New Classes of Finite Dimensional Filters With Non-Maximal Rank

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***Abstract*—Ever since the technique of Kalman filter was popularized, there has been an intense interest in finding new classes of finite dimensional recursive filters. In the late seventies, the idea of using estimation algebra to construct finite-dimensional nonlinear filters was first proposed by Brockett and Mitter independently. It has been proven to be an invaluable tool in the study of nonlinear filtering problem. For all known finite dimensional estimation algebras, the Wong's Ω-matrix has been proven to be a constant matrix. However, the Wong's Ω-matrix is shown not necessary to be a constant matrix in this letter when we consider finite dimensional estimation algebras with state dimension 3 and rank equal to 1. Several easily satisfied conditions are established for an estimation algebra of a special class of filtering systems to be finite-dimensional. Finally, we give the construction of finite dimensional filters of non-maximal rank.**

***Index Terms*—Algebraic/geometric methods, filtering, estimation, nonlinear drift, non-maximal rank.**

I. Introduction

In the early 1960s, Kalman and Bucy first established the finite dimensional filters (FDF) for linear-filtering systems with Gaussian initial distributions [18], [19]. Since then there are numerous research activities in the nonlinear filtering (NLF) problems, and solving the Duncan-Mortensen-Zakai (DMZ) equation which is satisfied by the unnormalized conditional density of the system state has long been the research focus of general NLF problems. In the late 1960s and early 1970s, the basic approach to nonlinear filtering theory was via the "innovations method" originally proposed by Kailath and subsequently rigorously developed by Fujisaki *et al.* [15]. However, the weakness of this method is that in general it is not explicit computable. In the late 1970s, Brockett and Clark [4], Brockett [5], and Mitter [23] proposed the idea of using estimation algebras to construct finite dimensional nonlinear filters independently. The motivation came from the Wei-Norman approach [30] of using the Lie algebraic method for solving time varying linear differential equations. For more details about the Wei-Norman approach and its connection with the nonlinear filtering problem, we refer the readers to paper [13], [29] and the survey article by Marcus [22]. The advantages of the Lie algebraic approach lie in two aspects: one is that the approach always leads to finite dimensional recursive filters as long as the estimation algebra is finite dimensional; and besides, the filter so constructed is universal in the sense of [7] (see Section IV). In addition, the dimension of the sufficient statistics used in computing the conditional density function is linear in n, where n is the dimension of the state space. Therefore, it is very meaningful to study the estimation algebras method. There are also many other ways to solve the DMZ equation in an approximation manner. Yau and Yau [35] proposed a novel method to the "pathwise-robust" DMZ equation which was generalized to the most general settings in [20]. Subsequently a real time algorithm was developed in one dimension, using the Hermite spectral method at every timestep [21]. Recently, based on the "Direct method" and an original Gaussian approximation algorithm, the approximate solution of the robust DMZ equation for a special class of NLF problems was given in terms of ordinary differential equations [28].

In 1981, Benés established exact finite-dimensional filters for certain diffusions with nonlinear drift, which is the first important breakthrough in the Lie algebra approach [1]. Later, Wong [31] constructed some new finite dimensional estimation algebras and used the Wei-Norman approach to construct FDF. By gauge transformation method Charalambous and Elliott [8] found another class of FDF where Benés exact filtering systems was extended by inserting linear combinations of $\mathrm{d}x(t)$ in the observations. There are also some results about new FDF with respect to various background scattered in [3], [11], [14], and [26]. However, not all NLF problems allow FDF, e.g., there exists no FDF for the cubic sensor problem [16]. Actually, only a few nonlinear filtering problems allow FDF. In [24] the existence and the nonexistence of FDF were discussed and the sufficient condition for FDF in discrete time partially observable systems was studied in [27].

Due to the practical importance of the estimation algebra method, Brockett [6] proposed the problem of classifying all finite dimensional estimation algebras at the 1983 International Congress of Mathematics, since this problem is helpful for finding new classes of FDF besides the Benés exact filtering. Since then, a lot of effort have been devoted to classifying

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finite dimensional estimation algebras. Under quite severe conditions, Wong [32] proved that all finite dimensional estimation algebras of (2.1) are solvable and the observation $h(x)$ is a polynomial of degree one. Besides, he was able to describe the structure of finite dimensional estimation algebras under these conditions. In Wong [33], the Wong Ω matrix concept was established which plays an important role in subsequent research. De Lara [10] gave a characterization of the finite dimensional estimation algebra with maximal rank under the assumption that the estimation algebra has one and only one operator with order greater or equal to two. Since the 1990s, Yau and his coworkers studied the algebraic structure of several general classes of estimation algebras in a series of research works. On the one hand, they have classified finite dimensional estimation algebras of maximal rank with arbitrary state space dimension [36] which included both Kalman-Bucy and Benés filtering systems as special cases. On the other hand, they were able to classify all finite dimensional estimation algebras with dimension at most six [17].

