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FINITE DIMENSIONAL ESTIMATION ALGEBRAS WITH STATE DIMENSION 3 AND RANK 2, I: LINEAR STRUCTURE OF WONG MATRIX*

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Abstract. In this paper we study the structure of finite dimensional estimation algebras with state dimension 3 and rank 2 arising from a nonlinear filtering system by using the theories of the Euler operator and underdetermined partial differential equations. The structure of the Wong Ω-matrix is shown to be linear. The fundamental strategy we use in this paper to prove these results is to show that if they were not true, then infinite sequences could be constructed in the finite dimensional estimation algebra.

Key words. finite dimensional filter, estimation algebra, nonlinear drift, nonmaximal rank

AMS subject classifications. 17B30, 35K15, 60G35, 93E11

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1. Introduction.

Ever since 1960, after Kalman-Bucy first established finite dimensional filters for linear filtering systems with Gaussian initial distributions, there have been numerous research activities in nonlinear filtering problems. In the late 1960s and early 1970s, the basic approach to nonlinear filtering theory was via the "innovations method" originally purposed by Kailath and subsequently rigorously developed by Fujisaki, Kallianpur, and Kunita [12]. However, the weakness of this method is that in general it is not explicit computable. In the late 1970s, Brockett and Clark [3], Brockett [4], and Mitter [17] proposed the idea of using estimation algebras to construct finite dimensional nonlinear filters independently. The motivation came from the Wei–Norman approach [20] of using a Lie algebraic method for solving time-varying linear differential equations. For more details about the Wei–Norman approach and its connection with the nonlinear filtering problem, we refer the reader to paper [10], [19], and the survey article by Marcus [16]. Other more direct approaches seek the solution of the well-known Duncan–Mortensen–Zakai (DMZ) equation or its pathwise robust version [2]. Recently, Yau and his collaborators have developed a direct method for a general class of nonlinear filtering systems [14, 15]. The advantages of a Lie algebraic approach are that, as long as the estimation algebra is finite dimensional, the approach always leads to finite dimensional recursive filters, and the filter so constructed is universal in the sense of [6]. In addition, the dimension of the sufficient statistics used in computing the conditional density function is linear in n, where n is the dimension of the state space. Therefore, it is very meaningful to study the estimation algebras method.

In 1981 Benés established exact finite dimensional filters for certain diffusions with nonlinear drift, which is the first important breakthrough in the Lie algebra approach [1]. Later, Wong [21] constructed some new finite dimensional estimation algebras and used the Wei–Norman approach to construct finite dimensional filters.

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Another class of finite dimensional filters was found by Charalambous and Elliott [7], where Benés exact filtering systems were extended by inserting linear combinations of $\mathrm{d}x(t)$ in the observations.

Due to the practical importance of the estimation algebra method, Brockett [5] proposed the problem of classifying all finite dimensional estimation algebra at the 1983 International Congress of Mathematics in order to find new classes of finite dimensional filters besides the Benés exact filtering. Since then a lot of effort has been devoted to classifying finite dimensional estimation algebras. Under quite severe conditions, Wong [22] proved that all finite dimensional estimation algebras of (2.1) are solvable and the observation $h(x)$ is a polynomial of degree 1. Also, he was able to describe the structure of finite dimensional estimation algebras under these conditions. In Wong [23], the Wong Ω-matrix concept was established, which has played an important role in subsequent research. Since the 1990s, in a series of research works, the second author and his coworkers gave the algebraic structure of several general classes of estimation algebras. On the one hand, they were able to classify all finite dimensional estimation algebras with dimension at most six [9, 13, 27]. On the other hand, they had classified the finite dimensional estimation algebras of maximal rank with arbitrary state space dimension [28, 29] which included both Kalman-Bucy and Benés filtering systems as special cases.

However, when the rank is not maximal, much more needs to be done. Wu and Yau [24] have classified finite dimensional estimation algebras with state dimension 2. For higher state dimensions $n \geq 3$, the question is still open. One of the key steps that Yau and his coworkers were able to classify all finite dimensional maximal rank estimation algebras is that they were able to show that Wong's Ω-matrix is a matrix with polynomial degree 1. Recently, Shi et al. [25] gave new classes of finite dimensional filters for state dimension n = 3 and rank 1, in which case the Wong Ω-matrix is unnecessary to be a constant matrix. In this paper we consider finite dimensional estimation algebras with state dimension 3 and rank equal to 2. The following is our main theorem:

Main theorem: Let E be the finite dimensional estimation algebra of (2.1) *with state dimension* 3 *and rank* 2*. Then the Ω-matrix has linear structure; i.e., all the entries in the Ω-matrix are degree* 1 *polynomials.*

The main theorem in this paper plays a fundamental role of our forthcoming paper, in which we shall prove that the Wong Ω-matrix is a constant matrix if there exists a degree 2 polynomial in the estimation algebras and the Mitter conjecture holds.

The paper is organized as follows. Section 2 describes some basic concepts about estimation algebras and some known results. The proof that the structure of the Wong Ω-matrix is linear is given in section 3.

2. Basic concepts and results.

2.1. Basic concepts. The filtering problem we consider is based on the following signal observation model:

$$\begin{cases} dx(t) = f(x(t))dt + g(x(t))dv(t) & x(0) = x_0 \\ dy(t) = h(x(t))dt + dw(t) & y(0) = 0, \end{cases} \tag{2.1}$$

where x, v, y, w are, respectively, R^n, R^p, R^m, R^m valued process and v and w are independent, standard Brownian motion. Assume f and h are C^∞ smooth and g is an orthogonal matrix. $x(t)$ is referred to as the state of the system at time t and $y(t)$ as the observation at time t.

Let $\rho(t,x)$ denote the conditional probability density of the state $x(t)$ given the observation $\{y(s) : 0 \leq s \leq t\}$. $\rho(t,x)$ is the normalized version of $\sigma(t,x)$ which satisfies the following DMZ equation:

$$d\sigma(t,x) = L_0\sigma(t,x)dt + \sum_{i=1}^{m} L_i\sigma(t,x)dy_i(t), \qquad \sigma(0,x) = \sigma_0,$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i} - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2$$

and for $i = 1, \ldots, m$, L_i is the zero degree differential operator of multiplication by h_i. σ_0 is the probability density of the initial point x_0.

Define $D_i = \frac{\partial}{\partial x_i} - f_i$, $\quad \eta = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^n f_i^2 + \sum_{i=1}^m h_i^2$. Then

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^{n} D_i^2 - \eta\right).$$

Definition 2.1. *If X and Y are differential operators, the Lie bracket of X and Y, $[X,Y]$ is defined by $[X,Y]\phi = X(Y\phi) - Y(X\phi)$ for any C^∞ function ϕ.*

Recall that a vector space $\mathcal{F}$ with the Lie bracket operation $\mathcal{F}\times\mathcal{F}\to\mathcal{F}$ denoted by $(x,y)\mapsto[x,y]$ is called a Lie algebra if the following axioms are satisfied:

(1) The Lie bracket operation is bilinear;
(2) $[x,x] = 0$ for all $x \in \mathcal{F}$;
(3) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \quad (x,y,z \in \mathcal{F})$.

Definition 2.2. *The estimation algebra E of a filtering problem (2.1) is defined to be the Lie algebra generated by $\{L_0, L_1, \ldots, L_m\}$. E is said to be an estimation algebra of maximal rank if, for any $1 \leq i \leq n$, there exists a constant c_i such that $x_i + c_i \in E$.*

The linear rank concept of estimation algebra was introduced by Wu and Yau [24].

Definition 2.3. *Let $L(E) \subset E$ be the vector space consisting of all the homogeneous degree 1 polynomials in E. Then the linear rank of estimation algebra E is defined by $r := \dim L(E)$. So estimation algebra of maximal rank is in fact linear rank n estimation algebra.*

Definition 2.4. *The Wong matrix is the matrix $\Omega = (\omega_{ij})$, where*

$$\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j} \quad \forall 1 \leq i,j \leq n.$$

Clearly $\omega_{ij} = -\omega_{ji}$.

Definition 2.5. *Let l be a positive integer such that $l \leq n$. The Euler operator $E_l(\cdot)$ is defined to be a differential operator such that*

$$E_l(\phi) = \sum_{i=1}^{l} x_i\frac{\partial\phi}{\partial x_i}$$

for any $\phi \in C^\infty(R^n)$.

DEFINITION 2.6. *Let U be the set of differential operators in the form*

$$A = \sum_{(i_1,\dots,i_n)\in I_A} a_{i_1,\dots,i_n} D_1^{i_1} \dots D_n^{i_n},$$

where nonzero functions $a_{i_1,\dots,i_n} \in C^\infty(R^n)$ and $I_A \subseteq N^n$ is the finite index set of A. For $i = (i_1, \dots, i_n) \in N^n$, denote $|i| := \sum_{k=1}^n i_k$. The order of A is denoted by ord(A):=$\max_i |i|$. If A=0, ord(A) is defined to be $-\infty$.

Mitter conjecture: Let E be a finite dimensional estimation algebra. If ϕ is a function in E, then ϕ is affine in x.

The following notations are used in this paper.

