2020 American Control Conference
Denver, CO, USA, July 1-3, 2020

Novel Classification of Finite Dimensional Filters with Non-maximal Rank Estimation Algebra

Wenhui Dong*, Xiuqiong Chen†, Stephen S.-T. Yau*
* Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, China
† Yau Mathematical Sciences Center, Tsinghua University, Beijing, 100084, China
Email: yau@uic.edu

Abstract— Ever since the technique of Kalman-Bucy filter was popularized, due to its limitations of linear assumption and Gaussian initial condition in model, there has been an intense interest in finding new classes of finite dimensional recursive filters. The idea of using estimation algebra was first proposed to construct finite-dimensional nonlinear filters by Brockett and Mitter independently in the late seventies, and it has rapidly been proven to be an invaluable tool in the study of nonlinear filtering (NLF) problem. For all known finite dimensional estimation algebras (FDEAs), the Wong's matrix has to be constant. However, as shown in this paper, the Wong's matrix is a polynomial when we consider FDEAs with state dimension 4 and linear rank equal to 2. Several mild conditions are established for finding a special class of NLF system. Finally, we give the construction of finite dimensional filters for this class of NLF system by Wei–Norman approach.

I. Introduction

Filtering is a technique of obtaining estimate of the quantities associated with a stochastic process $\{x_t\}$ based on the information accumulated from a related process $\{y_t\}$. The process $\{x_t\}$ is called the signal or state process and $\{y_t\}$ is the observation process. Ever since 1960, after Kalman-Bucy first established the finite dimensional filters for linear-filtering systems with Gaussian initial distributions, there are intensive research interests in the NLF problems. Numerous research activities are focused on the NLF problems, and one of the most significant research focuses of general NLF problems is to solve the Duncan-Mortensen-Zakai (DMZ) equation or its pathwise robust version [2], which is satisfied by the unnormalized conditional density of the states. In late 1960s and early 1970s, the basic approach to NLF theory was via the "innovations method" originally purposed by Kailath and subsequently rigorously developed by Fujisaki, Kallianpur, and Kunita [12]. However, the weakness of this method is generally not explicitly computable. In the late 1970s, the idea of using estimation algebras was independently proposed by Brockett and Clark [3], Brockett [4], and Mitter [15], which is used to construct finite dimensional nonlinear filters. The motivation came from the Wei–Norman approach [17] of using a Lie algebraic method for solving time varying linear differential equations. More details about the Wei-Norman approach and its connection with the NLF problems are scattered in other research works, such as papers [10], [16] and the survey article by Marcus [14]. There are many advantages of Lie algebraic approach for the NLF problems. Most importantly, as long as the estimation algebra is finite dimensional, the approach always leads to finite dimensional recursive filters and the filter so constructed is universal in the sense of [6]. In addition, dimension of the sufficient statistics used in computing the conditional density function is linear in n, where n is the dimension of the state space. Therefore, it is vitally important and meaningful to study the estimation algebra method for NLF problems.

In 1981, Beneš established exact finite-dimensional filters for certain diffusions with nonlinear drift which is the first important breakthrough in Lie algebra approach [1]. Later, Wong in [18] constructed some new FDEAs and used the Wei-Norman approach to construct finite dimensional filters. Another class of finite dimensional filters was found by Charalambous and Elliott [7], where Beneš exact filtering systems was extended by inserting linear combinations of $dx(t)$ in the observations.

In the Wei–Norman approach, one has to know explicitly the basis of the estimation algebra, which further can reduce the DMZ equation to a finite group of ordinary differential equations driven by observations and a Kolmogorov equation independent of observations. Due to the practical importance of the estimation algebra method, it is significant to find out the basis of FDEA. In 1983 International Congress of Mathematics, Brockett [5] proposed the problem of classifying all FDEAs in order to find new classes of finite dimensional filters. Since then, a lot of efforts have been devoted to classifying FDEAs. Under quite severe conditions, Wong [19] proved that all FDEAs of (1) are solvable and the observation $h(x)$ is a polynomial of degree one. The concept of Wong's Ω-matrix was established in Wong [20] which has been proven that it plays an important role in subsequent research. Since the 1990s, in a series of research works, Yau and his coworkers gave the algebraic structure of several general classes of estimation algebras. On the one hand, they were able to classify all FDEAs with dimension at most six [8], [13], [22]. On the other hand, they have classified the FDEAs of maximal rank with arbitrary state space dimension [23], [24] which included both Kalman-Bucy and Beneš filtering systems. One of the key steps that Yau and his coworkers were able to classify all finite dimensional maximal rank estimation algebras is that they were able to show that Wong's Ω-matrix is a constant matrix.

However, under the condition of the FDEA with non-maximal rank, the classification problem remains to be solved. Wu and Yau [21] have classified FDEAs with state dimension 2. For state dimension 3, Shi et al. in [26]–[28] studied the classification of non-maximal rank equal to 1 and 2. For higher state dimensions $n \geq 4$, Dong et al. in [11] have discussed a new class of finite dimensional filters with non-maximal rank estimation algebra on state dimension four and rank of one, they have found a novel class of polynomial nonlinear systems but with finite dimensional filters. In this paper, we give the construction of FDEAs with non-maximal rank when we consider FDEA with state dimension 4 and linear rank equal to 2. Under several easily satisfied conditions, a class of finite dimensional filters derived by Wei–Norman approach is presented.

The paper is organized as follows: Section II describes some basic concepts about estimation algebras. In section III the construction of new classes of finite dimensional filters is given. In section IV, we will use the structure results to derive finite dimensional filters for the robust-DMZ equation by the Wei-Norman approach. And we finally arrive at conclusion in the last section.

