Direct Method for Time-Varying Nonlinear Filtering Problems

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This paper discusses how to solve a filtering problem for a class of continuous nonlinear time-varying systems via the Duncan–Mortensen–Zakai (DMZ) equation. In this paper, the original DMZ equation is changed into the Kolmogorov forward equation (KFE) by exponential transformations in each time interval, and then, under some assumptions, the KFE can be transformed into a time-varying Schrödinger equation, which can be solved explicitly. The novelty of this paper lies in how to transform the KFE into the Schrödinger equation. As a direct application, the results of the paper "Nonlinear filtering and time varying Schrodinger equation" are extended for time-varying Yau systems.

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I. INTRODUCTION

The problem of estimating the state of a stochastic dynamical system from noisy observations taken on the state is of central importance in engineering, which known as the filtering problem. The continuous time-varying filtering problem considered in this paper can be stated as follows:

$$\begin{cases} dx_t = f(x_t, t)dt + g(t)dv_t \\ dy_t = h(x_t, t)dt + dw_t \end{cases} \tag{1}$$

where $x_t,\ f \in \mathbb{R}^{n\times 1}$, g is an $n \times r$ matrix, v_t is an r-vector Brownian motion process with $E[dv_t dv_t^T] = \tilde{Q}(t)dt$ and $\tilde{Q}(t) > 0$, $y_t, h \in \mathbb{R}^{m\times 1}$, and w_t is an m-vector Brownian motion process with $E[dw_t dw_t^T] = S(t)dt$ and $S(t) > 0$. Here, we refer x_t as the state of the system at time t, $f(x_t, t)$ as the drift term, $\tilde{Q}(t)$, $S(t)$ as the variance of the noises, and y_t as the observation at time t with $y_0 = 0$.

System (1) can model most of the practical physical situations. Taking a falling body in a constant field as an example, it can be modeled by a noise-disturbed second-order system

$$\ddot{z} = g_a + \tilde{v}_t, \qquad t \geq 0 \tag{2}$$

where the scalar z is the position, g_a is the gravitational acceleration, and $\tilde{v}_t$ is the white noise due to air friction. Let the position be $z = x_1$ and the velocity $\dot{z} = x_2$. Then, defining the state vector $x_t = [x_1, x_2]^T$, (2) can be written in system form

$$\dot{x}_t = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_t + \begin{bmatrix} 0 \\ g_a \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{v}_t. \tag{3}$$

We get scalar observations of position

$$\tilde{y}_t = [1,\ 0]x_t + \tilde{w}_t \tag{4}$$

where $\tilde{w}_t$ is the white noise due to measuring error. Since the white noise can be regarded as the derivative of the Brownian motion in the Itô sense [8], we can rewrite the mathematical model as follows:

$$\begin{cases} dx_t = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_t + \begin{bmatrix} 0 \\ g_a \end{bmatrix}\right) dt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} dv_t \\ dy_t = [1,\ 0]x_t dt + dw_t \end{cases} \tag{5}$$

where $dv_t = \tilde{v}_t dt, dw_t = \tilde{w}_t dt,$ and $dy_t = \tilde{y}_t dt$.

It is widely acknowledged that the most influential work in filtering theory is the classical Kalman filter (KF) [6], which was published in 1960, and its continuous counterpart Kalman–Bucy filter [7]. Since then, there have been many derivatives of the KF in filtering theory and control area. These methods approximate certain statistical quantities, such as mean and variance, among which the extended Kalman filter (EKF) is one of the most famous ones for the nonlinear filtering (NLF) problems. Till now, KF and its derivatives are still widely used in various industrial applications and scientific problems such as tracking, communications, economics, etc. Another possible way that is called global approach in [10] is to solve for the conditional density function of the states either directly or numerically. It is known that the unnormalized density function of the states satisfies the Duncan–Mortensen–Zakai (DMZ) equation [3], [11], [21]. Recently, the second and the third author of this paper have developed a real-time algorithm for the general NLF problems without memory based on the DMZ equation [9]. Its effectiveness has been numerically verified in very low dimensional problems. For more results on this direction, we refer interested readers to the survey paper [10].

In some sense, the NLF problems are said to be completely solved if one can solve the DMZ equation in real time and in a memoryless way, since all the statistical information can be extracted from the conditional density function of the states. For the past quarter of a century, as far as we know, there are two methods to solve the DMZ equation explicitly. The first one is to use the Lie algebraic method to solve the DMZ equation via the Wei–Norman approach. The basic idea is to reduce the DMZ equation to a finite system of ordinary differential equations (ODE), the Kolmogorov equation, and several first-order linear partial differential equations (PDE). However, one must know the basis of the estimation algebra. The third author and his coworkers [4], [17] have completely classified all finite-dimensional estimation algebras of maximal rank. In particular, they have proved that all the observation terms $h_i(x)$, $1 \leq i \leq m$, must be polynomials of degree 1. Another approach is the direct method introduced in [14] and [16]. Compared with the Lie algebraic method, it does not need to know the basis. Furthermore, it is unnecessary to integrate n first-order linear PDEs, which is inevitable in the Lie algebraic method. We need to remark that all the direct methods are for the Yau systems [19], i.e., $f(x,t)$ in (1) is of the form $f(x,t) = Lx + l + \nabla\phi(x)$, where L and l are matrices with proper dimensions, and $\phi(x)$ is a C^∞ function. Though it seems restrictive, it includes Kalman–Bucy [7] and Beneš [1] filtering systems as its special cases. Under the assumption that the noises' covariances of state and observation are identity matrices and the system is time invariant, the direct method has been extensively studied in [5], [15], [16], [18], and [19].

The time-invariant system can only be seen as an ideal model of practical applications. Thus, it is more meaningful to solve time-varying NLF problems. In this paper, we shall consider filtering problems for the time-varying Yau systems with time-varying covariances, which extends the results of that in [19]. The novelty of this paper lies in how to transform the Kolmogorov forward equation (KFE) into a time-varying Schrödinger equation with respect to (w.r.t.) the time-varying nonlinear systems, since the corresponding DMZ equation is much more complicated than that in [19].

This paper is organized as follows. The basic model and some preliminary results are stated in Section II. In Section III, we shall construct a transformation to change the Kolmogorov equation into the one without drift term, which is stated in Theorem 1. With further assumption on potential, we solve the KFE formally and directly by the power series method in Section IV. In Section V, we use the direct method to solve a numerical example and compare it with the EKF. We arrive at the conclusion in Section VI.

