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NEW CLASSES OF FINITE DIMENSIONAL FILTERS WITH NONMAXIMAL RANK ESTIMATION ALGEBRA ON STATE DIMENSION n AND LINEAR RANK n−2*

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Dedicated to Professor Roger Brockett on the occasion of his 82nd Birthday

Abstract. Ever since the Kalman filter technique was popularized, there has been an abundance of interest in finding new classes of finite dimensional recursive filters. In this paper, by applying Wong's theorem [W. S. Wong, *Systems Control Lett*, 9 (1987), pp. 79–83], we construct a new class of finite dimensional filters with arbitrary state space dimension n and linear rank $n-2$. Importantly, we show that in the new class of nonlinear filtering systems, the entries of Wong's Ω-matrix need not be constants or polynomials and can be C^∞ functions.

Key words. finite dimensional filters, estimation algebra, Wong's Ω-matrix, nonmaximal rank

AMS subject classifications. 17B30, 35K15, 60G35, 93E11

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1. Introduction.

The filtering problem is referred for estimating the state of a stochastic dynamical system under noisy observations. In the 1960s, Kalman and Bucy first proposed linear filtering theory for linear systems with Gaussian initial conditions, which motivated numerous research activities in nonlinear filtering problems. In the 1970s, Brockett and Clark [4], Brockett [3], and Mitter [11] independently constructed finite dimensional filters by using the estimation algebra method. The advantage of the estimation algebra approach is that if the estimation algebra is finite dimensional, finite dimensional recursive filters can always be constructed and are universal in the sense of [10]. Therefore, the estimation algebra approach is quite meaningful in nonlinear filtering theory. In filtering problems, probability density function of the state conditioned on observations plays an important role; its unnormalized form is described by the well-known Duncan–Mortensen–Zakai (DMZ) equation or its pathwise robust version [2]. Traditionally, the DMZ equation can be solved by the Wei–Norman approach [17] if one knows the explicit basis of estimation algebra. The advantage of the Wei–Norman approach is to simplify the DMZ equation to a Kolmogorov equation, a system of ordinary differential equations (ODEs), and several first-order linear partial differential equations (PDEs). There are many other ways to solve the DMZ equation. For example, Yau and Yau [24] proposed a new effective method to solve the "pathwise-robust" DMZ equation. Recently, an approximate real time filtering algorithm was proposed to solve the robust DMZ equation based on the "Direct method" and Gaussian approximation [6, 13].

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Great progress concerning the estimation algebra approach appeared in 1981. Benés first studied exact finite dimensional filtering systems for certain diffusions with nonlinear drift [1]. Since then, there has been more and more research interest in finding new classes of finite dimensional filters. However, not all nonlinear filtering systems possess finite dimensional estimation algebra. For example, there do not exist finite dimensional filters for the cubic sensor problem [9]. In 1987, Wong [18] found some new classes of solvable finite dimensional filters which contain both the linear and Benés filters. Meanwhile, Wong [19] proposed an important sufficient condition which can be effectively used to construct nonlinear filtering systems with finite dimensional estimation algebra. Wong's theorem [19] describes the structure of finite dimensional estimation algebras under some assumptions.

Due to the importance of the estimation algebra method in nonlinear filtering problems, a lot of work focused on classification of finite dimensional estimation algebra. Since the 1990s, through a series of work [5, 7, 16, 21, 22, 23], Yau and his collaborators have completely classified all finite dimensional estimation algebra of maximal rank with arbitrary state space dimension, including both Kalman–Bucy and Benés filtering systems as special cases. One of the critical steps that Yau and his co-workers can finish in the classification is proving the entries of Wong's Ω-matrix are all degree 1 polynomials. However, general classification of finite dimensional estimation algebras with nonmaximal rank is still an open problem. In [20], Wu and Yau finished the complete classification of finite dimensional estimation algebra with state dimension 2 and linear rank 1. In [12], Shi et al. constructed a new class of finite dimensional filters with state dimension 3 and linear rank 1 in which Wong's Ω-matrix need not be a constant matrix. In [14, 15], Shi and Yau studied the structure of finite dimensional estimation algebra with state dimension 3 and linear rank 2 and proved Wong's Ω-matrix has linear structure and Mitter conjecture holds in this case. Recently, Dong, Chen, and Yau [8] constructed a new class of finite dimensional filtering systems with state space dimension 4 and linear rank 1, in which entries of Wong's Ω-matrix can be polynomials of any degree.

In this paper, we construct a novel class of finite dimensional filters with any state space dimension n, $n \geq 3$ and linear rank $n-2$ by applying Wong's theorem [19] of constructing finite dimensional filters. In section 3, we first construct state evolutionary stochastic differential equations and calculate the corresponding Wong's Ω-matrix. Inspired by Wong's theorem, we define corresponding observation equations. In order to construct filters which satisfy all assumptions of Wong's theorem, we make an orthogonal transformation for H_i to obtain a new class of finite dimensional filters with any state dimension n and linear rank $n-2$. Matrix H_i is defined in section 2. More significantly, we can prove in such constructed filters entries of Wong's Ω-matrix need not be constants or polynomials and can be C^∞ functions.

This paper is organized as follows. In section 2, the nonlinear filtering problem and basic concepts of estimation algebras are presented. In section 3, the construction of the new class of finite dimensional filters and structure of corresponding finite dimensional estimation algebras are given.