When the rank of finite dimensional estimation algebra is not maximal, the problem is wide open. Wu and Yau [34] have classified finite dimensional estimation algebras with state dimension 2. For higher state dimensions $n \geq 3$, the question remains to be solved. One of the key steps that Yau and his coworkers were able to classify all finite dimensional maximal rank estimation algebras is that they were able to show that Wong's Ω-matrix is a constant matrix. However, the Wong's Ω-matrix is not necessary to be a constant matrix as is shown in this letter when we consider finite dimensional estimation algebras with state dimension 3 and rank equal to 1. The classes of FDF given by the conditions in this letter are more general than those classes considered in [25].

This letter is organized as follows: Section II formulates the nonlinear filtering problem and describes some basic concepts about estimation algebras. The construction of new classes of FDF is given in Section III. In Section IV, we use the structure results to derive the FDF for the robust-DMZ equation by the Wei-Norman approach. The conclusion is given in Section V.

II. Problem Formulation and Basic Concepts

In this letter we consider the following signal observation model:

$$\begin{cases} dx(t) = f(x(t))dt + g(x(t))dv(t) & x(0) = x_0 \\ dy(t) = h(x(t))dt + dw(t) & y(0) = 0, \end{cases} \tag{1}$$

where x, v, y, w are respectively R^n, R^p, R^m, R^m valued process, and v and w are independent, standard Brownian motion. Besides, we assume that f and h are C^∞ smooth, and g is an orthogonal matrix. $x(t)$ is referred to as the state of the system at time t and $y(t)$ as the observation at time t. Let $\rho(t, x)$ denote the conditional probability density of the state $x(t)$ given the observation history $\{y(s) : 0 \leq s \leq t\}$. The nonlinear filtering problem is to determine $\rho(t, x)$.

It is well known that the unnormalized conditional density $\sigma(t, x)$ of the state $x(t)$ satisfy the following Duncan-Mortensen-Zakai(DMZ) equation:

$$\begin{cases} \sigma(t, x) = L_0\sigma(t, x)dt + \sum_{i=1}^m L_i\sigma(t, x)dy_i(t), \\ \sigma(0, x) = \sigma_0, \end{cases} \tag{2}$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^n f_i \frac{\partial}{\partial x_i} - \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^m h_i^2$$

and for $i = 1, \ldots, m$, L_i is the zero degree differential operator of multiplication by h_i. σ_0 is the probability density of the initial state x_0. If we define $D_i = \frac{\partial}{\partial x_i} - f_i$, $\quad \eta = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^n f_i^2 + \sum_{i=1}^m h_i^2$, then we have

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^n D_i^2 - \eta\right).$$

The normalized conditional density $\rho(t, x)$ is then given by

$$\rho(t, x) = \frac{\sigma(t, x)}{\int \sigma(t, x)}.$$

Therefore in the subsequent sections we aim to solve the DMZ equation (2). We need the following preliminary definitions.

Definition 1: If X and Y are differential operators, the Lie bracket of X and Y, [X,Y] is defined by $[X, Y]\phi = X(Y\phi) - Y(X\phi)$ for any C^∞ function ϕ.

Recall that a vector space $\mathcal{F}$ with the Lie bracket operation $\mathcal{F} \times \mathcal{F} \to \mathcal{F}$ denoted by (x,y) $\mapsto$ [x,y] is called a Lie algebra if the following axioms are satisfied:

(1) The Lie bracket operation is bilinear;

(2) [x,x]=0 for all x$\in \mathcal{F}$;

(3) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad (\forall x, y, z \in \mathcal{F})$.

Definition 2: The estimation algebra E of the nonlinear filtering system (1) is defined to be the Lie algebra generated by the operators occur in the DMZ equation (2), i.e.,

$$E := < L_0, L_1, \ldots, L_m >_{L.A.} .$$

Definition 3: Let $L(E) \subset E$ be the vector space consisting of all the homogeneous degree one polynomials in E. Then the linear rank r of estimation algebra E is defined by $r := \dim L(E)$.