II. Basic concepts and Preliminary results

The filtering problem we consider is based on the following signal observation model:

$$\begin{cases} dx(t) = f(x(t))dt + g(x(t))dv(t) & x(0) = x_0 \\ dy(t) = h(x(t))dt + dw(t) & y(0) = 0, \end{cases} \quad (1)$$

where x, v, y, w are respectively R^n, R^p, R^m, R^m valued process, and v and w are independent, standard Brownian motions. Assume that f and h are C^∞ smooth functions, and g is an orthogonal matrix. $x(t)$ is referred to as the state of the system at time t and $y(t)$ as the observation at time t.

Let $\rho(t, x)$ denote the conditional probability density of the state $x(t)$ given the observation $\{y(s) : 0 \leq s \leq t\}$. And $\sigma(t, x)$ is the unnormalized version of $\rho(t, x)$, which satisfies the following Duncan-Mortensen-Zakai(DMZ) equation:

$$\begin{cases} \sigma(t,x) = L_0\sigma(t,x)dt + \sum_{i=1}^{m} L_i\sigma(t,x)dy_i(t), \\ \sigma(0,x) = \sigma_0, \end{cases} \quad (2)$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2$$

and L_i is the zero degree differential operator of multiplication by h_i for $i = 1, \cdots, m$. σ_0 is the probability density of the initial state x_0.

If we define

$$D_i = \frac{\partial}{\partial x_i} - f_i, \quad \eta = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{m} h_i^2, \quad (3)$$

then we have a more compact form of L_0 as

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^{n} D_i^2 - \eta\right).$$

In real applications, we are interested in considering robust state estimator from observed sample paths with some properties of robustness. In [9], Davis proposed some robust algorithms. In our case, this basic idea is reduced to define a new unnormalized density

$$u(t,x) = \exp\left(-\sum_{i=1}^{m} h_i(x)y_i(t)\right)\sigma(t,x), \quad (4)$$

then $u(t, x)$ satisfies the following equation

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) &= L_0u(t,x) + \sum_{i=1}^{m} y_i(t)[L_0, L_i]u(t,x) \\ &\quad + \frac{1}{2}\sum_{i,j=1}^{m} y_i(t)y_j(t)[[L_0, L_i], L_j]u(t,x), \\ u(0,x) &= \sigma_0(x), \end{cases} \quad (5)$$

where $[\cdot, \cdot]$ is the Lie bracket in Definition 1 below, and (5) is also called robust-DMZ equation.

Definition 1: If X and Y are differential operators, $[X, Y]$ is the Lie bracket of X and Y defined by $[X, Y]\phi = X(Y\phi) - Y(X\phi)$ for any C^∞ function ϕ.

Recall that a vector space $\mathcal{F}$ with the Lie bracket operation $\mathcal{F} \times \mathcal{F} \to \mathcal{F}$ denoted by $(x, y) \mapsto [x, y]$ is called a Lie algebra if the following axioms are satisfied:
(1) The Lie bracket operation is bilinear;
(2) $[x, x] = 0$ for any $x \in \mathcal{F}$;
(3) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for any $x, y, z \in \mathcal{F}$.

Definition 2: The estimation algebra E of a filtering problem (1) is defined to be the Lie algebra generated by $\{L_0, L_1, \ldots, L_m\}$. E is said to be an estimation algebra of maximal rank if, for any $1 \leq i \leq n$, there exists a constant c_i such that $x_i + c_i \in E$.

The linear rank concept of estimation algebra was introduced by X. Wu and Yau in [21].

Definition 3: Let $L(E) \subset E$ be the vector space consisting of all the homogeneous degree one polynomials in E. Then the linear rank of estimation algebra E is defined by $r := dimL(E)$. So estimation algebra of maximal rank is in fact linear rank n estimation algebra.

Definition 4: The Wong's matrix is the matrix $\Omega = (\omega_{ij})$, where

$$\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}, \quad \forall 1 \leq i, j \leq n.$$

Clearly $\omega_{ij} = -\omega_{ji}$.

Definition 5: Let U be the set of differential operators in the form

$$A = \sum_{|(i_1, i_2, \ldots, i_n)| = l+1} a_{i_1, i_2, \ldots, i_n} D_1^{i_1} D_2^{i_2} \ldots D_n^{i_n} \quad (6)$$

where nonzero functions $a_{i_1,i_2,...,i_n} \in \mathbb{C}^\infty(\mathbb{R}^n)$ and I_A is the finite index set of A. Each element of the index set is an n-tuple $(i_1, i_2, ..., i_n)$ of nonnegative integers. The norm of an index $i = (i_1, i_2, ..., i_n)$ is defined by $|i| = \sum_{l=1}^{n} i_l$. The order of A is denoted by $ord(A) = \max_{i=(i_1,i_2,...,i_n)\in I_A} |i|$. If $A = 0$, $ord(A)$ is defined to be $-\infty$. It is clear that for A, $B \in U$

$$ord(AB) = ord(BA) = ord(A) + ord(B), \tag{7}$$

$$ord(A \pm B) \leq \max(ord(A), ord(B)). \tag{8}$$

U is a Lie algebra under the Lie bracket $[\cdot,\cdot]$ defined earlier. Two differential operators A and B in U are equal if they have the identical index sets $I_A = I_B$ and $a_{i_1,i_2,...,i_n} = b_{i_1,i_2,...,i_n}$, $\forall a_{i_1,i_2,...,i_n} \in I_A$. Let U_k denote the subspace of U consisting of the elements with order less than or equal to k. In particular, $U_0 = \mathbb{C}^\infty(\mathbb{R}^n)$. Generally, *mod* is used to denote the equivalence class, i.e., if V is a subspace of U,

$$A = B,\ mod\, V \Leftrightarrow A - B \in V.$$

If $A, B \in U$, define

$$Ad_A B = [A, B],\ Ad_A^l B = [A, Ad_A^{l-1} B],\ l \geq 1, \tag{9}$$

where Ad_A^0 is the identity operator by standard convention.