2. Basic concepts and preliminary results.

2.1. Basic concepts. In this paper, we consider the following filtering system:

$$\begin{cases} dx(t) = f(x(t))dt + Gdw(t), & x(0) \in R^n, \\ dy(t) = Hx(t)dt + dv(t), & y(0) \in R^m, \end{cases} \tag{2.1}$$

where x, w, y, v are, respectively, R^n, R^n, R^m, R^m valued processes. $x(t)$ represents the state of the system and $y(t)$ represents observation. $w(t), v(t)$ are standard Brownian motions independent of the initial conditions $x(0), y(0)$. We assume that f is a vector valued C^∞ function and G, H are constant matrices. In addition, G is an orthogonal matrix.

Next, we denote $H = (H_1, H_2, \ldots, H_m)^T$, where $H_i = (H_{i1}, \ldots, H_{in})^T, 1 \le i \le m$. We define $L_i = h_i = H_i^T x,\ 1 \le i \le m$ as the zero degree differential operator of multiplication by h_i, and define the second order elliptic differential operator

$$L_0 = \frac{1}{2}\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^n f_i \frac{\partial}{\partial x_i} - \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^m h_i^2. \tag{2.2}$$

Let

$$\begin{aligned} D_i =& \frac{\partial}{\partial x_i} - f_i, \quad 1 \le i \le n, \\ \eta =& \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^n f_i^2 + \sum_{i=1}^m h_i^2; \end{aligned} \tag{2.3}$$

then we can obtain a more compact form of L_0,

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^n D_i^2 - \eta\right). \tag{2.4}$$

DEFINITION 2.1. *If X and Y are differential operators, the Lie bracket of X and Y, $[X,Y]$, is defined by $[X,Y]\phi = X(Y\phi) - Y(X\phi)$ for any C^∞ function ϕ.*

DEFINITION 2.2. *A vector space $\mathcal{F}$ with the Lie bracket operation $\mathcal{F} \times \mathcal{F} \to \mathcal{F}$ denoted by $(x,y) \longmapsto [x,y]$ is called a Lie algebra if the following axioms are satisfied:*

(1) *The Lie bracket operation is bilinear.*

(2) $[x,x] = 0$ *for all* $x \in \mathcal{F}$.

(3) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0, \quad x,y,z \in \mathcal{F}$.

DEFINITION 2.3. *The estimation algebra E of a filtering system* (2.1) *is defined to be the Lie algebra generated by $\{L_0, L_1, \ldots, L_m\}$, i.e., $E = \langle L_0, h_1, \ldots, h_m\rangle_{L.A.}$.*

DEFINITION 2.4 (see Wu and Yau [20]). *Let $L(E) \subset E$ be the vector space consisting of all the homogeneous degree* 1 *polynomials in E. Then the linear rank of estimation algebra E is defined by $r := \dim L(E)$.*

DEFINITION 2.5. *Wong's Ω-matrix is the matrix $\Omega = (\omega_{ij})$, where*

$$\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j} \quad \forall 1 \le i,j \le n. \tag{2.5}$$

Obviously, $\omega_{ij} = -\omega_{ji}$*, i.e.,* Ω *is an antisymmetric matrix.*

Let $J_\eta = \left(\frac{\partial^2 \eta}{\partial x_i \partial x_j}\right)$ be the Hessian matrix of η, which is a symmetric matrix.

DEFINITION 2.6. *Let* U *be the vector space of differential operators in the form*

$$A = \sum_{(i_1,i_2,\ldots,i_n)\in I_A} a_{i_1,i_2,\ldots,i_n} D_1^{i_1} D_2^{i_2} \cdots D_n^{i_n}, \tag{2.6}$$

where nonzero functions $a_{i_1,i_2,\ldots,i_n} \in C^\infty(R^n)$ *and* $I_A \subseteq N^n$ *are the finite sets of* A*. For* $i = (i_1, i_2, \ldots, i_n) \in N^n$*, denote* $|i| := \sum_{k=1}^n i_k$*. The order of* A *is defined by* $ord(A) := \max_i |i|$*.*

DEFINITION 2.7. *Let* R *be a linear space of number field* F*. Multiplication is defined in* R*:* $(\alpha, \beta) \to \alpha\beta$ *and satisfies*

$$\begin{cases} \alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma, \\ (\alpha+\beta)\gamma = \alpha\gamma + \beta\gamma, \\ (k\alpha)\beta = \alpha(k\beta) = k(\alpha\beta), \\ \alpha(\beta\gamma) = (\alpha\beta)\gamma, \end{cases} \tag{2.7}$$

where $\alpha, \beta, \gamma \in R$ *and* $k \in F$*. Then* R *is an associative algebra on* F*.*

2.2. Preliminaries. In this section, we first introduce Wong's theorem about finite dimensional estimation algebra.

THEOREM 2.8 (see Wong [19]). *Let* U *denote the associative algebra of* n *by* n *matrix valued functions generated by* $\{\Omega C, J_\eta C, I\}$*, where* I *stands for the identity matrix and* $C = GG^T$*. If* $H_i^T C \Gamma$ *is a constant vector for any* $1 \le i \le m$ *and any* $\Gamma \in U$*, then estimation algebra of system* (2.1) *is finite dimensional and* $\dim E \le 2n + m + 2$*.*

Some basic calculations of the Lie bracket are also given.