(1) Let U_k denote the subspace of E consisting of elements with order less than or equal to k. In particular, $U_0 = C^\infty(R^n)$.
(2) As usual, if V is a subspace of E, $A = B \bmod V \iff A - B \in V$. If $A, B \in U$, define $Ad_A B = [A, B]$, $\quad Ad_A^l B = [A, Ad_A^{l-1} B]$, $\quad l \geq 1$.
(3) $P_k(x_{i_1}, \dots, x_{i_m})$ denotes the space consisting of polynomials of degree at most k in $x_{i_1}, \dots, x_{i_m}$, and $pol_k(x_{i_1}, \dots, x_{i_m})$ denotes a polynomial in $P_k(x_{i_1}, \dots, x_{i_m})$.

2.2. Preliminary. In this section, we give some known results which are used in this paper.

THEOREM 2.7 (Ocone [18]). *Let E be a finite dimensional estimation algebra. If a function ϕ is in E, then ϕ is a polynomial of degree less than or equal to 2.*

THEOREM 2.8 (see [26]). *Let E be an estimation algebra of system* (2.1). *Suppose that $\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$ are constants for all $1 \leq i, j \leq n$.*

(1) *If η is a polynomial of degree at most 2, then E is finite dimensional and has a basis consisting of $E_0 = L_0$, differential operators $E_1, \dots, E_p$ (for some p) of the form*

$$\sum_{j=1}^n \alpha_{ij} D_j + \beta_j, 1 \leq i \leq p,$$

where α_{ij}'s are constants and β_j's are affine in x, and zero differential operators $E_{p+1}, \dots, E_q, 1$ (for some $q > p$), where E_i's are affine in x for $p + 1 \leq i \leq q$. Moreover the quadratic part of $\eta - \sum_{i=1}^m h_i^2$ is positive semidefinite.

(2) *Conversely, if E is finite dimensional, then $h_1, \dots, h_m$ are affine in x; i.e., the observation matrix is a constant matrix. Furthermore, if the observation matrix has rank n (in particular $m \geq n$), then η is a polynomial of degree at most 2.*

THEOREM 2.9 (see [29], [24]). *Let E be a finite dimensional estimation algebra, and let the D_i's be defined as above. If $l \geq 0$ and*

$$A = \sum_{|i|=l+1} a_{i_1,\dots,i_n} D_1^{i_1} \dots D_n^{i_n}, \quad \bmod U_l,$$

is in E, then $a_{i_1,\dots,i_n}$'s are polynomials.

THEOREM 2.10 (see [26]). *Let $F(x_1, \dots, x_n)$ be a C^∞-function on R^n. Suppose that there exists a path $c : R \to R^n$ and $\delta > 0$ such that $\lim_{t\to\infty} \|c(t)\| = \infty$ and*

$\lim_{t\to\infty}\sup_{B_\delta(c(t))} F = -\infty$, *where* $B_\delta(c(t)) = \{x \in R^n : \|x - c(t)\| < \delta\}$. *Then there are no* C^∞*-functions* $f_1, f_2, \ldots, f_n$ *on* R^n *satisfying*

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = F.$$

COROLLARY 2.11.

(i) *Suppose* $\eta = a_{30}x_1^3 + a_{21}x_1^2x_2 + a_{12}x_1x_2^2 + a_{03}x_2^3 + \psi$, *where* a_{ij}*'s are* C^∞*-functions of* x_3, ψ *is a polynomial of degree at most* 2 *in* x_1, x_2 *with* $C^\infty(x_3)$ *coefficients. If there exist functions* $f_1, f_2, \ldots, f_n$ *on* R^n *satisfying*

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = \eta, \tag{2.2}$$

then $a_{30} = a_{21} = a_{12} = a_{03} = 0$.

(ii) *If* η *is a polynomial of degree at most* 3 *in* x_1 *(or* x_2*) with* $C^\infty(x_2, x_3)$ *(or* x_1, x_3 *correspondingly) coefficients, then the coefficient of* x_1^3*(or* x_2^3*) in* η *must be zero.*

Proof.

(i) If $a_{30} \neq 0$, then fix x_2, x_3, and set x_1=-sgn$(a_{30})t$, where sgn$(\cdot)$ is the sign function. Let $t \to \infty$; then $\eta \to -\infty$. By Theorem 2.10, there is no solution to the above partial differential equation (2.2). Hence, $a_{30} = 0$. Similarly, we have $a_{03} = 0$.
Fix x_3; then a_{12}, a_{21} are constants. If $a_{21} \cdot a_{12} \neq 0$, we can always choose constants $l_1 \cdot l_2 > 0$ such that $a_{21}l_1 + a_{12}l_2 < 0$. Let $x_1 = l_1t, x_2 = l_2t$ and $t \to +\infty$; then $\eta \to -\infty$. From Theorem 2.10, the above partial differential equation (2.2) has no solution, a contradiction. Hence, $a_{21} \cdot a_{12} = 0$. It follows easily that $a_{21} = a_{12} = 0$.

(ii) Suppose $\eta = a_3x_1^3 + a_2x_1^2 + a_1x_1 + a_0$, where a_i's are C^∞-functions of x_2, x_3. If $a_3 \neq 0$, then fix x_2, x_3, and set x_1=-sgn(a_3)t. By letting $t \to +\infty$, we have $\eta \to -\infty$. $a_3 = 0$ follows as above. ∎

LEMMA 2.12 (see [24]). *Let* g, $h \in C^\infty(R^n)$, *and let* $i_1, \ldots, i_n, j_1, \ldots, j_n$ *be nonnegative integers with* $\sum_{l=1}^n i_l = r$, $\sum_{l=1}^n j_l = s$, *and* $r + s \geq 2$. *Let* δ_{ij} *be the Kronecker symbol; then*

$$\begin{aligned}&\left[gD_1^{i_1}\ldots D_n^{i_n}, hD_1^{j_1}\ldots D_n^{j_n}\right]\\ &\quad= \sum_{l=1}^{n}\left(i_l g\frac{\partial h}{\partial x_l} - j_l h\frac{\partial g}{\partial x_l}\right) D_1^{i_1+j_1-\delta_{1l}}\ldots D_n^{i_n+j_n-\delta_{nl}}, \quad \mathrm{mod}\ U_{r+s-2}.\end{aligned}$$

LEMMA 2.13 (see [26], [8]). *Let* E *be an estimation algebra for the filtering problem* (2.1). $\Omega = (\omega_{ij})$ *is defined as in Definition* 2.4. *Assume* $X, Y, Z \in E$ *and* $g, h \in C^\infty(R^n)$. *Then*

(1) $[XY, Z] = X[Y, Z] + [X, Z]Y$;

(2) $[gD_i, h] = g\frac{\partial h}{\partial x_i}$;

(3) $[gD_i, hD_j] = gh\omega_{ji} + g\frac{\partial h}{\partial x_i}D_j - h\frac{\partial g}{\partial x_j}D_i$, *where* $\omega_{ji} = [D_i, D_j]$;

(4) $[gD_i^2, h] = 2g\frac{\partial h}{\partial x_i}D_i + g\frac{\partial^2 h}{\partial x_i^2}$;

(5) $[D_i^2, hD_j] = 2\frac{\partial h}{\partial x_i}D_iD_j + 2h\omega_{ji}D_i + \frac{\partial^2 h}{\partial x_i^2}D_j + h\frac{\partial \omega_{ji}}{\partial x_i}$;

(6) $[D_i^2, D_j^2] = 4\omega_{ji}D_jD_i + 2\frac{\partial\omega_{ji}}{\partial x_j}D_i + 2\frac{\partial\omega_{ji}}{\partial x_i}D_j + \frac{\partial^2\omega_{ji}}{\partial x_i\partial x_j} + 2\omega_{ji}^2$;

(7) $[D_k^2, hD_iD_j] = 2\frac{\partial h}{\partial x_k}D_kD_iD_j + 2h\omega_{jk}D_iD_k + 2h\omega_{ik}D_kD_j + \frac{\partial^2 h}{\partial x_k^2}D_iD_j + 2h\frac{\partial\omega_{jk}}{\partial x_i}D_k + h\frac{\partial\omega_{jk}}{\partial x_k}D_i + h\frac{\partial\omega_{ik}}{\partial x_k}D_j + h\frac{\partial^2\omega_{jk}}{\partial x_i\partial x_k}$;

(8) $[gD_iD_j, hD_k] = g\frac{\partial h}{\partial x_j}D_iD_k + g\frac{\partial h}{\partial x_i}D_jD_k - h\frac{\partial g}{\partial x_k}D_iD_j + gh\omega_{kj}D_i + gh\omega_{ki}D_j + g\frac{\partial^2 h}{\partial x_i\partial x_j}D_k + gh\frac{\partial\omega_{kj}}{\partial x_i}$.

Assumption: In this paper, we consider state dimension $n = 3$ estimation algebra E of system (2.1) with linear rank 2, also $\dim(E) < \infty$. Without loss of generality, we assume there exist constants $c_i, 1 \le i \le 2$, such that $x_i + c_i \in E, 1 \le i \le 2$ and for any constant c, $x_3 + c \notin E$.