Theorem 1: ([25]) Let E be a finite dimensional estimation algebra. If a function ϕ is in E, then ϕ is a polynomial of degree less than or equal to 2.

Theorem 2: ([24]) Let E be a finite dimensional estimation algebra. And D_i is defined as in (3). If $l \geq 0$ and

$$A = \sum_{|(i_1,i_2,...,i_n)|=l+1} a_{i_1,i_2,...,i_n} D_1^{i_1} D_2^{i_2} ... D_n^{i_n}, mod\, U_l$$

is in E, then $a_{i_1,i_2,...,i_n}$ are polynomials.

Lemma 1: ([21]) Let $g, h \in \mathbb{C}^\infty(\mathbb{R}^n)$ and $i_1, \cdots, i_n$, $j_1, \cdots, j_n$ be nonnegative integers with $\sum_{l=1}^{n} i_l = r$, $\sum_{l=1}^{n} j_l = s$, and $r + s \geq 2$, then

$$\begin{aligned}
&[gD_1^{i_1}...D_n^{i_n}, hD_1^{j_1}...D_n^{j_n}] \\
&= \sum_{l=1}^{n} (i_l g \frac{\partial h}{\partial x_l} - j_l h \frac{\partial g}{\partial x_l}) D_1^{i_1+j_1-\delta_{1l}} ... D_n^{i_n+j_n-\delta_{nl}}, \\
&mod\ U_{r+s-2}
\end{aligned}$$

where δ_{ij} is the kronecker symbol with $\delta_{ij} = 1$, if $i = j$; otherwise $\delta_{ij} = 0$.

III. Construction of new class of finite dimensional filters

In this section, we consider that, in state space with dimension four, the finite dimensional filtering systems of which the estimation algebra, denoted by E, is finite dimensional and has linear rank of 2. Without loss of generality, we may assume that there exist constants c_i such that $x_i + c_i \in E$, $1 \leq i \leq 2$ and for any constants c_j, $x_j + c_j \notin E$, $3 \leq j \leq 4$.

Before we give the following elementary lemma, we introduce some notations. Let $P_k(x_{i_1}, \cdots, x_{i_n})$ denote all the polynomials of degree up to k about the variables $x_{i_1}, \cdots, x_{i_n}$.

Lemma 2:

$$\begin{aligned}
&[L_0, x_i + c_i] = D_i \in E,\ 1 \leq i \leq 2, \\
&[D_2, D_1] = \omega_{12} \in E, \\
&[D_1, x_1 + c_1] = 1 \in E,
\end{aligned} \tag{10}$$

$$\begin{aligned}
&Y_1 := [L_0, D_1] \\
&= \omega_{12} D_2 + \omega_{13} D_3 + \omega_{14} D_4 + \\
&\quad \frac{1}{2}\frac{\partial \omega_{12}}{\partial x_2} + \frac{1}{2}\frac{\partial \omega_{13}}{\partial x_3} + \frac{1}{2}\frac{\partial \omega_{14}}{\partial x_4} + \frac{1}{2}\frac{\partial \eta}{\partial x_1} \in E \\
&= \omega_{12} D_2 + \omega_{13} D_3 + \omega_{14} D_4\ mod\ U_0,
\end{aligned} \tag{11}$$

$$\begin{aligned}
&Y_2 := [L_0, D_2] \\
&= \omega_{21} D_1 + \omega_{23} D_3 + \omega_{24} D_4 + \\
&\quad \frac{1}{2}\frac{\partial \omega_{21}}{\partial x_1} + \frac{1}{2}\frac{\partial \omega_{23}}{\partial x_3} + \frac{1}{2}\frac{\partial \omega_{24}}{\partial x_4} + \frac{1}{2}\frac{\partial \eta}{\partial x_2} \in E \\
&= \omega_{21} D_2 + \omega_{23} D_3 + \omega_{24} D_4\ mod\ U_0,
\end{aligned} \tag{12}$$

therefore, $P_1(x_1, x_2) \subseteq E$.

If we impose the following assumptions:
(i) $\omega_{12} = \omega_{13} = \omega_{14} = \omega_{23} = \omega_{24} = 0$,
(ii) $\eta = P_2(x_1, x_2) + \phi(x_3, x_4)$,
(iii) h_i, $1 \leq i \leq m$ are degree one polynomials of x_1, x_2,
then the estimation algebra E is finite dimensional with basis given by $\{1, x_1, x_2, D_1, D_2, L_0\}$.

Next we construct a class of NLF systems which satisfy the assumptions above. Combined the Assumption (iii) with Assumption (ii), i.e.,

$$\eta = \sum_{i=1}^{4} \left(f_i^2 + \frac{\partial f_i}{\partial x_i} \right) + \sum_{i=1}^{m} h_i^2, \tag{13}$$

is a polynomial of degree 2 at most with respect to x_1, x_2, then we may assume that f_i are at most degree 1 polynomials with respect to x_1, x_2, i.e., we assume that for $1 \leq i \leq 4$,

$$f_i = a_i(x_3, x_4) x_1 + b_i(x_3, x_4) x_2 + \phi_i(x_3, x_4), \tag{14}$$

where $a_i(x_3, x_4), b_i(x_3, x_4)$ and $\phi_i(x_3, x_4)$ are $\mathbb{C}^\infty$ functions of x_3, x_4. By assumption (i), we have