Definition 4: The Wong matrix is the matrix $\Omega = (\omega_{ij})$, where

$$\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}, \quad \forall 1 \leq i, j \leq n.$$

Clearly $\omega_{ij} = -\omega_{ji}$.

III. Construction of New Class of Finite Dimensional Filters

In this section, we consider the finite dimensional estimation algebra E corresponding to (1) with state dimension $n = 3$ and linear rank $r = 1$. Without loss of generality, we may assume that $x_1 \in E$, $x_2, x_3 \notin E$.

It is easy to see that

$$[L_0, x_1] = D_1 \in E, \quad [D_1, x_1] = 1 \in E,$$
$$[L_0, D_1] = \omega_{12}D_2 + \omega_{13}D_3 + \frac{1}{2}\frac{\partial \omega_{12}}{\partial x_2} + \frac{1}{2}\frac{\partial \omega_{13}}{\partial x_3} + \frac{1}{2}\frac{\partial \eta}{\partial x_1}. \quad (3)$$

If we impose the following conditions:

(I) $\omega_{12} = \omega_{13} = 0$,

(II) $\eta = P_2(x_1) + \phi(x_2, x_3)$,

where $P_2(x_1)$ denotes degree at most two polynomial of x_1 and $\phi(x_2, x_3)$ is a C^∞ function of x_2, x_3. Then from (3) it is easy to see that the estimation algebra E is finite dimensional with basis given by $\{1, x_1, D_1, L_0\}$.

Next we construct a class of nonlinear filtering systems which satisfy conditions (I) and (II). By condition (II),

$$\eta = \sum_{i=1}^{3}\left(f_i^2 + \frac{\partial f_i}{\partial x_i}\right) + \sum_{i=1}^{m} h_i^2, \quad (4)$$

is a polynomial of degree at most 2 with respect to x_1, then we may assume that f_i, $1 \le i \le 3$ are polynomials of degree at most 1 with respect to x_1, i.e., we assume that for $1 \le i \le 3$,

$$f_i = a_i(x_2, x_3)x_1 + \phi_i(x_2, x_3), \quad (5)$$

where $a_i(x_2, x_3)$ and $\phi_i(x_2, x_3)$ are C^∞ function of x_2, x_3. By condition (I), we have

$$\omega_{12} = \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} = a_2 - \left(\frac{\partial a_1}{\partial x_2}x_1 + \frac{\partial \phi_1}{\partial x_2}\right) = 0,$$
$$\omega_{13} = \frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3} = a_3 - \left(\frac{\partial a_1}{\partial x_3}x_1 + \frac{\partial \phi_1}{\partial x_3}\right) = 0, \quad (6)$$

hence we have

$$\begin{cases} \dfrac{\partial a_1}{\partial x_2} = 0, & \dfrac{\partial a_1}{\partial x_3} = 0, \\ \dfrac{\partial \phi_1}{\partial x_2} = a_2, & \dfrac{\partial \phi_1}{\partial x_3} = a_3. \end{cases} \quad (7)$$

From (7), a_1 must be a constant. Now

$$\begin{aligned} \eta &= \sum_{i=1}^{3}\left(f_i^2 + \frac{\partial f_i}{\partial x_i}\right) + \sum_{i=1}^{m} h_i^2 \\ &= \left(\sum_{i=1}^{3} a_i^2\right)x_1^2 + \left(2\sum_{i=1}^{3} a_i \cdot \phi_i + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}\right)x_1 \\ &\quad + \sum_{i=1}^{3}\phi_i^2 + a_1 + \frac{\partial \phi_2}{\partial x_2} + \frac{\partial \phi_3}{\partial x_3} + \sum_{i=1}^{m} h_i^2. \end{aligned} \quad (8)$$

Since the estimation algebra has linear rank 1, we assume that h_i, $1 \le i \le m$ are degree one polynomials of x_1. Now condition (II) implies

(II.1) $\sum_{i=1}^3 a_i^2(x_2, x_3)$ is a constant;

(II.2) $2\sum_{i=1}^3 a_i\phi_i + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}$ is a constant.

To summarize, in order to satisfy conditions (I) and (II), it suffices to satisfy the following conditions:

(i) $f_i = a_i x_1 + \phi(x_2, x_3)$, $1 \le i \le 3$,

(ii) $\frac{\partial \phi_1}{\partial x_2} = a_2$, $\frac{\partial \phi_1}{\partial x_3} = a_3$,

(iii) a_1 is constant, $\sum_{i=1}^3 a_i^2$ is a positive constant,

(iv) $2\sum_{i=1}^3 a_i \cdot \phi_i + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}$ is constant,

(v) h_i's are degree at most 1 polynomials of x_1, and then the estimation algebra E is finite dimensional with basis $\{L_0, D_1, x_1, 1\}$.

