymbol{l}|_\infty - m} \prod_{j=1}^d \left(1 - \frac{l_j}{n_j}\right)^{-1} \cdots \left(1 - \frac{m-1}{n_j}\right)^{-1}
\end{aligned} \tag{2.28}$$

and

$$\max_{\boldsymbol{n}\in\boldsymbol{\Omega}^c_{N,m}}\left\{\prod_{j=1}^d\left(1-\frac{l_j}{n_j}\right)^{-1}\cdots\left(1-\frac{m-1}{n_j}\right)^{-1}\right\}\le\max_{\boldsymbol{n}\in\boldsymbol{\Omega}^c_{N,m}}\left\{\prod_{j=1}^d\left(1-\frac{m-1}{n_j}\right)^{l_j-m}\right\}$$

$$\le\prod_{j=1}^d m^{m-l_j}=m^{dm-|\boldsymbol{l}|_1},\tag{2.29}$$

we arrive at the conclusion that

$$II_1\le\left(\frac{m}{2}\right)^{dm-|\boldsymbol{l}|_1}\prod_{j=1}^d\alpha_j^{2(l_j-m)}N^{|\boldsymbol{l}|_\infty-m}\left|\left|\mathcal{D}_{\boldsymbol{x}}^{m\cdot\boldsymbol{1}}u\right|\right|^2.\tag{2.30}$$

For II_2: The index set $\boldsymbol{\Omega}^c_{N,\boldsymbol{l}}\setminus\boldsymbol{\Omega}^c_{N,\boldsymbol{m}}$ is

$$\boldsymbol{\Omega}^c_{N,\boldsymbol{l}}\setminus\boldsymbol{\Omega}^c_{N,\boldsymbol{m}}=\{\boldsymbol{n}\in\mathbb{N}_0^d:\ |\boldsymbol{n}|_{\text{mix}}>N\text{ and }\boldsymbol{n}\ge\boldsymbol{l},\ \exists j,\text{ such that }n_j<m\}.$$

Let us divide the index $1\le j\le d$ into two parts:

$$\mathcal{N}:=\{j:\ l_j\le n_j<m,\ 1\le j\le d\}\text{ and }\quad\mathcal{N}^c:=\{j:\ n_j\ge m,\ 1\le j\le d\}.\tag{2.31}$$

It is easy to see that neither $\mathcal{N}$ nor $\mathcal{N}^c$ is empty set. We denote

$$\tilde{\boldsymbol{\mu}}_{\boldsymbol{n},\boldsymbol{l},m}=\left(\prod_{j\in\mathcal{N}}\mu_{n_j,l_j}\right)\left(\prod_{i\in\mathcal{N}^c}\mu_{n_i,m}\right):=\boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{k}},\tag{2.32}$$

where $\boldsymbol{k}$ is a d-dimensional index consisting of l_j for $j\in\mathcal{N}$ and m for $j\in\mathcal{N}^c$. Now, we treat II_2 as

$$II_2\overset{(2.32)}{\le}\max_{\boldsymbol{n}\in\boldsymbol{\Omega}^c_{N,\boldsymbol{l}}\setminus\boldsymbol{\Omega}^c_{N,m}}\left\{\frac{\boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{l}}}{\boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{k}}}\right\}\sum_{\boldsymbol{n}\in\boldsymbol{\Omega}^c_{N,\boldsymbol{l}}\setminus\boldsymbol{\Omega}^c_{N,m}}\boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{k}}\left|\hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}\right|^2$$

$$\le\max_{\boldsymbol{n}\in\boldsymbol{\Omega}^c_{N,\boldsymbol{l}}\setminus\boldsymbol{\Omega}^c_{N,m}}\left\{\frac{\boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{l}}}{\boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{k}}}\right\}|u|^2_{\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)},\tag{2.33}$$

since $|\boldsymbol{k}|_\infty=m$. It remains to estimate the maximum in (2.33). We have that

$$\frac{\boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{l}}}{\boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{k}}}=2^{|\boldsymbol{l}|_1-|\boldsymbol{k}|_1}\prod_{j\in\mathcal{N}^c}\alpha_j^{2(l_j-m)}\frac{1}{(n_j-l_j)\cdots(n_j-m+1)},$$

$$=2^{|\boldsymbol{l}|_1-|\boldsymbol{k}|_1}\prod_{j\in\mathcal{N}^c}\alpha_j^{2(l_j-m)}\prod_{j\in\mathcal{N}^c}n_j^{l_j-m}\prod_{j\in\mathcal{N}^c}\left(1-\frac{l_j}{n_j}\right)^{-1}\cdots\left(1-\frac{m-1}{n_j}\right)^{-1}.\tag{2.34}$$

Observe that $j\in\mathcal{N}^c$ implies $n_j\ge m>\boldsymbol{l}\ge0$. That is, $n_j\ge1$. Hence, $\bar{n}_j=n_j$ for all $j=1,\ldots,d$. In view of $|\boldsymbol{n}|_{\text{mix}}>N$, we deduce that

$$\prod_{j\in\mathcal{N}^c}\bar{n}_j>\frac{N}{\prod_{j\in\mathcal{N}}\bar{n}_j}>\frac{N}{\prod_{j\in\mathcal{N}}m}.$$

With the same estimate in (2.29) and the fact that

$$2^{|\boldsymbol{l}|_1-|\boldsymbol{k}|_1}=2^{\sum_{j\in\mathcal{N}^c}(l_j-m)}\le2^{|\boldsymbol{l}|_\infty-m},\tag{2.35}$$

we get that

$$\max_{\boldsymbol{n}\in\boldsymbol{\Omega}^c_{N,\boldsymbol{l}}\setminus\boldsymbol{\Omega}^c_{N,m}}\left\{\frac{\boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{l}}}{\boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{k}}}\right\}\le C_{\boldsymbol{\alpha},\boldsymbol{l},m}2^{|\boldsymbol{l}|_\infty-m}m^{(2d-1)m-|\boldsymbol{l}|_1-(d-1)|\boldsymbol{l}|_\infty}N^{|\boldsymbol{l}|_\infty-m}, \tag{2.36}$$

where $C_{\boldsymbol{\alpha},\boldsymbol{l},m}$ denotes some constant depending on $\boldsymbol{\alpha}$, $\boldsymbol{l}$, and m. The desired result follows immediately from (2.30), (2.33), and (2.36). ∎

COROLLARY 2.3. *We have that*

$$\left|\left|P_N^{\boldsymbol{\alpha},\boldsymbol{\beta}}u-u\right|\right|_{\mathcal{K}^l_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)}\le C_{\boldsymbol{\alpha},l,m,d}N^{\frac{l-m}{2}}|u|_{\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)}\quad\forall\, 0\le l\le m,$$

where $C_{\boldsymbol{\alpha},l,m,d}$ is some constant depending on $\boldsymbol{\alpha}$, l, m, and d.

Remark 2.2. Recall that $M=\mathrm{card}(\boldsymbol{\Omega}_N)=\mathcal{O}(N(\ln N)^{d-1})\le CN^{1+\epsilon(d-1)}$, for arbitrary small $\epsilon>0$. Then

$$\left|\left|P_N^{\boldsymbol{\alpha},\boldsymbol{\beta}}u-u\right|\right|_{\mathcal{K}^l_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)}\le C_{\boldsymbol{\alpha},l,m,d}M^{\frac{l-m}{2(1+\epsilon(d-1))}}|u|_{\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)}\quad\forall\, 0\le l\le m,$$

where $C_{\boldsymbol{\alpha},l,m,d}$ is some constant depending on $\boldsymbol{\alpha}$, l, m, and d. It is clear to see that the convergence rate deteriorates slightly with increasing d.

2.3.3. OHC approximation. In order to completely break the curse of dimensionality, we consider the index set introduced in [12],

$$\boldsymbol{\Omega}_{N,\gamma}:=\{\boldsymbol{n}\in\mathbb{N}_0^d:\ |\boldsymbol{n}|_{\mathrm{mix}}|\boldsymbol{n}|_\infty^{-\gamma}\le N^{1-\gamma}\},\quad -\infty\le\gamma<1. \tag{2.37}$$

The cardinality of $\boldsymbol{\Omega}_{N,\gamma}$ is $\mathcal{O}(N)$, for $\gamma\in(0,1)$, where the dependence of dimension is in the big-O; see [12]. The family of spaces are defined as

$$X_{N,\gamma}^{\boldsymbol{\alpha},\boldsymbol{\beta}}:=\mathrm{span}\{\boldsymbol{\mathcal{H}}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}:\ \boldsymbol{n}\in\boldsymbol{\Omega}_{N,\gamma}\}. \tag{2.38}$$

Remark 2.3. In particular, we have $X_{N,0}^{\boldsymbol{\alpha},\boldsymbol{\beta}}=X_N^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ in RHC (2.23), and $X_{N,-\infty}^{\boldsymbol{\alpha},\boldsymbol{\beta}}=\mathrm{span}\{\mathcal{H}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}:\ |\boldsymbol{n}|_\infty\le N\}$, i.e., the full grid.

We denote the projection operator as $P_{N,\gamma}^{\boldsymbol{\alpha},\boldsymbol{\beta}}:\ L^2(\mathbb{R}^d)\to X_{N,\gamma}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$. In this case, the $\boldsymbol{k}$-complement of index set of $\boldsymbol{\Omega}_{N,\gamma}$ is

$$\boldsymbol{\Omega}^c_{N,\gamma,\boldsymbol{k}}=\{\boldsymbol{n}\in\mathbb{N}_0^d:\ \boldsymbol{n}\in\boldsymbol{\Omega}^c_{N,\gamma}\ \text{and}\ \boldsymbol{n}\ge\boldsymbol{k}\}\quad\forall\,\boldsymbol{k}\in\mathbb{N}_0^d. \tag{2.39}$$

Although [25] obtains the similar result for Jacobi polynomials as Theorem 2.4 below, we believe that there is a gap in their error analysis of OHC, namely Theorem 2.3 [25]. We circumvent it with more delicate analysis.