3. Linear structure of the Wong Ω-matrix. In this section we will prove that the entries of the Ω-matrix are polynomials of degree ≤ 1.

We give the following elementary lemma.

LEMMA 3.1.

$$[L_0, x_i + c_i] = D_i \in E, 1 \le i \le 2, \tag{3.1}$$
$$[D_2, D_1] = \omega_{12} \in E, \quad [D_1, x_1 + c_1] = 1 \in E$$

$$Y_1 := [L_0, D_1] = \omega_{12}D_2 + \omega_{13}D_3 + \frac{1}{2}\frac{\partial\omega_{12}}{\partial x_2} + \frac{1}{2}\frac{\partial\omega_{13}}{\partial x_3} + \frac{1}{2}\frac{\partial\eta}{\partial x_1} \in E \tag{3.2}$$

$$= \omega_{12}D_2 + \omega_{13}D_3 \mod U_0 \tag{3.3}$$

$$Y_2 := [L_0, D_2] = \omega_{21}D_1 + \omega_{23}D_3 + \frac{1}{2}\frac{\partial\omega_{21}}{\partial x_1} + \frac{1}{2}\frac{\partial\omega_{23}}{\partial x_3} + \frac{1}{2}\frac{\partial\eta}{\partial x_2} \in E \tag{3.4}$$

$$= \omega_{21}D_1 + \omega_{23}D_3 \mod U_0. \tag{3.5}$$

So, $P_1(x_1, x_2) \subseteq E$.

We will use the notations Y_1, Y_2 in equations (3.3) and (3.5) throughout this paper.

The next lemma is very useful for the subsequent proof.

LEMMA 3.2. *Suppose* $i = (i_1, \ldots, i_n)$ *and* $|i| = \sum_{l=1}^n i_l \ge 2$; *then*

$$gD_1^{i_1} \ldots D_n^{i_n} = gD_{k_1}^{i_{k_1}} \ldots D_{k_n}^{i_{k_n}} \mod U_{|i|-2},$$

where g *is a* C^∞*-function of* $x_1, \ldots, x_n$ *and* $k = (k_1, \ldots, k_n)$ *is a permutation of* $(1, 2, \ldots, n)$.

Proof. First,

$$\begin{aligned} D_jD_i(\cdot) &= \left[\left(\frac{\partial}{\partial x_j} - f_j\right)\left(\frac{\partial}{\partial x_i} - f_i\right)\right](\cdot) \\ &= \frac{\partial^2(\cdot)}{\partial x_i\partial x_j} - f_j\frac{\partial(\cdot)}{\partial x_i} - \left[\frac{\partial f_i}{\partial x_j}(\cdot) + f_i\frac{\partial(\cdot)}{\partial x_j}\right] + f_jf_i(\cdot) \\ &= \left[\frac{\partial^2}{\partial x_i\partial x_j} - f_j\frac{\partial}{\partial x_i} - \frac{\partial f_i}{\partial x_j} - f_i\frac{\partial}{\partial x_j} + f_jf_i\right](\cdot). \end{aligned}$$

Similarly,

$$D_iD_j = \frac{\partial^2}{\partial x_i\partial x_j} - f_i\frac{\partial}{\partial x_j} - \frac{\partial f_j}{\partial x_i} - f_j\frac{\partial}{\partial x_i} + f_jf_i.$$

Recall that $\omega_{ji} = \frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i}$; then $D_i D_j = D_j D_i + \omega_{ji} = D_j D_i \mod U_0$. Without loss of generality, $k_1 \neq 1$, therefore, by induction,

$$\begin{aligned}
gD_1^{i_1}\dots D_n^{i_n} &= gD_1^{i_1}\dots D_{k_1-1}^{i_{k_1-1}} D_{k_1}^{i_{k_1}}\dots D_n^{i_n}\\
&= gD_1^{i_1}\dots \left(D_{k_1}^{i_{k_1}} D_{k_1-1}^{i_{k_1-1}} \mod U_{i_{k_1}+i_{k_1-1}-2}\right) D_{k_1+1}^{i_{k_1+1}}\dots D_n^{i_n}\\
&= \cdots\\
&= g\left(D_{k_1}^{i_{k_1}} D_1^{i_1}\dots D_{k_1-1}^{i_{k_1-1}} \mod U_{\sum_{l=1}^{k_1} i_l-2}\right) D_{k_1+1}^{i_{k_1+1}}\dots D_n^{i_n}\\
&= gD_{k_1}^{i_{k_1}} D_1^{i_1}\dots D_{k_1-1}^{i_{k_1-1}} D_{k_1+1}^{i_{k_1+1}}\dots D_n^{i_n} \mod U_{|i|-2}\\
&= gD_{k_1}^{i_{k_1}}\left(D_{k_2}^{i_{k_2}} D_1^{i_1}\dots D_{i_n}^{i_{k_n}} \mod U_{\sum_{|i|-i_{k_1}-2}}\right) \mod U_{|i|-2}\\
&= \cdots\\
&= gD_{k_1}^{i_{k_1}}\dots D_{k_n}^{i_{k_n}} \mod U_{|i|-2}. \qquad \square
\end{aligned}$$

LEMMA 3.3. *For any function $\phi \in E$, ϕ does not contain x_1x_3, x_2x_3 terms.*

Proof. By Theorem 2.7, every function in estimation algebra E is a polynomial of degree at most 2. Since $P_1(x_1, x_2) \subseteq E$, without loss of generality, assume ϕ in E be

$$\phi = ax_1^2 + bx_2^2 + cx_3^2 + dx_1x_2 + ex_1x_3 + fx_2x_3 + gx_3,$$

where a, b, c, d, e, f, g are constants:

$$\begin{aligned}
[D_1, \phi] &= \frac{\partial \phi}{\partial x_1} = 2ax_1 + dx_2 + ex_3 \in E,\\
[D_2, \phi] &= \frac{\partial \phi}{\partial x_2} = 2bx_2 + dx_1 + fx_3 \in E.
\end{aligned}$$

So $ex_3, fx_3 \in E$. By assumption, $x_3 \notin E$; hence, $e = f = 0$. $\square$

THEOREM 3.4. *ω_{12} is a degree no more than 1 polynomial of x_1, x_2.*

Proof. Step [1]: We prove that the degree 2 part of ω_{12} can only be $const \cdot x_3^2$, where *const* means a constant.

From Theorem 2.9 and equations (3.3), (3.5), $\omega_{12}, \omega_{13}, \omega_{23}$ are polynomials. From Lemma 3.3, we may assume any $\phi \in E$ is of the following form:

$$\phi = ax_1^2 + bx_2^2 + cx_3^2 + dx_1x_2 + gx_3,$$

where a, b, c, d are constants. Consider

$$\begin{aligned}
Z := [L_0, \phi] &= (2ax_1 + dx_2)D_1 + (2bx_2 + dx_1)D_2 + (2cx_3 + g)D_3 + a + b + c \in E,\\
[D_1, Z] &= 2aD_1 + dD_2 + (2bx_2 + dx_1)\cdot\omega_{21} + (2cx_3 + g)\cdot\omega_{31} \in E,\\
[D_2, Z] &= dD_1 + 2bD_2 + (2ax_1 + dx_2)\cdot\omega_{12} + (2cx_3 + g)\cdot\omega_{32} \in E.
\end{aligned}$$

Since $D_1 \in E, D_2 \in E$, we have

$$(2bx_2 + dx_1)\cdot\omega_{21} + (2cx_3 + g)\cdot\omega_{31} \in E \tag{3.6}$$

$$(2ax_1 + dx_2)\cdot\omega_{12} + (2cx_3 + g)\cdot\omega_{32} \in E. \tag{3.7}$$

Case (1): There exists $\phi \in E$ in which a, b, d are not all 0. Note in Theorem 2.7 and Lemma 3.3 that any function in E is a degree no more than 2 polynomial and does not contain x_1x_3, x_2x_3 terms.

Case (1.1): If $c \neq 0$, then ω_{13}, ω_{23} are degree at most 2 polynomials. If $a \neq 0$, then from equation (3.7) the degree 2 part of ω_{12} cannot contain x_1^2, x_1x_2, x_2^2 terms; that is, the degree 2 part of ω_{12} can only be $const \cdot x_3^2$. If $b \neq 0$ or $d \neq 0$, we can easily find the same conclusion holds.

Case (1.2): If $c = g = 0$, then from equations (3.6), (3.7), we can easily find that the degree 2 part of ω_{12} can only be $const \cdot x_3^2$.

Case (1.3): If $c = 0, g \neq 0$, then

$$\begin{aligned} Z = [L_0, \phi] &= (2ax_1 + dx_2)D_1 + (2bx_2 + dx_1)D_2 + gD_3 + a + b \in E, \\ [Z, \phi] &= (4a^2 + d^2)x_1^2 + (4b^2 + d^2)x_2^2 + 4(a+b)dx_1x_2 + g^2 \in E, \\ &\Rightarrow \psi := \hat{a}x_1^2 + \hat{b}x_2^2 + \hat{d}x_1x_2 \in E, \end{aligned}$$

where $\hat{a} = 4a^2 + d^2, \hat{b} = 4b^2 + d^2, \hat{d} = 4(a+b)d$ are not all 0. By the above case (1.2), the degree 2 part of ω_{12} can only be $const \cdot x_3^2$, where const means constant (hereinafter).

