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The novel classes of finite dimensional filters with non-maximal rank estimation algebra on state dimension four and rank of one*

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ABSTRACT

Ever since the technique of Kalman-Bucy filter was popularised, finding new classes of finite dimensional recursive filters has drawn much concern. The idea of using estimation algebra to construct finite-dimensional nonlinear filters was first proposed by Brockett and Mitter independently in the late 1970s, which has been proven an invaluable tool in tackling nonlinear filtering (NLF) problems. Once the estimation algebra is finite dimensional, one can construct the finite dimensional filters (FDFs) for NLF problems by Wei–Norman approach. In this paper, we give the construction of finite dimensional estimation algebra (FDEA) with state space dimension 4 and linear rank equal to 1, and further obtain a new class of NLF systems with FDFs. Importantly, we show that there is a class of polynomial FDF system in state space dimension 4 with linear rank one, but the coefficients in Wong's Ω-matrix are polynomials of degree two, or higher. In particular, these are the first examples of polynomial filtering systems not of Yau type (i.e. the drift term is not gradient plus affine functions) but with FDFs. Furthermore, we write down several easily satisfied sufficient conditions for the construction of more special classes of FDFs. Additionally, we derive the FDFs for the proposed NLF systems by using the Wei–Norman approach.

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Finite-dimensional nonlinear filter; Wong's Ω-matrix; finite dimensional estimation algebra; Wei–Norman approach

1. Introduction

The NLF problem involves how to estimate the states of a stochastic dynamical system from noisy observations, partially or fully associated with the states. This problem is of central significance in science and engineering, and has been widely used in many fields including navigation, radar tracking, sonar ranging, satellite and airplane orbit determination, and forecasting in weather, econometrics and mathematical finance. Generally, the filtering problem we consider is based on the following signal observation model

$$\begin{cases} \mathrm{d}x_t = f(x_t)\,\mathrm{d}t + G(x_t)\,\mathrm{d}v_t, \\ \mathrm{d}y_t = h(x_t)\,\mathrm{d}t + \mathrm{d}w_t, \end{cases} \tag{1}$$

where $x_t \in \mathbb{R}^n$ is referred to as the state vector of the system at time t, $y_t \in \mathbb{R}^m$ is the observation at the instant t. Assume that $\{v_t,\ t \geq 0\}, \{w_t,\ t \geq 0\}$ are Brownian motion processes taking values in $\mathbb{R}^p$, $\mathbb{R}^m$, with the covariance matrices $Q(t)$ and $S(t)$ respectively. We further assume that $f : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}^m$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times p}$ are $\mathbb{C}^\infty$ smooth functions, and $n = p$, G is an orthogonal matrix. Moreover, $y_0 = 0$, and $\{v_t,\ t \geq 0\}, \{w_t,\ t \geq 0\}$, the initial state x_0 are mutually independent.

In 1960, Kalman firstly devised a linear filtering that has great impact on modern industry, which is so-called Kalman filter (KF) (Kalman, 1960). One year later, the continuous version of KF was investigated by Kalman and Bucy (1961). Since then, the Kalman and Kalman-Bucy filter have been widely used in science and engineering. However, the Kalman and Kalman-Bucy filter have limited application due to the linearity assumptions of the drift term $f(x_t)$ and the observation term $h(x_t)$ with respect to the states x_t, and with the Gaussian assumption of the initial state x_0. The main objective of NLF, in general, is to determine the conditional expectations, or perhaps even to compute the conditional density $\rho(x_t\,|\,\mathcal{Y}_t)$ of states x_t based on the time history of observations (or filtration) $\mathcal{Y}_t := \sigma\{y_s : 0 \leq s \leq t\}$. On the account that $\rho(x_t\,|\,\mathcal{Y}_t)$ embodies all the statistical information of x_t, such as its conditional expectation, conditional covariance and all its high order moments, it is clear that $\rho(x_t\,|\,\mathcal{Y}_t)$ is the complete solution of the filtering problem. Whereas, the NLF problems is an essentially more difficult problem since the existing optimal filter is generally infinite-dimensional, i.e. the conditional density $\rho(x_t\,|\,\mathcal{Y}_t)$ depends on all its moments. The success of KF for the linear Gaussian estimation problem encouraged many researchers to generalise Kalman's results to nonlinear dynamical systems, and the existing approximate filters for NLF problems include the extended Kalman filter (EKF), the unscented Kalman filter (UKF), the ensemble Kalman filter (EnKF), particle filter and the splitting up method; see Crisan and Lyons (1999), Gillijns et al. (2006), Kan (2008) and Wan and Van der Merwe (2000). All of these methods have their own deficiencies. UKF and EnKF shall need to assume that the probability density of the states is Gaussian. Since re-sampling step in PF is applied at every iteration, which leads to a rapid shortfall of diversity in particles, PF could be heavy computational load