THEOREM 2.4. *For any $u\in\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)$, $d\ge 2$, and $0\le|\boldsymbol{l}|_1<m$,*

(2.40)

$$\left|\left|\boldsymbol{\mathcal{D}}_{\boldsymbol{x}}^{\boldsymbol{l}}\left(P_{N,\gamma}^{\boldsymbol{\alpha},\boldsymbol{\beta}}u-u\right)\right|\right|\le C_{\boldsymbol{\alpha},\boldsymbol{l},m,d,\gamma}|u|_{\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)}\begin{cases}N^{\frac{|\boldsymbol{l}|_1-m}{2}} & \text{if } 0<\gamma\le\dfrac{|\boldsymbol{l}|_1}{m},\\ N^{\frac{(1-\gamma)[|\boldsymbol{l}|_1-(d-1)m]}{d-1-\gamma}} & \text{if } \dfrac{|\boldsymbol{l}|_1}{m}\le\gamma<1,\end{cases}$$

where $C_{\boldsymbol{\alpha},\boldsymbol{l},m,d,\gamma}$ is some constant depending on $\boldsymbol{\alpha}$, $\boldsymbol{l}$, m, d, and γ. In particular, if $\boldsymbol{\alpha}=\boldsymbol{1}$, then

$$C_{\boldsymbol{1},\boldsymbol{l},m,d,\gamma}=m^{dm-|\boldsymbol{l}|_1}\begin{cases}2^{|\boldsymbol{l}|_\infty-m}m^{\frac{(d-1)(\gamma m-|\boldsymbol{l}|_1)}{1-\gamma}} & \text{if } 0<\gamma\le\dfrac{|\boldsymbol{l}|_1}{m},\\ 2^{|\boldsymbol{l}|_1-dm} & \text{if } \dfrac{|\boldsymbol{l}|_1}{m}\le\gamma<1.\end{cases}$$

Proof. As argued in the proof of Theorem 2.2, we arrive at

$$\begin{aligned}\left|\left|\boldsymbol{\mathcal{D}}_{\boldsymbol{x}}^{\boldsymbol{l}}\left(P_{N,\gamma}^{\boldsymbol{\alpha},\boldsymbol{\beta}}u-u\right)\right|\right|^2 \le& \max_{\boldsymbol{n}\in\boldsymbol{\Omega}_{N,\gamma,m}^c}\left\{\frac{\boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{l}}}{\boldsymbol{\mu}_{\boldsymbol{n},m}}\right\}\sum_{\boldsymbol{n}\in\boldsymbol{\Omega}_{N,\gamma,m}^c}\boldsymbol{\mu}_{\boldsymbol{n},m}\left|\hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}\right|^2\\ &+\max_{\boldsymbol{n}\in\boldsymbol{\Omega}_{N,\gamma,\boldsymbol{l}}^c\setminus\boldsymbol{\Omega}_{N,\gamma,m}^c}\left\{\frac{\boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{l}}}{\tilde{\boldsymbol{\mu}}_{\boldsymbol{n},\boldsymbol{l},m}}\right\}\sum_{\boldsymbol{n}\in\boldsymbol{\Omega}_{N,\gamma,\boldsymbol{l}}^c\setminus\boldsymbol{\Omega}_{N,\gamma,m}^c}\tilde{\boldsymbol{\mu}}_{\boldsymbol{n},\boldsymbol{l},m}\left|\hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}\right|^2\\ :=& III_1+III_2, \end{aligned}\tag{2.41}$$

where $\tilde{\boldsymbol{\mu}}_{\boldsymbol{n},\boldsymbol{l},m}$ is defined as in (2.32). To estimate III_1, like in (2.28), we have

$$\begin{aligned}\frac{\boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{l}}}{\boldsymbol{\mu}_{\boldsymbol{n},m}} &= 2^{|\boldsymbol{l}|_1-dm}\prod_{j=1}^d\alpha_j^{2(l_j-m)}\prod_{j=1}^d\left(1-\frac{l_j}{n_j}\right)^{-1}\cdots\left(1-\frac{m-1}{n_j}\right)^{-1}\prod_{j=1}^d n_j^{l_j-m}\\ &:= D_1\prod_{j=1}^d n_j^{l_j-m}.\end{aligned}\tag{2.42}$$

The estimate of $\max_{\boldsymbol{n}\in\boldsymbol{\Omega}_{N,\gamma,m}^c} D_1$ is followed by the similar argument in (2.29), i.e.,

$$\max_{\boldsymbol{n}\in\boldsymbol{\Omega}_{N,\gamma,m}^c} D_1 \le \left(\frac{m}{2}\right)^{dm-|\boldsymbol{l}|_1}\prod_{j=1}^d\alpha_j^{2(l_j-m)}.\tag{2.43}$$

Notice that for any $\boldsymbol{n}\in\boldsymbol{\Omega}_{N,\gamma}^c$,

$$|\boldsymbol{n}|_{\text{mix}}|\boldsymbol{n}|_\infty^{-\gamma} > N^{1-\gamma} \Rightarrow \left(\frac{|\boldsymbol{n}|_\infty^\gamma}{|\boldsymbol{n}|_{\text{mix}}}\right)^{\frac{1}{1-\gamma}} < \frac{1}{N},\tag{2.44}$$

and furthermore, if $\boldsymbol{n}\in\boldsymbol{\Omega}_{N,\gamma,m}^c$,

$$\frac{|\boldsymbol{n}|_\infty}{|\boldsymbol{n}|_{\text{mix}}} \le \frac{1}{m^{d-1}}.\tag{2.45}$$

Moreover,

$$|\boldsymbol{n}|_\infty^{d-\gamma} \ge |\boldsymbol{n}|_{\text{mix}}|\boldsymbol{n}|_\infty^{-\gamma} > N^{1-\gamma} \Rightarrow |\boldsymbol{n}|_\infty > N^{\frac{1-\gamma}{d-\gamma}}.\tag{2.46}$$

Let us estimate the product on the right-hand side of (2.42):

$$\prod_{j=1}^d n_j^{l_j-m} = \left(\prod_{j=1}^d n_j^{l_j}\right)\left(\prod_{j=1}^d n_j\right)^{-m} \le \left(\prod_{j=1}^d |\boldsymbol{n}|_\infty^{l_j}\right)|\boldsymbol{n}|_{\text{mix}}^{-m} = |\boldsymbol{n}|_\infty^{|\boldsymbol{l}|_1}|\boldsymbol{n}|_{\text{mix}}^{-m}.\tag{2.47}$$

If $0<\gamma\le\frac{|\boldsymbol{l}|_1}{m}$, then

$$\begin{aligned}\max_{\boldsymbol{n}\in\boldsymbol{\Omega}_{N,\gamma,m}^c}\prod_{j=1}^d n_j^{l_j-m} &\overset{(2.47)}{\le} \max_{\boldsymbol{n}\in\boldsymbol{\Omega}_{N,\gamma,m}^c}\left\{\left(\frac{|\boldsymbol{n}|_\infty^\gamma}{|\boldsymbol{n}|_{\text{mix}}}\right)^{\frac{m-|\boldsymbol{l}|_1}{1-\gamma}}\left(\frac{|\boldsymbol{n}|_\infty}{|\boldsymbol{n}|_{\text{mix}}}\right)^{\frac{|\boldsymbol{l}|_1-\gamma m}{1-\gamma}}\right\}\\ &\overset{(2.44),(2.45)}{<} m^{\frac{(d-1)(\gamma m-|\boldsymbol{l}|_1)}{1-\gamma}}N^{|\boldsymbol{l}|_1-m}.\end{aligned}\tag{2.48}$$

Otherwise, if $\frac{|\boldsymbol{l}|_1}{m} \le \gamma < 1$, then

$$\max_{\boldsymbol{n}\in\boldsymbol{\Omega}^c_{N,\gamma,m}} \prod_{j=1}^{d} n_j^{l_j-m} \overset{(2.47)}{\le} \max_{\boldsymbol{n}\in\boldsymbol{\Omega}^c_{N,\gamma,m}} \left\{ \left(\frac{|\boldsymbol{n}|_\infty^\gamma}{|\boldsymbol{n}|_{\text{mix}}} \right)^m |\boldsymbol{n}|_\infty^{|\boldsymbol{l}|_1-\gamma m} \right\} \overset{(2.44),(2.46)}{\le} N^{\frac{1-\gamma}{d-\gamma}(|\boldsymbol{l}|_1-\gamma m)-(1-\gamma)m}. \tag{2.49}$$

Combine (2.43), (2.48), and (2.49), the first term on the right-hand side of (2.41) has the upper bound

$$III_1 \le \left(\frac{m}{2}\right)^{dm-|\boldsymbol{l}|_1} \prod_{j=1}^{d} \alpha_j^{2(l_j-m)} \left|\left|\boldsymbol{\mathcal{D}}_{\boldsymbol{x}}^{m\cdot\boldsymbol{1}} u\right|\right|^2 \begin{cases} m^{\frac{(d-1)(\gamma m-|\boldsymbol{l}|_1)}{1-\gamma}} N^{|\boldsymbol{l}|_1-m} & \text{if } 0<\gamma\le \dfrac{|\boldsymbol{l}|_1}{m}, \\ N^{\frac{1-\gamma}{d-\gamma}(|\boldsymbol{l}|_1-\gamma m)-(1-\gamma)m} & \text{if } \dfrac{|\boldsymbol{l}|_1}{m}\le\gamma<1. \end{cases} \tag{2.50}$$

Next, we consider III_2. Define $\mathcal{N}$ and $\mathcal{N}^c$ as in (2.31). Like in (2.33), we obtain that

$$III_2 \le \max_{\boldsymbol{n}\in\boldsymbol{\Omega}^c_{N,\gamma,\boldsymbol{l}}\setminus\boldsymbol{\Omega}^c_{N,\gamma,m}} \left\{ \frac{\boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{l}}}{\boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{k}}} \right\} |u|^2_{\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)}. \tag{2.51}$$

We need to estimate the maximum similarly as in (2.34):

$$\begin{aligned} \frac{\boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{l}}}{\boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{k}}} &= 2^{|\boldsymbol{l}|_1-|\boldsymbol{k}|_1} \prod_{j\in\mathcal{N}^c} \alpha_j^{2(l_j-m)} \prod_{j\in\mathcal{N}^c} n_j^{l_j-m} \prod_{j\in\mathcal{N}^c} \left(1-\frac{l_j}{n_j}\right)^{-1} \cdots \left(1-\frac{m-1}{n_j}\right)^{-1} \\ &:= D_2 \prod_{j\in\mathcal{N}^c} n_j^{l_j-m}. \end{aligned} \tag{2.52}$$

Similar argument as in (2.29) yields that

$$\max_{\boldsymbol{n}\in\boldsymbol{\Omega}^c_{N,\gamma,\boldsymbol{l}}\setminus\boldsymbol{\Omega}^c_{N,\gamma,m}} D_2 \le 2^{|\boldsymbol{l}|_1-|\boldsymbol{k}|_1} \prod_{j\in\mathcal{N}^c} \alpha_j^{2(l_j-m)} m^{dm-|\tilde{\boldsymbol{l}}|_1}, \tag{2.53}$$

where

$$\tilde{\boldsymbol{l}} = (l_1,\ldots,l_d) = \begin{cases} l_j & \text{if } j\in\mathcal{N}^c, \\ 0 & \text{otherwise.} \end{cases} \tag{2.54}$$