Case (2): For any $\phi \in E$, $a = b = d = 0$. In this case since $\omega_{12} \in E$, the degree 2 part of ω_{12} can only be be $const \cdot x_3^2$.

From case (1) and case (2), we can assume $\omega_{12} = \frac{1}{2}kx_3^2 + gx_3 + mx_1 + nx_2 + l \in E$.

Step [2]: In this step we prove that ω_{12} is a degree at most one polynomial in x_1, x_2.

If $k = 0$, then $g = 0$ by assumption, and the conclusion holds.

If $k \neq 0$, without loss of generality assume $k = 1$; then

$$\omega_{12} \in E \Rightarrow \frac{1}{2}x_3^2 + gx_3 \in E \tag{3.8}$$

$$[L_0, \omega_{12}] = mD_1 + nD_2 + (x_3 + g)D_3 + \frac{1}{2} \in E \Rightarrow (x_3 + g)D_3 \in E \tag{3.9}$$

$$[D_1, (x_3 + g)D_3] = (x_3 + g)\omega_{31} \in E \tag{3.10}$$

$$[D_2, (x_3 + g)D_3] = (x_3 + g)\omega_{32} \in E. \tag{3.11}$$

By Theorem 2.7 and Lemma 3.3, ω_{13}, ω_{23} are degree at most 1 polynomials of x_3.

(i) If ω_{31}, ω_{32} are both degree 1, without loss of generality, assume $\omega_{31} = x_3 + \alpha, \omega_{32} = x_3 + \beta$, where α, β are constants. From (3.10),

$$(x_3 + g)(x_3 + \alpha) = x_3^2 + (g + \alpha)x_3 + g\alpha \in E \Rightarrow x_3^2 + (g + \alpha)x_3 \in E,$$

combining this with (3.8) $\Rightarrow (g - \alpha)x_3 \in E \Rightarrow \alpha = g$. Similarly, β=g. So $\omega_{31} = \omega_{32} = x_3 + g$. Recall that

$$\begin{aligned} Y_1 &= \omega_{12}D_2 + \omega_{13}D_3 \mod U_0 = \omega_{12}D_2 - (x_3 + g)D_3 \mod U_0 \in E, \\ Y_2 &= \omega_{21}D_1 + \omega_{23}D_3 \mod U_0 = \omega_{21}D_1 - (x_3 + g)D_3 \mod U_0 \in E. \end{aligned}$$

Combining Y_1, Y_2 with (3.9) we have

$$\omega_{12}D_2 \mod U_0 \in E, \quad \omega_{12}D_1 \mod U_0 \in E. \tag{3.12}$$

(ii) Only one of ω_{13}, ω_{23} is degree 1. Without loss of generality, assume $\omega_{31} = x_3 + \alpha, \omega_{32} = \beta$. The proof in (i) shows that $\omega_{12}D_2 \mod U_0 \in E$. From (3.11), $\omega_{32} = \beta = 0$. From Y_2 we have $\omega_{12}D_1 \mod U_0 \in E$. Therefore, (3.12) holds.

(iii) If ω_{31}, ω_{32} are all constants, from the proof of (ii) we can see that (3.12) also holds.

Namely, (3.12) always holds. Consider

$$
\begin{aligned}
N_0 &= [(x_3+g)D_3, \omega_{12}D_2 \mod U_0] = (x_3+g)^2 D_2 \mod U_0 \in E,\\
M_1 &= [L_0, N_0] = 2(x_3+g)D_2D_3 \mod U_1 \in E,\\
N_1 &= [M_1, N_0] = 2^2(x_3+g)^2 D_2^2 \mod U_1 \in E,\\
&\cdots\\
M_n &= [L_0, N_{n-1}] = 2^{2n-1}(x_3+g)D_2^n D_3 \mod U_n \in E,\\
N_n &= [M_n, N_0] = 2^{2n}(x_3+g)^2 D_2^{n+1} \mod U_n \in E,\\
&\cdots
\end{aligned}
$$

Continuing this procedure, we can gain an infinite sequence in E which contradicts with the finite dimensionality of E. Hence, ω_{12} must be a degree 1 polynomial of x_1, x_2. □

Lemma 3.5. *Suppose that*

$$
\begin{aligned}
K :&= cD_3^{n+1} + (2ax_1 + dx_2 + e)D_1D_3^n\\
&\quad + (2bx_2 + dx_1 + f)D_2D_3^n + \cdots \mod U_n \in E,\\
A :&= (2ax_1 + dx_2 + e)D_3^l + \cdots \mod U_{l-1} \in E,\\
B :&= (2bx_2 + dx_1 + f)D_3^l + \cdots \mod U_{l-1} \in E,
\end{aligned}
$$

where a, b, c, d, e, f are constants, $n \geq 1, l \geq 1$. The $(\cdots)$ part means terms with highest order but lower order in D_3. Then $a = b = d = 0$.

Proof. If

$$
\det \begin{pmatrix} 2a & d \\ d & 2b \end{pmatrix} = 4ab - d^2 \neq 0,
$$

then a, d are not all zero, and from A and B we have

$$
\begin{cases}
C_{11} := (x_1 + \tilde{c}_1)D_3^l + \cdots \mod U_{l-1} \in E,\\
C_{12} := (x_2 + \tilde{c}_2)D_3^l + \cdots \mod U_{l-1} \in E,\\
B_{21} = [K, C_{11}] = (2ax_1 + dx_2 + e)D_3^{l+n} + \cdots U_{l+n-1} \in E,\\
B_{22} = [K, C_{12}] = (dx_1 + 2bx_2 + f)D_3^{l+n} + \cdots U_{l+n-1} \in E.
\end{cases}
$$

For the same reason, we have

$$
\begin{cases}
C_{21} := (x_1 + \tilde{c}_1)D_3^{l+n} + \cdots \mod U_{l+n-1} \in E,\\
C_{22} := (x_2 + \tilde{c}_2)D_3^{l+n} + \cdots \mod U_{l+n-1} \in E.
\end{cases}
$$

Continuing this procedure, we can gain an infinite sequence in E, a contradiction! Hence, $d^2 = 4ab$.

Suppose $a \neq 0$, and let $d = k_1 \cdot 2a$, where $k_1 = \frac{d}{2a}$; then $2b = k_1 \cdot d$.

If $a + b \neq 0$, then

$$
\begin{aligned}
K &= cD_3^{n+1} + (2ax_1 + dx_2 + e)D_1D_3^n\\
&\quad + (k_1 \cdot (2ax_1 + dx_2 + e) + c')D_2D_3^n + \cdots \mod U_n,
\end{aligned}
$$

where $c' = f - k_1 \cdot e$:

$$\begin{aligned} T_1 = [K, A] &= \big((2a + k_1 \cdot d)(2ax_1 + dx_2 + e) + d \cdot c'\big)D_3^{l+n} + \cdots \quad \bmod U_{l+n-1}, \\ &= \big(2(a+b)(2ax_1 + dx_2 + e) + d \cdot c'\big)D_3^{l+n} + \cdots \quad \bmod U_{l+n-1} \in E, \\ &\cdots \\ T_n = [K, T_{n-1}] &= (2(a+b))^{n-1}\big((2(a+b)(2ax_1 + dx_2 + e) + d \cdot c'\big)D_3^{l+k\cdot n} \\ &\quad + \cdots \quad \bmod U_{l+k\cdot n-1}, \\ &\cdots \end{aligned}$$

Continuing this procedure, we can gain an infinite sequence $\{T_n\}$ in E, a contradiction! Hence, $a + b = 0$; thus, a, b have the opposite sign. However, this contradicts with $d^2 = 4ab$. So $a = 0$, and therefore $d = 0$. Similarly, $b = 0$. ∎

LEMMA 3.6. *Since ω_{13} is a polynomial of x_1, x_2, x_3, we may assume that*

$$\omega_{13} = a_l x_3^l + \cdots + a_1 x_3 + a_0, \tag{3.13}$$

where $a_i, 0 \le i \le l$ are polynomials of x_1, x_2, $a_l \neq 0$. If $l \ge 1$, then $a_l \in P_1(x_1, x_2)$.