$$\begin{aligned}
\omega_{12} &= \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} = a_2 - b_1 = 0, \\
\omega_{13} &= \frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3} = a_3 - (\frac{\partial a_1}{\partial x_3} x_1 + \frac{\partial b_1}{\partial x_3} x_2 + \frac{\partial \phi_1}{\partial x_3}) = 0, \\
\omega_{14} &= \frac{\partial f_4}{\partial x_1} - \frac{\partial f_1}{\partial x_4} = a_4 - (\frac{\partial a_1}{\partial x_4} x_1 + \frac{\partial b_1}{\partial x_4} x_2 + \frac{\partial \phi_1}{\partial x_4}) = 0, \\
\omega_{23} &= \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} = b_3 - (\frac{\partial a_2}{\partial x_3} x_1 + \frac{\partial b_2}{\partial x_3} x_2 + \frac{\partial \phi_2}{\partial x_3}) = 0, \\
\omega_{24} &= \frac{\partial f_4}{\partial x_2} - \frac{\partial f_2}{\partial x_4} = b_4 - (\frac{\partial a_2}{\partial x_4} x_1 + \frac{\partial b_2}{\partial x_4} x_2 + \frac{\partial \phi_2}{\partial x_4}) = 0,
\end{aligned} \tag{15}$$

hence we have

$$\begin{cases} a_2 = b_1, \\ \dfrac{\partial a_1}{\partial x_3} = 0,\ \dfrac{\partial a_1}{\partial x_4} = 0,\ \dfrac{\partial a_2}{\partial x_3} = 0,\ \dfrac{\partial a_2}{\partial x_4} = 0, \\ \dfrac{\partial b_1}{\partial x_3} = 0,\ \dfrac{\partial b_1}{\partial x_4} = 0,\ \dfrac{\partial b_2}{\partial x_3} = 0,\ \dfrac{\partial b_2}{\partial x_4} = 0, \\ \dfrac{\partial \phi_1}{\partial x_3} = a_3,\ \dfrac{\partial \phi_1}{\partial x_4} = a_4,\ \dfrac{\partial \phi_2}{\partial x_3} = a_3,\ \dfrac{\partial \phi_2}{\partial x_4} = a_4. \end{cases} \tag{16}$$

From (16), a_1, b_1, a_2, b_2 must be constants and $a_2 = b_1$. Now

$$\begin{aligned} \eta &= \sum_{i=1}^{4}\left(f_i^2 + \frac{\partial f_i}{\partial x_i}\right) + \sum_{i=1}^{m} h_i^2 \\ &= \left(a_1^2 + a_2^2 + \left(\frac{\partial \phi_1}{\partial x_3}\right)^2 + \left(\frac{\partial \phi_1}{\partial x_4}\right)^2\right) x_1^2 \\ &\quad + \left(b_1^2 + b_2^2 + \left(\frac{\partial \phi_2}{\partial x_3}\right)^2 + \left(\frac{\partial \phi_2}{\partial x_4}\right)^2\right) x_2^2 \\ &\quad + \left(2a_1b_1 + 2a_2b_2 + 2\frac{\partial \phi_1}{\partial x_3}\cdot\frac{\partial \phi_2}{\partial x_3} + 2\frac{\partial \phi_1}{\partial x_4}\cdot\frac{\partial \phi_2}{\partial x_4}\right) x_1x_2 \\ &\quad + \left(2\sum_{i=1}^{2} a_i\phi_i + \sum_{j=3}^{4}\left(2\phi_j\frac{\partial \phi_1}{\partial x_j} + \frac{\partial^2 \phi_1}{\partial x_j^2}\right)\right) x_1 \\ &\quad + \left(2\sum_{i=1}^{2} b_i\phi_i + \sum_{j=3}^{4}\left(2\phi_j\frac{\partial \phi_2}{\partial x_j} + \frac{\partial^2 \phi_2}{\partial x_j^2}\right)\right) x_2 \\ &\quad + \sum_{i=1}^{4}\phi_i^2 + a_1 + b_2 + \frac{\partial \phi_3}{\partial x_3} + \frac{\partial \phi_4}{\partial x_4} + \sum_{i=1}^{m} h_i^2. \end{aligned} \tag{17}$$

Now assumption (ii) implies:

(a) $\left(\dfrac{\partial \phi_1}{\partial x_3}\right)^2 + \left(\dfrac{\partial \phi_1}{\partial x_4}\right)^2$ is a constant;

(b) $\left(\dfrac{\partial \phi_2}{\partial x_3}\right)^2 + \left(\dfrac{\partial \phi_2}{\partial x_4}\right)^2$ is a constant;

(c) $\dfrac{\partial \phi_1}{\partial x_3}\cdot\dfrac{\partial \phi_2}{\partial x_3} + \dfrac{\partial \phi_1}{\partial x_4}\cdot\dfrac{\partial \phi_2}{\partial x_4}$ is a constant;

(d) $2\sum_{i=1}^{2} a_i\phi_i + \sum_{j=3}^{4}\left(2\phi_j\dfrac{\partial \phi_1}{\partial x_j} + \dfrac{\partial^2 \phi_1}{\partial x_j^2}\right)$ is a constant;

(e) $2\sum_{i=1}^{2} b_i\phi_i + \sum_{j=3}^{4}\left(2\phi_j\dfrac{\partial \phi_2}{\partial x_j} + \dfrac{\partial^2 \phi_2}{\partial x_j^2}\right)$ is a constant;

(f) h_i, $1 \le i \le m$ are degree one polynomials of x_1, x_2.