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*Dedicate to Professor Roger W. Brockett on the occasion of his 81st birthday.

and is sensitive to outliers. Moreover, PF are more applicable at low-dimensional and moderate high-dimensional systems, while PF has obstacles to high dimensional cases shown by Budhiraja, Chen, and Lee (2007). However, the splitting up method requires G and h in the system (1) to be bounded, which even excludes the linear case.

It is shown in Kushner (1967) that $\rho(x_t \mid \mathcal{Y}_t)$ satisfies an Itô stochastic differential equation (SDE), which is called Kushner's equation. After suitable change of probability measure, the unnormalised conditional density $\sigma(x_t \mid \mathcal{Y}_t)$ satisfies a linear Itô SDE, which is the so-called Duncan-Mortensen-Zakai (DMZ) equation (Duncan, 1967; Mortensen, 1966; Zakai, 1969). Apparently, solving DMZ equation for general dynamic systems is the more preferable one. And the solution to the Kushner's equation $\rho(x_t \mid \mathcal{Y}_t)$ and that to the DMZ equation $\sigma(x_t \mid \mathcal{Y}_t)$ is one-to-one correspondence. During the last few decades, various approaches are available to directly or numerically solve the DMZ equation for general NLF problems. The idea of estimation algebra method to directly and globally solve DMZ equation originated from the Wei–Norman approach (Wei & Norman, 1964), which took advantage of the Lie algebraic method to solve time-varying linear differential equations. This idea is due to Brockett (1981), Brockett and Clark (1980), and Mitter (1979) independently. More details about the Wei–Norman approach and its connection with the NLF problems can be seen in Dong, Tam, Wong, and Yau (1991) and Tam, Wong, and Yau (1990) and the survey article Marcus (1984). The most important advantage of the estimation algebra approach for NLF problems is that as long as the estimation algebra is finite-dimensional, not only can the finite dimensional recursive filters be constructed, but also the filter so constructed is universal in the sense of Maurel and Michel (1984). When applying the Wei–Norman approach to the NLF problems, however, we need to explicitly know the basis of the Lie algebra generated by the operators of the DMZ equation in order to reduce the DMZ equation to a finite system of ordinary differential equations (ODEs), Kolmogorov equation, and several first-order linear partial differential equations (PDEs). In Wong (1987), the nonlinear filtering system (1) with states valued in $\mathbb{R}^3$ is considered, where the components of $f(x)$ are defined by $f_1(x) = x_1 + x_2 + x_3 + \gamma(x_1, x_2, x_3)$, $f_2(x) = x_1 + x_3$, $f_3(x) = x_2 - x_3$ respectively and the observation valued in $\mathbb{R}^1$ with $h(x) = x_2 - x_3$, where γ is a $\mathbb{C}^\infty$ function with a bounded, non-zero first derivative and $v = (v_1, v_2, v_3)$ and w are independent, standard Brownian processes. As is shown in Section 2.1, the FDEA of this filtering system is a four-dimensional Lie algebra with basis given by $\{L_0, D_2 - D_3, x_2 - x_3, 1\}$. Therefore, it is very meaningful to clarify the classification of FDEAs in order to construct FDFs for NLF problems.