Proof. Using Lemma 2.12 and Lemma 3.2, we have

$$\begin{aligned} Ad_{L_0} Y_1 &= \frac{\partial \omega_{13}}{\partial x_3} D_3^2 + \frac{\partial \omega_{13}}{\partial x_1} D_1 D_3 + \frac{\partial \omega_{13}}{\partial x_2} D_2 D_3 + \cdots \quad \bmod U_1, \\ Ad_{L_0}^2 Y_1 &= \frac{\partial^2 \omega_{13}}{\partial x_3^2} D_3^3 + 2\frac{\partial^2 \omega_{13}}{\partial x_1 \partial x_3} D_1 D_3^2 + 2\frac{\partial^2 \omega_{13}}{\partial x_2 \partial x_3} D_2 D_3^2 + \cdots \quad \bmod U_2, \\ &\cdots \\ Ad_{L_0}^l Y_1 &= \frac{\partial^l \omega_{13}}{\partial x_3^l} D_3^{l+1} + l \cdot \frac{\partial^l \omega_{13}}{\partial x_1 \partial x_3^{l-1}} D_1 D_3^l + l \cdot \frac{\partial^l \omega_{13}}{\partial x_2 \partial x_3^{l-1}} D_2 D_3^l + \cdots \quad \bmod U_l, \\ Ad_{L_0}^{l+1} Y_1 &= (l+1) \cdot \frac{\partial^{l+1} \omega_{13}}{\partial x_1 \partial x_3^l} D_1 D_3^{l+1} + (l+1) \cdot \frac{\partial^{l+1} \omega_{13}}{\partial x_2 \partial x_3^l} D_2 D_3^{l+1} + \cdots \quad \bmod U_{l+1}, \end{aligned}$$

where $(\cdots)$ in the above equations means terms with highest order but lower order in D_3.

Define

$$M_1 = \frac{1}{l!} Ad_{L_0}^l Y_1 = a_l D_3^{l+1} + \cdots \quad \bmod U_l \tag{3.14}$$

$$M_2 = \frac{1}{(l+1)!} Ad_{L_0}^{l+1} Y_1 = \frac{\partial a_l}{\partial x_1} D_1 D_3^{l+1} + \frac{\partial a_l}{\partial x_2} D_2 D_3^{l+1} + \cdots \quad \bmod U_{l+1}. \tag{3.15}$$

Suppose $\deg(a_l) = k \ge 2$, where $\deg(a_l)$ means the degree of the polynomial a_l. Assume that the homogeneous degree k part of a_l is

$$a_l^{(k)} = b_0 x_1^k + b_1 x_1^{k-1} x_2 + \cdots + b_k x_2^k, \tag{3.16}$$

where $b_0, b_1, \ldots, b_k$ are not all zero constants:

$$
\begin{aligned}
A_1 &:= Ad_{D_1}^{k-i-2}(Ad_{D_2}^i M_1) = \frac{\partial^{k-2} a_l}{\partial x_1^{k-i-2}\partial x_2^i} D_3^{l+1} + \cdots \mod U_l \\
&= \left(\frac{1}{2} i!(k-i)! b_i x_1^2 + (i+1)!(k-i-1)! b_{i+1} x_1 x_2 \right. \\
&\quad \left. + \frac{1}{2}(i+2)!(k-i-2)! b_{i+2} x_2^2 + pol_1(x_1, x_2)\right) D_3^{l+1} + \cdots \mod U_l \\
&:= p(x_1, x_2) D_3^{l+1} + \cdots \mod U_l,
\end{aligned}
\tag{3.17}
$$

where

$$
\begin{aligned}
p(x_1, x_2) &= \frac{1}{2} i!(k-i)! b_i x_1^2 + (i+1)!(k-i-1)! b_{i+1} x_1 x_2 \\
&\quad + \frac{1}{2}(i+2)!(k-i-2)! b_{i+2} x_2^2 + pol_1(x_1, x_2) \\
&:= a x_1^2 + b x_2^2 + d x_1 x_2 + pol(x_1, x_2),
\end{aligned}
\tag{3.18}
$$

with $a = \frac{1}{2} i!(k-i)! b_i$, $b = \frac{1}{2}(i+2)!(k-i-2)! b_{i+2}$, $d = (i+1)!(k-i-1)! b_{i+1}$, $i = 0, 1, \ldots, k-2$. Consider

$$
\begin{aligned}
A_2 &:= Ad_{D_1}^{k-i-2}(Ad_{D_2}^i M_2) \\
&= \frac{\partial^{k-1} a_l}{\partial x_1^{k-i-1}\partial x_2^i} D_1 D_3^{l+1} + \frac{\partial^{k-1} a_l}{\partial x_1^{k-i-2}\partial x_2^{i+1}} D_2 D_3^{l+1} + \cdots \mod U_{l+1} \\
&= \frac{\partial p(x_1, x_2)}{\partial x_1} D_1 D_3^{l+1} + \frac{\partial p(x_1, x_2)}{\partial x_2} D_2 D_3^{l+1} + \cdots \mod U_{l+1} \\
&= (2a x_1 + d x_2 + c_1) D_1 D_3^{l+1} + (d x_1 + 2b x_2 + c_2) D_2 D_3^{l+1} + \cdots \mod U_{l+1},
\end{aligned}
\tag{3.19}
$$

where c_1, c_2 are constants. Consider

$$
\begin{cases}
B := [D_1, A_1] = (2a x_1 + d x_2 + c_1) D_3^{l+1} + \cdots \mod U_l \in E, \\
C := [D_2, A_1] = (d x_1 + 2b x_2 + c_2) D_3^{l+1} + \cdots \mod U_l \in E.
\end{cases}
\tag{3.20}
$$

Note $A_2, B, C \in E$ satisfy the assumption of Lemma 3.5, and we have $a = b = d = 0$. That is, $b_i = b_{i+1} = b_{i+2} = 0, 0 \le i \le k-2$, which contradict with that $b_i, i = 0, 1, \ldots, k$ are not all zero. So we have proved that a_l must be a polynomial of x_1, x_2 with degree no more than 1. ∎

Lemma 3.7. *Suppose*

$$\omega_{13} = \alpha_k x_1^k + \cdots + \alpha_1 x_1 + \alpha_0, k \ge 1, \alpha_k \ne 0,$$

where $\alpha_i, 0 \le i \le k$ are polynomials of x_2, x_3. Then $\alpha_k \in P_1(x_2, x_3)$.

Proof. From equation (3.3), we have

$$
\begin{aligned}
Ad_{D_1}^k Y_1 &= \frac{\partial^k \omega_{12}}{\partial x_1^k} D_2 + \frac{\partial^k \omega_{13}}{\partial x_1^k} D_3 \mod U_0 \\
&= const \cdot D_2 + \frac{\partial^k \omega_{13}}{\partial x_1^k} D_3 \mod U_0 \in E
\end{aligned}
\tag{3.21}
$$

$$
\Longrightarrow \frac{\partial^k \omega_{13}}{\partial x_1^k} D_3 \mod U_0 = k! \cdot \alpha_k D_3 \mod U_0 \in E.
\tag{3.22}
$$

Define

$$M_0 = \alpha_k D_3 \mod U_0 \in E,$$
$$M_1 = [L_0, M_0] = \frac{\partial \alpha_k}{\partial x_2} D_2 D_3 + \frac{\partial \alpha_k}{\partial x_3} D_3^2 \mod U_1 \in E.$$

We first prove that when α_k is a degree 2 polynomial of x_2, x_3, there exists a contradiction in Part (I). When the degree of α_k is higher than 2, we will reduce it to degree 2 case in Part (II). Therefore, α_k must be degree less than 2 polynomial of x_2, x_3.

Part (I): Suppose $\deg(\alpha_k) = 2$. We may assume that $\alpha_k^{(2)} = ax_2^2 + bx_3^2 + dx_2x_3$, where a, b, d are not all zero:

$$M_1 = [L_0, M_0] = (2ax_2 + dx_3 + c_1)D_2D_3 + (2bx_3 + dx_2 + c_2)D_3^2 \mod U_1,$$

where c_1, c_2 are constants.

Step [1]. We claim that $a = 0$.

If $a \neq 0$, then

$$Ad_{D_2}^2 M_0 = \frac{\partial^2 \alpha_k}{\partial x_2^2} D_3 \mod U_0 \in E \Longrightarrow D_3 \mod U_0 \in E.$$

Define

$$\begin{cases} G_1 := [D_2, M_0] = (2ax_2 + dx_3 + c_1)D_3 \mod U_0 \in E, \\ G_2 := [D_3 \mod U_0, M_0] = (2bx_3 + dx_2 + c_2)D_3 \mod U_0 \in E. \end{cases} \tag{3.23}$$

Step [1.a]: We claim that $d^2 = 4ab$.

If $d^2 \neq 4ab$, then from (3.23) we have

$$\begin{cases} (x_2 + e)D_3 \mod U_0 \in E, \\ (x_3 + f)D_3 \mod U_0 \in E, \end{cases} \tag{3.24}$$

where e and f are constants:

$$\begin{aligned} &[D_2, M_1] = dD_3^2 + 2aD_2D_3 \mod U_1 \in E, \\ &[[D_2, M_1], (x_2 + e)D_3 \mod U_0] = 2aD_3^2 \mod U_1 \in E, \\ &\Longrightarrow T_1 := D_3^2 \mod U_1 \in E, \quad K := D_2D_3 \mod U_1 \in E \end{aligned} \tag{3.25}$$

$$\begin{cases} A_{11} := [T_1, M_0] = 2(2bx_3 + dx_2 + c_2)D_3^2 \mod U_1 \in E, \\ A_{12} := [K, A_{11}] \Rightarrow dD_3^3 + 2bD_2D_3^2 \mod U_2 \in E, \\ A_{13} := [A_{12}, (x_2 + e)D_3 \mod U_0] \Rightarrow bD_3^3 \mod U_2 \in E. \end{cases} \tag{3.26}$$

If $b \neq 0$, define $T_2 = D_3^3 \mod U_2 \in E$, continuing the same procedure as in (3.26); we can obtain an infinite sequence $\{T_n\}$ in E, a contradiction! Hence, $b = 0$.