To summarize, if the following conditions are satisfied,

(1) a_1, b_1, a_2, b_2 must be constants and $a_2 = b_1$;

(2) $\dfrac{\partial \phi_1}{\partial x_3} = a_3$, $\dfrac{\partial \phi_1}{\partial x_4} = a_4$, $\dfrac{\partial \phi_2}{\partial x_3} = b_3$, $\dfrac{\partial \phi_2}{\partial x_4} = b_4$;

(3) $\left(\dfrac{\partial \phi_1}{\partial x_3}\right)^2 + \left(\dfrac{\partial \phi_1}{\partial x_4}\right)^2$, $\left(\dfrac{\partial \phi_2}{\partial x_3}\right)^2 + \left(\dfrac{\partial \phi_2}{\partial x_4}\right)^2$ and $\dfrac{\partial \phi_1}{\partial x_3}\cdot\dfrac{\partial \phi_2}{\partial x_3} + \dfrac{\partial \phi_1}{\partial x_4}\cdot\dfrac{\partial \phi_2}{\partial x_4}$ are constants;

(4) $2\sum_{i=1}^{2} a_i\phi_i + \sum_{j=3}^{4}\left(2\phi_j\dfrac{\partial \phi_1}{\partial x_j} + \dfrac{\partial^2 \phi_1}{\partial x_j^2}\right)$ is a constant;

(5) $2\sum_{i=1}^{2} b_i\phi_i + \sum_{j=3}^{4}\left(2\phi_j\dfrac{\partial \phi_2}{\partial x_j} + \dfrac{\partial^2 \phi_2}{\partial x_j^2}\right)$ is a constant;

(6) $h_i, 1 \le i \le m$'s are degree at most 1 polynomials of x_1, x_2.

Then the assumptions (i–iii) are satisfied, and the corresponding estimation algebra E is finite dimensional with basis of $\{L_0, D_2, D_1, x_2, x_1, 1\}$.

Example 1. We give a NLF system example which satisfies the conditions above. Let all the a_i's be constants, for example, we take $a_1 = 1, a_2 = 2, b_1 = 2, b_2 = -3$, then condition (1) is satisfied and from (2-3) we may assume that ϕ_1, ϕ_2 is degree at most 1 polynomial of x_3, x_4. Thus we can take $\phi_1 = 0$, $\phi_2 = 0$. Now the condition (4-5) are constants respectively can be easily satisfied. For example, if we take $\phi_3 = x_3^3 + 6x_3^2x_4 + x_4^2$, $\phi_4 = x_3^2 + 3x_3x_4^2 + x_4^4$, then the condition (4-5) are satisfied. Condition (6) is easily satisfied by letting the observation term $h(x) = c_1x_1 + c_2x_2$, where c_1, c_2 are arbitrary constants. Therefore, the NLF system can be as follows

$$\begin{cases} dx_1(t) = (x_1 + 2x_2)dt + dv_1(t), \\ dx_2(t) = (2x_1 - 3x_2)dt + dv_2(t), \\ dx_3(t) = (x_3^3 + 6x_3^2x_4 + x_4^2)dt + dv_3(t), \\ dx_4(t) = (x_3^2 + 3x_3x_4^2 + x_4^4)dt + dv_4(t), \\ dy(t) = (c_1x_1 + c_2x_2)dt + dw(t), \end{cases} \tag{18}$$

Now the corresponding Ω-matrix is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_{34} \\ 0 & 0 & \omega_{43} & 0 \end{pmatrix}$$

where $\omega_{34} = 6x_3^2 + 3x_4^2 + 2x_3 - 2x_4$ and $\omega_{43} = -\omega_{34}$, $\eta = 4x_1^2 + 2x_1 + \gamma(x_2, x_3)$. Then the estimation algebra corresponding to this class of NLF system is finite dimensional with basis given by $\{L_0, D_2, D_1, x_2, x_1, 1\}$.

IV. Finite dimensional Filters by Wei-Norman Approach

In this section, we will use the structure results of section III to derive finite-dimensional filters for the robust-DMZ equation by the Wei-Norman approach. The following theorem gives the solution of the robust-DMZ equation (5) by the basis of the corresponding estimation algebra in terms of ordinary differential equations.

Theorem 3: If the NLF system (1) satisfies the conditions (1)-(6), and then we assume $\eta = a_5x_1^2 + a_4x_2^2 + a_3x_1x_2 + a_2x_2 + a_1x_1 + a_0(x_3, x_4)$, $h_i = c_{i2}x_2 + c_{i1}x_1 + c_{i0}, 1 \le i \le m$, where $c_{j2}, c_{j1}, c_{j0}, a_5, a_4, a_3, a_2, a_1$ are constants and $a_0(x_3, x_4)$ is a $\mathbb{C}^\infty$ function of x_3, x_4, then its robust DMZ equation (5) has a solution of the form:

$$\xi(t, x) = e^{r_0(t)}e^{r_1(t)x_1}e^{r_2(t)x_2}e^{r_3(t)D_1}e^{r_4(t)D_2}e^{tL_0}\sigma_0,$$

where r_i's satisfy the following ordinary differential equations for any $t \geq 0$

$$
\begin{aligned}
\dot{r}_1(t) &= -a_2 r_2(t), \\
\dot{r}_2(t) &= r_1(t) + \sum_{i=1}^{m} c_{i1} y_i(t), \\
\dot{r}_0(t) &= \frac{r_1(t)^2}{2} - \frac{a_2}{2} r_2(t)^2 - \frac{1}{2} a_1 r_2(t) \\
&\quad + \sum_{i=1}^{m} c_{i1} y_i(t) r_1(t) + \frac{1}{2} \sum_{i=1}^{m} c_{i1} c_{j1} y_i(t) y_j(t).
\end{aligned} \tag{19}
$$