II. PRELIMINARIES

First, we give some assumptions in terms of system (1). We assume that $G(t) \triangleq g(t)\tilde{Q}(t)g^T(t)$ is C^∞ smooth, and $f(x,t)$ and $h(x,t)$ are C^∞ smooth in both state and time. For the sake of clarity, we state some notations first: $*_{ij}$ denotes the ij-entry of the matrix $*$, $*_i$ denotes the ith element of the vector $*$, and $*^T$ denotes the transposition of $*$.

The unnormalized density function $\sigma(t,x)$ of x_t conditioned on the observation history $\mathscr{F}_t \triangleq \{y_s : 0 \leq s \leq t\}$ satisfies the DMZ equation [3]:

$$\begin{cases} d\sigma(t,x) = L\sigma(t,x)dt + \sigma(t,x)h^T(x,t)S^{-1}(t)dy_t \\ \sigma(0,x) = \sigma_0(x) \end{cases} \tag{6}$$

where $\sigma_0(x)$ is the probability density of the initial state x_0, and

$$L(*) \equiv \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\left[G_{ij}(t)*\right] - \sum_{i=1}^{n} \frac{\partial (f_i *)}{\partial x_i}. \tag{7}$$

For each arrived observation, making an invertible exponential transformation [12]

$$u(t,x) = \exp\left[-h^T(x,t)S^{-1}(t)y_t\right]\sigma(t,x) \tag{8}$$

the DMZ equation is transformed into a deterministic PDE with stochastic coefficients

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = & \frac{1}{2}\sum_{i,j=1}^{n} G_{ij}(t)\frac{\partial^2 u}{\partial x_i \partial x_j}(t,x) - \sum_{i=1}^{n} f_i \frac{\partial u}{\partial x_i}(t,x) \\ & + \left(-\frac{\partial}{\partial t}\left(h^T S^{-1}\right)^T y_t \right. \\ & + \frac{1}{2}\sum_{i,j=1}^{n} G_{ij}(t)\left[\frac{\partial^2 \tilde{K}}{\partial x_i \partial x_j} + \frac{\partial \tilde{K}}{\partial x_i}\frac{\partial \tilde{K}}{\partial x_j}\right] \\ & - \sum_{i=1}^{n} f_i \frac{\partial \tilde{K}}{\partial x_i}(t,x) - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(t,x) \\ & \left. - \frac{1}{2}\left(h^T S^{-1} h\right)\right) u(t,x) \\ u(0,x) = & \sigma_0(x) \end{cases} \tag{9}$$

in which

$$\tilde{K}(x,t) = h^T(x,t)S^{-1}(t)y_t. \tag{10}$$

We shall call (9) "pathwise-robust" DMZ equation in this paper. However, the exact solution to (9), generally speaking, does not have a closed form. Therefore, many mathematicians try to seek an efficient algorithm to construct a good approximation. Let us assume that the observations arrive at discrete instants; therefore, we construct the approximation as in [9] and get the robust DMZ equation (11) in each time interval.

Let us denote the observation time sequence as $\mathcal{P}_k = \{0 = \tau_0 < \tau_1 < \cdots < \tau_N = T\}$. Let u_k be the solution of the robust DMZ equation with $y_t = y_{\tau_{k-1}}$ on the time interval $\tau_{k-1} \le t \le \tau_k,\ k = 1, 2, \ldots, N,$

$$\begin{cases} \dfrac{\partial u_k}{\partial t}(t,x) = \dfrac{1}{2}\sum_{i,j=1}^{n} G_{ij}(t)\dfrac{\partial^2 u_k}{\partial x_i \partial x_j}(t,x) - \sum_{i=1}^{n} f_i \dfrac{\partial u_k}{\partial x_i}(t,x) \\ \qquad + \left(-\dfrac{\partial}{\partial t}\left(h^T S^{-1}\right)^T y_{\tau_{k-1}} \right. \\ \qquad + \dfrac{1}{2}\sum_{i,j=1}^{n} G_{ij}(t)\left[\dfrac{\partial^2 K}{\partial x_i \partial x_j} + \dfrac{\partial \tilde{K}}{\partial x_i}\dfrac{\partial \tilde{K}}{\partial x_j}\right] \\ \qquad - \sum_{i=1}^{n} f_i \dfrac{\partial \tilde{K}}{\partial x_i}(t,x) - \sum_{i=1}^{n}\dfrac{\partial f_i}{\partial x_i}(t,x) \\ \qquad \left. - \dfrac{1}{2}\left(h^T S^{-1} h\right)\right) u_k(t,x) \\ u_1(0,x) = \sigma_0(x) \\ u_k(\tau_{k-1},x) = u_{k-1}(\tau_{k-1},x),\ k = 2, 3, \ldots, N \end{cases} \tag{11}$$

with

$$\tilde{K}(x,t) = h^T(x,t)S^{-1}(t)y_{\tau_{k-1}}. \tag{12}$$

Define the norm of $\mathcal{P}_k$ by $|\mathcal{P}_k| = \sup_{1\le k\le N}(\tau_k - \tau_{k-1})$. By [20], we know that in both pointwise sense and L^2 sense

$$u(\tau,x) = \lim_{|\mathcal{P}_k|\to 0} u_k(\tau,x). \tag{13}$$

Therefore, $u_k(t,x)$ is a good approximation of $u(t,x)$ in the interval $[\tau_{k-1}, \tau_k]$. We only need to seek the solution of DMZ equation (11).

In [9], the second and third authors proposed an on- and offline algorithm to solve the NLF problems in real time, which has been verified numerically as an effective tool in very low dimension. The key observation is that the heavy computation of solving PDE can be moved to offline by the following proposition.