LEMMA 2.9 (see [7, 21]). *Let* E *be an estimation algebra for the filtering problem* (2.1)*.* $\Omega = (\omega_{ij})$ *is defined as in Definition* 2.5*. Assume* $X, Y, Z \in E$ *and* $g, h \in C^\infty(R^n)$*. Then*

(1) $[XY, Z] = X[Y, Z] + [X, Z]Y$;

(2) $[gD_i, h] = g\frac{\partial h}{\partial x_i}$;

(3) $[gD_i, hD_j] = gh\omega_{ji} + g\frac{\partial h}{\partial x_i}D_i - h\frac{\partial g}{\partial x_j}D_i$;

(4) $[gD_i^2, h] = 2g\frac{\partial h}{\partial x_i}D_i + g\frac{\partial^2 h}{\partial x_i^2}$;

(5) $[D_i^2, hD_j] = 2\frac{\partial h}{\partial x_i}D_iD_j + 2h\omega_{ji}D_i + \frac{\partial^2 h}{\partial x_i^2}D_j + h\frac{\partial \omega_{ji}}{\partial x_i}$;

(6) $[D_i^2, D_j^2] = 4\omega_{ji}D_jD_i + 2\frac{\partial \omega_{ji}}{\partial x_j}D_i + 2\frac{\partial \omega_{ji}}{\partial x_i}D_j + \frac{\partial^2 \omega_{ji}}{\partial x_i \partial x_j} + 2\omega_{ji}^2$;

(7) $[D_k^2, hD_iD_j] = 2\frac{\partial h}{\partial x_k}D_kD_iD_j + 2h\omega_{jk}D_iD_k + 2h\omega_{ik}D_kD_j + \frac{\partial^2 h}{\partial x_k^2}D_iD_j + 2h\frac{\partial \omega_{jk}}{\partial x_i}D_k + h\frac{\partial \omega_{jk}}{\partial x_k}D_i + h\frac{\partial \omega_{ik}}{\partial x_k}D_j + h\frac{\partial^2 \omega_{jk}}{\partial x_i \partial x_k}$;

(8) $[gD_iD_j, hD_k] = g\frac{\partial h}{\partial x_j}D_iD_k + g\frac{\partial h}{\partial x_i}D_jD_k - h\frac{\partial g}{\partial x_k}D_iD_j + gh\omega_{kj}D_i + gh\omega_{ki}D_j + g\frac{\partial^2 h}{\partial x_i \partial x_j}D_k + gh\frac{\partial \omega_{kj}}{\partial x_i}$.

3. Construction of new classes of finite dimensional filters. In this section, we consider finite dimensional filters with arbitrary state space dimension n and linear rank $n-2$. In order to construct a finite dimensional filter, our main method is based on Theorem 2.8, which is used for finding proper f, G, H in (2.1).

First, we define

$$\begin{cases} f_1 &= x_1 + x_2 + \cdots + x_n + \gamma(x_1 + x_2 + \cdots + x_n), \\ f_2 &= x_1 + x_3 + \cdots + x_n, \\ f_3 &= x_1 + x_2 + x_4 + \cdots + x_n, \\ \cdots \\ f_n &= x_1 + x_2 + \cdots + x_{n-1}, \end{cases} \tag{3.1}$$

where γ is a C^∞ function. Let G be the identity matrix; then the corresponding state evolutionary equation is

$$\begin{cases} dx_1 = (x_1 + x_2 + \cdots + x_n + \gamma(x_1 + x_2 + \cdots + x_n))dt + dw_1, \\ dx_2 = (x_1 + x_3 + \cdots + x_n)dt + dw_2, \\ dx_3 = (x_1 + x_2 + x_4 + \cdots + x_n)dt + dw_3, \\ \cdots \\ dx_n = (x_1 + x_2 + \cdots + x_{n-1})dt + dw_n, \end{cases} \tag{3.2}$$

where $w_i,\ 1 \le i \le n$ are independent standard Brownian motions.

Wong's Ω-matrix can be calculated by Definition 2.5,

$$\Omega = \begin{pmatrix} 0 & -1 & -1 & \cdots & -1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \cdot \gamma'(x_1 + x_2 + \cdots + x_n). \tag{3.3}$$

Then we can calculate eigenvalues and corresponding eigenvectors of Ω^T,

$$\begin{aligned} &\lambda_1 = \lambda_2 = \cdots = \lambda_{n-2} = 0, \quad \lambda_{n-1} = \sqrt{n-1}\gamma'\mathrm{i}, \quad \lambda_n = -\sqrt{n-1}\gamma'\mathrm{i}, \\ &H_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, H_{n-2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ -1 \end{pmatrix}, \\ &H_{n-1} = \begin{pmatrix} \sqrt{n-1}\mathrm{i} \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad H_n = \begin{pmatrix} -\sqrt{n-1}\mathrm{i} \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}. \end{aligned} \tag{3.4}$$

Since the selection of H is restricted to the real matrix, we only consider real eigenvectors $H_1, H_2, \ldots, H_{n-2}$, which are $n-2$ linearly independent eigenvectors corresponding to eigenvalue 0.