Proof: As described in section III, the estimation algebra E of (1) satisfies conditions (i) and (ii) with basis of $\{L_0, D_2, D_1, x_2, x_1, 1\}$. By differentiating $\xi(t,x)$, we have

$$
\begin{aligned}
\frac{\partial \xi}{\partial t} =& e^{r_0(t)} e^{r_1(t)x_1} e^{r_2(t)x_1} e^{r_3(t)D_1} e^{r_4(t)D_2} L_0 e^{tL_0} \sigma_0 \\
&+ \dot{r}_4(t) e^{r_0(t)} e^{r_1(t)x_1} e^{r_2(t)x_1} e^{r_3(t)D_1} D_2 e^{r_4(t)D_2} e^{tL_0} \sigma_0 \\
&+ \dot{r}_3(t) e^{r_0(t)} e^{r_1(t)x_1} e^{r_2(t)x_1} D_1 e^{r_3(t)D_1} e^{r_4(t)D_2} e^{tL_0} \sigma_0 \\
&+ (\dot{r}_2(t) x_2 + \dot{r}_1(t) x_1 + \dot{r}_0(t)) \xi(t,x) \\
=& A + B + C + (\dot{r}_2(t) x_2 + \dot{r}_1(t) x_1 + \dot{r}_0(t)) \xi(t,x),
\end{aligned} \tag{20}
$$

where

$$
A := e^{r_0(t)} e^{r_1(t)x_1} e^{r_2(t)x_1} e^{r_3(t)D_1} e^{r_4(t)D_2} L_0 e^{tL_0} \sigma_0, \tag{21}
$$

$$
B := \dot{r}_4(t) e^{r_0(t)} e^{r_1(t)x_1} e^{r_2(t)x_1} e^{r_3(t)D_1} D_2 e^{r_4(t)D_2} e^{tL_0} \sigma_0, \tag{22}
$$

$$
C := \dot{r}_3(t) e^{r_0(t)} e^{r_1(t)x_1} e^{r_2(t)x_1} D_1 e^{r_3(t)D_1} e^{r_4(t)D_2} e^{tL_0} \sigma_0. \tag{23}
$$

By applying Proposition 1 of [16], we have

$$
\begin{aligned}
A :=& e^{r_0(t)} e^{r_1(t)x_1} e^{r_2(t)x_2} e^{r_3(t)D_1} \Big(L_0 + r_4(t)[D_2, L_0] \\
&+ \frac{1}{2} r_4(t)^2 [D_2, [D_2, L_0]] \Big) e^{r_4(t)D_2} e^{tL_0} \sigma_0 \\
=& e^{r_0(t)} e^{r_1(t)x_1} e^{r_2(t)x_2} e^{r_3(t)D_1} L_0 e^{r_4(t)D_2} e^{tL_0} \sigma_0 \\
&- \left(r_4 \left(a_4 x_2 + \frac{1}{2} a_3 x_1 + \frac{1}{2} a_2 \right) - \frac{r_4(t)^2}{2} a_4 \right) \xi(t,x),
\end{aligned} \tag{24}
$$

by further computing

$$
\begin{aligned}
& e^{r_0(t)} e^{r_1(t)x_1} e^{r_2(t)x_2} e^{r_3(t)D_1} L_0 e^{r_4(t)D_2} e^{tL_0} \sigma_0 \\
=& e^{r_0(t)} e^{r_1(t)x_1} e^{r_2(t)x_2} \Big[L_0 - r_3(t)(a_5 x_1 + \frac{1}{2} a_3 x_2 + \frac{a_1}{2}) \\
&+ \frac{r_3(t)^2}{2} a_5 \Big] e^{r_3(t)D_1} e^{r_4(t)D_2} e^{tL_0} \sigma_0, \\
=& e^{r_0(t)} e^{r_1(t)x_1} e^{r_2(t)x_2} L_0 e^{r_3(t)D_1} e^{r_4(t)D_2} e^{tL_0} \sigma_0 \\
&- \left[r_3(t) \left(a_5 x_1 + \frac{1}{2} a_3 x_2 + \frac{a_1}{2} \right) - \frac{r_3(t)^2}{2} a_5 \right] \xi(t,x),
\end{aligned} \tag{25}
$$

and

$$
\begin{aligned}
& e^{r_0(t)} e^{r_1(t)x_1} e^{r_2(t)x_2} L_0 e^{r_3(t)D_1} e^{r_4(t)D_2} e^{tL_0} \sigma_0 \\
=& e^{r_0(t)} e^{r_1(t)x_1} \left(L_0 - r_2(t) D_2 + \frac{r_2(t)^2}{2} \right) e^{r_2(t)x_2} \\
& e^{r_3(t)D_1} e^{r_4(t)D_2} e^{tL_0} \sigma_0 \\
=& L_0 \xi(t,x) - r_1(t) D_1 \xi(t,x) - r_2(t) D_2 \xi(t,x) \\
&+ \frac{r_1(t)^2}{2} \xi(t,x) + \frac{r_2(t)^2}{2} \xi(t,x).
\end{aligned} \tag{26}
$$

Putting (25) and (26) into (24), we have

$$
\begin{aligned}
A :=& L_0 \xi(t,x) - r_1(t) D_1 \xi(t,x) - r_2(t) D_2 \xi(t,x) \\
&+ \Bigg[-r_3 \left(a_5 x_1 + \frac{1}{2} a_3 x_2 + \frac{a_1}{2} \right) \\
&- r_4 \left(a_4 x_2 + \frac{1}{2} a_3 x_1 + \frac{a_2}{2} \right) + \frac{r_1(t)^2}{2} \\
&+ \frac{r_2(t)^2}{2} + \frac{r_3(t)^2}{2} a_5 + \frac{r_4(t)^2}{2} a_4 \Bigg] \xi(t,x).
\end{aligned} \tag{27}
$$