Example: Now we give an nonlinear filtering system example which satisfy conditions (i)-(v). We let all the a_i's be constants, e.g., we take $a_1 = 1, a_2 = 1, a_3 = -1$, then condition (iii) is satisfied and from (ii) we can see that ϕ_1 is degree at most 1 polynomial of x_2, x_3. Thus we can take $\phi_1 = x_2 - x_3$. Now the condition (iv) which says

$$\phi_1 + \phi_2 - \phi_3 = \phi_2 - \phi_3 + x_2 - x_3$$

is a constant can be easily satisfied. For example, if we take $\phi_2 = x_2^2 + x_3^2 + x_3 - x_2 + 1$, $\phi_3 = x_2^2 + x_3^2$, then the condition (iv) is satisfied. Condition (v) is easily satisfied by letting the observation term $h(x) = x_1$. Now the Ω-matrix is given by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2x_2 - 2x_3 - 1 \\ 0 & 2x_3 - 2x_2 + 1 & 0 \end{pmatrix}$$

and $\eta = 4x_1^2 + 2x_1 + \gamma(x_2, x_3)$. Then the estimation algebra corresponding to this class of nonlinear filtering systems is finite dimensional with basis $\{L_0, D_1, x_1, 1\}$.

IV. Finite Dimensional Filters

In this section, we use the structure results to derive finite-dimensional filters for the robust-DMZ equation by the Wei-Norman approach.

In real applications, we are interested in considering robust state estimator from observed sample paths with some properties of robustness. Davis [12] considered this problem and proposed some robust algorithms. In our case, his basic idea reduced to define a new unnormalized density

$$u(t, x) = \exp\left(-\sum_{i=1}^{m} h_i(x)y_i(t)\right)\sigma(t, x), \quad (9)$$

then $u(t, x)$ satisfies the following equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) = L_0 u(t, x) + \sum_{i=1}^m y_i(t)[L_0, L_i]u(t, x) \\ \qquad\qquad + \frac{1}{2}\sum_{i,j=1}^m y_i(t)y_j(t)[[L_0, L_i], L_j]u(t, x), \\ u(0, x) = \sigma_0(x), \end{cases} \quad (10)$$

where $[\cdot, \cdot]$ is the Lie bracket. Equation (10) is called robust-DMZ equation.

The objective of constructing a robust finite-dimensional filter to (1) is equivalent to finding a smooth manifold M and complete C^∞ vector fields μ_i on M and C^∞ functions ν on $M \times \mathbb{R} \times \mathbb{R}^n$ and ω_i's on $\mathbb{R}^m$, such that $u(t, x)$ can be represented in the form:

$$\begin{cases} \dfrac{dz(t)}{dt} = \sum_{i=1}^k \mu_i(z(t))\omega_i(y(t)), & z(0) \in M, \\ u(t, x) = \nu(z(t), t, x). \end{cases} \quad (11)$$

Following [7], we say that system (1) has a robust universal finite-dimensional filter if for each initial probability density

TABLE I
LIE BRACKET MULTIPLICATION OF E

	L_0	D_1	x_1	1
L_0	0	$a_2x_1 + \frac{a_1}{2}$	D_1	0
D_1	$-a_2x_1 - \frac{a_1}{2}$	0	1	0
x_1	$-D_1$	-1	0	0
1	0	0	0	0

σ_0 there exists a z_0, such that (11) holds if $z(0) = z_0$, and μ_i, ω_i are independent of σ_0.

The following theorem gives the solution of the above robust-DMZ equation by the basis of the corresponding estimation algebra in terms of ordinary differential equations.