We define $H = (H_1, H_2, \dots, H_{n-2})^T$ and obtain the observation equation,

$$\begin{cases} dy_1 = H_1^T x dt + dv_1 = (x_2 - x_3)dt + dv_1, \\ dy_2 = H_2^T x dt + dv_2 = (x_2 - x_4)dt + dv_2, \\ dy_3 = H_3^T x dt + dv_2 = (x_2 - x_5)dt + dv_3, \\ \cdots \\ dy_{n-2} = H_{n-2}^T x dt + dv_2 = (x_2 - x_n)dt + dv_{n-2}, \end{cases} \tag{3.5}$$

where $v_i,\ 1 \le i \le n-2$ are independent standard Brownian motions.

By combining state evolutionary equation (3.2) and observation equation (3.5), we obtain the filtering system (F1),

$$\text{(F1):} \begin{cases} dx_1 = (x_1 + x_2 + \cdots + x_n + \gamma(x_1 + x_2 + \cdots + x_n))dt + dw_1, \\ dx_2 = (x_1 + x_3 + \cdots + x_n)dt + dw_2, \\ \cdots \\ dx_n = (x_1 + x_2 + \cdots + x_{n-1})dt + dw_n, \\ dy_1 = (x_2 - x_3)dt + dv_1, \\ dy_2 = (x_2 - x_4)dt + dv_2, \\ \cdots \\ dy_{n-2} = (x_2 - x_n)dt + dv_{n-2}. \end{cases} \tag{3.6}$$

In filtering system (F1), $H_i,\ 1 \le i \le n-2$ is the real eigenvector of Ω^T but not J_η. In the following calculations, we aim to make an orthogonal transformation for $H_i,\ 1 \le i \le n-2$ and obtain a filtering system satisfying the assumptions of Theorem 2.8.

First, η can be calculated by definition,

$$\begin{aligned} \eta &= \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^n f_i^2 + \sum_{i=1}^{n-2} h_i^2 \\ &= (1+\gamma') + \left(\sum_{i=1}^n x_i + \gamma\right)^2 + \sum_{k=2}^n \left(\sum_{i=1}^n x_i - x_k\right)^2 + \sum_{i=3}^n (x_2 - x_i)^2. \end{aligned} \tag{3.7}$$

The gradient of η can be calculated as follows:

$$\begin{cases} \eta_{x_1} = \delta_0 + 2\left[(n-1)x_1 + (n-2)\sum_{i=2}^n x_i\right], \\ \eta_{x_2} = \delta_0 + 2\left[(n-2)x_1 + 2(n-2)x_2 + (n-4)\sum_{i=3}^n x_i\right], \\ \eta_{x_k} = \delta_0 + 2\left[(n-2)x_1 + (n-4)x_2 + (n-3)\sum_{i=3}^n x_i + 2x_k\right], \quad 3 \le k \le n, \end{cases} \tag{3.8}$$

where $\delta_0 = \gamma'' + 2(x_1 + x_2 + \cdots + x_n + \gamma)(1 + \gamma')$.

Then we can calculate Hessian matrix J_η,

$$J_\eta = \eta_0 \times \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} + 2\begin{pmatrix} n-1 & n-2 & n-2 & \cdots & n-2 & n-2 \\ n-2 & 2(n-2) & n-4 & \cdots & n-4 & n-4 \\ n-2 & n-4 & n-1 & \cdots & n-3 & n-3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n-2 & n-4 & n-3 & \cdots & n-3 & n-1 \end{pmatrix}, \tag{3.9}$$

where $\eta_0 = \gamma''' + 2(x_1 + \cdots + x_n + \gamma)\gamma'' + 2(1+\gamma')^2$. Next, we calculate $J_\eta H_i,\ 1 \leq i \leq n-2$,

$$\begin{cases} J_\eta H_1 = (0, 2n, -6, -2, \ldots, -2)^T = 6H_1 + 2H_2 + \cdots + 2H_{n-2}, \\ J_\eta H_2 = (0, 2n, -2, -6, \ldots, -2)^T = 2H_1 + 6H_2 + \cdots + 2H_{n-2}, \\ \quad \cdots \\ J_\eta H_{n-2} = (0, 2n, -2, -2, \ldots, -6)^T = 2H_1 + 2H_2 + \cdots + 6H_{n-2}. \end{cases} \tag{3.10}$$

The above (3.10) can be expressed as the matrix form

$$J_\eta(H_1, H_2, \ldots, H_{n-2}) = (H_1, H_2, \ldots, H_{n-2})A, \tag{3.11}$$

where

$$A = \begin{pmatrix} 6 & 2 & 2 & \cdots & 2 \\ 2 & 6 & 2 & \cdots & 2 \\ 2 & 2 & 6 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \cdots & 6 \end{pmatrix}. \tag{3.12}$$

Since A is a real symmetric matrix, it can be orthogonally diagonalized. First, we calculate eigenvalues and eigenvectors of A. Due to $\det(A - \lambda I) = (2n-\lambda)(4-\lambda)^{n-3}$, we get eigenvalues of A,

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{n-3} = 4, \quad \lambda_{n-2} = 2n. \tag{3.13}$$