PROPOSITION 1 (see [9, Proposition 2.1]) For each $\tau_{k-1} \le t \le \tau_k,\ k = 1, 2, \ldots, N$, $u_k(t,x)$ satisfies (11) if and only if

$$\tilde{u}_k(t,x) = \exp\left[h^T(x,t)S^{-1}(t)y_{\tau_{k-1}}\right] u_k(t,x) \tag{14}$$

satisfies the KFE

$$\frac{\partial \tilde{u}_k}{\partial t}(t,x) = \left(L - \frac{1}{2}h^T S^{-1} h\right)\tilde{u}_k(t,x) \tag{15}$$

where L is defined in (7), that is, $\tilde{u}_k(t,x)$ satisfies

$$\begin{cases} \dfrac{\partial \tilde{u}_k}{\partial t}(t,x) = \dfrac{1}{2}\sum_{i,j=1}^{n} G_{ij}(t)\dfrac{\partial^2 \tilde{u}_k}{\partial x_i \partial x_j}(t,x) - \sum_{i=1}^{n} f_i \dfrac{\partial \tilde{u}_k}{\partial x_i}(t,x) \\ \qquad - \left(\sum_{i=1}^{n}\dfrac{\partial f_i}{\partial x_i}(t,x) + \dfrac{1}{2}h^T S^{-1} h\right)\tilde{u}_k(t,x) \\ \tilde{u}_1(0,x) = \sigma_0(x), \\ \tilde{u}_k(\tau_{k-1},x) = \exp\left[h^T(x,\tau_{k-1})S^{-1}(\tau_{k-1})(y_{\tau_{k-1}} - y_{\tau_{k-2}})\right] \\ \qquad \cdot\, \tilde{u}_{k-1}(\tau_{k-1},x),\ k = 2, 3, \ldots, N. \end{cases} \tag{16}$$

III. SCHRÖDINGER EQUATION

As mentioned in Section I, all the direct methods are for the time-invariant Yau system [19]. Even though they include a large class of systems such as time-invariant Kalman–Bucy [7] and Beneš [1] filtering systems, time-varying systems are more general in real applications. In this section, we aim to extend the results to the more general time-varying Yau systems

$$f(x,t) = L(t)x + l(t) + \nabla_x \phi(t,x) \tag{17}$$

where $L(t) = (l_{ij}(t)),\ 1 \le i, j \le n,\ l^T(t) = (l_1(t), \ldots, l_n(t))$ and $\phi(t,x)$ is a C^∞ function on $\mathbb{R}^n$. For the conciseness of notation, we shall omit the t in l and L in the following if no confusion will arise.

PROPOSITION 2 Suppose $\tilde{u}_k(t,x)$ is the solution to (16) in the interval $[\tau_{k-1}, \tau_k],\ k = 1, 2, \ldots, N$ and $f(x,t)$ is of the form (17). Let

$$\tilde{u}_k(t,x) = e^{\phi(t,x)}\tilde{v}_k(t,x). \tag{18}$$

Then, we have the following equation for $\tilde{v}_k(t,x)$:

$$\begin{cases} \dfrac{\partial \tilde{v}_k}{\partial t}(t,x) = \dfrac{1}{2}\sum_{i,j=1}^{n} G_{ij}(t)\dfrac{\partial^2 \tilde{v}_k}{\partial x_i \partial x_j}(t,x) \\ \qquad - (Lx + l)^T \nabla \tilde{v}_k(t,x) - \dfrac{1}{2}q(t,x)\tilde{v}_k(t,x) \\ \tilde{v}_1(0,x) = \sigma_0(x)e^{-\phi(0,x)} \\ \tilde{v}_k(\tau_{k-1},x) = \exp\left[h^T(x,\tau_{k-1})S^{-1}(\tau_{k-1})(y_{\tau_{k-1}} - y_{\tau_{k-2}})\right] \\ \qquad \cdot\, \tilde{v}_{k-1}(\tau_{k-1},x),\ k = 2, 3, \ldots, N \end{cases} \tag{19}$$

where

$$\begin{aligned} q(t,x) = &\sum_{i,j=1}^{n} G_{ij}(t)\frac{\partial^2 \phi}{\partial x_i \partial x_j}(t,x) \\ &+ \nabla_x \phi^T(t,x)G(t)\nabla_x\phi(t,x) \\ &+ 2(Lx + l)^T \nabla\phi_x(t,x) \\ &+ \sum_{p,l=1}^{n} S_{pl}^{-1}(t)h_p(x,t)h_l(x,t) + 2tr(L). \end{aligned} \tag{20}$$

PROOF Direct computations yield the following:

$$\frac{\partial \tilde{u}_k}{\partial t}(t,x) = e^{\phi(t,x)}\frac{\partial \tilde{v}_k}{\partial t}(t,x) \tag{21}$$

$$\frac{\partial \tilde{u}_k}{\partial x_i}(t,x) = e^{\phi(t,x)}\left[\frac{\partial \phi}{\partial x_i}(t,x)\tilde{v}_k(t,x) + \frac{\partial \tilde{v}_k}{\partial x_i}(t,x)\right] \tag{22}$$

and

$$\begin{aligned} \frac{\partial^2 \tilde{u}_k}{\partial x_i \partial x_j}(t,x) = &\, e^{\phi(t,x)}\frac{\partial \phi}{\partial x_j}(t,x)\left[\frac{\partial \phi}{\partial x_i}(t,x)\tilde{v}_k(t,x) \right. \\ &\left. + \frac{\partial \tilde{v}_k}{\partial x_i}(t,x)\right] + e^{\phi(t,x)}\left[\frac{\partial^2 \phi}{\partial x_i \partial x_j}(t,x)\tilde{v}_k(t,x) \right. \\ &\left. + \frac{\partial \phi}{\partial x_i}(t,x)\frac{\partial \tilde{v}_k}{\partial x_j}(t,x) + \frac{\partial^2 \tilde{v}_k}{\partial x_i \partial x_j}(t,x)\right]. \end{aligned} \tag{23}$$

Putting (18), (21)–(23), into (16), we arrive at (19). The initial conditions of $\tilde{v}_k$ follow from those in (16). ■

By (13), (14), and (18), it can be easily concluded that the filtering problem for the time-varying Yau system (17) becomes solving the Kolmogorov equation (19). In [19], the third author and his coworker changed the KFE (19) into the Schrödinger equation. However, the transformation is much more difficult here, since the coefficients G_{ij} in front of the second derivative in (19) are time varying rather than the identity matrix I. Some assumptions on the system are stated below, and the transformation will be introduced in Theorem 1.

Assumption 1 $G(t)$ is a positive-definite matrix.

Since $G(t)$ is positive definite, then we can find a positive-definite matrix $F(t) > 0$ such that

$$G(t) = F(t)F^T(t) \tag{24}$$

according to the Cholesky decomposition.

Assumption 2 *$L(t)$ in (17)* can be expressed as follows:

$$L(t) = G(t)\Omega(t) + \frac{dF(t)}{dt}F^{-1}(t) \tag{25}$$

where $\Omega(t) \in \mathbb{R}^{n\times n}$ is an arbitrary symmetric matrix.

REMARK 1 If the state of system (1) is scalar or the state is a vector and $G(t)$, $L(t)$ are diagonal, it is obvious that Assumption 2 is naturally satisfied.