Theorem 1: If the nonlinear filtering system (1) satisfies the conditions (i)-(v), then we can assume $\eta = a_2x_1^2 + a_1x_1 + a_0(x_2, x_3)$, $h_i = c_{i1}x_1 + c_{i0}$, $1 \le i \le m$, where c_{i1}, c_{i0}, a_2, a_1 are constants and $a_0(x_2, x_3)$ is a C^∞ function of x_2, x_3. Then the robust DMZ equation (10) has a solution for all t of the form:

$$u(t, x) = e^{r_0(t)}e^{r_1(t)x_1}e^{r_2(t)D_1}e^{tL_0}\sigma_0, \tag{12}$$

where r_i's satisfy the following ordinary differential equations for all $t \ge 0$,

$$\begin{aligned}\dot r_1(t) &= a_2r_2(t),\\ \dot r_2(t) &= r_1(t) + \sum_{i=1}^m c_{i1}y_i(t),\\ \dot r_0(t) &= \frac{r_1(t)^2}{2} + \frac{a_2}{2}r_2(t)^2 + \sum_{i=1}^m c_{i1}y_i(t)r_1(t)\\ &\quad + \frac{1}{2}a_1r_2(t) + \frac{1}{2}\sum_{i,j=1}^m c_{i1}c_{j1}y_i(t)y_j(t).\end{aligned} \tag{13}$$

The solution is unique up to a constant and a universal finite dimensional filter exists for (1).

Proof: As described in Section III, the estimation algebra E of (2) satisfies conditions (I) and (II) with basis of $\{L_0, D_1, x_1, 1\}$. First, we give the basis calculations of estimation algebra E in Table I. By differentiating $u(t, x)$, we have

$$\begin{aligned}\frac{\partial u}{\partial t} &= e^{r_0(t)}e^{r_1(t)x_1}e^{r_2(t)D_1}L_0e^{tL_0}\sigma_0\\ &\quad + \dot r_2(t)\cdot e^{r_0(t)}e^{r_1(t)x_1}D_1e^{r_2(t)D_1}e^{tL_0}\sigma_0\\ &\quad + (\dot r_0(t) + \dot r_1(t)\cdot x_1)\cdot u(t, x)\\ &= A + B + (\dot r_0(t) + \dot r_1(t)\cdot x_1)\cdot u(t, x),\end{aligned} \tag{14}$$

where we denote

$$\begin{aligned}A &:= e^{r_0(t)}e^{r_1(t)x_1}e^{r_2(t)D_1}L_0e^{tL_0}\sigma_0,\\ B &:= \dot r_2(t)\cdot e^{r_0(t)}e^{r_1(t)x_1}D_1e^{r_2(t)D_1}e^{tL_0}\sigma_0.\end{aligned} \tag{15}$$

Recall the classical Baker-Campbell-Hausdorff type relation, i.e.,

$$\begin{aligned}e^{r(t)E_i}E_ke^{s(t)E_j} &= (E_k + r(t)[E_i, E_k]\\ &\quad + \frac{r(t)^2}{2!}[E_i, [E_i, E_k]] + \cdots)e^{r(t)E_i}e^{s(t)E_j},\end{aligned} \tag{16}$$

where E_i, E_k, E_j are elements of a lie algebra. The following calculations basically come from (16), we have

$$\begin{aligned}A &:= e^{r_0(t)}e^{r_1(t)x_1}(L_0 + r_2(t)[D_1, L_0]\\ &\quad + \frac{r_2(t)^2}{2}[D_1, [D_1, L_0]] + \cdots)\cdot e^{r_2(t)D_1}e^{tL_0}\sigma_0\\ &= e^{r_0(t)}e^{r_1(t)x_1}\left(L_0 - r_2(t)\left(a_2x_1 + \frac{1}{2}a_1\right)\right.\\ &\qquad \left. - \frac{r_2(t)^2}{2}a_2\right)e^{r_2(t)D_1}e^{tL_0}\sigma_0\\ &= e^{r_0(t)}e^{r_1(t)x_1}L_0e^{r_2(t)D_1}e^{tL_0}\sigma_0\\ &\quad - \left(r_2(t)\left(a_2x_1 + \frac{1}{2}a_1\right) + \frac{r_2(t)^2}{2}a_2\right)u(t, x),\end{aligned} \tag{17}$$

$$\begin{aligned}&e^{r_0(t)}e^{r_1(t)x_1}L_0e^{r_2(t)D_1}e^{tL_0}\sigma_0\\ &= e^{r_0(t)}\left(L_0 + r_1(t)[x_1, L_0] + \frac{r_1(t)^2}{2}[x_1, [x_1, L_0]]\right)\\ &\quad \cdot e^{r_1(t)x_1}e^{r_2(t)D_1}e^{tL_0}\sigma_0\\ &= e^{r_0(t)}\left(L_0 - r_1(t)D_1 + \frac{r_1(t)^2}{2}\right)\cdot e^{r_1(t)x_1}e^{r_2(t)D_1}e^{tL_0}\sigma_0,\end{aligned} \tag{18}$$

and

$$\begin{aligned}e^{r_0(t)}L_0e^{r_1(t)x_1}e^{r_2(t)D_1}e^{tL_0}\sigma_0 &= L_0u(t, x),\\ e^{r_0(t)}r_1(t)D_1e^{r_1(t)x_1}e^{r_2(t)D_1}e^{tL_0}\sigma_0 &= r_1(t)D_1u(t, x).\end{aligned} \tag{19}$$