Eigenvectors corresponding to the first $n-3$ eigenvalues are given,

$$\alpha_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \alpha_{n-3} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ -1 \end{pmatrix}. \tag{3.14}$$

By using Schmidt orthogonalization for $\{\alpha_1, \alpha_2, \ldots, \alpha_{n-3}\}$, normalized orthogonal eigenvectors can be obtained,

$$v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\\0\\\vdots\\0\end{pmatrix}, v_2 = \sqrt{\frac{2}{3}}\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\-1\\0\\\vdots\\0\end{pmatrix}, v_3 = \frac{\sqrt{3}}{2}\begin{pmatrix}\frac{1}{3}\\\frac{1}{3}\\\frac{1}{3}\\-1\\\vdots\\0\end{pmatrix}, \dots, v_{n-3} = \sqrt{\frac{n-3}{n-2}}\begin{pmatrix}\frac{1}{n-3}\\\frac{1}{n-3}\\\frac{1}{n-3}\\\vdots\\\frac{1}{n-3}\\-1\end{pmatrix}. \tag{3.15}$$

The eigenvector corresponding to $\lambda_{n-2} = 2n$ is

$$v_{n-2} = \frac{1}{\sqrt{n-2}}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}. \tag{3.16}$$

We define $P = (v_1, v_2, \dots, v_{n-2})$, which is an orthogonal matrix. Then we get the diagonal decomposition $A = P\Lambda P^T$, where $\Lambda = \mathrm{diag}(4, 4, \dots, 4, 2n)$. Thus (3.11) becomes

$$\begin{aligned} &J_\eta(H_1, H_2, \dots, H_{n-2}) = (H_1, H_2, \dots, H_{n-2})P\Lambda P^T \\ \Longrightarrow\ &J_\eta(H_1, H_2, \dots, H_{n-2})P = (H_1, H_2, \dots, H_{n-2})P\Lambda. \end{aligned} \tag{3.17}$$

Next, we define $(\widetilde{H}_1, \widetilde{H}_2, \dots, \widetilde{H}_{n-2}) = (H_1, H_2, \dots, H_{n-2})P$, which is given by

$$\begin{aligned} &\widetilde{H}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\-1\\1\\0\\\vdots\\0\end{pmatrix}, \quad \widetilde{H}_2 = \sqrt{\frac{2}{3}}\begin{pmatrix}0\\0\\-\frac{1}{2}\\-\frac{1}{2}\\1\\\vdots\\0\end{pmatrix}, \dots, \\ &\widetilde{H}_{n-3} = \sqrt{\frac{n-3}{n-2}}\begin{pmatrix}0\\0\\-\frac{1}{n-3}\\-\frac{1}{n-3}\\\vdots\\-\frac{1}{n-3}\\1\end{pmatrix}, \quad \widetilde{H}_{n-2} = \frac{1}{\sqrt{n-2}}\begin{pmatrix}0\\n-2\\-1\\\vdots\\-1\end{pmatrix}. \end{aligned} \tag{3.18}$$

Clearly, $\widetilde{H}_i,\ 1 \le i \le n-2$ is the eigenvector of J_η.

We define $\widetilde{H} = (\widetilde{H}_1, \widetilde{H}_2, \dots, \widetilde{H}_{n-2})^T$ and keep f, G unchanged. Then we consider the following filtering system (F2):

$$
\text{(3.19)}\quad \text{(F2)}:\begin{cases}
dx_1 =(x_1+x_2+\cdots+x_n+\gamma(x_1+x_2+\cdots+x_n))dt+dw_1,\\
dx_2 =(x_1+x_3+\cdots+x_n)dt+dw_2,\\
dx_3 =(x_1+x_2+x_4+\cdots+x_n)dt+dw_3,\\
\quad\cdots\\
dx_n =(x_1+x_2+\cdots+x_{n-1})dt+dw_n,\\
dy_1 =\widetilde{H}_1^T x dt+dv_1\\
\quad =\frac{1}{\sqrt{2}}(-x_3+x_4)dt+dv_1,\\
dy_2 =\widetilde{H}_2^T x dt+dv_2\\
\quad =\sqrt{\frac{2}{3}}\left(-\frac{1}{2}x_3-\frac{1}{2}x_4+x_5\right)dt+dv_2,\\
\quad\cdots\\
dy_{n-3} =\widetilde{H}_{n-3}^T x dt+dv_{n-3}\\
\quad =\sqrt{\frac{n-3}{n-2}}\left(-\frac{1}{n-3}x_3-\frac{1}{n-3}x_4-\cdots-\frac{1}{n-3}x_{n-1}+x_n\right)dt\\
\qquad +dv_{n-3},\\
dy_{n-2} =\widetilde{H}_{n-2}^T x dt+dv_{n-2}\\
\quad =\frac{1}{\sqrt{n-2}}((n-2)x_2-x_3-\cdots-x_n)dt+dv_{n-2}.
\end{cases}
$$