Under Assumptions 1 and 2, we introduce a transformation to eliminate the drift term $\nabla\tilde{v}_k(t,x)$ in (19) so that the Schrödinger equation can be naturally connected to the NLF problems later; see Section IV for details.

THEOREM 1 Under Assumptions 1 and 2, suppose $\tilde{v}_k(t,x)$ is a solution of (19) and let

$$\tilde{v}_k(t,x) = e^{x^T D(t)x} v_k(t,z) \tag{26}$$

where

$$\begin{aligned} z &= B(t)x + b(t)\\ B(t) &= F^{-1}(t)\\ b(t) &= \int_0^t B(s)l(s)ds \end{aligned} \tag{27}$$

and

$$D(t) = \frac{1}{2}\Omega(t). \tag{28}$$

Then, $v_k(t,z)$ is the solution of the following equation:

$$\begin{cases} \dfrac{\partial v_k}{\partial t}(t,z) = \dfrac{1}{2}\Delta v_k(t,z)\\ \qquad\qquad -\dfrac{1}{2}\tilde{q}\left(t, F(t)z - F(t)b(t)\right) v_k(t,z)\\ v_1(0,x) = \sigma_0(F(0)x)\cdot \exp\big[-\phi\left(0, F(0)x\right)\\ \qquad\qquad -\left(F(0)x\right)^T D(0)\left(F(0)x\right)\big]\\ v_k(\tau_{k-1},x) = \exp\big[h^T(F(\tau_{k-1})x - F(\tau_{k-1})b(\tau_{k-1}),\tau_{k-1})\\ \qquad\qquad S^{-1}(\tau_{k-1})(y_{\tau_{k-1}} - y_{\tau_{k-2}})\big]\, v_{k-1}(\tau_{k-1},x) \end{cases} \tag{29}$$

$k = 2, 3, \dots, N,$ where

$$\begin{aligned} \tilde{q}(t,x) &= q(t,x) + 2x^T\frac{dD(t)}{dt}x - tr\big(G(t)(D(t)+D^T(t))\big)\\ &\quad - x^T\left(D(t)+D^T(t)\right)G(t)\left(D(t)+D^T(t)\right)x\\ &\quad + 2(L(t)x+l)^T\left(D(t)+D^T(t)\right)x. \end{aligned} \tag{30}$$

PROOF Through direct computations, we have

$$\begin{aligned} \frac{\partial\tilde{v}_k}{\partial t}(t,x) &= e^{x^T D(t)x}\Bigg[x^T\frac{dD(t)}{dt}x v_k(t,z) + \frac{\partial v_k}{\partial t}(t,z)\\ &\quad + \sum_{i,j=1}^n \frac{\partial v_k}{\partial z_i}(t,z)\left(\frac{dB_{ij}(t)}{dt}x_j + \frac{db_i(t)}{dt}\right)\Bigg]\\ &= e^{x^T D(t)x}\Bigg[x^T\frac{dD(t)}{dt}x v_k(t,z) + \frac{\partial v_k}{\partial t}(t,z)\\ &\quad + \left(\frac{dB(t)}{dt}x + \frac{db(t)}{dt}\right)^T \nabla v_k(t,z)\Bigg] \end{aligned} \tag{31}$$

$$\begin{aligned} \frac{\partial\tilde{v}_k}{\partial x_i}(t,x) &= e^{x^T D(t)x}\Bigg[\sum_{l=1}^n (D_{il}+D_{li})x_l v_k(t,z)\\ &\quad + \sum_{l=1}^n \frac{\partial v_k}{\partial z_l}(t,z)b_{li}\Bigg]\\ &= e^{x^T D(t)x}\Bigg[\sum_{l=1}^n (D_{il}+D_{li})x_l v_k(t,z)\\ &\quad + \left(B^T(t)\nabla v_k(t,z)\right)_i\Bigg] \end{aligned} \tag{32}$$

and

$$\begin{aligned} \frac{\partial^2\tilde{v}_k}{\partial x_i\partial x_j}(t,x) &= e^{x^T D(t)x}\sum_{l=1}^n (D_{jl}+D_{lj})x_l\\ &\quad \cdot\Bigg[\sum_{p=1}^n (D_{ip}+D_{pi})x_p v_k(t,z)\\ &\quad + \left(B^T(t)\nabla v_k(t,z)\right)_i\Bigg]\\ &\quad + e^{x^T D(t)x}\Bigg[(D_{ij}+D_{ij})v_k(t,z)\\ &\quad + \sum_{l=1}^n (D_{il}+D_{li})x_l\left(B^T(t)\nabla v_k(t,z)\right)_j\\ &\quad + \sum_{p,l=1}^n \frac{\partial^2 v_k}{\partial z_p\partial z_l}(t,z)b_{pi}b_{lj}\Bigg] \end{aligned} \tag{33}$$

where $*_{ij}$ denotes the ij-entry of the matrix $*$, and $*_i$ denotes the ith element of the vector $*$.

Let us write (32) and (33) compactly:

$$\begin{aligned} \nabla\tilde{v}_k(t,x) &= e^{x^T D(t)x}\big[\left(D(t)+D^T(t)\right)x v_k(t,z)\\ &\quad + B^T(t)\nabla v_k(t,z)\big] \end{aligned} \tag{34}$$

$$\sum_{i,j=1}^{n} G_{ij}(t)\frac{\partial^2 \tilde{v}_k}{\partial x_i \partial x_j}(t,x)$$
$$= e^{x^T D(t)x}\Bigg[\left(D(t)x + D^T(t)x\right)^T G(t)\left(D(t)x + D^T(t)x\right) v_k(t,z)$$
$$+ 2\left(D(t)x + D^T(t)x\right)^T B^T(t)\nabla v_k(t,z)$$
$$+ tr\left(G(t)(D(t)+D^T(t))\right) v_k(t,z)$$
$$+ \sum_{i,j=1}^{n}\sum_{p,l=1}^{n} B_{il}G_{lp}B_{jp}\frac{\partial^2 v_k}{\partial z_i \partial z_j}(t,z)\Bigg]. \quad (35)$$

Putting (26), (31), (34), and (35) into (19), we can obtain

$$\frac{\partial v_k}{\partial t}(t,z) = \frac{1}{2}tr\left(G(t)B^T(t)H(v_k(t,z))B(t)\right)$$
$$+ \Bigg[\left(B(t)G(t)\left(D(t)+D^T(t)\right) - B(t)L(t) - \frac{dB(t)}{dt}\right)x$$
$$- \left(\frac{db(t)}{dt} + B(t)l(t)\right)\Bigg]^T \nabla v_k(t,z)$$
$$- \frac{1}{2}\Bigg[q(t,x) + 2x^T\frac{dD(t)}{dt}x$$
$$- x^T(D(t)+D^T(t))G(t)(D(t)+D^T(t))x$$
$$- tr\left(G(t)(D(t)+D^T(t))\right)$$
$$+ 2(L(t)x + l(t))^T(D(t)+D^T(t))x\Bigg] v_k(t,z) \quad (36)$$

where $H(v_k(t,z))$ is the Hessian matrix of $v_k(t,z)$.