For the classification of FDEAs with maximal rank, in a series of research works (Chen & Yau, 1996; Chiou & Yau, 1995; Tam et al., 1990; Yau, 1994, 2003; Yau & Hu, 2005; Yau, Wu, & Wong, 1999), Yau and his co-workers have completely classified all FDEAs of maximal rank with arbitrary state space dimension, which included both Kalman-Bucy and Benés filtering systems as special cases (Yau & Hu, 2005). One of the key steps that Yau and his coworkers were able to classify all FDEAs with maximal rank, is that they were able to show that Wong's Ω-matrix is a matrix with polynomial degree 1. When the rank of FDEA is not maximal, the problem is still open. Wu and Yau have classified FDEAs of non-maximal rank with state dimension 2 (Wu & Yau, 2006). Recently, Shi et al. give new classes of FDFs for state dimension 3 and linear rank 1 (Shi, Chen, Dong, & Yau, 2017), in which the Wong's Ω-matrix is unnecessary to be a constant matrix. In Shi and Yau (2017), the authors consider FDEAs with state dimension 3 and linear rank 2, where the main theorem says that if E is the FDEA of system (1) with state dimension 3 and linear rank 2, then the Wong's Ω-matrix has linear structure; i.e. all the entries in the Wong's Ω-matrix are degree 1 polynomials at most.

In this paper, we find a novel class of FDF systems for NLF problems by estimation algebra method, and especially obtain a new class of polynomial filtering systems with FDFs. Firstly, we give the construction of FDEA with basis of $\{L_0, D_1, x_1, 1\}$ in state space dimension 4 and linear rank equal to 1. For the construction of more special classes of NLF systems with FDFs in state space dimension 4 and linear rank equal to one, we write down several easily satisfied sufficient conditions in Section 3. More importantly, we shall show that there exists a polynomial filtering system in state space dimension 4 with linear rank one, but the coefficients in Wong's Ω-matrix are polynomials of degree two, or higher. In particular, these are the first examples of polynomial filtering systems not of Yau type (i.e. the drift term is not gradient plus affine functions) but with FDFs. Moreover, we derive the explicit solution for the robust-DMZ equation by using the Wei–Norman approach, which solution is the FDFs for the proposed NLF systems. This derivation is presented in the main theorem of Section 4.

The paper is organised as follows: In Section 2 some basic concepts about estimation algebra and NLF problem are presented. In Section 3 new classes of FDF systems and structure results of estimation algebra in state space dimension four with non-maximal rank are exposited. In Section 4 the FDFs for the robust-DMZ equation of the proposed NLF systems are derived by the Wei–Norman approach. And we finally arrive at conclusion in the last section.

2. Basic concepts and preliminary results

2.1 Ducan–Mortensen–Zakai's equation for NLF problem

In the filtering problem, the infinitesimal generator L^* of the state $\{x_t,\ t \geq 0\}$ defined in the system (1) is considered,

$$L^*(o) := \frac{1}{2}\sum_{i,j=1}^{n}(GQG^\top(t, x_t))_{ij}\frac{\partial^2(o)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} f_i(x_t)\frac{\partial(o)}{\partial x_i},$$

where f_i and x_i are the ith component of the vector-value function f and the vector state x_t, respectively. The filtering problem is then able to be interpreted as how to find a recursive or finite-dimensional method to compute the conditional density of states x_t with the filtration $\mathcal{Y}_t$, i.e. $\rho(x_t \mid \mathcal{Y}_t)$.

Under certain mild conditions, the unnormalised conditional density $\sigma(x_t \mid \mathcal{Y}_t)$ of states x_t given observations $\mathcal{Y}_t$ for the system (1), defined as $\sigma(t, x) := \sigma(x_t \mid \mathcal{Y}_t)$ for simple notation, satisfies the DMZ equation (Duncan, 1967; Mortensen, 1966; Zakai, 1969)

$$\begin{cases} d\sigma(t, x) = L(\sigma(t, x))\, dt + \sigma(t, x)h^\top(x)S^{-1}(t)\, dy_t, \\ \sigma(0, x) = \sigma_0(x), \end{cases} \tag{2}$$

where $\sigma_0(x) \in \mathbb{C}^\infty(\mathbb{R}^n)$ is the probability density of the initial state x_0, and L is the adjoint operator of L^*, i.e.

$$L(o) := \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\left[(GQG^\top)_{ij}o\right] - \sum_{i=1}^{n}\frac{\partial(f_i o)}{\partial x_i}. \quad (3)$$

The normalised conditional density $\rho(x_t|\mathcal{Y}_t)$ is then given by

$$\rho(x_t \mid \mathcal{Y}_t) = \frac{\sigma(t,x)}{\int \sigma(t,x)}.$$

Remark 2.1: The covariance matrices $Q(t)$ and $S(t)$ of Brownian motion processes $\{v_t,\ t \geq 0\}, \{w_t,\ t \geq 0\}$ respectively in the nonlinear model (1), without loss of generality, are assumed to be identity matrices in the sequel.