And it is verified that

$$\prod_{j\in\mathcal{N}^c} n_j^{l_j-m} \le \left(\prod_{j\in\mathcal{N}^c} |\tilde{\boldsymbol{n}}|_\infty^{l_j}\right) \left(\prod_{j\in\mathcal{N}^c} n_j\right)^{-m} = |\tilde{\boldsymbol{n}}|_\infty^{|\tilde{\boldsymbol{l}}|_1} |\tilde{\boldsymbol{n}}|_{\text{mix}}^{-m} \le |\tilde{\boldsymbol{n}}|_\infty^{|\boldsymbol{l}|_1} |\tilde{\boldsymbol{n}}|_{\text{mix}}^{-m}, \tag{2.55}$$

where $\tilde{\boldsymbol{n}}$ is defined similarly as $\tilde{\boldsymbol{l}}$ in (2.54). With similar argument as in (2.44), we deduce that for any $\boldsymbol{n}\in\boldsymbol{\Omega}^c_{N,\gamma}$,

$$N^{1-\gamma} < |\boldsymbol{n}|_{\text{mix}} |\boldsymbol{n}|_\infty^{-\gamma} \le m^{d-1} |\tilde{\boldsymbol{n}}|_{\text{mix}} |\tilde{\boldsymbol{n}}|_\infty^{-\gamma} \Rightarrow \left(\frac{|\tilde{\boldsymbol{n}}|_\infty^\gamma}{|\tilde{\boldsymbol{n}}|_{\text{mix}}}\right)^{\frac{1}{1-\gamma}} < m^{\frac{d-1}{1-\gamma}} N^{-1}. \tag{2.56}$$

And similarly as in (2.45), we have for any $\boldsymbol{n} \in \boldsymbol{\Omega}_{N,\gamma,m}^c$,

$$\frac{|\tilde{\boldsymbol{n}}|_\infty}{|\tilde{\boldsymbol{n}}|_{\text{mix}}} \leq \frac{1}{m^{d-2}} \tag{2.57}$$

and

$$N^{1-\gamma} \overset{(2.57)}{<} m^{d-1}|\tilde{\boldsymbol{n}}|_{\text{mix}}|\tilde{\boldsymbol{n}}|_\infty^{-\gamma} \leq m^{d-1}|\tilde{\boldsymbol{n}}|_\infty^{d-1-\gamma} \Rightarrow |\tilde{\boldsymbol{n}}|_\infty > \left(\frac{N^{1-\gamma}}{m^{d-1}}\right)^{\frac{1}{d-1-\gamma}}. \tag{2.58}$$

If $0 < \gamma \leq \frac{|\boldsymbol{l}|_1}{m}$, then

$$\begin{aligned} &\max_{\boldsymbol{n}\in\boldsymbol{\Omega}_{N,\gamma,\boldsymbol{l}}^c\setminus\boldsymbol{\Omega}_{N,\gamma,m}^c} \prod_{j\in\mathcal{N}^c} n_j^{l_j-m} \\ &\overset{(2.55)}{<} \max_{\boldsymbol{n}\in\boldsymbol{\Omega}_{N,\gamma,\boldsymbol{l}}^c\setminus\boldsymbol{\Omega}_{N,\gamma,m}^c} \left\{ \left(\frac{|\tilde{\boldsymbol{n}}|_\infty^\gamma}{|\tilde{\boldsymbol{n}}|_{\text{mix}}}\right)^{\frac{m-|\boldsymbol{l}|_1}{1-\gamma}} \left(\frac{|\tilde{\boldsymbol{n}}|_\infty}{|\tilde{\boldsymbol{n}}|_{\text{mix}}}\right)^{\frac{|\boldsymbol{l}|_1-\gamma m}{1-\gamma}} \right\} \\ &\overset{(2.56),(2.57)}{\leq} m^{\frac{1}{1-\gamma}\{[(\gamma+1)d-(2\gamma+1)]m-(2d-3)|\boldsymbol{l}|_1\}} N^{|\boldsymbol{l}|_1-m}. \end{aligned} \tag{2.59}$$

Otherwise, if $\frac{|\boldsymbol{l}|_1}{m} \leq \gamma < 1$, then

$$\begin{aligned} \max_{\boldsymbol{n}\in\boldsymbol{\Omega}_{N,\gamma,\boldsymbol{l}}^c\setminus\boldsymbol{\Omega}_{N,\gamma,m}^c} \prod_{j\in\mathcal{N}^c} n_j^{l_j-m} &\overset{(2.55)}{<} \max_{\boldsymbol{n}\in\boldsymbol{\Omega}_{N,\gamma,\boldsymbol{l}}^c\setminus\boldsymbol{\Omega}_{N,\gamma,m}^c} \left\{ \left(\frac{|\tilde{\boldsymbol{n}}|_\infty^\gamma}{|\tilde{\boldsymbol{n}}|_{\text{mix}}}\right)^m |\tilde{\boldsymbol{n}}|_\infty^{|\boldsymbol{l}|_1-\gamma m} \right\} \\ &\overset{(2.56),(2.58)}{\leq} m^{(d-1)\left[m-\frac{|\boldsymbol{l}|_1-\gamma m}{d-1-\gamma}\right]} N^{\frac{(1-\gamma)[|\boldsymbol{l}|_1-(d-1)m]}{d-1-\gamma}}. \end{aligned} \tag{2.60}$$

Combining (2.34), (2.51), (2.53), (2.59), and (2.60), we arrive at

$$\begin{aligned} III_2 \leq\ & 2^{|\boldsymbol{l}|_\infty-m} \prod_{j\in\mathcal{N}^c} \alpha_j^{2(l_j-m)} m^{dm-|\boldsymbol{l}|_1} |u|^2_{\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)} \\ & \times \begin{cases} m^{\frac{1}{1-\gamma}\{[(\gamma+1)d-(2\gamma+1)]m-(2d-3)|\boldsymbol{l}|_1\}} N^{|\boldsymbol{l}|_1-m} & \text{if } 0 < \gamma \leq \dfrac{|\boldsymbol{l}|_1}{m}, \\ m^{(d-1)\left[m-\frac{|\boldsymbol{l}|_1-\gamma m}{d-1-\gamma}\right]} N^{\frac{(1-\gamma)[|\boldsymbol{l}|_1-(d-1)m]}{d-1-\gamma}} & \text{if } \dfrac{|\boldsymbol{l}|_1}{m} \leq \gamma < 1. \end{cases} \end{aligned} \tag{2.61}$$

Therefore, the desired result follows immediately from (2.50) and (2.61). ∎

Corollary 2.5. *For any $u \in \mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)$, $0 \leq l < m$, and $0 < \gamma \leq \frac{l}{m}$,*

$$\left|\left|P_{N,\gamma}^{\boldsymbol{\alpha},\boldsymbol{\beta}} u - u\right|\right|_{\mathcal{W}^l_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)} \leq C_{\boldsymbol{\alpha},\boldsymbol{l},m,d,\gamma} N^{\frac{l-m}{2}} |u|_{\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)},$$

where $C_{\boldsymbol{\alpha},\boldsymbol{l},m,d,\gamma}$ is some constant depending on $\boldsymbol{\alpha}$, $\boldsymbol{l}$, m, d, and γ.

Remark 2.4. Due to the fact that $M = \text{card}(\boldsymbol{\Omega}_{N,\gamma}) = \mathcal{O}(N) \leq CN$, we obtain that

$$\left|\left|P_{N,\gamma}^{\boldsymbol{\alpha},\boldsymbol{\beta}} u - u\right|\right|_{\mathcal{W}^l_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)} \leq C_{\boldsymbol{\alpha},\boldsymbol{l},m,d,\gamma} M^{\frac{l-m}{2}} |u|_{\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)},$$

where $C_{\boldsymbol{\alpha},\boldsymbol{l},m,d,\gamma}$ is some constant depending on $\boldsymbol{\alpha}$, $\boldsymbol{l}$, m, d, and γ. It is clear to see that the convergence rate does not deteriorate with respect to d anymore. The effect of the dimension goes into the constant in front.

3. Application to linear parabolic PDE. In this section, we shall study the Galerkin HSM with the HC approximation applying to high dimensional linear parabolic PDE. Let us consider the linear parabolic PDE of the general form:

$$\begin{cases} \partial_t u(\boldsymbol{x}) + Lu(\boldsymbol{x}) = f(\boldsymbol{x}, t), & \boldsymbol{x} \in \mathbb{R}^d,\ t \in [0, T], \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \end{cases} \tag{3.1}$$

where

$$Lu = -\nabla \cdot (\boldsymbol{A}\nabla u) + \boldsymbol{b} \cdot \nabla u + cu, \tag{3.2}$$

with $\boldsymbol{A} = (a_{ij})_{i,j=1}^d : \mathbb{R}^d \mapsto \mathbb{R}^{d\times d}$, $\boldsymbol{b} = (b_i)_{i=1}^d : \mathbb{R}^d \mapsto \mathbb{R}^d$, and $c : \mathbb{R}^d \mapsto \mathbb{R}$. The aim of HSM is to find $u_N \in X$, such that

$$\langle \partial_t u_N, \varphi \rangle - \mathcal{A}(u_N, \varphi) = \langle f, \varphi \rangle \quad \forall\, \varphi \in X, \tag{3.3}$$

where X is some approximate space, and $\mathcal{A}(u, v)$ is a bilinear form given by

$$\mathcal{A}(u, v) = \int_{\mathbb{R}^d} (\nabla u)^T \boldsymbol{A}\nabla v + v\boldsymbol{b} \cdot \nabla u + cuv\, d\boldsymbol{x}. \tag{3.4}$$

In our content, X could be chosen as $X_N^{\boldsymbol{\alpha},\boldsymbol{\beta}}$, $X_{N,\gamma}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ in the previous section.

To guarantee the existence and regularity of the solution to (3.1), we assume the following:

(C_1) The bilinear form is continuous, i.e., there is a constant $C > 0$ such that

$$|\mathcal{A}(u, v)| \le C||u||_{H_0^1(\mathbb{R}^d)}||v||_{H_0^1(\mathbb{R}^d)} \quad \forall\, u, v \in H_0^1(\mathbb{R}^d). \tag{3.5}$$

(C_2) The bilinear form is coercive, i.e., there exists some $c > 0$ such that

$$\mathcal{A}(u, u) \ge c||u||_{H_0^1(\mathbb{R}^d)}^2 \quad \forall\, u \in H_0^1(\mathbb{R}^d). \tag{3.6}$$

(C_3) The coefficients a_{ij}, b_i, and c are smooth.

Here, $H_0^1(\mathbb{R}^d)$ denotes the normal Sobolev space with the functions decaying to zero at infinity. More generally, $H_0^m(\mathbb{R}^d)$ is defined as, for any $u \in H^m(\mathbb{R}^d)$, it satisfies $|u| \to 0$, as $|\boldsymbol{x}| \to \infty$ and

$$||u||_{H^m(\mathbb{R}^d)}^2 = \sum_{0\le|\boldsymbol{k}|_1\le m} \left|\left|\boldsymbol{\partial}_{\boldsymbol{x}}^{\boldsymbol{k}} u\right|\right|^2 < \infty. \tag{3.7}$$

Let us first show some relation between the Sobolev-type space $W_{\boldsymbol{\alpha},\boldsymbol{\beta}}^l(\mathbb{R}^d)$ (see (2.18)) and the normal Sobolev space $H^l(\mathbb{R}^d)$.