The form

$$\frac{\partial v_k}{\partial t}(t,z) = \frac{1}{2}\Delta v_k(t,z) - \frac{1}{2}\tilde{q}(t,x)v_k(t,z) \quad (37)$$

can be obtained by choosing $B(t)$, $b(t)$, $D(t)$ such that

$$B(t)G(t)B^T(t) = I_{n\times n}$$
$$\frac{db(t)}{dt} + B(t)l(t) = 0$$
$$B(t)G(t)\left(D(t)+D^T(t)\right) - B(t)L(t) - \frac{dB(t)}{dt} = 0. \quad (38)$$

Then, we can easily obtain $B(t)$ and $b(t)$ from (24) and the first two equations of (38). The last equation of (38) is equivalent to

$$D(t)+D^T(t) = G^{-1}B^{-1}\left(BL + \frac{dB}{dt}\right)$$
$$= G^{-1}L + G^{-1}B^{-1}\frac{dB}{dt}$$
$$= G^{-1}L - G^{-1}\frac{dF}{dt}F^{-1}. \quad (39)$$

Under Assumption 2, we can obtain

$$D(t)+D^T(t) = G^{-1}\left(G\Omega + \frac{dF}{dt}F^{-1}\right) - G^{-1}\frac{dF}{dt}F^{-1}$$
$$= \Omega(t). \quad (40)$$

Without loss of generality, we choose $D(t)$ to be symmetric, i.e., $D(t) = D^T(t) = \frac{1}{2}\Omega(t)$. Equation (29) follows from (37) immediately by noting that $x = B^{-1}(t)z - B^{-1}b(t)$. ■

IV. FILTERING PROBLEM

If $q(t,x)$ in (29) is quadratic in x, then it is called the Schrödinger equation. Though it feels very restrictive, it includes Kalman–Bucy [7] and Beneš [1] filtering.

Assumption 3 $\tilde{q}(t,x)$ defined in (20) is quadratic w.r.t. x.

Note that the observation term $h_i(x,t)$ can be nonlinear, which extends the Kalman–Bucy filtering system. Since $q(t,x)$ is quadratic, $h_i(x,t)$, $1 \le i \le m$, are of linear growth w.r.t. the state x, i.e., $h_i^2(x,t) \le M(t)(1+|x|^2)$ for some $M(t)$ from (20).

Since $\tilde{q}(t,x)$ is quadratic in x by (30) under Assumption 3, we thus can assume that

$$\tilde{q}(t,x) = x^T Q(t)x + p^T(t)x + r(t). \quad (41)$$

THEOREM 2 Let $K(t,x,y)$ be the fundamental solution of

$$\frac{\partial v_k}{\partial t}(t,x) = \frac{1}{2}\Delta v_k(t,x) - \frac{1}{2}\tilde{q}\left(t, F(t)x - F(t)b(t)\right)v_k(t,x) \quad (42)$$

where

$$\tilde{q}\left(t, F(t)x - F(t)b(t)\right) = x^T F^T(t)Q(t)F(t)x$$
$$- \left[2b^T(t)F^T(t)Q(t)F(t) - p^T(t)F(t)\right]x$$
$$+ b^T(t)F^T(t)Q(t)F(t)b(t) - p^T(t)F(t)b(t) + r(t). \quad (43)$$

Assume that the fundamental solution $K(t,x,y)$ can be written as

$$K(t,x,y) = (2\pi t)^{-n/2} exp\left\{x^T\tilde{A}(t)x + x^T\tilde{B}(t)y + y^T\tilde{C}(t)y + \tilde{D}^T(t)x + \tilde{E}^T(t)y + s(t)\right\} \quad (44)$$

where $\tilde{A}(t)$ and $\tilde{C}(t)$ are $n\times n$ symmetric matrices, $\tilde{B}(t)$ is an $n\times n$ matrix, and $\tilde{D}(t)$ and $\tilde{E}(t)$ are column n-vectors. Then, the coefficients $\tilde{A}(t)$, $\tilde{B}(t)$, $\tilde{C}(t)$, $\tilde{D}(t)$, and $\tilde{E}(t)$ satisfy the following ODEs:

$$\frac{d\tilde{A}}{dt}(t) = 2\tilde{A}^2(t) - \frac{1}{2}F^T(t)Q(t)F(t) \quad (45)$$
$$\frac{d\tilde{B}}{dt}(t) = 2\tilde{A}(t)\tilde{B}(t) \quad (46)$$
$$\frac{d\tilde{C}}{dt}(t) = \frac{1}{2}\tilde{B}^T(t)\tilde{B}(t) \quad (47)$$
$$\frac{d\tilde{D}}{dt}(t) = 2\tilde{A}(t)\tilde{D}(t) + F^T(t)Q(t)F(t)b(t) - \frac{1}{2}F^T(t)p(t) \quad (48)$$

$$\frac{d\tilde{E}}{dt}(t) = \tilde{B}^T(t)\tilde{D}(t) \tag{49}$$

$$\frac{ds}{dt}(t) = \frac{1}{2}\tilde{D}^T(t)\tilde{D}(t) + tr(\tilde{A}(t)) - \frac{1}{2}\left[b^T(t)F^T(t)Q(t)F(t)b(t) - p^T(t)F(t)b(t) + r(t)\right] + \frac{n}{2t}. \tag{50}$$

PROOF The proof is the same as that of [19, Th. 4]. ■

Since $G(t)$ is C^∞ smooth, $f(x,t)$ and $h(x,t)$ are C^∞ smooth in both state and time, it can be easily concluded that $F(t)$, $b(t)$, $Q(t)$, $p(t)$, and $r(t)$ are analytic. Therefore, we shall solve the ODEs (45)–(50) formally by the power series method.