In the subsequent sections the aim is to solve the DMZ equation (2) for the NLF problems. The DMZ equation (2) can be equivalently expressed as the following form

$$\begin{cases} d\sigma(t,x) = L_0\sigma(t,x)\,dt + \sum_{i=1}^m L_i\sigma(t,x)\,dy_i(t), \\ \sigma(0,x) = \sigma_0(x), \end{cases} \quad (4)$$

where L_0 is the second-order differential operator defined as

$$L_0 := \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2, \quad (5)$$

and L_i is the zero-order differential operator of multiplication by h_i, for $i = 1, \ldots, m$.

For each arrived observation, taking an invertible exponential transformation (Davis & Marcus, 1981)

$$u(t,x) = \exp\left[-h^\top(x)S^{-1}(t)y_t\right]\sigma(t,x), \quad (6)$$

$u(t,x)$ satisfies the following 'pathwise-robust' DMZ equation (7) which involves y_t only in the coefficients of the PDE.

$$\begin{cases} \frac{d}{dt}u(t,x) = L_0 u(t,x) + \sum_{i=1}^m y_i(t)[L_0, L_i]u(t,x) \\ \qquad + \frac{1}{2}\sum_{i=1}^m\sum_{j=1}^m y_i(t)y_j(t)[[L_0,L_i],L_j]u(t,x), \\ u(0,x) = \sigma_0(x), \end{cases} \quad (7)$$

where $[\cdot,\cdot]$ is the Lie bracket. Furthermore, the estimation algebra E of a filtering problem (1), specifically associated with the robust-DMZ equation (7), is defined to be the Lie algebra generated by $\{L_0, L_1, \ldots, L_m\}$.

2.2 Preliminaries

Definition 2.1: If X and Y are differential operators, the Lie bracket of X and Y, $[X,Y]$, is defined by $[X,Y]\phi = X(Y\phi) - Y(X\phi)$ for any $\mathbb{C}^\infty$ function ϕ.

Definition 2.2: A vector space $\mathcal{F}$ with the Lie bracket operation $\mathcal{F}\times\mathcal{F} \to \mathcal{F}$ denoted by $(x,y) \mapsto [x,y]$ is called a Lie algebra if the following axioms are satisfied

(1) The Lie bracket operation is bilinear;
(2) $[x,x] = 0$ for all $x \in \mathcal{F}$;
(3) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0\ (x,y,z \in \mathcal{F})$.

Definition 2.3 (Wong, 1987): The Wong's Ω-matrix is the matrix $\Omega = (\omega_{ij})$, where

$$\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}, \quad \forall\, 1 \leq i,\ j \leq n.$$

Obviously, Ω is skew symmetric and $\frac{\partial \omega_{jk}}{\partial x_i} + \frac{\partial \omega_{ki}}{\partial x_j} + \frac{\partial \omega_{ij}}{\partial x_k} = 0$, for every $1 \leq i,j,k \leq n$. If we define

$$D_i := \frac{\partial}{\partial x_i} - f_i, \quad \eta := \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^n f_i^2 + \sum_{i=1}^m h_i^2, \quad (8)$$

then we have a more compact form of L_0,

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^n D_i^2 - \eta\right). \quad (9)$$

Definition 2.4: Let U be the vector space of differential operators in the form

$$A = \sum_{(i_1,i_2,\ldots,i_n)\in I_A} a_{i_1,i_2,\ldots,i_n} D_1^{i_1} D_2^{i_2}\cdots D_n^{i_n},$$

where nonzero functions $a_{i_1,i_2,\ldots,i_n} \in \mathbb{C}^\infty(\mathbb{R}^n)$, $I_A := \{(i_1, i_2, \ldots, i_n)\}$ is the finite index set of A and i_l are nonnegative integers. The order of A is denoted by $\mathrm{ord}(A) := \max_{i\in I_A}|i|$, where $i := (i_1, i_2, \ldots, i_n)$, $|i| := \sum_{l=1}^n i_l$.