In the filtering system (F2), we denote

$$
\text{(3.20)}\quad
\begin{aligned}
\widetilde{f}_i &= f_i, \quad 1\le i\le n,\\
\widetilde{h}_i &= \widetilde{H}_i^T x, \quad 1\le i\le n,\\
\widetilde{\Omega} &= (\widetilde{\omega}_{ij}), \quad \widetilde{\omega}_{ij}=\frac{\partial \widetilde{f}_j}{\partial x_i}-\frac{\partial \widetilde{f}_i}{\partial x_j}, \quad 1\le i,j\le n,\\
\widetilde{\eta} &= \sum_{i=1}^n \frac{\partial \widetilde{f}_i}{\partial x_i}+\sum_{i=1}^n \widetilde{f}_i^2+\sum_{i=1}^{n-2}\widetilde{h}_i^2.
\end{aligned}
$$

Due to $\widetilde{f}_i = f_i,\ 1\le i\le n$, and

$$
\text{(3.21)}\quad
\begin{aligned}
\sum_{i=1}^{n-2}\widetilde{h}_i^2 &= \sum_{i=1}^{n-2}(\widetilde{H}_i^T x)^2\\
&= x^T\widetilde{H}^T\widetilde{H}x\\
&= x^T H^T P P^T H x\\
&= x^T H^T H x\\
&= \sum_{i=1}^{n-2}(H_i^T x)^2=\sum_{i=1}^{n-2}h_i^2,
\end{aligned}
$$

we then have $\widetilde{\eta}=\eta$. By (3.17), we obtain

$$
\text{(3.22)}\quad J_{\widetilde{\eta}}(\widetilde{H}_1,\widetilde{H}_2,\ldots,\widetilde{H}_{n-2})=(\widetilde{H}_1,\widetilde{H}_2,\ldots,\widetilde{H}_{n-2})\Lambda,
$$

which means $\widetilde{H}_i$ is the eigenvector of $J_{\widetilde{\eta}}$. Due to $\Omega^T H_i = 0,\ 1 \le i \le n-2,\ \widetilde{\Omega} = \Omega$, and the definition of $\widetilde{H}_i$, then $\widetilde{\Omega}^T \widetilde{H}_i = 0,\ 1 \le i \le n-2$, which means that $\widetilde{H}_i$ is the eigenvector of $\widetilde{\Omega}^T$. Therefore, $\widetilde{H}_i$'s are $n-2$ linearly independent common real eigenvectors of $\widetilde{\Omega}^T$ and $J_{\widetilde{\eta}}$.

Let U be the associative algebra of n by n matrix-valued functions generated by $\{\widetilde{\Omega}, J_{\widetilde{\eta}}, I\}$, where I stands for the identity matrix. The filtering system (F2) satisfies all assumptions of Theorem 2.8. Therefore, estimation algebra of the filtering system (F2) is finite dimensional.

In the following theorem, we show the structure of the estimation algebra and Wong's Ω-matrix of filtering system (F2).

THEOREM 3.1. *The nonlinear filtering system is given by*

$$\begin{cases} dx_1 = (x_1 + x_2 + \cdots + x_n + \gamma(x_1 + x_2 + \cdots + x_n))dt + dw_1, \\ dx_2 = (x_1 + x_3 + \cdots + x_n)dt + dw_2, \\ dx_3 = (x_1 + x_2 + x_4 + \cdots + x_n)dt + dw_3, \\ \cdots \\ dx_n = (x_1 + x_2 + \cdots + x_{n-1})dt + dw_n, \\ dy_1 = \frac{1}{\sqrt{2}}(-x_3 + x_4)dt + dv_1, \\ dy_2 = \sqrt{\frac{2}{3}}(-\frac{1}{2}x_3 - \frac{1}{2}x_4 + x_5)dt + dv_2, \\ \cdots \\ dy_{n-3} = \sqrt{\frac{n-3}{n-2}}(-\frac{1}{n-3}x_3 - \frac{1}{n-3}x_4 - \cdots - \frac{1}{n-3}x_{n-1} + x_n)dt + dv_{n-3}, \\ dy_{n-2} = \frac{1}{\sqrt{n-2}}((n-2)x_2 - x_3 - \cdots - x_n)dt + dv_{n-2}, \end{cases} \tag{3.23}$$

where γ is a C^∞ function. w, v are vector-valued independent standard Brownian motions independent of the initial conditions. Then in this filtering system, entries of Wong's Ω-matrix need not be constants or polynomials. Dimension of estimation algebra is $2n-2$, and linear rank of estimation algebra is $n-2$.

Proof. For convenience, we omit the tilde in (3.20) and denote

$$\begin{cases} f_1 = x_1 + x_2 + \cdots + x_n + \gamma(x_1 + x_2 + \cdots + x_n), \\ f_k = \sum_{i=1}^{n} x_i - x_k, \quad 2 \le k \le n, \\ h_k = \sqrt{\frac{k}{k+1}}\left(-\frac{1}{k}x_3 - \frac{1}{k}x_4 - \cdots - \frac{1}{k}x_{k+2} + x_{k+3}\right), \quad 1 \le k \le n-3, \\ h_{n-2} = \frac{1}{\sqrt{n-2}}((n-2)x_2 - x_3 - \cdots - x_n). \end{cases} \tag{3.24}$$

Wong's Ω-matrix is given by (3.3),

$$\Omega = \begin{pmatrix} 0 & -1 & -1 & \cdots & -1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \cdot \gamma'(x_1 + x_2 + \cdots + x_n). \tag{3.25}$$

Due to the arbitrariness of γ, entries of Ω are not meant necessarily to be constants or polynomials. Next, we calculate elements in estimation algebra E of the filtering system. By definition, $L_0, h_1, h_2, \ldots, h_{n-2} \in E$.