THEOREM 3 Under Assumptions 1–3, the solution $v_k(t,z)$ in $\tau_{k-1} \le t \le \tau_k$ of (29) with $\tilde{q}\,(t, F(t)x - F(t)b(t))$ in (43) is given by

$$v_k(t,x) = \int_{\mathbb{R}^n} K(t,x,y)v_k(\tau_{k-1},y)dy \tag{51}$$

where

$$\begin{aligned} K(t,x,y) = {}& (2\pi(t-\tau_{k-1}))^{-n/2} \\ & \cdot \exp\Bigg\{-\frac{|x-y|^2}{2(t-\tau_{k-1})} + x^T\tilde{A}(t-\tau_{k-1})x \\ & + x^T\tilde{B}(t-\tau_{k-1})y + y^T\tilde{C}(t-\tau_{k-1})y \\ & + \tilde{D}^T(t-\tau_{k-1})x + \tilde{E}^T(t-\tau_{k-1})y \\ & + s(t-\tau_{k-1})\Bigg\} \end{aligned} \tag{52}$$

with $\tilde{A}(t-\tau_{k-1}) = \sum_{\nu=1}^\infty \tilde{A}_\nu(t-\tau_{k-1})^\nu$, $\tilde{B}(t-\tau_{k-1}) = \sum_{\nu=1}^\infty \tilde{B}_\nu(t-\tau_{k-1})^\nu$, $\tilde{C}(t-\tau_{k-1}) = \sum_{\nu=1}^\infty \tilde{C}_\nu(t-\tau_{k-1})^\nu$, $\tilde{D}(t-\tau_{k-1}) = \sum_{\nu=1}^\infty \tilde{D}_\nu(t-\tau_{k-1})^\nu$, $\tilde{E}(t-\tau_{k-1}) = \sum_{\nu=1}^\infty \tilde{E}_\nu(t-\tau_{k-1})^\nu$, $s(t-\tau_{k-1}) = \sum_{\nu=1}^\infty s_\nu(t-\tau_{k-1})^\nu$, $b(t-\tau_{k-1}) = \sum_{\nu=0}^\infty b_\nu(t-\tau_{k-1})^\nu$, $F(t-\tau_{k-1}) = \sum_{\nu=0}^\infty F_\nu(t-\tau_{k-1})^\nu$, $Q(t-\tau_{k-1}) = \sum_{\nu=0}^\infty Q_\nu(t-\tau_{k-1})^\nu$, $p(t-\tau_{k-1}) = \sum_{\nu=0}^\infty p_\nu(t-\tau_{k-1})^\nu$, $r(t-\tau_{k-1}) = \sum_{\nu=0}^\infty r_\nu(t-\tau_{k-1})^\nu$, where

$$\tilde{A}_{\nu+1} = \frac{2}{\nu+3}\sum_{i=0}^{\nu}\tilde{A}_i\tilde{A}_{\nu-i} - \frac{1}{2(\nu+3)}\sum_{j=0}^{\nu}\sum_{i=0}^{j}F_i^TQ_{j-i}F_{\nu-j} \tag{53}$$

$$\tilde{B}_{\nu+1} = \frac{2}{\nu+2}\sum_{i=0}^{\nu+1}\tilde{A}_i\tilde{B}_{\nu-i} \tag{54}$$

$$\tilde{C}_{\nu+1} = \frac{1}{2(\nu+1)}\sum_{i=-1}^{\nu+1}\tilde{B}_i^T\tilde{B}_{\nu-i} \tag{55}$$

$$\tilde{D}_{\nu+1} = \frac{2}{\nu+2}\sum_{i=0}^{\nu+1}\tilde{A}_i\tilde{D}_{\nu-i} - \frac{1}{2(\nu+2)}\sum_{i=0}^{\nu}F_i^Tp_{\nu-i} - \frac{1}{2(\nu+2)}\sum_{j=0}^{\nu}\sum_{i=0}^{j}\sum_{l=0}^{i}F_l^TQ_{i-l}F_{j-i}b_{\nu-j} \tag{56}$$

$$\tilde{E}_{\nu+1} = \frac{2}{\nu+1}\sum_{i=-1}^{\nu+1}\tilde{B}_i\tilde{D}_{\nu-i} \tag{57}$$

$$\begin{aligned} s_{\nu+1} = {}& \frac{1}{2(\nu+1)}\sum_{i=-1}^{\nu+1}\tilde{D}_i^T\tilde{D}_{\nu-i} + \frac{1}{\nu+1}tr(\tilde{A}_\nu) \\ & - \frac{1}{2(\nu+1)}\Bigg[\sum_{i=0}^{\nu}\sum_{j=0}^{i}\sum_{m=0}^{j}\sum_{l=0}^{m}b_l^TF_{m-l}^TQ_{j-m}F_{i-j}b_{\nu-i} \\ & - \sum_{j=0}^{\nu}\sum_{i=0}^{j}p_i^TF_{j-i}b_{\nu-j} + r_\nu\Bigg] \end{aligned} \tag{58}$$

with

$$\begin{aligned} & \tilde{A}_{-1} = \tilde{C}_{-1} = -\frac{1}{2}I,\ \tilde{B}_{-1} = I, \\ & \tilde{D}_{-1} = \tilde{E}_{-1} = s_{-1} = 0, \\ & \tilde{A}_0 = \tilde{B}_0 = \tilde{C}_0 = \tilde{D}_0 = \tilde{E}_0 = s_0 = 0. \end{aligned} \tag{59}$$

PROOF Suppose that all matrices in (45)–(50) can be expanded as follows:

$$\begin{aligned} & \tilde{A}(t) = \sum_{i=-1}^\infty \tilde{A}_it^i, \quad \tilde{B}(t) = \sum_{i=-1}^\infty \tilde{B}_it^i, \quad \tilde{C}(t) = \sum_{i=-1}^\infty \tilde{C}_it^i \\ & \tilde{D}(t) = \sum_{i=-1}^\infty \tilde{D}_it^i, \quad \tilde{E}(t) = \sum_{i=-1}^\infty \tilde{E}_it^i, \quad s(t) = \sum_{i=-1}^\infty s_it^i \\ & b(t) = \sum_{i=0}^\infty b_it^i, \quad F(t) = \sum_{i=0}^\infty F_it^i, \quad Q(t) = \sum_{i=0}^\infty Q_it^i \\ & p(t) = \sum_{i=0}^\infty p_it^i, \quad r(t) = \sum_{i=0}^\infty r_it^i. \end{aligned} \tag{60}$$

Putting (60) into (45)–(50), we can easily know the following.