Remark 2.2: Let U_k be the subspace of U consisting of the elements with order less than or equal to k, where $k \leq 0$. In particular, $U_0 := \mathbb{C}^\infty(\mathbb{R}^n)$.

Lemma 2.5 (Wu & Yau, 2006, Lemma 3.1): *Let $g, h \in \mathbb{C}^\infty(\mathbb{R}^n)$ and let $i_1,\ldots,i_n, j_1,\ldots,j_n$ be nonnegative integers with $\sum_{l=1}^n i_l = r$, $\sum_{l=1}^n j_l = s$, and $r+s \geq 2$. Let δ_{ij} be the kronecker symbol, then*

$$\begin{aligned}[gD_1^{i_1}\cdots D_n^{i_n}, hD_1^{j_1}\cdots D_n^{j_n}] &= \sum_{l=1}^n \left(i_l g\frac{\partial h}{\partial x_l} - j_l h\frac{\partial g}{\partial x_l}\right) D_1^{i_1+j_1-\delta_{1l}}\cdots D_n^{i_n+j_n-\delta_{nl}} \\ &\quad \bmod U_{r+s-2}. \end{aligned} \quad (10)$$

Remark 2.3: Here, $(\cdot) \bmod U_l$ represents a member of the affine class of operators obtained by adding members of U_l to the argument. Generally, *mod* is used to denote the equivalence class, i.e. if U_l is a subspace of U,

$$A = B \mod U_l \Leftrightarrow A - B \in U_l.$$

Remark 2.4: We give an example to illustrate how to obtain the estimation algebra with finite dimension, once the NLF system is given.

In Wong (1987), the following filtering system defined in $\mathbb{R}^3$ is considered,

$$\begin{cases} dx_1(t) = (x_1 + x_2 + x_3 + \gamma(x_1, x_2, x_3))\, dt + dv_1(t), \\ dx_2(t) = (x_1 + x_3)\, dt + dv_2(t), \\ dx_3(t) = (x_1 + x_2)\, dt + dv_3(t), \\ dy(t) = (x_2 - x_3)\, dt + dw(t), \end{cases} \tag{11}$$

where γ is a $\mathbb{C}^\infty$ function with a bounded, non-zero first derivative and $v = (v_1, v_2, v_3)$ and w are independent, standard Brownian processes.

Then its corresponding Wong's Ω-matrix and drift terms are

$$\Omega = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \gamma'(x_1 + x_2 + x_3),$$

$$\begin{aligned} f_1(x) &= x_1 + x_2 + x_3 + \gamma(x_1, x_2, x_3), \\ f_2(x) &= x_1 + x_3, \\ f_3(x) &= x_1 + x_2, \\ h(x) &= x_2 - x_3, \\ D_i &= \frac{\partial}{\partial x_i} - f_i(x), \quad 1 \le i \le 3, \\ L_0 &= \frac{1}{2}\left(\sum_{i=1}^{3} D_i^2 - \eta\right), \end{aligned} \tag{12}$$

where

$$\begin{aligned} \eta &= \sum_{i=1}^{3} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{3} f_i^2 + h(x)^2 \\ &= 1 + \gamma'(x_1 + x_2 + x_3) \\ &\quad + [x_1 + x_2 + x_3 + \gamma(x_1 + x_2 + x_3)]^2 \\ &\quad + (x_1 + x_3)^2 + (x_1 + x_2)^2 + (x_2 - x_3)^2. \end{aligned} \tag{13}$$