First, we calculate elements $[L_0, h_i],\ 1 \le i \le n-2$,

$$
\begin{aligned}
[L_0, h_1] =& \frac{1}{2}\left[\sum_{i=1}^{n} D_i^2 - \eta, \frac{1}{\sqrt{2}}(-x_3 + x_4)\right] \\
=& \frac{1}{\sqrt{2}}(-D_3 + D_4) \in E, \\
[L_0, h_2] =& \frac{1}{2}\left[\sum_{i=1}^{n} D_i^2 - \eta, \sqrt{\frac{2}{3}}\left(-\frac{1}{2}x_3 - \frac{1}{2}x_4 + x_5\right)\right] \\
=& \sqrt{\frac{2}{3}}\left(-\frac{1}{2}D_3 - \frac{1}{2}D_4 + D_5\right) \in E, \\
& \cdots \\
[L_0, h_k] =& \frac{1}{2}\left[\sum_{i=1}^{n} D_i^2 - \eta, \sqrt{\frac{k}{k+1}}\left(-\frac{1}{k}\sum_{i=3}^{k+2} x_i + x_{k+3}\right)\right] \\
=& \sqrt{\frac{k}{k+1}}\left(-\frac{1}{k}\sum_{i=3}^{k+2} D_i + D_{k+3}\right) \in E, \quad 1 \le k \le n-3, \\
& \cdots \\
[L_0, h_{n-3}] =& \sqrt{\frac{n-3}{n-2}}\left(-\frac{1}{n-3}\sum_{i=3}^{n-1} D_i + D_n\right) \in E, \\
[L_0, h_{n-2}] =& \frac{1}{2}\left[\sum_{i=1}^{n} D_i^2 - \eta, \frac{1}{\sqrt{n-2}}\left((n-2)x_2 - \sum_{i=3}^{n} x_i\right)\right] \\
=& \frac{1}{\sqrt{n-2}}\left((n-2)D_2 - \sum_{i=3}^{n} D_i\right) \in E.
\end{aligned} \tag{3.26}
$$

Next, we can calculate the Lie bracket of L_0 and the above first order differential operators obtained by (3.26),

$$
\begin{aligned}
[L_0, -D_3 + D_4] =& \frac{1}{2}\left(-\frac{\partial \eta}{\partial x_3} + \frac{\partial \eta}{\partial x_4}\right) \in E, \\
\left[L_0, -\frac{1}{2}D_3 - \frac{1}{2}D_4 + D_5\right] =& \frac{1}{2}\left(-\frac{1}{2}\frac{\partial \eta}{\partial x_3} - \frac{1}{2}\frac{\partial \eta}{\partial x_4} + \frac{\partial \eta}{\partial x_5}\right) \in E, \\
& \cdots \\
\left[L_0, -\frac{1}{k}\sum_{i=3}^{k+2} D_i + D_{k+3}\right] =& \frac{1}{2}\left[\sum_{i=1}^{n} D_i^2 - \eta, -\frac{1}{k}\sum_{j=3}^{k+2} D_j + D_{k+3}\right] \\
=& -\frac{1}{2k}\sum_{j=3}^{k+2}\sum_{i=1}^{n}[D_i^2, D_j] + \frac{1}{2}\sum_{i=1}^{n}[D_i^2, D_{k+3}] \\
& + \frac{1}{2}\left[-\frac{1}{k}\sum_{j=3}^{k+2} D_j + D_{k+3}, \eta\right]
\end{aligned} \tag{3.27}
$$

$$
\begin{aligned}
&= -\frac{1}{2k}\sum_{j=3}^{k+2}\sum_{i=1}^{n}\left(2\omega_{ji}D_i + \frac{\partial \omega_{ji}}{\partial x_i}\right) \\
&\quad + \frac{1}{2}\sum_{i=1}^{n}\left(2\omega_{k+3,i}D_i + \frac{\partial \omega_{k+3,i}}{\partial x_i}\right) \\
&\quad - \frac{1}{2k}\sum_{j=3}^{k+2}\frac{\partial \eta}{\partial x_j} + \frac{1}{2}\frac{\partial \eta}{\partial x_{k+3}} \\
&= -\frac{1}{2k}\sum_{j=3}^{k+2}\left(2\omega_{j1}D_1 + \frac{\partial \omega_{j1}}{\partial x_1}\right) \\
&\quad + \frac{1}{2}\left(2\omega_{k+3,1}D_1 + \frac{\partial \omega_{k+3,1}}{\partial x_1}\right) \\
&\quad - \frac{1}{2k}\sum_{j=3}^{k+2}\frac{\partial \eta}{\partial x_j} + \frac{1}{2}\frac{\partial \eta}{\partial x_{k+3}} \\
&= \left(\sum_{j=3}^{k+2} -\frac{1}{k}\omega_{j1} + \omega_{k+3,1}\right) D_1 \\
&\quad + \frac{1}{2}\left(\sum_{j=3}^{k+2} -\frac{1}{k}\frac{\partial \omega_{j1}}{\partial x_1} + \frac{\partial \omega_{k+3,1}}{\partial x_1}\right) \\
&\quad + \frac{1}{2}\left(-\frac{1}{k}\sum_{j=3}^{k+2}\frac{\partial \eta}{\partial x_j} + \frac{\partial \eta}{\partial x_{k+3}}\right) \\
&= \left(\sum_{j=3}^{k+2} -\frac{1}{k} + 1\right)\gamma' D_1 + \frac{1}{2}\left(\sum_{j=3}^{k+2} -\frac{1}{k} + 1\right)\gamma'' \\
&\quad + \frac{1}{2}\left(-\frac{1}{k}\sum_{j=3}^{k+2}\frac{\partial \eta}{\partial x_j} + \frac{\partial \eta}{\partial x_{k+3}}\right) \\
&= \frac{1}{2}\left(-\frac{1}{k}\sum_{j=3}^{k+2}\frac{\partial \eta}{\partial x_j} + \frac{\partial \eta}{\partial x_{k+3}}\right) \in E, \quad 1 \le k \le n-3,
\end{aligned}
$$