Similarly, we have

$$
\begin{aligned}
B :=& \dot{r}_4(t) e^{r_0(t)} e^{r_1(t)x_1} e^{r_2(t)x_1} e^{r_3(t)D_1} D_2 e^{r_4(t)D_2} e^{tL_0} \sigma_0 \\
=& \dot{r}_4(t) e^{r_0(t)} e^{r_1(t)x_1} e^{r_2(t)x_1} \Big(D_2 + r_3(t)[D_1, D_2] \\
&+ \frac{r_3^2}{2} [D_1, [D_1, D_2]] \Big) e^{r_3(t)D_1} e^{r_4(t)D_2} e^{tL_0} \sigma_0 \\
=& \dot{r}_4(t) D_2 \xi(t,x) - \dot{r}_4(t) r_2(t) \xi(t,x).
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
C :=& \dot{r}_3(t) e^{r_0(t)} e^{r_1(t)x_1} e^{r_2(t)x_1} D_1 e^{r_3(t)D_1} e^{r_4(t)D_2} e^{tL_0} \sigma_0 \\
=& \dot{r}_3(t) D_1 \xi(t,x) - \dot{r}_3(t) r_1(t) \xi(t,x).
\end{aligned} \tag{29}
$$

Putting (27)-(29) into (20), we have

$$
\begin{aligned}
\frac{\partial \xi}{\partial t} =& L_0 \xi(t,x) + (\dot{r}_3(t) - r_1(t)) D_1 \xi(t,x) \\
&+ (\dot{r}_4(t) - r_2(t)) D_2 \xi(t,x) \\
&\Big[-r_3 (a_5 x_1 + \frac{1}{2} a_3 x_2 + \frac{a_1}{2}) \\
&- r_4 (a_4 x_2 + \frac{1}{2} a_3 x_1 + \frac{a_2}{2}) \\
&- \dot{r}_4(t) r_2(t) - \dot{r}_3(t) r_1(t) + \dot{r}_2(t) x_2 + \dot{r}_1(t) x_1 + \dot{r}_0(t) \\
&+ \frac{r_1(t)^2}{2} + \frac{r_2(t)^2}{2} + \frac{r_3(t)^2}{2} a_5 + \frac{r_4(t)^2}{2} a_4 \Big] \xi(t,x).
\end{aligned} \tag{30}
$$

Note that L_i is the zero degree differential operator of multiplication by h_i, then (5) is

$$
\begin{aligned}
\frac{\partial \xi}{\partial t} =& L_0 \xi(t,x) + \left(\sum_{i=1}^{m} c_{i1} y_i(t) \right) D_1 \xi(t,x) \\
&+ \left(\sum_{i=1}^{m} c_{i2} y_i(t) \right) D_2 \xi(t,x) \\
&\left(\frac{1}{2} \sum_{i,j=1}^{m} (c_{i1} c_{j1} + c_{i2} c_{j2}) y_i(t) y_j(t) \right) \xi(t,x).
\end{aligned} \tag{31}
$$

Comparing (30) and (31), we have

$$\begin{aligned}
&\dot r_3(t) - r_1(t) = \textstyle\sum_{i=1}^m c_{i1} y_i(t),\\
&\dot r_4(t) - r_2(t) = \textstyle\sum_{i=1}^m c_{i2} y_i(t),\\
&-r_3(a_5x_1 + \tfrac{1}{2}a_3x_2 + \tfrac{a_1}{2}) - r_4(a_4x_2 + \tfrac{1}{2}a_3x_1 + \tfrac{a_2}{2})\\
&-\dot r_4(t)r_2(t) - \dot r_3(t)r_1(t) + \dot r_2(t)x_2 + \dot r_1(t)x_1\\
&+\dot r_0(t) + \tfrac{r_1(t)^2}{2} + \tfrac{r_2(t)^2}{2} + \tfrac{r_3(t)^2}{2}a_5 + \tfrac{r_4(t)^2}{2}a_4\\
&\qquad\qquad = \tfrac{1}{2}\textstyle\sum_{i,j=1}^m (c_{i1}c_{j1} + c_{i2}c_{j2}) y_i(t) y_j(t).
\end{aligned} \tag{32}$$

From (32) we have

$$\begin{aligned}
\dot r_1(t) = \;& a_5 r_3(t) + \tfrac{1}{2}a_3 r_4(t),\\
\dot r_2(t) = \;& \tfrac{1}{2}a_3 r_3(t) + \tfrac{1}{2}a_4 r_4(t),\\
\dot r_3(t) = \;& r_1(t) + \textstyle\sum_{i=1}^m c_{i1} y_i(t),\\
\dot r_4(t) = \;& r_2(t) + \textstyle\sum_{i=1}^m c_{i2} y_i(t),\\
\dot r_0(t) = \;& \frac{1}{2}\textstyle\sum_{i,j=1}^m (c_{i1}c_{j1} + c_{i2}c_{j2})\, y_i(t) y_j(t)\\
& + \left(\textstyle\sum_{i=1}^m c_{i1} y_i(t)\right) r_1(t)\\
& + \left(\textstyle\sum_{j=1}^m c_{j2} y_j(t)\right) r_2(t)\\
& + \frac{r_1(t)^2}{2} + \frac{r_2(t)^2}{2} - \frac{a_3}{2} r_3(t)^2\\
& - \frac{a_4}{2} r_4(t)^2 + \frac{a_1}{2} r_3(t) + \frac{a_2}{2} r_4(t).
\end{aligned} \tag{33}$$

It is clear that (33) have solutions for all $t \geq 0$. ■

The finite dimensional filters of the Example 1 in section III can be constructed by Theorem 3.