From $d^2 \neq 4ab$, we have $d \neq 0$; let

$$\begin{cases} B_{11} := [T_1, M_1] = 2dD_2D_3^2 \mod U_2 \Rightarrow D_2D_3^2 \mod U_2 \in E, \\ B_{12} := [D_2D_3^2 \mod U_2, (x_2 + e)D_3 \mod U_0] = D_3^3 \mod U_2 \in E, \\ T_2 := D_3^3 \mod U_2 \in E. \end{cases} \tag{3.27}$$

Continuing the procedure in (3.27), we can obtain an infinite sequence in E, a contradiction. Therefore, $d^2 = 4ab$.

Step [1.b]: We claim that $b = d = 0$.

If $b \neq 0$, we have

$$\begin{aligned} C_1 : &= [[D_2, M_1], M_0] = \big(2d(2bx_3 + dx_2 + c_2) + 2a(2ax_2 + dx_3 + c_1)\big)D_3^2 \\ &\quad + 2a(2bx_3 + dx_2 + c_2)D_2D_3 \mod U_1 \\ &= \big((4a^2 + 2d^2)x_2 + (2ad + 4bd)x_3 + const\big)D_3^2 \\ &\quad + 2a(2bx_3 + dx_2 + c_2)D_2D_3 \mod U_1, \end{aligned}$$

$$[D_2, C_1] = (4a^2 + 2d^2)D_3^2 + 2adD_2D_3 \mod U_1 \in E. \tag{3.28}$$

Recall from equation (3.25) that $[D_2, M_1] = dD_3^2 + 2aD_2D_3 \mod U_1 \in E$; then

$$(3.28) - d \cdot [D_2, M_1] = (4a^2 + d^2)D_3^2 \mod U_1 \in E.$$

So we have $H_1 := D_3^2 \mod U_1 \in E, \quad D_2D_3 \mod U_1 \in E$:

$$\begin{cases} R_{11} := [H_1, M_1] = 2(2bD_3^3 + dD_2D_3^2) \mod U_2 \in E, \\ R_{12} := \dfrac{1}{2}[R_{11}, G_2] = (12b^2 + d^2)D_3^3 + 4bdD_2D_3^2 \mod U_2 \in E, \\ R_{12} - 2b \cdot R_{11} = (4b^2 + d^2)D_3^3 \mod U_2 \in E, \\ \Longrightarrow H_2 := D_3^3 \mod U_2 \in E. \end{cases} \tag{3.29}$$

Continuing the same procedure as in (3.29), we can get a infinite sequence$\{H_n\}$ in E, a contradiction. Hence, $b = 0$, and by Step [1.a] we have $d = 0$.

Now we have

$$\begin{aligned} M_1 &= (2ax_2 + c_1)D_2D_3 \mod U_1 \in E, \\ [D_2, M_1] &= aD_2D_3 \mod U_1 \in E \Rightarrow F_1 := D_2D_3 \mod U_1 \in E, \\ [F_1, M_1] &= 2aD_2D_3^2 \mod U_2 \in E \Rightarrow F_2 := D_2D_3^2 \mod U_2 \in E, \\ [F_2, M_1] &= 2aD_2D_3^3 \mod U_2 \in E \Rightarrow F_3 := D_2D_3^3 \mod U_2 \in E. \\ &\dots \end{aligned}$$

Continuing this procedure, we get a contradiction as usual; hence, $a = 0$. The claim of Step [1] is proved.

Step [2]. We claim that $d = 0$.

Now

$$M_1 = (dx_3 + c_1)D_2D_3 + (dx_2 + 2bx_3 + c_2)D_3^2 \mod U_1 \in E.$$

If $d \neq 0$, consider

$$\begin{aligned} &[D_2, M_0] = (dx_3 + c_1)D_3 \mod U_0 \in E, \\ &N_0 : = [M_1, [D_2, M_0]] = (2d^2x_2 + 2bdx_3 + const)D_3^2 \mod U_1 \in E, \end{aligned} \tag{3.30}$$

where *const* means constant term:

$$N_1 := \frac{1}{2d^2}[D_2, N_0] = D_3^2 \mod U_1 \in E.$$

If $b \neq 0$, then $b \cdot d \neq 0$,

$$N_2 := [N_1, N_0] = 4bdD_3^3 \mod U_2 \Rightarrow D_3^3 \mod U_2 \in E,$$
$$N_3 := [N_2, N_0] = 6bdD_3^4 \mod U_3 \Rightarrow D_3^4 \mod U_3 \in E,$$
$$\cdots$$

Continuing the procedure, we can gain an infinite sequence $\{N_n\}$ in E, a contradiction. Hence, $b = 0$. Now from equation (3.30) we have $(x_2 + const)D_3^2 \mod U_1 \in E$. Consider

$$\begin{cases} T_1 := [L_0, (x_2 + const)D_3^2 \mod U_1] = D_2D_3^2 \mod U_2 \in E, \\ [T_1, (x_2 + const)D_3^2 \mod U_1] = D_3^4 \mod U_3 \in E, \\ T_2 := \dfrac{1}{4d}[D_3^4 \mod U_3, M_1] = D_2D_3^4 \mod U_4 \in E. \end{cases} \tag{3.31}$$

Continuing the procedure (3.31), we get a contradiction; hence, $d = 0$.

Step [3]. We claim that $b = 0$.

Now

$$M_1 = (2bx_3 + c_2)D_3^2 + c_1D_2D_3 \mod U_1 \in E.$$

If $b \neq 0$, consider

$$\begin{cases} [L_0, M_1] = 2bD_3^3 \mod U_2 \in E \Rightarrow D_3^3 \mod U_2 \in E, \\ [D_3^3 \mod U_2, M_1] = 6bD_3^4 \mod \in E \Rightarrow D_3^4 \mod \in E, \\ \cdots \end{cases}$$

Continuing the procedure, we get a contradiction; hence, $b = 0$.

However, $a = b = d = 0$ contradict with the assumption that a, b, d are not all zero! Therefore, α_k cannot be degree 2 polynomial of x_2, x_3.

Part (II): Assume that $\deg(\alpha_k) = l > 2$. We may assume that the homogeneous degree l part of α_k is

$$\alpha_k^{(l)} = b_0x_2^l + b_1x_2^{l-1}x_3 + \cdots + b_lx_3^l,$$

where $b_0, \ldots, b_l$ are not all zero constants. Consider

$$Ad_{D_2}^{l-2}M_0 = \left(\frac{1}{2}l!b_0x_2^2 + (l-1)!b_1x_2x_3 + (l-2)!b_2x_3^2 + pol_1(x_2, x_3)\right) D_3 \mod U_0 \in E,$$

where $pol_1(x_2, x_3)$ comes from the homogeneous degree l part of α_k. By Part (I), we have $b_0 = b_1 = b_2 = 0$.

If $b_3 \neq 0$, then $\alpha_k^{(l)} = b_3x_2^{l-3}x_3^3 + \cdots + b_lx_3^l$. Consider

$$\begin{cases} Ad_{D_2}^{l-3}M_0 = \big((l-3)!b_3x_3^3 + pol_2(x_2, x_3)\big)D_3 \mod U_0 \in E, \\ \Rightarrow N_0 := \big(x_3^3 + pol_2(x_2, x_3)\big)\, D_3 \mod U_0 \in E, \\ T_0 := \dfrac{1}{6}Ad_{L_0}^2N_0 = (x_3 + const)D_3^3 + \cdots \mod U_2 \in E, \\ T_1 := Ad_{L_0}T_0 = D_3^4 + \cdots \mod U_3 \in E, \\ T_2 := \dfrac{1}{4}[T_1, T_0] = D_3^6 + \cdots \mod U_5 \in E, \\ \cdots \end{cases}$$

where $(\cdots)$ in $\{T_n\}$ means terms with highest order but lower order in D_3. Continuing this procedure, we get a contradiction. Hence, $b_3 = 0$; similarly, repeating the above procedure we can get that $b_4 = \cdots = b_l = 0$. That is, $b_0, \ldots, b_l$ are all zero, a contradiction!

From Part(I) and (II) we see that α_k can only be degree at most 1 polynomial of x_2, x_3. ∎

LEMMA 3.8. *Suppose that* $\omega_{13} = a_l x_3^l + \cdots + a_1 x_3 + a_0, (l \geq 1), a_l \neq 0,$ *where* $a_i, 0 \leq i \leq l$ *are polynomials of* x_1, x_2. *Then* $l < 2$.

Proof. Assume that $\omega_{13} = \alpha_k x_1^k + \cdots + \alpha_1 x_1 + \alpha_0, k \geq 0, \alpha_k \neq 0$. Suppose $l \geq 2$.

Part (I): $k \geq 1$ case. We claim that $l < 2$.