Lemma 3.1. *For $u \in \mathcal{W}_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{|\boldsymbol{k}|_1+|\boldsymbol{r}|_1}(\mathbb{R}^d)$, for any $\boldsymbol{r}, \boldsymbol{k} \in \mathbb{N}_0^d$, we have*

$$\left|\left|\boldsymbol{x}^{\boldsymbol{r}}\boldsymbol{\partial}_{\boldsymbol{x}}^{\boldsymbol{k}} u\right|\right| \lesssim \left(\prod_{i=1}^d \alpha_i^{-r_i}\right) |\boldsymbol{k} + \boldsymbol{r}|_{\text{mix}}^{\frac{1}{2}} \cdot ||u||_{\mathcal{W}_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{|\ bk|_1+|\boldsymbol{r}|_1}(\mathbb{R}^d)}. \tag{3.8}$$

Proof. For clarity, we show it holds for $d = 1$ in detail. Let us start from the left-hand sided of (3.8).

$$\begin{aligned} \left|\left|x^r\partial_x^k u\right|\right|^2 &= \left|\left|\sum_{n=0}^{\infty} \hat{u}_n^{\alpha,\beta} x^r \partial_x^k \mathcal{H}_n^{\alpha,\beta}(x)\right|\right|^2 \\ &\overset{(2.5),(2.6)}{=} \alpha^{-2r} \left|\left|\sum_{n=0}^{\infty} \hat{u}_n^{\alpha,\beta} \sum_{i=-(k+r)}^{k+r} \eta_{n,i}\mathcal{H}_{n+i}^{\alpha,\beta}(x)\right|\right|^2, \end{aligned} \tag{3.9}$$

where, for each n, $\eta_{n,i}$ is a product of $k+r$ factors of $(\pm\frac{\sqrt{\lambda_{n+i}}}{2})$ or $\frac{\beta}{2}$ with $-(k+r)\leq i\leq k+r$. Notice that

$$\lambda_{n+i}\sim\lambda_{n+j}, \tag{3.10}$$

provided that $\lambda_{n+i},\lambda_{n+j}\neq 0$, for all $-(k+r)\leq i,j\leq k+r$. In fact, it is equivalent to show that $\lambda_n\sim\lambda_{n+l}$, for all $0\leq l\leq 2(k+r)$. By convention, $\lambda_n=0$ if $n\leq 0$. Notice that

$$\frac{\lambda_n}{\lambda_{n+l}}=\frac{n}{n+l}\leq 1\quad\text{and}\quad\frac{n}{n+l}\geq\frac{1}{1+l}\geq\frac{1}{1+2(k+r)}\ \forall\, n\geq 1.$$

Meanwhile, $\lim_{n\to\infty}\frac{n}{n+l}=1$ for all $0\leq l\leq 2(k+r)$. Therefore, $\frac{\lambda_n}{\lambda_{n+l}}\sim 1$. Hence, $\eta_{n,j}\lesssim\sqrt{\mu_{n,k+r}}$, by (2.8), (3.10). Thus,

$$\begin{aligned}(3.11)\quad &\left|\left|x^r\partial_x^k u\right|\right|^2\\ &\overset{(3.9)}{\sim}\alpha^{-2r}\left|\left|\sum_{n=0}^{\infty}\hat{u}_n^{\alpha,\beta}\sqrt{\mu_{n,k+r}}\sum_{i=-(k+r)}^{k+r}\mathcal{H}_{n+i}^{\alpha,\beta}(x)\right|\right|^2\\ &=\alpha^{-2r}\sum_{n=0}^{\infty}\hat{u}_n^{\alpha,\beta}\sqrt{\mu_{n,k+r}}\sum_{i=-(k+r)}^{k+r}\sum_{l=0}^{\infty}\hat{u}_l^{\alpha,\beta}\sqrt{\mu_{l,k+r}}\left\langle\mathcal{H}_{n+i}^{\alpha,\beta}(x),\sum_{j=-(k+r)}^{k+r}\mathcal{H}_{l+j}^{\alpha,\beta}(x)\right\rangle.\end{aligned}$$

It is clear that the scalar product in (3.11) is nonzero only if $l=n+i-j$. And $\mu_{n,k+r}\sim\mu_{n+i-j,k+r}$ for all $-(k+r)\leq i,j\leq k+r$. It can be verified by (2.8) and (3.10). Therefore,

$$\begin{aligned}&\left|\left|x^r\partial_x^k u\right|\right|^2\\ &\overset{(3.11)}{\sim}\ \alpha^{-2r}\sum_{n=0}^{\infty}\mu_{n,k+r}\hat{u}_n^{\alpha,\beta}\sum_{\tilde{l}=-2(k+r)}^{2(k+r)}\hat{u}_{n+\tilde{l}}^{\alpha,\beta}\\ &\leq\ \alpha^{-2r}\sum_{n=0}^{\infty}\mu_{n,k+r}\sum_{\tilde{l}=-2(k+r)}^{2(k+r)}\left|\hat{u}_n^{\alpha,\beta}\right|\left|\hat{u}_{n+\tilde{l}}^{\alpha,\beta}\right|\\ &\leq\ \alpha^{-2r}\sum_{n=0}^{\infty}\mu_{n,k+r}\frac{1}{2}\sum_{\tilde{l}=-2(k+r)}^{2(k+r)}\left(\left|\hat{u}_n^{\alpha,\beta}\right|^2+\left|\hat{u}_{n+\tilde{l}}^{\alpha,\beta}\right|^2\right)\\ &=\ \alpha^{-2r}\sum_{n=0}^{\infty}\mu_{n,k+r}\left[2(k+r)\left|\hat{u}_n^{\alpha,\beta}\right|^2+\frac{1}{2}\sum_{\tilde{l}=-2(k+r)}^{2(k+r)}\left|\hat{u}_{n+\tilde{l}}^{\alpha,\beta}\right|^2\right]\\ &=\ 2(k+r)\alpha^{-2r}\sum_{n=0}^{\infty}\mu_{n,k+r}\left|\hat{u}_n^{\alpha,\beta}\right|^2+\frac{1}{2}\alpha^{-2r}\sum_{\tilde{n}=0}^{\infty}\sum_{\tilde{l}=-2(k+r)}^{2(k+r)}\mu_{\tilde{n}-\tilde{l},k+r}\left|\hat{u}_{\tilde{n}}^{\alpha,\beta}\right|^2\\ &\sim\ \alpha^{-2r}4(k+r)\sum_{n=0}^{\infty}\mu_{n,k+r}\left|\hat{u}_n^{\alpha,\beta}\right|^2\lesssim\alpha^{-2r}(k+r)||u||^2_{\mathcal{W}_{\alpha,\beta}^{k+r}(\mathbb{R})}.\end{aligned}$$

Until now, we have shown that (3.12) holds for $d=1$. For $d\geq 2$, we shall proceed the argument similarly as for $d = 1$. We then have

$$\begin{aligned}
\left|\left|\boldsymbol{x}^{\boldsymbol{r}}\boldsymbol{\partial}_{\boldsymbol{x}}^{\boldsymbol{k}}u\right|\right|^2 &= \left(\prod_{\tilde{i}=1}^{d}\alpha_{\tilde{i}}^{-2r_{\tilde{i}}}\right)\left|\left|\sum_{\boldsymbol{n}\in\mathbb{N}_0^d}\hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}\sum_{-(\boldsymbol{k}+\boldsymbol{r})\le\boldsymbol{i}\le\boldsymbol{k}+\boldsymbol{r}}\boldsymbol{\eta}_{\boldsymbol{n},\boldsymbol{i}}\boldsymbol{\mathcal{H}}_{\boldsymbol{n}+\boldsymbol{i}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{x})\right|\right|^2 \\
&\sim \left(\prod_{\tilde{i}=1}^{d}\alpha_{\tilde{i}}^{-2r_{\tilde{i}}}\right)\left|\left|\sum_{\boldsymbol{n}\in\mathbb{N}_0^d}\hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}\sqrt{\boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{k}+\boldsymbol{r}}}\sum_{-(\boldsymbol{k}+\boldsymbol{r})\le\boldsymbol{i}\le(\boldsymbol{k}+\boldsymbol{r})}\boldsymbol{\mathcal{H}}_{\boldsymbol{n}+\boldsymbol{i}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{x})\right|\right|^2 \\
&\lesssim \left(\prod_{\tilde{i}=1}^{d}\alpha_{\tilde{i}}^{-2r_{\tilde{i}}}\right)\sum_{\boldsymbol{n}\in\mathbb{N}_0^d}\boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{k}+\boldsymbol{r}}\sum_{-2(\boldsymbol{k}+\boldsymbol{r})\le\tilde{\boldsymbol{l}}\le2(\boldsymbol{k}+\boldsymbol{r})}\left(\left|\hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}\right|^2+\left|\hat{u}_{\boldsymbol{n}+\tilde{\boldsymbol{l}}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}\right|^2\right) \\
&\sim \left(\prod_{\tilde{i}=1}^{d}\alpha_{\tilde{i}}^{-2r_{\tilde{i}}}\right)|\boldsymbol{k}+\boldsymbol{r}|_{\mathrm{mix}}\sum_{\boldsymbol{n}\in\mathbb{N}_0^d}\boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{k}+\boldsymbol{r}}\left|\hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}\right|^2 \\
&\lesssim \left(\prod_{\tilde{i}=1}^{d}\alpha_{\tilde{i}}^{-2r_{\tilde{i}}}\right)|\boldsymbol{k}+\boldsymbol{r}|_{\mathrm{mix}}\cdot||u||^2_{\mathcal{W}_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{|\boldsymbol{k}|_1+|\boldsymbol{r}|_1}(\mathbb{R}^d)}.
\end{aligned}$$

Therefore, we obtain the desired result. ∎

COROLLARY 3.2. *For $u\in\mathcal{W}_{\boldsymbol{\alpha},\boldsymbol{\beta}}^m(\mathbb{R}^d)$, we have $||u||_{H^m(\mathbb{R}^d)}\lesssim||u||_{\mathcal{W}_{\boldsymbol{\alpha},\boldsymbol{\beta}}^m(\mathbb{R}^d)}$ for all $m\ge0$.*

Proof. Compared the definitions of $W_{\boldsymbol{\alpha},\boldsymbol{\beta}}^m(\mathbb{R}^d)$ and $H^m(\mathbb{R}^d)$ in (2.19) and (3.7), it remains to show that

$$\left|\left|\boldsymbol{\partial}_{\boldsymbol{x}}^{\boldsymbol{k}}u\right|\right|^2\lesssim\left|\left|\boldsymbol{\mathcal{D}}_{\boldsymbol{x}}^{\boldsymbol{k}}u\right|\right|^2\overset{(2.15)}{=}\sum_{\boldsymbol{n}\in\mathbb{N}_0^d}\boldsymbol{\mu}_{\boldsymbol{n},\boldsymbol{k}}\left|\hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}\right|^2 \tag{3.12}$$

for all $0\le|\boldsymbol{k}|_1\le m$. The desired result is followed immediately from Lemma 3.1 by letting $\boldsymbol{r}=\boldsymbol{0}$, i.e.,

$$\left|\left|\boldsymbol{\partial}_{\boldsymbol{x}}^{\boldsymbol{k}}u\right|\right|^2\lesssim|\boldsymbol{k}|_{\mathrm{mix}}\cdot\left|\left|\boldsymbol{\mathcal{D}}_{\boldsymbol{x}}^{\boldsymbol{k}}u\right|\right|^2. \qquad ∎$$

The convergence rate of the HSM with the HC approximation under the assumptions (C_1)–(C_3) is as follows.