1) Equation (45) is equivalent to

$$-\tilde{A}_{-1} = 2\tilde{A}_{-1}^2 \tag{61}$$

$$0 = 2(\tilde{A}_{-1}\tilde{A}_0 + \tilde{A}_0\tilde{A}_{-1}) \tag{62}$$

$$(\nu+1)\tilde{A}_{\nu+1} = 2\sum_{i=-1}^{\nu+1}\tilde{A}_i\tilde{A}_{\nu-i} - \frac{1}{2}\sum_{j=0}^{\nu}\sum_{i=0}^{j}F_i^TQ_{j-i}F_{\nu-j}, \quad \nu \ge 0. \tag{63}$$

2) Equation (46) is equivalent to

$$-\tilde{B}_{-1} = 2\tilde{A}_{-1}\tilde{B}_{-1} \tag{64}$$

$$0 = 2(\tilde{A}_{-1}\tilde{B}_0 + \tilde{A}_0\tilde{B}_{-1}) \tag{65}$$

$$(\nu+1)\tilde{B}_{\nu+1} = 2\sum_{i=-1}^{\nu+1}\tilde{A}_i\tilde{B}_{\nu-i}, \quad \nu \geq 0. \tag{66}$$

3) Equation (47) is equivalent to

$$-\tilde{C}_{-1} = \frac{1}{2}\tilde{B}_{-1}^T\tilde{B}_{-1} \tag{67}$$

$$0 = \frac{1}{2}(\tilde{B}_{-1}^T\tilde{B}_0 + \tilde{B}_0^T\tilde{B}_{-1}) \tag{68}$$

$$(\nu+1)\tilde{C}_{\nu+1} = \frac{1}{2}\sum_{i=-1}^{\nu+1}\tilde{B}_i^T\tilde{B}_{\nu-i}, \quad \nu \geq 0. \tag{69}$$

4) Equation (48) is equivalent to

$$-\tilde{D}_{-1} = 2\tilde{A}_{-1}\tilde{D}_{-1} \tag{70}$$

$$0 = 2(\tilde{A}_{-1}\tilde{D}_0 + \tilde{A}_0\tilde{D}_{-1}) \tag{71}$$

$$(\nu+1)\tilde{D}_{\nu+1} = 2\sum_{i=-1}^{\nu+1}\tilde{A}_i\tilde{D}_{\nu-i} - \frac{1}{2}\sum_{i=0}^{\nu}F_i^T p_{\nu-i} - \frac{1}{2}\sum_{j=0}^{\nu}\sum_{i=0}^{j}\sum_{l=0}^{i}F_l^T Q_{i-l}F_{j-i}b_{\nu-j}, \ \nu \geq 0. \tag{72}$$

5) Equation (49) is equivalent to

$$-\tilde{E}_{-1} = \tilde{B}_{-1}^T\tilde{D}_{-1} \tag{73}$$

$$0 = (\tilde{B}_{-1}^T\tilde{D}_0 + \tilde{B}_0^T\tilde{D}_{-1}) \tag{74}$$

$$(\nu+1)\tilde{E}_{\nu+1} = 2\sum_{i=-1}^{\nu+1}\tilde{B}_i\tilde{D}_{\nu-i}, \quad \nu \geq 0. \tag{75}$$

6) Equation (50) is equivalent to

$$-s_{-1} = \frac{1}{2}\tilde{D}_{-1}^T\tilde{D}_{-1} \tag{76}$$

$$0 = \frac{1}{2}(\tilde{D}_{-1}^T\tilde{D}_0 + \tilde{D}_0^T\tilde{D}_{-1}) + \mathrm{tr}(\tilde{A}_{-1}) + \frac{n}{2} \tag{77}$$

$$(\nu+1)s_{\nu+1} = \frac{1}{2}\sum_{i=-1}^{\nu+1}\tilde{D}_i^T\tilde{D}_{\nu-i} + \mathrm{tr}(\tilde{A}_\nu) - \frac{1}{2}\left[\sum_{i=0}^{\nu}\sum_{j=0}^{i}\sum_{m=0}^{j}\sum_{l=0}^{m}b_l^T F_{m-l}^T Q_{j-m}F_{i-j}b_{\nu-i} - \sum_{j=0}^{\nu}\sum_{i=0}^{j}p_i^T F_{j-i}b_{\nu-j} + r_\nu\right], \ \nu \geq 0. \tag{78}$$

The verification of $K(t, x, y)$ in (52) is a fundamental solution of (29), which is the same as that of [19, Th. 5]. ■

TABLE I
Numerical Implementation of the Direct Method

Algorithm Numerical Implementation of the Direct Method for (1)	
1:	**Initialization**: give T_0, T, Δ, $\sigma_0(x)$, $M \geq 0$
2:	Calculate $N = (T - T_0)/\Delta$
3:	Calculate $F(t)$, $B(t)$, $b(t)$, $D(t)$ by (24), (25), (27), and (28)
4:	Calculate $Q(t)$, $p(t)$, $r(t)$ by (20), (30), and (41)
5:	Calculate $\tilde{A}_\nu$, $\tilde{B}_\nu$, $\tilde{C}_\nu$, $\tilde{D}_\nu$, $\tilde{E}_\nu$, s_ν, $0 \leq \nu < M$ by (53)–(58)
6:	Calculate $\hat{x}_{t_0}$
7:	**for** $k = 1$ to N **do**
8:	Calculate $v_k(t_{k-1}, x)$, $v_k(t_k, x)$ by (29) and (51)
9:	Calculate $\tilde{v}_k(t_k, x)$, $\tilde{u}_k(t_k, x)$ by (18) and (26)
10:	Calculate $u_k(t_k, x)$, $\sigma(t_k, x)$ by (8) and (14)
11:	Calculate $\hat{x}_{t_k}$
12:	Assign $k := k + 1$
13:	**end for**

$[T_0, T]$ is the appointed time period, Δ is the discrete time step size, M is the assumed truncate order, and $\sigma_0(x)$ is the assumed probability density of the initial state x_0.

V. SIMULATIONS

In this section, we use an example to show the efficiency of the proposed direct method and compare it with the EKF.