By Lemma 3.7, we can assume that $\alpha_k = a x_2 + b x_3 + c$, where a, b, c are not all zero constants. From Lemma 3.6, a_l is a degree ≤ 1 polynomial of x_1, x_2; assume $a_l = c_1 x_1 + c_2 x_2 + c_0$, where c_0, c_1, c_2 are not all zero. Note that ω_{12} is a degree ≤ 1 polynomial of x_1, x_2; we have

$$
\begin{aligned}
&\frac{1}{k!} Ad_{D_1}^k Y_1 = const \cdot D_2 + \frac{1}{k!}\frac{\partial^k \omega_{13}}{\partial x_1^k} D_3 \mod U_0 \Rightarrow Z := \alpha_k D_3 \mod U_0 \in E,\\
&T_1 := [L_0, Z] = a D_2 D_3 + b D_3^2 \mod U_1 \in E,\\
&T_2 := [T_1, Z] = (a^2 + 2b^2) D_3^2 + ab D_2 D_3 \mod U_1 \in E,\\
&T_2 - b \cdot T_1 = (a^2 + b^2) D_3^2 \mod U_1 \in E.
\end{aligned}
$$

Step [1]. We claim that when a, b are not all zero, then $l < 2$ holds.

If a, b are not all zero, then $K_0 := D_3^2 \mod U_1 \in E$. Consider

$$
\begin{aligned}
Ad_{K_0}^l Y_1 &= 2^l \cdot \frac{\partial^l \omega_{13}}{\partial x_3^l} D_3^{l+1} \mod U_l \in E\\
&= 2^l \cdot l! a_l D_3^{l+1} \mod U_l \in E \Rightarrow a_l D_3^{l+1} \mod U_l \in E.
\end{aligned}
$$

If $c_1 \neq 0$, then $[D_1, a_l D_3^{l+1}] = c_1 D_3^{l+1} \mod U_l \in E$. If $c_2 \neq 0$, then $[D_2, a_l D_3^{l+1}] = c_2 D_3^{l+1} \mod U_l \in E$. If $c_1 = c_2 = 0$; then $c_0 \neq 0$, and we have $c_0 D_3^{l+1} \mod U_l \in E$. In both cases, we have $K_1 := D_3^{l+1} \mod U_l \in E$. Consider

$$
\begin{aligned}
Ad_{K_1}^l Y_1 &= (l+1)^l \cdot \frac{\partial^l \omega_{13}}{\partial x_3^l} D_3^{l^2+1} \mod U_{l^2} \in E\\
&= (l+1)^l \cdot l! \cdot a_l D_3^{l^2+1} \mod U_{l^2} \in E \Rightarrow a_l D_3^{l^2+1} \mod U_{l^2} \in E.
\end{aligned}
$$

For the same reason as above, we have $K_2 := D_3^{l^2+1} \mod U_{l^2} \in E$. Repeat the above procedure; we can get a infinite sequence $\{K_n\}$ in E, a contradiction. Hence, $l < 2$ holds.

Step [2]. We claim that when $a = b = 0$, $l < 2$ holds.

If $a = b = 0$, then $\alpha_k = c \neq 0$. Without loss of generality, we can assume $\alpha_k = 1$. Then $Z = D_3 \mod U_0 \in E$. Consider

$$
\begin{aligned}
&M_0 := \frac{1}{(l-1)!} Ad_Z^{(l-1)} Y_1 = (l a_l x_3 + a_{l-1}) D_3 \mod U_0 \in E,\\
&N_0 := \frac{1}{l!} Ad_Z^l Y_1 = a_l D_3 \mod U_0 \in E,\\
&[L_0, N_0] = c_1 D_1 D_3 + c_2 D_2 D_3 \mod U_1 \in E,\\
&[[L_0, N_0], N_0] = (c_1^2 + c_2^2) D_3^2 \mod U_1 \in E.
\end{aligned}
$$

Step [2.a]. If c_1, c_2 are not all zero, then $D_3^2 \mod U_1 \in E$. Just like the proof in Step [1], we have $l < 2$ in this case.

Step [2.b]. If $c_1 = c_2 = 0$, without loss of generality, we may assume that $a_l = 1$. Then $M_0 = (lx_3 + a_{l-1})D_3 \mod U_0 \in E$. Since a_{l-1} is a polynomial of x_1, x_2, suppose $\deg(a_{l-1}) = r$.

(*). $r \geq 2$ case. We may assume that the homogeneous degree r part of a_{l-1} is

$$a_{l-1}^{(r)} = b_s x_1^{r-s} x_2^s + \cdots + b_t x_1^{r-t} x_2^t,$$

where $b_s, \ldots, b_t$ are constants and $0 \leq s \leq t \leq r, b_s \neq 0, b_t \neq 0$. Consider

$$\begin{aligned}
&Ad_{D_1}^{r-s-1}(Ad_{D_2}^s M_0) \\
&\quad = (s!(r-s)!b_s x_1 + (s+1)!(r-s-1)!b_{s+1}x_2 + const)D_3 \mod U_0 \in E, \\
&\Longrightarrow R := (ex_1 + fx_2 + const)D_3 \mod U_0 \in E, \\
&T := [L_0, R] = eD_1D_3 + fD_2D_3 \mod U_1 \in E, \\
&[T, R] = (e^2 + f^2)D_3^2 \mod U_1 \in E \Rightarrow D_3^2 \mod U_1 \in E,
\end{aligned}$$

where $e = s!(r-s)!b_s \neq 0, f = (s+1)!(r-s-1)!b_{s+1}$. From the proof in Step [1], we have $l < 2$ in this case.

(**). $r \leq 1$ case. Assume $a_{l-1} = k_1x_1 + k_2x_2 + k_0$, where k_1, k_2, k_0 are constants. Consider

$$\begin{aligned}
&M_1 = [L_0, M_0] = lD_3^2 + k_1D_1D_3 + k_2D_2D_3 \mod U_1 \in E, \\
&M_2 = [M_1, M_0] = (2l^2 + k_1^2 + k_2^2)D_3^2 + lk_1D_1D_3 + lk_2D_2D_3 \mod U_1 \in E, \\
&M_2 - l \cdot M_1 = (l^2 + k_1^2 + k_2^2)D_3^2 \mod U_1 \in E \Rightarrow D_3^2 \mod U_1 \in E.
\end{aligned}$$

From the proof in Step [1], we have $l < 2$ in this case.

By Steps [1] and [2], we have proved that for $k \geq 1$ case, $l < 2$ holds.

Part (II): $k = 0$ case. In this case, ω_{13} is a polynomial of x_2, x_3.

(1) Suppose the degree of ω_{13} with respect to x_2 is at least 1 since x_2 plays the same role as x_1 in ω_{13}; by the argument of Part (I), we have $l < 2$ also holds.
(2) Suppose ω_{13} is a polynomial of x_3 and is irrelevant with x_1, x_2 variables. Then $a_l, \ldots, a_0$ are constants. Without loss of generality, assume $a_l = 1$. Consider

$$\begin{aligned}
&A_1 := \frac{1}{l!} Ad_{L_0}^{l-1} Y_1 = (x_3 + const)D_3^l + \cdots \mod U_{l-1} \in E, \\
&A_2 := \frac{1}{l!} Ad_{L_0}^{l} Y_1 = D_3^{l+1} + \cdots \mod U_l \in E, \\
&A_3 := [A_2, A_1] = (l+1)D_3^{2l} + \cdots \mod U_{2l-1} \in E, \\
&\cdots
\end{aligned}$$

Continuing this procedure, we get a contradiction as above. Hence, $l < 2$ holds.

From Part (I) and Part (II), we have proved $l < 2$. ∎

Lemma 3.9. *By Lemma* 3.8, $\omega_{13} = a_1x_3 + a_0$, *where* $a_1 \in P_1(x_1, x_2)$. *Then* $a_0 \in P_2(x_1, x_2)$.

Proof. Assume $a_1 = k_1x_1 + k_2x_2 + k_0$, where k_0, k_1, k_2 are constants. Suppose $\deg(a_0) = k > 2$ and the homogeneous degree k part of a_0 is

$$a_0^{(k)} = b_0x_1^k + b_1x_1^{k-1}x_2 + \cdots + b_kx_2^k,$$

where $b_0, \ldots, b_k$ are not all zero constants. Consider

$$\begin{aligned} Ad_{L_0} Y_1 &= \frac{\partial \omega_{13}}{\partial x_3} D_3^2 + \frac{\partial \omega_{13}}{\partial x_1} D_1 D_3 + \frac{\partial \omega_{13}}{\partial x_2} D_2 D_3 + \cdots \mod U_1 \\ &= a_1 D_3^2 + \left(k_1 x_3 + \frac{\partial a_0}{\partial x_1} \right) D_1 D_3 + \left(k_2 x_3 + \frac{\partial a_0}{\partial x_2} \right) D_2 D_3 + \cdots \mod U_1 \in E. \end{aligned}$$