THEOREM 3.3. *Assume that conditions (C_1)–(C_3) are satisfied, and the solution $u\in L^\infty(0,T;\mathcal{K}_{\boldsymbol{\alpha},\boldsymbol{\beta}}^m(\mathbb{R}^d))\cap L^2(0,T;\mathcal{K}_{\boldsymbol{\alpha},\boldsymbol{\beta}}^m(\mathbb{R}^d))$, for $m>1$. Let u_N be the approximate solution obtained by HSM (3.3), then*

$$||u-u_N||(t)\lesssim c^*N^{\frac{1-m}{2}},$$

where c^* depends on $\boldsymbol{\alpha}$, the norms of $L^2(0,T;\mathcal{K}_{\boldsymbol{\alpha},\boldsymbol{\beta}}^m(\mathbb{R}^d))$ and $L^\infty(0,T;\mathcal{K}_{\boldsymbol{\alpha},\boldsymbol{\beta}}^m(\mathbb{R}^d))$.

Proof. For the notational convenience, we denote $U_N=P_N^{\boldsymbol{\alpha},\boldsymbol{\beta}}u$. It is readily verified that

$$\langle\partial_t(u-U_N),\varphi\rangle=0 \quad\Rightarrow\quad \langle\partial_tU_N,\varphi\rangle=\langle-Lu+f,\varphi\rangle \quad \forall\,\varphi\in X_N^{\boldsymbol{\alpha},\boldsymbol{\beta}}. \tag{3.13}$$

Combined with the formulation of Hermite spectral method (3.3), we have

$$\begin{aligned}
\langle\partial_t(U_N-u_N),\varphi\rangle =&\langle-Lu+f,\varphi\rangle+\mathcal{A}(u_N,\varphi)+\langle f,\varphi\rangle=\mathcal{A}(u_N-u,\varphi) \\
=&-\mathcal{A}(u-U_N,\varphi)-\mathcal{A}(U_N-u_N,\varphi) \quad \forall\,\varphi\in X_N^{\boldsymbol{\alpha},\boldsymbol{\beta}}.
\end{aligned}$$

Take $\varphi = 2(U_N - u_N) \in X_N^{\boldsymbol{\alpha},\boldsymbol{\beta}}$, then

$$\begin{aligned}\partial_t ||U_N - u_N||^2 &= -2\mathcal{A}(u - U_N, U_N - u_N) - 2\mathcal{A}(U_N - u_N, U_N - u_N) \\ &\overset{(3.5),(3.6)}{\leq} 2C||u - U_N||_{H_0^1(\mathbb{R}^d)}||U_N - u_N||_{H_0^1(\mathbb{R}^d)} - 2c||U_N - u_N||^2_{H_0^1(\mathbb{R}^d)} \\ &\lesssim ||u - U_N||^2_{H_0^1(\mathbb{R}^d)}, \quad \text{by Young's inequality.}\end{aligned}$$

With Corollaries 3.2 and 2.5 (if OHC approximation is considered), we have

$$\begin{aligned}&\partial_t ||U_N - u_N||^2 \lesssim ||u - U_N||^2_{\mathcal{W}^1_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)} \lesssim N^{1-m}|u|^2_{\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)} \\ \Rightarrow \quad &||U_N - u_N||^2(t) \lesssim N^{\frac{1-m}{2}}\left[\int_0^t |u|^2_{\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)}(s)ds\right]^{\frac{1}{2}}.\end{aligned}$$

The same estimate holds for RHC approximation with Corollary 2.5 replaced by Corollary 2.3. And then, it yields that

$$\begin{aligned}||u - u_N||(t) &\leq ||u - U_N||(t) + ||U_N - u_N||(t) \\ &\lesssim N^{-\frac{m}{2}}|u|_{\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)}(t) + N^{\frac{1-m}{2}}\left[\int_0^t |u|^2_{\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)}(s)ds\right]^{\frac{1}{2}} \lesssim c^* N^{\frac{1-m}{2}},\end{aligned}$$

where c^* depends on $\boldsymbol{\alpha}$, the norms of $L^2(0,T;\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d))$ and $L^\infty(0,T;\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d))$. ∎

However, the assumptions (C_1) and (C_2) are not easy to verify. In what follows, we make assumptions on the operator L and the convergence rate of the HSM is investigated under the conditions below. Assume the following:

(C_4) The operator L (c.f. (3.2)) is strongly elliptic and uniformly bounded, i.e.,

$$\sum_{i,j=1}^d a_{ij}(\boldsymbol{x})\xi_i\xi_j \geq \theta|\xi|^2 \quad \forall \xi \in \mathbb{R}^d \quad \text{and} \quad ||\boldsymbol{A}||_\infty = \max_{i,j=1,\cdots,d}||a_{ij}||_\infty < \infty,$$

for $\boldsymbol{x} \in \mathbb{R}^d$, where $\theta > 0$.

(C_5) There exists some constant $C > 0$, such that

$$c(\boldsymbol{x}) - \frac{1}{2}\nabla \cdot \boldsymbol{b}(\boldsymbol{x}) \geq -C$$

for all $\boldsymbol{x} \in \mathbb{R}^d$.

(C_6) There exist some integer indices $\boldsymbol{\gamma}, \boldsymbol{\delta} \in \mathbb{N}_0^d$, such that

$$c(\boldsymbol{x}) \lesssim 1 + \boldsymbol{x}^{2\boldsymbol{\gamma}} \quad \text{and} \quad b_i(\boldsymbol{x}) \lesssim 1 + \boldsymbol{x}^{2\boldsymbol{\delta}} \quad \forall\, i = 1, 2, \ldots, d,$$

for all $\boldsymbol{x} \in \mathbb{R}^d$.

THEOREM 3.4. *Assume that conditions* (C_3)–(C_6) *are satisfied and the solution to* (3.1) $u \in L^2(0,T;\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d))$, *for some integer* $m > \max\{|\boldsymbol{\gamma}|_1, |\boldsymbol{\delta}|_1 + 1\}$, *and let* u_N *be the approximate solution obtained by HSM* (3.3), *then*

$$||u - u_N||(t) \lesssim c^\sharp N^{\frac{\max\{|\boldsymbol{\gamma}|_1, |\boldsymbol{\delta}|_1+1\}-m}{2}},$$

where $c^\sharp$ *depends on* $\boldsymbol{\alpha}$, T, *and the norm of* $L^2(0,T;\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d))$.

Proof. Similarly as we did in the proof of Theorem 3.3, denote $U_N = P_N^{\boldsymbol{\alpha},\boldsymbol{\beta}} u$ for convenience, and let $\varphi = 2(U_N - u_N) \in X_N^{\boldsymbol{\alpha},\boldsymbol{\beta}}$, then

$$\partial_t ||U_N - u_N||^2 = -2\mathcal{A}(u - U_N, U_N - u_N) - 2\mathcal{A}(U_N - u_N, U_N - u_N) := V_1 + V_2, \tag{3.14}$$

where $\mathcal{A}$ is defined in (3.4). For V_2,

$$\begin{aligned} -\frac{1}{2}V_2 &= \int_{\mathbb{R}^d} (\nabla(U_N - u_N))^T \boldsymbol{A}(\nabla(U_N - u_N)) + \int_{\mathbb{R}^d} (U_N - u_N)\boldsymbol{b} \cdot \nabla(U_N - u_N) \\ &\quad + \int_{\mathbb{R}^d} c(U_N - u_N)^2 \\ &= \int_{\mathbb{R}^d} (\nabla(U_N - u_N))^T \boldsymbol{A}(\nabla(U_N - u_N)) + \int_{\mathbb{R}^d} \left(c - \frac{1}{2}\nabla \cdot \boldsymbol{b}\right)(U_N - u_N)^2 \\ &\overset{(\boldsymbol{C_4}),(\boldsymbol{C_5})}{\geq} \theta||\nabla(U_N - u_N)||^2 - C||U_N - u_N||^2. \end{aligned} \tag{3.15}$$

Meanwhile for V_1,

$$\begin{aligned} |V_1| &= 2\left[\int_{\mathbb{R}^d} (\nabla(u - U_N))^T \boldsymbol{A}(\nabla(U_N - u_N)) + \int_{\mathbb{R}^d} (U_N - u_N)\boldsymbol{b} \cdot \nabla(u - U_N)\right. \\ &\quad \left. + \int_{\mathbb{R}^d} c(u - U_N)(U_N - u_N)\right] \\ &\leq 2[||\boldsymbol{A}||_\infty ||\nabla(u - U_N)|| \cdot ||\nabla(U_N - u_N)|| + ||\boldsymbol{b} \cdot \nabla(u - U_N)|| \cdot ||U_N - u_N|| \\ &\quad + ||c(u - U_N)|| \cdot ||U_N - u_N||] \\ &\lesssim C_{||\boldsymbol{A}||_\infty,\theta} ||\nabla(u - U_N)||^2 + 2\theta||\nabla(U_N - u_N)||^2 \\ &\quad + ||\boldsymbol{b} \cdot \nabla(u - U_N)||^2 + ||c(u - U_N)||^2 \\ &\quad + ||U_N - u_N||^2. \end{aligned} \tag{3.16}$$