According to the Definition 2.1 of Lie bracket, (9) and Lemma 2.5, it is easy to compute that

$$[L_0, h(x)] = [L_0, x_2] - [L_0, x_3] = D_2 - D_3,$$

and

$$\begin{aligned} [L_0, D_2 - D_3] &= [L_0, D_2] - [L_0, D_3] \\ &= \sum_{i=1}^{3}\left(\omega_{2i} D_i + \frac{1}{2}\frac{\partial \omega_{2i}}{\partial x_i}\right) + \frac{1}{2}\frac{\partial \eta}{\partial x_2} \\ &\quad - \sum_{i=1}^{3}\left(\omega_{3i} D_i + \frac{1}{2}\frac{\partial \omega_{3i}}{\partial x_i}\right) - \frac{1}{2}\frac{\partial \eta}{\partial x_3} \\ &= \omega_{21} D_1 + \frac{1}{2}\frac{\partial \omega_{21}}{\partial x_1} + \frac{1}{2}\frac{\partial \eta}{\partial x_2} - \omega_{31} D_1 - \frac{1}{2}\frac{\partial \omega_{31}}{\partial x_1} \\ &\quad - \frac{1}{2}\frac{\partial \eta}{\partial x_3} \\ &= \frac{1}{2}\frac{\partial \eta}{\partial x_2} - \frac{1}{2}\frac{\partial \eta}{\partial x_3} = 3(x_2 - x_3). \end{aligned} \tag{14}$$

Therefore, the estimation algebra E of this filtering system (11) is a four-dimensional Lie algebra with basis given by $\{L_0, D_2 - D_3, x_2 - x_3, 1\}$.

3. Novel classes of NLF systems with FDFs

3.1 The derivation of a new class of FDF system

Definition 3.1 (Wu & Yau, 2006, Definition 3.4): Let $L(E) \subset E$ be the vector space consisting of all the homogeneous degree one polynomials in E. Then the linear rank of estimation algebra E is defined by $r := \dim L(E)$.

In this section, we consider the finite dimensional NLF systems which is constructed from the non-maximal rank estimation algebra, denoted by E, with state space dimension of four and linear rank equal to 1.

Lemma 3.2: *Assume E with state space dimension $n = 4$ and $r = 1$, without loss of generality, we may assume that there exists constant c_1, such that $x_1 + c_1 \in E$, and for any constants c_j, $x_j + c_j \notin E$, $2 \le j \le 4$, then we have*

$$\begin{aligned} &[L_0, x_1 + c_1] = D_1 \in E, \\ &[D_1, x_1 + c_1] = 1 \in E, \\ &[L_0, D_1] = \omega_{12} D_2 + \omega_{13} D_3 + \omega_{14} D_4 \mod U_0 \in E. \end{aligned} \tag{15}$$

Proof: According to the Definition 2.1 of Lie brackets, the compact form of the two-order differential operator L_0 and Lemma 2.5, it can be computed that

$$\begin{aligned} [L_0, x_1 + c_1] &= \left[\frac{1}{2}\left(\sum_{i=1}^{4} D_i^2 - \eta\right), x_1 + c_1\right] \\ &= \frac{1}{2}[D_1^2, x_1 + c_1] \\ &= D_1 \in E, \end{aligned} \tag{16}$$

since $L_0 \in E$. Furthermore,

$$[D_1, x_1 + c_1] = 1 \in E, \tag{17}$$

$$\begin{aligned} Y_1 &:= [L_0, D_1] \\ &= \left[\frac{1}{2}\left(\sum_{i=1}^{4} D_i^2 - \eta\right), D_1\right] \\ &= \omega_{12} D_2 + \omega_{13} D_3 + \omega_{14} D_4 + \frac{1}{2}\frac{\partial \omega_{12}}{\partial x_2} \\ &\quad + \frac{1}{2}\frac{\partial \omega_{13}}{\partial x_3} + \frac{1}{2}\frac{\partial \omega_{14}}{\partial x_4} + \frac{1}{2}\frac{\partial \eta}{\partial x_1} \in E \\ &= \omega_{12} D_2 + \omega_{13} D_3 + \omega_{14} D_4 \mod U_0 \in E. \end{aligned} \tag{18}$$

where $U_0 = \mathbb{C}^\infty(\mathbb{R}^n)$. ■

Remark 3.1: On the assumptions of Lemma 3.2, it is noted that $P_1(x_1) \subseteq E$, where $P_1(x_1)$ is degree one polynomial of x_1.

Lemma 3.3: *Under the assumptions of Lemma 3.2, if in addition, we impose the following conditions:*

(i) $\omega_{12} = \omega_{13} = \omega_{14} = 0$,
(ii) $\eta = P_2(x_1) + \phi(x_2, x_3, x_4)$,

where $P_2(x_1)$ denotes the polynomial at most degree two with respect to x_1, then the estimation algebra E is finite dimensional with basis given by $\{L_0, D_1, x_1, 1\}$.