$$\cdots$$

$$\left[L_0, -\frac{1}{n-3}\sum_{i=3}^{n-1} D_i + D_n\right] = \frac{1}{2}\left(-\frac{1}{n-3}\sum_{i=3}^{n-1}\frac{\partial \eta}{\partial x_i} + \frac{\partial \eta}{\partial x_n}\right) \in E,$$

$$\left[L_0, (n-2)D_2 - \sum_{i=3}^{n} D_i\right] = \frac{1}{2}\left((n-2)\frac{\partial \eta}{\partial x_2} - \sum_{i=3}^{n}\frac{\partial \eta}{\partial x_i}\right) \in E,$$

where η is given by (3.7). By using (3.8), the above (3.27) can be simplified as

$$\left[L_0, -\frac{1}{k}\sum_{i=3}^{k+2} D_i + D_{k+3}\right] \tag{3.28}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(-\frac{1}{k}\sum_{j=3}^{k+2}\frac{\partial\eta}{\partial x_j}+\frac{\partial\eta}{\partial x_{k+3}}\right)\\
&= -\frac{1}{2k}\sum_{j=3}^{k+2}\left[\delta_0+2\left((n-2)x_1+(n-4)x_2+(n-3)\sum_{i=3}^{n}x_i+2x_j\right)\right]\\
&\quad+\frac{1}{2}\left[\delta_0+2\left((n-2)x_1+(n-4)x_2+(n-3)\sum_{i=3}^{n}x_i+2x_{k+3}\right)\right]\\
&= 2\left(-\frac{1}{k}\sum_{j=3}^{k+2}x_j+x_{k+3}\right)\in E,\quad 1\le k\le n-3,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left[L_0,(n-2)D_2-\sum_{i=3}^{n}D_i\right] \\
&= \frac{1}{2}\left((n-2)\frac{\partial\eta}{\partial x_2}-\sum_{i=3}^{n}\frac{\partial\eta}{\partial x_i}\right)\\
&= \left((n-2)^2x_1+2(n-2)^2x_2+(n-2)(n-4)\sum_{i=3}^{n}x_i\right)\\
&\quad-\left((n-2)^2x_1+(n-2)(n-4)x_2+((n-2)(n-3)+2)\sum_{i=3}^{n}x_i\right)\\
&= n\left((n-2)x_2-\sum_{i=3}^{n}x_i\right)\in E.
\end{aligned} \tag{3.29}
$$

Due to

$$
\left[\frac{1}{\sqrt{2}}(-D_3+D_4),\frac{1}{\sqrt{2}}(-x_3+x_4)\right]=1\in E, \tag{3.30}
$$

all constants are in E. Finally, we calculate the Lie bracket between first order differential operators obtained by (3.26),

$$
\begin{aligned}
&\left[-\frac{1}{k_1}\sum_{j=3}^{k_1+2}D_j+D_{k_1+3},-\frac{1}{k_2}\sum_{j=3}^{k_2+2}D_j+D_{k_2+3}\right]=0,\quad 1\le k_1,k_2\le n-3,\\
&\left[-\frac{1}{k}\sum_{j=3}^{k+2}D_j+D_{k+3},(n-2)D_2-\sum_{i=3}^{n}D_i\right]=0,\quad 1\le k\le n-3.
\end{aligned} \tag{3.31}
$$

Therefore, estimation algebra E of the filtering system (3.23) is a $2n-2$ dimensional Lie algebra with basis given by

$$
\begin{aligned}
&\Big\{1,-x_3+x_4,-\frac{1}{2}x_3-\frac{1}{2}x_4+x_5,\dots,-\frac{1}{n-3}x_3-\frac{1}{n-3}x_4-\cdots-\frac{1}{n-3}x_{n-1}+x_n,\\
&\quad(n-2)x_2-x_3-\cdots-x_n,-D_3+D_4,-\frac{1}{2}D_3-\frac{1}{2}D_4+D_5,\dots,\\
&\quad-\frac{1}{n-3}D_3-\frac{1}{n-3}D_4-\cdots-\frac{1}{n-3}D_{n-1}+D_n,(n-2)D_2-D_3-\cdots-D_n,L_0\Big\}.
\end{aligned} \tag{3.32}
$$

Clearly, the linear rank of estimation algebra E is $n-2$. □

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