On the right-hand side of (3.16), the third and fourth terms are to be estimated:

$$\begin{aligned} ||c(u - U_N)||^2 &\overset{(\boldsymbol{C_6})}{\lesssim} ||(1 + \boldsymbol{x}^{2\boldsymbol{\gamma}})(u - U_N)||^2 \lesssim ||u - U_N||^2 + ||\boldsymbol{x}^{2\boldsymbol{\gamma}}(u - U_N)||^2 \\ &\lesssim ||u - U_N||^2 + \left(\prod_{i=1}^d \alpha_i^{-4\gamma_i}\right) |\boldsymbol{\gamma}|_{\mathrm{mix}} \cdot ||u - U_N||^2_{\mathcal{W}^{|\boldsymbol{\gamma}|_1}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)} \end{aligned} \tag{3.17}$$

by Lemma 3.1. Similarly, from (C_6) again, we deduce that

$$\begin{aligned} &||\boldsymbol{b} \cdot \nabla(u - U_N)||^2 \\ &\leq \sum_{i=1}^d ||b_i(\boldsymbol{x})\partial_{x_i}(u - U_N)||^2 \lesssim \sum_{i=1}^d \left|\left|(1 + \boldsymbol{x}^{2\boldsymbol{\delta}})\partial_{x_i}(u - U_N)\right|\right|^2 \\ &\leq \sum_{i=1}^d ||\partial_{x_i}(u - U_N)||^2 + \sum_{i=1}^d \left|\left|\boldsymbol{x}^{2\boldsymbol{\delta}}\partial_{x_i}(u - U_N)\right|\right|^2 \\ &\lesssim ||u - U_N||^2_{\mathcal{W}^1_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)} + \sum_{i=1}^d \left(\prod_{i=1}^d \alpha_i^{-4\delta_i}\right) |\boldsymbol{\delta} + \boldsymbol{e}_i|_{\mathrm{mix}} \cdot ||u - U_N||^2_{\mathcal{W}^{|\boldsymbol{\delta}|_1+1}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)} \\ &\lesssim ||u - U_N||^2_{\mathcal{W}^1_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)} + d\left(\prod_{i=1}^d \alpha_i^{-4\delta_i}\right) |\boldsymbol{\delta} + 1|_{\mathrm{mix}} \cdot ||u - U_N||^2_{\mathcal{W}^{|\boldsymbol{\delta}|_1+1}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)}. \end{aligned} \tag{3.18}$$

Combining (3.14)–(3.16), we have

$$\begin{aligned}\partial_t||u_N-U_N||^2 &\lesssim ||\nabla(u-U_N)||^2\\ &\qquad +||\boldsymbol{b}\cdot\nabla(u-U_N)||^2+||c(u-U_N)||^2+C||u_N-U_N||^2\\ &\overset{(3.17),(3.18)}{\lesssim} ||\nabla(u-U_N)||^2+C||u_N-U_N||^2+||u-U_N||^2_{\mathcal{W}^1_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)}\\ &\qquad +||u-U_N||^2_{\mathcal{W}^{|\boldsymbol{\delta}|_1+1}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)}+||u-U_N||^2_{\mathcal{W}^{|\boldsymbol{\gamma}|_1}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)}\\ &\lesssim C||u_N-U_N||^2+N^{\max\{|\boldsymbol{\gamma}|_1,|\boldsymbol{\delta}|_1+1\}-m}|u|^2_{\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)},\end{aligned}$$

by Corollary 2.3 or Corollary 2.5. Hence,

$$\begin{aligned}||u_N-U_N||^2(t)&\le e^{Ct}||u_N-U_N||^2(0)\\ &\qquad +N^{\max\{|\boldsymbol{\gamma}|_1,|\boldsymbol{\delta}|_1+1\}-m}e^{Ct}\int_0^t e^{-Cs}|u|^2_{\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)}(s)ds\\ &\le N^{\max\{|\boldsymbol{\gamma}|_1,|\boldsymbol{\delta}|_1+1\}-m}\int_0^t e^{C(t-s)}|u|^2_{\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)}(s)ds.\end{aligned}$$

Therefore,

$$\begin{aligned}||u-u_N||^2(t)&\le ||u-U_N||^2(t)+||u_N-U_N||^2(t)\\ &\lesssim N^{1-m}|u|^2_{\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)}(t)+N^{\max\{|\boldsymbol{\gamma}|_1,|\boldsymbol{\delta}|_1+1\}-m}\int_0^t e^{C(t-s)}|u|^2_{\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)}(s)ds\\ &\lesssim N^{\max\{|\boldsymbol{\gamma}|_1,|\boldsymbol{\delta}|_1+1\}-m}\int_0^T |u|^2_{\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)}(s)ds.\end{aligned}$$

The desired result is obtained. □

4. Numerical results.

4.1. HC approximations with Hermite functions. In Figure 4.1, we display the indices of RHC and OHC (with $\gamma = 0.5$) in dimension 2 with $N = 31$. It is clear to see that the indices of OHC is a subset of RHC. Furthermore, we list the number of indices of $N = 31$ with dimensions ranging from 2 to 5.

dim	2	3	4	5
# of indices in RHC	176	712	2485	7922
# of indices in OHC ($\gamma = 0.5$)	136	440	1264	3392

It is well known that the abscissas of Hermite polynomials are nonnested, except the origin. It will lead to more numbers of points than those nested quadrature, such as Chebyshev polynomials. However, the number is still dramatically reduced, compared to the full grids. We list the abscissas of RHC, OHC, and the full grid of $N = 31$ with dimensions ranging from 2 to 4.

dim	2	3	4
# of abscissas in OHC ($\gamma = 0.5$)	108	3348	28944
# of abscissas in RHC	298	6612	82704
# of abscissas in full grid	961	29791	923521

It is clear that the abscissa in RHC/OHC is much fewer than those in the full grid.

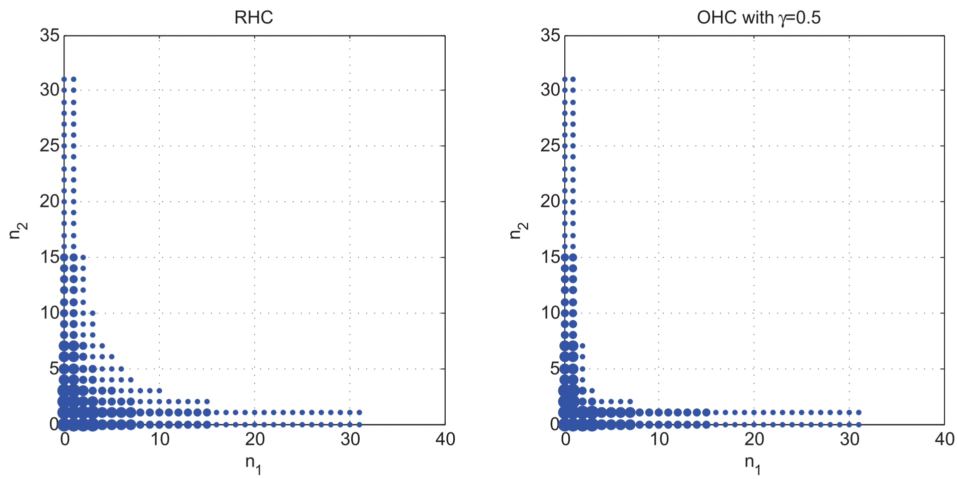

FIG. 4.1. *For $d = 2$, $N = 31$. Left: the index set $\mathbf{\Omega}_N$ of RHC. Right: the index set $\mathbf{\Omega}_{N,\gamma}$ of OHC with $\gamma = 0.5$.*

4.2. HSM with sparse grid. Although the HC approximation is theoretically feasible, it is not suitable for practical implementations, due to the unclarity "combining effecting" of the product rules, i.e., how to determine the weights from different combinations of one-dimensional (1-D) Gauss–Hermite quadrature. Thus, in this subsection, we stick to Smolyak's algorithm [27] to test the accuracy of high-dimensional HSM applying to linear parabolic PDE.

Let us recall that Smolyak's algorithm is given as

$$\mathcal{I}(L,d) = \sum_{L-d+1 \le |\boldsymbol{i}|_1 \le L} (-1)^{L-|\boldsymbol{i}|_1} \binom{d-1}{L-|\boldsymbol{i}|_1} (\mathcal{U}^{i_1} \otimes \cdots \otimes \mathcal{U}^{i_d}),$$

where $\mathcal{U}^i$ is an indexed family of 1D quadrature, $\boldsymbol{i}$ is the 1D level, $\boldsymbol{i} = (i_1, \ldots, i_d)$ is the level vector, and L is the max level. The sparse grid is formed by weighted combinations of those product rules whose product level $|\boldsymbol{i}|_1$ falls between $L - d + 1$ and L.

In Figure 4.2, we display the abscissas of the Hermite functions and the index set with level L ranging from 2 to 4 in $d = 2$.

Let us test the accuracy with the following linear parabolic PDE:

$$\begin{cases} \partial_t u = \triangle u - \displaystyle\sum_{i=1}^{d} x_i^2 u + f(\boldsymbol{x}, t), \\ u(\boldsymbol{x}, 0) = \left(\displaystyle\sum_{i=1}^{d} x_i \right) e^{-\frac{1}{2}(x_1^2 + \cdots + x_d^2)}, \end{cases}$$

where $\triangle$ is the Laplacian operator,

$$f(\boldsymbol{x}, t) = \left[\cos t + d \sin t + (d+2) \sum_{i=1}^{d} x_i \right] e^{-\frac{1}{2}(x_1^2 + \cdots x_d^2)}.$$

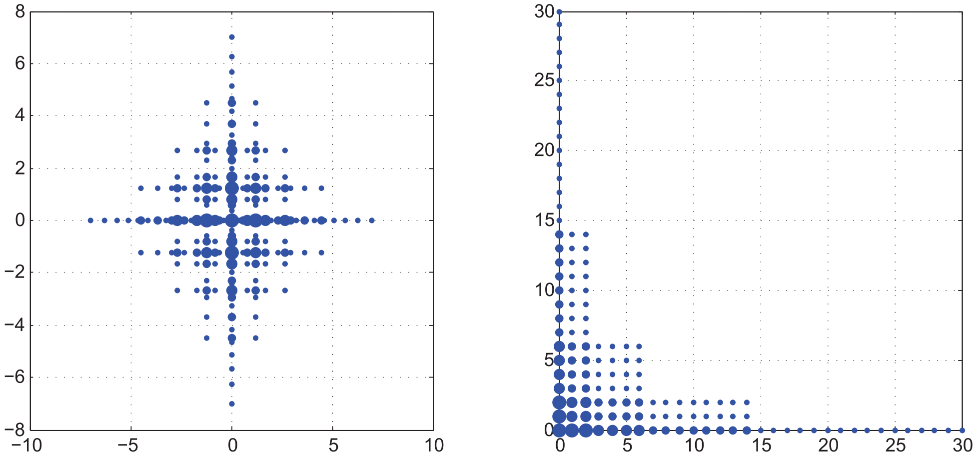

FIG. 4.2. *In $d = 2$, level L ranging from 2 to 4. Left: the abscissas of Hermite functions. Right: the indices in the index set. The larger the dot is, the lower level it belongs to.*

By direct computations, the exact solution to this PDE is

$$u(\boldsymbol{x}, t) = \left(\sum_{i=1}^{d} x_i + \sin t \right) e^{-\frac{1}{2}(x_1^2 + \cdots + x_d^2)}.$$

It is known from [19] that the best scaling factor is $\boldsymbol{\alpha} = \mathbf{1}$ in this case, since the first two Hermite functions will resolve the exact solution perfectly only with the round-off errors (around 10^{-16} on my computer). To make the convergence rate observable with respect to the level L, we shall choose the scaling factor $\boldsymbol{\alpha}$ to be $1.01 \times \mathbf{1}$.