Putting (18) and (19) into (17), we have

$$\begin{aligned}A = {}& L_0u(t, x) - r_1(t)D_1u(t, x)\\ &+ \left(\frac{r_1(t)^2}{2} - \frac{r_2(t)^2}{2}a_2 - r_2(t)(a_2x_1 + \frac{1}{2}a_1)\right)u(t, x).\end{aligned} \tag{20}$$

Similarly, we have

$$\begin{aligned}B &:= \dot r_2(t)e^{r_0(t)}e^{r_1(t)x_1}D_1e^{r_2(t)D_1}e^{tL_0}\sigma_0\\ &= \dot r_2(t)e^{r_0(t)}(D_1 - r_1(t))e^{r_1(t)x_1}e^{r_2(t)D_1}e^{tL_0}\sigma_0\\ &= \dot r_2(t)D_1u(t, x) - \dot r_2(t)r_1(t)u(t, x).\end{aligned} \tag{21}$$

Putting (20) and (21) into (13), we have

$$\begin{aligned}\frac{\partial u}{\partial t} &= L_0u(t, x) + (\dot r_2(t) - r_1(t))D_1u(t, x)\\ &\quad + (\frac{r_1(t)^2}{2} - r_2(t)(a_2x_1 + \frac{1}{2}a_1) - \frac{r_2(t)^2}{2}a_2\\ &\quad + \dot r_0(t) + \dot r_1(t)x_1 - \dot r_2(t)r_1(t))u(t, x).\end{aligned} \tag{22}$$

Note that L_i is the zero degree differential operator of multiplication by h_i, then (10) becomes

$$\begin{aligned}\frac{\partial u}{\partial t} &= L_0u(t, x) + \left(\sum_{i=1}^m c_{i1}y_i(t)\right)D_1u(t, x)\\ &\quad + \left(\frac{1}{2}\sum_{i,j=1}^m c_{i1}c_{j1}y_i(t)y_j(t)\right)u(t, x).\end{aligned} \tag{23}$$

Comparing (22) and (23), we have

$$\dot r_2(t) - r_1(t) = \sum_{i=1}^m c_{i1}y_i(t), \tag{24}$$

and

$$\big(\dot r_1(t) - a_2 r_2(t)\big)x_1 + \dot r_0(t) - \frac{a_1}{2} r_2(t) - \frac{a_2}{2} r_2(t)^2 + \frac{r_1(t)^2}{2} - \dot r_2(t) r_1(t) = \frac{1}{2}\sum_{i,j=1}^{m} c_{i1}c_{j1}y_i(t)y_j(t). \quad (25)$$

From (24) and (25) we have

$$\dot r_1(t) = a_2 r_2(t),$$
$$\dot r_2(t) = r_1(t) + \sum_{i=1}^{m} c_{i1} y_i(t),$$
$$\dot r_0(t) = \frac{r_1(t)^2}{2} + \frac{a_2}{2} r_2(t)^2 + \sum_{i=1}^{m} c_{i1}y_i(t) r_1(t) + \frac{1}{2} a_1 r_2(t) + \frac{1}{2}\sum_{i,j=1}^{m} c_{i1}c_{j1}y_i(t)y_j(t). \quad (26)$$

It is clear that (26) have solutions for all $t \geq 0$. Note that $r_1(t), r_2(t)$ is uniquely determined by the first two equations of (26), then by the last equation of (26) $r_0(t)$ is unique up to a constant.

Let $r_i(t), 0 \leq i \leq 2$ play the role of $z(t)$ in (11), then it is easy to check that (13) and (12) are of the form (11), i.e., a universal finite dimensional filter exists for (1). ■

The FDF of the *Example* in Section III can be constructed by Theorem 1.

V. Conclusion

The estimation algebra method has been proven to be an invaluable tool in the study of nonlinear filtering problem. In this letter, we establish several conditions for an estimation algebra of special classes of nonlinear filtering systems to be finite-dimensional. The Wong's Ω-matrix is shown not necessary to be a constant matrix when we consider finite dimensional estimation algebras with state dimension 3 and rank equal to 1. Moreover, the finite dimensional filter is constructed.

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