For $i = 0, \ldots, k-2$, denote

$$\begin{aligned} p_i(x_1, x_2) := \frac{\partial^{k-2} a_0}{\partial x_1^{k-i-2} \partial x_2^i} &= \frac{1}{2}(k-i)! \cdot i! b_i x_1^2 + (k-i-1)! \cdot (i+1)! b_{i+1} x_1 x_2 \\ &+ \frac{1}{2}(k-i-2)! \cdot (i+2)! b_{i+2} x_2^2 + pol_1(x_1, x_2) \\ &= a x_1^2 + b x_2^2 + d x_1 x_2 + pol_1(x_1, x_2), \end{aligned}$$

where $a = \frac{1}{2}(k-i)! \cdot i! b_i,\ b = \frac{1}{2}(k-i-2)! \cdot (i+2)! b_{i+2},\ d = (k-i-1)! \cdot (i+1)! b_{i+1}$. Consider

$$\begin{aligned} K :&= Ad_{D_2}^i \left(Ad_{D_1}^{k-i-2} (Ad_{L_0} Y_1) \right) \\ &= \frac{\partial^{k-2} a_1}{\partial x_1^{k-i-2} \partial x_2^i} D_3^2 + \frac{\partial p_i(x_1, x_2)}{\partial x_1} D_1 D_3 + \frac{\partial p_i(x_1, x_2)}{\partial x_2} D_2 D_3 + \cdots \mod U_1 \\ &= const \cdot D_3^2 + (2a x_1 + d x_2 + e) D_1 D_3 + (2b x_2 + d x_1 + f) D_2 D_3 \\ &\quad + \cdots \mod U_1 \in E, \end{aligned}$$

where e, f are constants:

$$\begin{aligned} B :&= Ad_{D_1}^{k-i-1} (Ad_{D_2}^i Y_1) = \frac{\partial^{k-1} \omega_{13}}{\partial x_1^{k-i-1} \partial x_2^i} D_3 \mod U_0 \\ &= \frac{\partial p_i(x_1, x_2)}{\partial x_1} D_3 \mod U_0 = (2a x_1 + d x_2 + e) D_3 \mod U_0, \\ C :&= Ad_{D_1}^{k-i-2} (Ad_{D_2}^{i+1} Y_1) = \frac{\partial^{k-1} \omega_{13}}{\partial x_1^{k-i-2} \partial x_2^{i+1}} D_3 \mod U_0 \\ &= \frac{\partial p_i(x_1, x_2)}{\partial x_2} D_3 \mod U_0 = (2b x_2 + d x_1 + f) D_3 \mod U_0. \end{aligned}$$

K, B, C satisfy the assumption of Lemma 3.5, so $a = b = d = 0$. That is, $b_i = b_{i+1} = b_{i+2} = 0, i = 0, \ldots, k-2$, a contradiction. Therefore, $a_0 \in P_2(x_1, x_2)$. ∎

THEOREM 3.10. *ω_{13}, ω_{23} are degree at most 1 polynomials of x_1, x_2, x_3.*

Proof. By Lemma 3.9, we may assume that $\omega_{13} = a_1 x_3 + a_0,\ a_1 = c_1 x_1 + c_2 x_2 + c_0,\ a_0^{(2)} = a x_1^2 + b x_2^2 + d x_1 x_2$, where c_0, c_1, c_2, a, b, d are constants. Consider

$$\begin{cases} [D_1, Y_1] = const \cdot D_2 + \dfrac{\partial \omega_{13}}{\partial x_1} D_3 \mod U_0, \\ \Rightarrow G_1 := (c_1 x_3 + 2a x_1 + d x_2 + e) D_3 \mod U_0 \in E, \\ [D_2, Y_1] = const \cdot D_1 + \dfrac{\partial \omega_{13}}{\partial x_2} D_3 \mod U_0, \\ \Rightarrow G_2 := (c_2 x_3 + 2b x_2 + d x_1 + f) D_3 \mod U_0 \in E, \end{cases} \tag{3.32}$$

where e, f are constants.

Step [1]. We claim that a_l is a constant, that is, $c_1 = c_2 = 0$.

If c_1, c_2 are not all zero, without loss of generality, we can assume that $c_1 \neq 0$. From (3.32) we have

$$
\begin{aligned}
T &:= \frac{c_2}{c_1} \cdot G_1 - G_2 = \left(\left(\frac{2a\cdot c_2}{c_1} - d\right)x_1 + \left(\frac{d\cdot c_2}{c_1} - 2b\right)x_2 + const\right)D_3 \mod U_0\\
&= (\alpha_1 x_1 + \alpha_2 x_2 + const)D_3 \mod U_0 \in E,\\
[L_0, T] &= \alpha_1 D_1 D_3 + \alpha_2 D_2 D_3 \mod U_1 \in E,\\
[[L_0, T], T] &= (\alpha_1^2 + \alpha_2^2) D_3^2 \mod U_1 \in E,
\end{aligned}
$$

where $\alpha_1 = \frac{2a\cdot c_2}{c_1} - d, \alpha_2 = \frac{d\cdot c_2}{c_1} - 2b$.

If α_1, α_2 are not all zero, denote $A_1 := D_3^2 \mod U_1 \in E$:

$$
(3.33) \qquad \begin{cases}
[A_1, Y_1] = 2a_1 D_3^2 \mod U_1 \in E \Rightarrow a_1 D_3^2 \mod U_1 \in E,\\
[L_0, a_1 D_3^2 \mod U_1] = c_1 D_1 D_3^2 + c_2 D_2 D_3^2 \mod U_2 \in E,\\
[[L_0, a_1 D_3^2 \mod U_1], a_1 D_3^2 \mod U_1] = (c_1^2 + c_2^2) D_3^4 \mod U_3 \in E,\\
\Rightarrow A_2 := D_3^4 \mod U_3 \in E,\\
\cdots
\end{cases}
$$

Continuing this procedure, we get a contradiction. Hence, $\alpha_1 = \alpha_2 = 0$, that is,

$$
\begin{aligned}
&\frac{2a\cdot c_2}{c_1} - d = 0 \Rightarrow 2a\cdot c_2 = d\cdot c_1,\\
&\frac{d\cdot c_2}{c_1} - 2b = 0 \Rightarrow 2b\cdot c_1 = d\cdot c_2.
\end{aligned}
$$

Using $2a\cdot c_2 = d\cdot c_1$ and recalling $c_1 \neq 0$, we have

$$
\begin{aligned}
N_0 : &= c_1 \cdot G_1 = c_1 \cdot (c_1 x_3 + 2a x_1 + d x_2 + e) D_3 \mod U_0\\
&= (c_1^2 x_3 + 2a\cdot c_1 x_1 + d\cdot c_1 x_2 + c_1\cdot e) D_3 \mod U_0\\
&= (c_1^2 x_3 + 2a\cdot (c_1 x_1 + c_2 x_2) + c_1 \cdot e) D_3 \mod U_0 \in E,\\
F : &= [L_0, N_0]\\
&= c_1^2 D_3^2 + 2ac_1 D_1 D_3 + 2ac_2 D_2 D_3 \mod U_1 \in E,\\
H : &= [F, N_0]\\
&= (2c_1^4 + 4a^2 c_1^2 + 4a^2 c_2^2) D_3^2 + 2ac_1^3 D_1 D_3 + 2ac_1^2 c_2 D_2 D_3 \mod U_1 \in E,\\
H - c_1^2 \cdot F &= \left(c_1^4 + 4a^2 c_1^2 + 4a^2 c_2^2\right) D_3^2 \mod U_1 \Rightarrow B_1 := D_3^2 \mod U_1 \in E.
\end{aligned}
$$

Replacing A_1 with B_1 in (3.33) and repeating the procedure, we can get an infinite sequence $\{B_n\}$ in E, a contradiction. Hence, a_1 is a constant.

Step [2]. We claim that $a_0 \in P_1(x_1, x_2)$.

Now (3.32) becomes

$$
\begin{aligned}
G_1 &= (2a x_1 + d x_2 + e) D_3 \mod U_0 \in E,\\
G_2 &= (2b x_2 + d x_1 + f) D_3 \mod U_0 \in E.
\end{aligned}
$$

Consider

$$
\begin{aligned}
K : = Ad_{L_0} Y_1 = c_0 D_3^2 &+ (2a x_1 + d x_2 + e) D_1 D_3\\
&+ (2b x_2 + d x_1 + f) D_2 D_3 + \cdots \mod U_1 \in E.
\end{aligned}
$$

Note $K, G_1, G_2 \in E$ satisfy the assumption of Lemma 3.5; hence, $a = b = d = 0$. That is, a_0 is a degree 1 polynomial of x_1, x_2.

By Steps [1] and [2] we have proved ω_{13} is a degree 1 polynomial of x_1, x_2, x_3. We can similarly prove that ω_{23} is a degree at most 1 polynomial of x_1, x_2, x_3. ∎

Corollary 3.11. *Suppose*

$$A = g(x_1, x_2, x_3)D_3 \mod U_0 \in E,$$

where $g(x_1, x_2, x_3)$ *is a polynomial. Then* $g(x_1, x_2, x_3) \in P_1(x_1, x_2, x_3)$.

Proof. Just replace ω_{13} in Y_1 by $g(x_1, x_2, x_3)$ in the proof of Theorem 3.10 and the related Lemmas 3.6–3.9. ∎

From Theorems 3.4 and 3.10, we have proved the linear structure of the Ω-matrix.

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