The corresponding spectral scheme (cf. (3.3), (3.4)) is as follows:

$$\begin{cases} \langle \partial_t u_N(t), \varphi \rangle = -\langle \nabla u_N, \nabla \varphi \rangle - \displaystyle\sum_{i=1}^{d} \langle x_i^2 u_N, \varphi \rangle + \langle f, \varphi \rangle \\ \qquad u_N(0) = P_N u_0 \end{cases} \tag{4.1}$$

for all $\varphi \in X_N$. Here, we choose $X_N = X_N^{\alpha,\beta} = \text{span}\{\mathcal{H}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} : \boldsymbol{\Omega}_N \text{ from Smolyak}\}$. Thus, we can write the numerical solution as

$$u_N(\boldsymbol{x}, t) = \sum_{\boldsymbol{n} \in \boldsymbol{\Omega}_N} a_{\boldsymbol{n}}(t) \mathcal{H}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{x}).$$

Taking $\varphi(\boldsymbol{x}) = \mathcal{H}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{x})$ in (4.1). Due to (2.6), (2.5), and (2.14), we arrive at an ODE,

$$\begin{cases} \partial_t a_{\boldsymbol{n}} = A a_{\boldsymbol{n}} + \hat{f}_{\boldsymbol{n}}, \\ a_{\boldsymbol{n}}(0) = (\hat{u}_0)_{\boldsymbol{n}}, \end{cases} \tag{4.2}$$

where $\hat{f}_{\boldsymbol{n}}$ (resp., $(\hat{u}_0)_{\boldsymbol{n}}$) is the Hermite coefficients of f (resp., u_0) and the matrix A comes from the Laplacian operator and the potential. We display the nonzero entries of the matrix A for dimensions 3 and 4 with level= 4 in Figure 4.3.

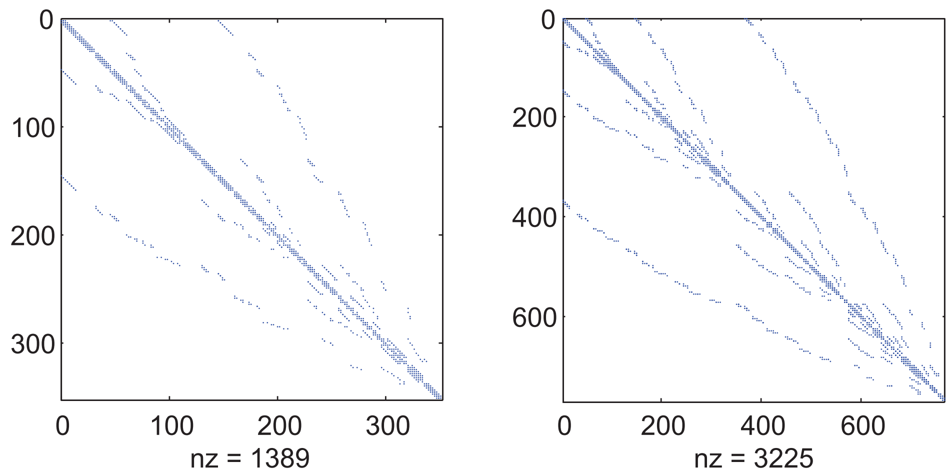

FIG. 4.3. *The nonzero entries in the matrix A (cf. (4.2)) are displayed with level $= 4$. Left: $d = 3$, Right: $d = 4$.*

We adopt the central difference scheme to solve (4.2) with $T = 0.1$, $dt = 10^{-5}$, $\boldsymbol{\alpha} = 1.01 \times \mathbf{1}$, and $\boldsymbol{\beta} = \mathbf{0}$. Figure 4.4 shows the L^2-norm of $(u_N - u_{\text{exact}})$ with respect to the level in dimension ranging from 2 to 4. It is exactly what we expect that in the semilog plot the error goes down almost along a straight line, which indicates that the convergence rate is nearly exponential decaying. However, with the dimension grows, the error becomes slightly larger. It reveals that the convergence rate still slightly deteriorates with the dimension increasing.

level/dim	2	3	4
2	2.24E-03	7.99E-03	n/a
3	3.99E-04	544E-03	2.10E-02
4	4.75E-06	1.93E-03	1.14E-02
5	2.72E-07	2.66E-04	4.11E-03

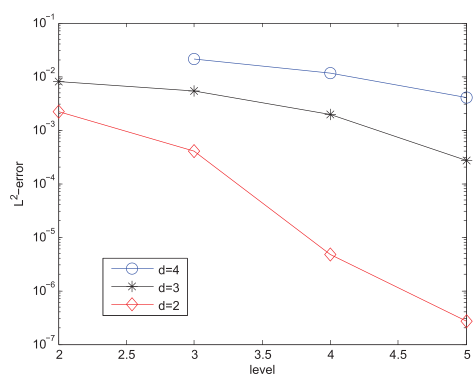

FIG. 4.4. *The L^2 error of u_N with respect to the level in $d = 2, 3$, and 4 is drawn.*

5. Conclusion. In this paper, we consider the HC approximation with generalized Hermite functions. We established the error estimate in the appropriate space for both RHC and OHC. Furthermore, the error estimate of the dimensional adaptive approximation is obtained with respect to the dependence of dimension. As an application, the HC approximation is applied to high-dimensional linear parabolic PDEs. We investigated the convergence rate of the Galerkin-type HSM in the suitable weighted Korobov space. It is shown to be exponential convergent. Moreover, the numerical simulation supports our theoretical proofs.

Appendix A. Proof of Theorem 2.1.

Proof of Theorem 2.1. Let $\mathbf{\Omega}_N^c = \{\boldsymbol{n} \in \mathbb{N}_0^d : \ |\boldsymbol{n}|_\infty > N\}$. By (2.17), (2.20), and (2.21),

$$\text{(A.1)} \qquad \left|P_N^{\boldsymbol{\alpha},\boldsymbol{\beta}} u - u\right|^2_{\mathcal{W}^l_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)} = \sum_{j=1}^d \sum_{\boldsymbol{n}\in\mathbf{\Omega}_N^c} \mu_{n_j,l} \left|\hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}\right|^2.$$

For any $1 \le j \le d$,

$$\text{(A.2)} \qquad \sum_{\boldsymbol{n}\in\mathbf{\Omega}_N^c} \mu_{n_j,l} \left|\hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}\right|^2 = \sum_{\boldsymbol{n}\in\mathbf{\Lambda}_N^{1,j}} \mu_{n_j,l} \left|\hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}\right|^2 + \sum_{\boldsymbol{n}\in\mathbf{\Lambda}_N^{2,j}} \mu_{n_j,l} \left|\hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}\right|^2 := I_1 + I_2,$$

where $\mathbf{\Lambda}_N^{1,j} = \{\boldsymbol{n} \in \mathbf{\Omega}_N^c : \ n_j > N\}$ and $\mathbf{\Lambda}_N^{2,j} = \{\boldsymbol{n} \in \mathbf{\Omega}_N^c : \ n_j \le N\}$. For I_1:

$$\text{(A.3)} \qquad I_1 \le \max_{\boldsymbol{n}\in\mathbf{\Lambda}_N^{1,j}} \left\{\frac{\mu_{n_j,l}}{\mu_{n_j,m}}\right\} \sum_{\boldsymbol{n}\in\mathbf{\Lambda}_N^{1,j}} \mu_{n_j,m} \left|\hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}\right|^2 \lesssim |\boldsymbol{\alpha}|_\infty^{2(l-m)} N^{l-m} |u|^2_{\mathcal{W}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)}.$$

In fact,

$$\begin{aligned}\max_{\boldsymbol{n}\in\mathbf{\Lambda}_N^{1,j}} \left\{\frac{\mu_{n_j,l}}{\mu_{n_j,m}}\right\} &= \max_{\boldsymbol{n}\in\mathbf{\Lambda}_N^{1,j}} \left\{\frac{2^{l-m}\alpha_j^{2(l-m)}}{(n_j-l)(n_j-l-1)\cdots(n_j-m+1)}\right\} \\ &\le 2^{l-m}|\boldsymbol{\alpha}|_\infty^{2(l-m)}(N-m+1)^{l-m}.\end{aligned}$$

For I_2, if $\boldsymbol{n} \in \mathbf{\Lambda}_N^{2,j}$, then there exists some $k \ne j$, such that $n_k > N$. We then have

$$\text{(A.4)} \quad I_2 \le \max_{\boldsymbol{n}\in\mathbf{\Lambda}_N^{2,j}} \left\{\frac{\mu_{n_j,l}}{\mu_{n_k,m}}\right\} \sum_{\boldsymbol{n}\in\mathbf{\Lambda}_N^{2,j}} \mu_{n_k,m} \left|\hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}\right|^2 \lesssim |\boldsymbol{\alpha}|_\infty^{2l} \left|\frac{1}{\boldsymbol{\alpha}}\right|_\infty^{2m} N^{l-m-2} |u|^2_{\mathcal{W}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)},$$

since

$$\begin{aligned}\max_{\boldsymbol{n}\in\mathbf{\Lambda}_N^{2,j}} \left\{\frac{\mu_{n_j,l}}{\mu_{n_k,m}}\right\} &= \max_{\boldsymbol{n}\in\mathbf{\Lambda}_N^{2,j}} \left\{2^{l-m}\frac{\alpha_j^{2l}}{\alpha_k^{2m}} \frac{\frac{n_j!}{(n_j-l)!}}{\frac{n_k!}{(n_k-m)!}}\right\} \le 2^{l-m}|\boldsymbol{\alpha}|_\infty^{2l}\left|\frac{1}{\boldsymbol{\alpha}}\right|_\infty^{2m} \frac{\frac{N!}{(N-l)!}}{\frac{(N+1)!}{(N+1-m)!}} \\ &= 2^{l-m}|\boldsymbol{\alpha}|_\infty^{2l}\left|\frac{1}{\boldsymbol{\alpha}}\right|_\infty^{2m} \frac{1}{N+1}\frac{1}{(N-l)(N-l-1)\cdots(N-m)} \\ &\le 2^{l-m}|\boldsymbol{\alpha}|_\infty^{2l}\left|\frac{1}{\boldsymbol{\alpha}}\right|_\infty^{2m}(N-m)^{l-m-2}.\end{aligned}$$

Combine (A.1)–(A.4), we obtain the result. Furthermore, the mix derivatives of the order equal to or less than m can be bounded by the seminorm $|u|_{\mathcal{W}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)}$. ∎

Appendix B. Dimensional adaptive approximation. The standard sparse grids are isotropic, treating all the dimensions equally. Many problems vary rapidly in only some dimensions, remaining less variable in other dimensions. In some situations, the highly changing dimensions can be recognized a priori. Consequently, it is advantageous to treat them accordingly. Without loss of generality, we assume the first d_1

dimensions are rapidly variable ones, and we wish to adopt the full grid. Meanwhile, the OHC approximation will be used in the rest of the $d_2 := d - d_1$ dimensions.