A. Algorithm

To implement the proposed direct method numerically, we need to truncate the higher order in (60), which means that we only need to compute $\tilde{A}_\nu$, $\tilde{B}_\nu$, $\tilde{C}_\nu$, $\tilde{D}_\nu$, $\tilde{E}_\nu$, s_ν, $0 \leq \nu < M$ by (53)–(58), where M is the assumed order. The numerical procedure of the direct method for the NLF problem (1) is listed in Table I.

B. Numerical Example

The numerical example considered here is as follows:

$$\begin{cases} dx_t = \frac{t}{40}\cdot x_t dt + dv_t \\ dy_1(t) = x_t \sin x_t dt + dw_1(t) \\ dy_2(t) = x_t \cos x_t dt + dw_2(t) \end{cases} \tag{79}$$

where $x_t \in \mathbb{R}$ is the state with the initial state x_0. Here, the true initial state x_0 has been chosen to be 0.1, $v_t \in R$ is the standard Brownian motion, and $y(t) = [y_1(t), y_2(t)]^T \in \mathbb{R}^2$ is the 2-D measurement, and $w_t = [w_1(t), w_2(t)]^T$ is the 2-D standard Brownian motion.

Numerical simulations are obtained through the construction of an exact stochastic realization of system (79) at discrete times $t_k = k\Delta$ with $\Delta = 0.01$ on the interval $[0, T]$ with $T = 10$ according to the Euler–Maruyama method. All ODEs and integrals in the simulations by two methods are solved by the Euler method. The initial values for the EKF are $\hat{x}_0$ and P_0, $\sigma_0(x)$ is the Gaussian distribution $\mathcal{N}(x_0, P_0)$, and the integral step size for the direct method is $h = 0.05$.

We shall take different initial to examine the effectiveness of our direct method. The initial value $\hat{x}_0$ is generated by posing various P_0. We observe that when $P_0 = 3$, the EKF blows up eight times in 100 simulations, and this number becomes 80 when $P_0 = 10$, while our method seems not so sensitive as the EKF.

TABLE II
Average of MSE by the Direct Method and the EKF

Algorithm	Direct method	EKF
$P_0 = 0.1$	1.9671	7.1694
$P_0 = 3$	2.1196	——
$P_0 = 10$	2.2753	——

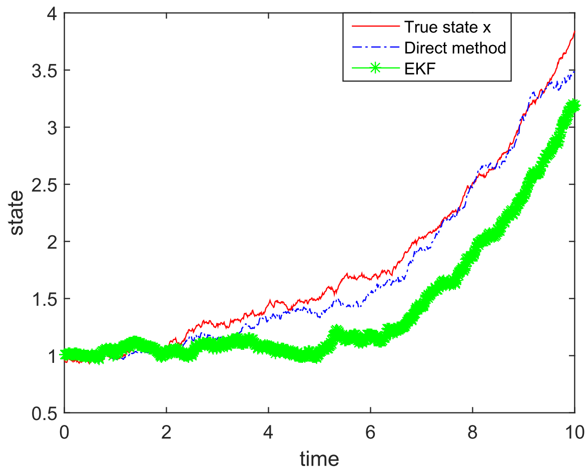

Fig. 1. Estimation results of state x with $\hat{x}_0 \sim \mathcal{N}(x_0, 0.1)$.

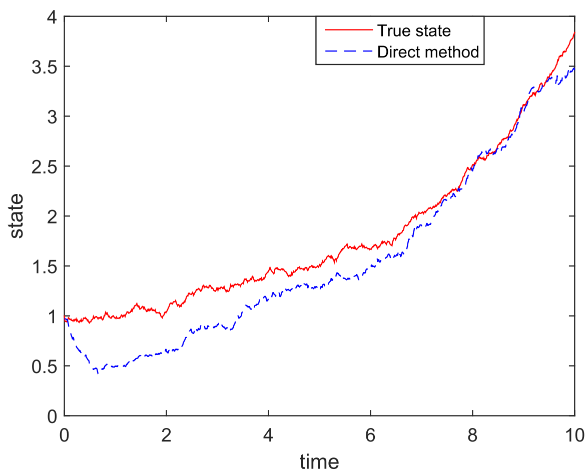

Fig. 2. Estimation results of state x with $\hat{x}_0 \sim \mathcal{N}(x_0, 3)$.

We define the mean of the squared estimation error (MSE) for one realization

$$\mu_x = \frac{1}{N+1}\sum_{k=0}^{N}(x_{t_k} - \hat{x}_{t_k})^2 \tag{80}$$

and the average of MSE over 100 simulations for different methods is listed in Table II. The average estimation results of EKF with $P_0 = 3, 10$ cannot be obtained due to the blow up.

The average results over 100 simulations of different methods are displayed in Figs. 1–3. It can be clearly seen that the proposed direct method always performs better than the EKF, and our method can trace the real state well even with large P_0, which means that the estimation initial value $\hat{x}_0$ is far from the real initial value x_0. Furthermore, comparing estimation results of the direct method in Figs. 1–3, we can observe that $\hat{x}_0$ only effects the initial estimation result, and the direct method has the same performance with different $\hat{x}_0$ after some time. This property is very useful, since in real applications, we can hardly know the real initial state and P_0 can be very large, so the direct method is superior from this point of view.

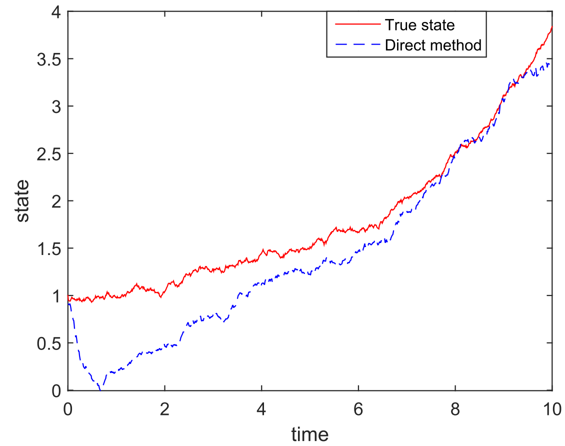

Fig. 3. Estimation results of state x with $\hat{x}_0 \sim \mathcal{N}(x_0, 10)$.

VI. CONCLUSION

In this paper, we extend the algorithm developed in [19] to solve the NLF problems for the time-varying Yau filtering system with arbitrary initial conditions. Under some mild assumptions, we obtain the corresponding time-varying Schrödinger equation by the transformation of the DMZ equation, and then, we write down its fundamental solution in terms of the solution of a system of ODEs, which are solved by the power series method when the potential is quadratic in state. Besides, the numerical simulation shows the efficiency of the proposed method.

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