V. Conclusion

The estimation algebra method has been proven to be an invaluable tool in the study of NLF problems. Once we obtain the FDEAs, we can construct FDFs by Wei–Norman approach for a class of NLF problems. By interpreting the DMZ equation or its robust form as a partial differential equation with time varying parameters, one derives a filtering approach based on Lie algebra as well as the theory of linear differential operators. In this paper, we establish several conditions for an estimation algebra of a special class of filtering systems to be finite-dimensional and construct the finite nonlinear filter. Moreover, the Wong's Ω-matrix is shown not necessary to be constant when we consider finite dimensional estimation algebras with state dimension 4 and rank equal to 2.

Acknowledgements

This work is supported by National Natural Science Foundation of China (11471184) and the start-up fund from Tsinghua University.

References

[1] V. E. Beneš, "Exact finite-dimensional filters for certain diffusions with nonlinear drift," Stochastics, vol. 5, pp. 65-92, 1981.

[2] A. Bensoussan, Stochastic Control of Partially Observable Systems, Cambridge University Press, Cambridge, 1992.

[3] R. W. Brockett and J. M. C. Clark, The geometry of the conditional density function, Academic Press, New York, pp. 299-309, 1980.

[4] R. W. Brockett, Nonlinear systems and nonlinear estimation theory, Springer, Dordrecht, 1981.

[5] R. W. Brockett, "Nonlinear control theory and differential geometry", in Proceedings of the International Congress of Mathematics, Warsaw, pp. 1357-1368, 1983.

[6] M. Chaleyat-Maurel and D. Michel, "Des resultats de non existence de filtre de dimension finie," Stochastics, vol. 13, pp. 83-102, 1984.

[7] C. D. Charalambous, R. J. Elliott, "New explicit filters and smoothers for diffusions with nonlinear drift and measurements," System Control Lett., vol. 33, pp. 89-103, 1998.

[8] W. L. Chiou, W. R. Chiueh and S. S. T. Yau, "Structure theorem for five-dimensional estimation algebras," Systems Control Lett. vol. 55, , pp. 275-281, 2006.

[9] M. H. A. Davis, "On a multiplicative functional transformation arising in nonlinear filtering theory," Z. Wahrsch Verw. Gebiete, vol. 54, pp. 125-139, 1980.

[10] R. T. Dong, L. F. Tam, W. S. Wong and S. S. T. Yau, "Structure and classification theorems of finite dimensional exact estimation algebras," SIAM J. Control Optim., vol. 29, no. 4, pp. 866-877, 1991.

[11] W. H. Dong, X. Q. Chen and S. S. T. Yau, "The novel classes of finite dimensional filters with non-maximal rank estimation algebra on state dimension four and rank of one," Inter. J. Contr., 2019, DOI: 10.1080/00207179.2019.1637543.

[12] M. Fujisaki, G. Kallianpur and H. Kunita, "Stochastic differential equations for the nonlinear filtering problems," Osaka J. Math., vol. 1, pp. 19-40, 1972.

[13] Y. Jiao, S. S. T. Yau and W. L. Chiou, "Mitter conjecture and structure theorem for six-dimensional estimation algebras," Inter. J. Contr., vol. 86, pp. 146-158, 2013.

[14] S. I. Marcus, "Algebraic and geometric methods in nonlinear filtering," SIAM J. Control Optim., vol. 22, pp.817-884, 1984.

[15] S. K. Mitter, "On the analogy between mathematical problems of nonlinear filtering and quantum physics," Ricerche Automat., vol. 10, pp. 163-216, 1979.

[16] L. F. Tam, W. S. Wong and S. S.-T. Yau, "On a necessary and sufficient condition for finite dimensionality of estimation algebra," SIAM J. Control Optim., vol. 28, no. 1, pp. 173-185, 1990.

[17] J. Wei and E. Norman, "On the global representation of the solutions of linear differential equations as a product of exponentials," Proc. Amer. Math. Sci., vol. 15 , pp. 327-334, 1964.

[18] W. S. Wong, "New classes of finite dimensional nonlinear filters," Systems Control Lett., vol. 3, pp. 155-164, 1983.

[19] W. S. Wong, "Theorems on the structure of finite dimensional estimation algebra," Systems Control Lett., vol. 9, pp. 117-124 1987.

[20] W. S. Wong, "On a new class of finite dimensional estimation algebras," Systems Control Lett., vol. 9, pp. 79-83, 1987.

[21] X. Wu and S. S. T. Yau, "Classification of estimation algebras with state dimension 2," SIAM J. Control Optim., vol. 45, pp. 1039-1073, 2006.

[22] S. S. T. Yau and Amid Rasoulian, "Classification of four-dimensional estimation algebras," IEEE Trans. on Automat. Contr., vol. 44, pp. 2312-2318, 1999.

[23] S. S. T. Yau, "Complete classification of finite-dimensional estimation algebras of maximal rank," Inter. J. Contr., vol. 76, pp. 657-677, 2003.

[24] S. S. T. Yau and G. Q. Hu, "Classification of finite-dimensional estimation algebras of maximal rank with arbitrary state-space dimension and mitter conjecture," Int. J. Contr., vol. 78, pp. 689-705, 2005.

[25] D. L. Ocone, Finite dimensional estimation algebras in nonlinear filtering, Springer Netherlands, 1981.

[26] J. Shi and S. S. T. Yau, "Finite dimensional estimation algebras with state dimension 3 and rank 2, I: linear structure of Wong matrix," SIAM J. Control Optim., vol. 55, no. 6, pp. 4227-4226, 2017.

[27] J. Shi, X. Q. Chen, W. H. Dong and S. S. T. Yau, "New classes of finite dimensional filters with non-maximal rank," IEEE Contr. Syst. Letters, vol. 1, no. 2, pp. 233-237, Oct. 2017.

[28] J. Shi and S. S. T. Yau, "Finite dimensional estimation algebras with state dimension 3 and rank 2, Mitter conjecture," Inter. J. Contr., 2018, DOI: 10.1080/00207179.2018.1550268.