Let us denote that $\boldsymbol{n} := \boldsymbol{n}_1 \bigoplus \boldsymbol{n}_2$, where $\boldsymbol{n}_1 = (n_1, \dots, n_{d_1})$ and $\boldsymbol{n}_2 = (n_{d_1+1}, \dots, n_d)$. The index set is

(B.1)
$$\boldsymbol{\Omega}_{N_1,N_2,\gamma} := \left\{\boldsymbol{n} \in \mathbb{N}_0^d : \ |\boldsymbol{n}_1|_\infty \le N_1, \ |\boldsymbol{n}_2|_{\text{mix}} |\boldsymbol{n}_2|_\infty^{-\gamma} \le N_2^{1-\gamma}\right\} \quad \forall -\infty < \gamma < 1.$$

The complement of the index set is

$$\boldsymbol{\Omega}^c_{N_1,N_2,\gamma} := \left\{\boldsymbol{n} \in \mathbb{N}_0^d : \ |\boldsymbol{n}_1|_\infty > N_1 \quad \text{or} \quad |\boldsymbol{n}_2|_{\text{mix}} |\boldsymbol{n}_2|_\infty^{-\gamma} > N_2^{1-\gamma}\right\},$$

and the $\boldsymbol{k}$-complement of $\boldsymbol{\Omega}_{N_1,N_2,\gamma}$ is defined similarly as in (2.39):

$$\boldsymbol{\Omega}^c_{N_1,N_2,\gamma,\boldsymbol{k}} := \left\{\boldsymbol{n} \in \boldsymbol{\Omega}^c_{N_1,N_2,\gamma} : \ \boldsymbol{n} \ge \boldsymbol{k}\right\} \ \forall \boldsymbol{k} \in \mathbb{N}_0^d.$$

And the subspace $X^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{N_1,N_2}$ is defined accordingly, i.e.,

$$X^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{N_1,N_2} := \text{span}\{\boldsymbol{\mathcal{H}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\boldsymbol{n}}(\boldsymbol{x}) : \ \boldsymbol{n} \in \boldsymbol{\Omega}_{N_1,N_2,\gamma}\}, \tag{B.2}$$

so defined the projection operator $P^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{N_1,N_2,\gamma} : \ L^2(\mathbb{R}^d) \to X^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{N_1,N_2}$.

THEOREM B.1. *For any $u \in \mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)$, for $0 < l \le m$, we have*

$$\left|P^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{N_1,N_2,\gamma} u - u\right|_{W^l_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)} \lesssim |\boldsymbol{\alpha}|_\infty^{l-m} \left(N_1^{l-m} + N_2^{\frac{1-\gamma}{d-d_1-\gamma}(l-m)}\right)^{\frac{1}{2}} |u|_{\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)}.$$

Proof. Before we proceed to prove, we divide the index set $\boldsymbol{\Omega}^c_{N_1,N_2,\gamma}$ into two subsets:

$$\Gamma_1 := \{\boldsymbol{n} \in \boldsymbol{\Omega}^c_{N_1,N_2,\gamma} : \ |\boldsymbol{n}_1|_\infty > N_1\},$$
$$\Gamma_2 := \{\boldsymbol{n} \in \boldsymbol{\Omega}^c_{N_1,N_2,\gamma} : \ |\boldsymbol{n}_1|_\infty \le N_1 \text{ and } |\boldsymbol{n}_2|_{\text{mix}} |\boldsymbol{n}_2|_\infty^{-\gamma} > N_2^{1-\gamma}\}.$$

Our proof mainly follows the proof of Theorem 2.1:

$$\begin{aligned}
\left|P^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{N_1,N_2,\gamma} u - u\right|^2_{\mathcal{W}^l_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)} &\overset{(2.21)}{=} \sum_{j=1}^d \sum_{\boldsymbol{n} \in \boldsymbol{\Omega}^c_{N_1,N_2,\gamma}} \mu_{n_j,l} \left|\hat{u}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\boldsymbol{n}}\right|^2 \\
&= \sum_{j=1}^d \sum_{\boldsymbol{n} \in \Gamma_1} \mu_{n_j,l} \left|\hat{u}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\boldsymbol{n}}\right|^2 + \sum_{j=1}^d \sum_{\boldsymbol{n} \in \Gamma_2} \mu_{n_j,l} \left|\hat{u}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\boldsymbol{n}}\right|^2 \\
&:= IV_1 + IV_2.
\end{aligned} \tag{B.3}$$

For IV_1, for any $1 \le j \le d$,

$$IV_1 = \sum_{\boldsymbol{n} \in \Lambda^{1,j}_{N_1}} \mu_{n_j,l} \left|\hat{u}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\boldsymbol{n}}\right|^2 + \sum_{\boldsymbol{n} \in \Lambda^{2,j}_{N_1}} \mu_{n_j,l} \left|\hat{u}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\boldsymbol{n}}\right|^2 := IV_{1,1} + IV_{1,2},$$

where

$$\Lambda^{1,j}_{N_1} := \{\boldsymbol{n} \in \Gamma_1 : \ n_j > N_1\}, \quad \Lambda^{2,j}_{N_1} := \{\boldsymbol{n} \in \Gamma_1 : \ n_j \le N_1\}.$$

For $IV_{1,1}$:

$$IV_{1,1} \le \max_{\boldsymbol{n}\in\Lambda_{N_1}^{1,j}} \left\{ \frac{\mu_{n_j,l}}{\mu_{n_j,m}} \right\} \sum_{\boldsymbol{n}\in\Lambda_{N_1}^{1,j}} \mu_{n_j,m} \left| \hat{u}_{\boldsymbol{n}}^{ba,\boldsymbol{\beta}} \right|^2$$

$$\overset{(A.3)}{\le} 2^{l-m}|\boldsymbol{\alpha}|_\infty^{2(l-m)}(N_1-m+1)^{l-m}|u|^2_{\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)}. \tag{B.4}$$

For $IV_{1,2}$, since $\boldsymbol{n} \in \Gamma_1$, there exists some $j_0 \in \{1,\dots,d_1\}$ such that $n_{j_0} > N_1$. We have that

$$IV_{1,2} \le \max_{\boldsymbol{n}\in\Lambda_{N_1}^{2,j}} \left\{ \frac{\mu_{n_j,l}}{\mu_{n_{j_0},m}} \right\} \sum_{\boldsymbol{n}\in\Lambda_{N_1}^{2,j}} \mu_{n_{j_0},m} \left| \hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} \right|^2$$

$$\overset{(A.4)}{\le} 2^{l-m}|\boldsymbol{\alpha}|_\infty^{2l} \left| \frac{1}{\boldsymbol{\alpha}} \right|_\infty^{2m} (N_1-m)^{l-m-2}|u|_{\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)}. \tag{B.5}$$

Hence, combining (B.4) and (B.5), we have

$$IV_1 \lesssim |\boldsymbol{\alpha}|_\infty^{2(l-m)} N_1^{l-m}|u|^2_{\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)}. \tag{B.6}$$

For IV_2, let us deduce as in (2.46) that

$$|\boldsymbol{n}_2|_{\text{mix}}|\boldsymbol{n}_2|_\infty^{-\gamma} > N_2^{1-\gamma} \Rightarrow |\boldsymbol{n}_2|_\infty > N_2^{\frac{1-\gamma}{d-d_1-\gamma}}. \tag{B.7}$$

With the similar argument for IV_1, we write

$$IV_2 = \sum_{\boldsymbol{n}\in\Lambda_{N_2}^{1,j}} \mu_{n_j,l} \left| \hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} \right|^2 + \sum_{\boldsymbol{n}\in\Lambda_{N_2}^{2,j}} \mu_{n_j,l} \left| \hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} \right|^2 := IV_{2,1} + IV_{2,2},$$

where

$$\Lambda_{N_2}^{1,j} := \left\{ \boldsymbol{n} \in \Gamma_2 :\ n_j > N_2^{\frac{1-\gamma}{d-d_1-\gamma}} \right\}, \quad \Lambda_{N_2}^{2,j} := \left\{ \boldsymbol{n} \in \Gamma_2 :\ n_j \le N_2^{\frac{1-\gamma}{d-d_1-\gamma}} \right\}.$$

Thus,

$$IV_{2,1} \le \max_{\boldsymbol{n}\in\Lambda_{N_2}^{1,j}} \left\{ \frac{\mu_{n_j,l}}{\mu_{n_j,m}} \right\} \sum_{\boldsymbol{n}\in\Lambda_{N_2}^{1,j}} \mu_{n_j,m} \left| \hat{u}_{\boldsymbol{n}}^{ba,\boldsymbol{\beta}} \right|^2$$

$$\le 2^{l-m}|\boldsymbol{\alpha}|_\infty^{2(l-m)} \left(N_2^{\frac{1-\gamma}{d-d_1-\gamma}} - m + 1 \right)^{l-m} |u|^2_{\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)}, \tag{B.8}$$

and by (B.7), there exists some $j_0 \in \{d_1+1,\dots,d\}$ such that $n_{j_0} > N_2^{\frac{1-\gamma}{d-d_1-\gamma}}$, then

$$IV_{2,2} \le \max_{\boldsymbol{n}\in\Lambda_{N_2}^{2,j}} \left\{ \frac{\mu_{n_j,l}}{\mu_{n_{j_0},m}} \right\} \sum_{\boldsymbol{n}\in\Lambda_{N_2}^{2,j}} \mu_{n_{j_0},m} \left| \hat{u}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} \right|^2$$

$$\le 2^{l-m}|\boldsymbol{\alpha}|_\infty^{2l} \left| \frac{1}{\boldsymbol{\alpha}} \right|_\infty^{2m} \left(\left\lfloor N_2^{\frac{1-\gamma}{d-d_1-\gamma}} \right\rfloor - m \right)^{l-m-2} |u|^2_{\mathcal{K}^m_{\boldsymbol{\alpha},\beta b}(\mathbb{R}^d)}, \tag{B.9}$$

where $\lfloor \cdot \rfloor$ denotes the largest integer smaller or equal to $\cdot$. The estimate of IV_2 follows immediately from (B.8) and (B.9):

$$IV_2 \lesssim |\boldsymbol{\alpha}|_\infty^{2(l-m)} N_2^{\frac{1-\gamma}{d-d_1-\gamma}(l-m)} |u|^2_{\mathcal{K}^m_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbb{R}^d)}. \tag{B.10}$$

The desired result follows from (B.6) and (B.10). ☐

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