Proof: This conclusion can be arrived at easily. From Lemma 3.2, we can know that $L_0, D_1, x_1, 1 \in E$. Since $\omega_{12} = \omega_{13} = \omega_{14} = 0$ and $\eta = P_2(x_1) + \phi(x_2, x_3, x_4)$, then $Y_1 = 0$, and further $0 \mod U_0 \in E$ means that E contains affine function of x_1 at most. Therefore, the estimation algebra E is formed by the basis of $\{L_0, D_1, x_1, 1\}$. ■

Next we construct a class of NLF system which satisfies the conditions all above. By condition (ii),

$$\eta = \sum_{i=1}^{4}\left(f_i^2 + \frac{\partial f_i}{\partial x_i}\right) + \sum_{i=1}^{m} h_i^2, \tag{19}$$

is degree 2 at most with respect to x_1, then we may assume that f_i's are polynomials at most degree 1 of x_1, i.e. we assume that for $1 \le i \le 4$,

$$f_i = a_i(x_2, x_3, x_4)x_1 + \phi_i(x_2, x_3, x_4), \tag{20}$$

where $a_i(x_2, x_3, x_4)$ and $\phi_i(x_2, x_3, x_4)$ are $\mathbb{C}^\infty$ functions of x_2, x_3, x_4. By condition (i), we have

$$\begin{aligned}
\omega_{12} &= \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} = a_2 - \left(\frac{\partial a_1}{\partial x_2}x_1 + \frac{\partial \phi_1}{\partial x_2}\right) = 0,\\
\omega_{13} &= \frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3} = a_3 - \left(\frac{\partial a_1}{\partial x_3}x_1 + \frac{\partial \phi_1}{\partial x_3}\right) = 0,\\
\omega_{14} &= \frac{\partial f_4}{\partial x_1} - \frac{\partial f_1}{\partial x_4} = a_4 - \left(\frac{\partial a_1}{\partial x_4}x_1 + \frac{\partial \phi_1}{\partial x_4}\right) = 0,
\end{aligned} \tag{21}$$

hence we have

$$\begin{cases}
\dfrac{\partial a_1}{\partial x_2} = 0, \quad \dfrac{\partial a_1}{\partial x_3} = 0, \quad \dfrac{\partial a_1}{\partial x_4} = 0,\\
\dfrac{\partial \phi_1}{\partial x_2} = a_2, \quad \dfrac{\partial \phi_1}{\partial x_3} = a_3, \quad \dfrac{\partial \phi_1}{\partial x_4} = a_4.
\end{cases} \tag{22}$$

From (22), we know that a_1 must be a constant. Now

$$\begin{aligned}
\eta &= \sum_{i=1}^{4}\left(f_i^2 + \frac{\partial f_i}{\partial x_i}\right) + \sum_{i=1}^{m} h_i^2\\
&= (a_1x_1 + \phi_1)^2 + \left(\frac{\partial \phi_1}{\partial x_2}x_1 + \phi_2\right)^2 + \left(\frac{\partial \phi_1}{\partial x_3}x_1 + \phi_3\right)^2\\
&\quad + \left(\frac{\partial \phi_1}{\partial x_4}x_1 + \phi_4\right)^2 + a_1 + \frac{\partial a_2}{\partial x_2}x_1 + \frac{\partial \phi_2}{\partial x_2}\\
&\quad + \frac{\partial a_3}{\partial x_3}x_1 + \frac{\partial \phi_3}{\partial x_3} + \frac{\partial a_4}{\partial x_4}x_1 + \frac{\partial \phi_4}{\partial x_4} + \sum_{i=1}^{m} h_i^2\\
&= \left(\sum_{i=1}^{4} a_i^2\right)x_1^2 + \left(2\sum_{i=1}^{4} a_i\phi_i + \frac{\partial^2 \phi_1}{\partial x_2^2} + \frac{\partial^2 \phi_1}{\partial x_3^2} + \frac{\partial^2 \phi_1}{\partial x_4^2}\right)x_1\\
&\quad + \sum_{i=1}^{4}\phi_i^2 + a_1 + \frac{\partial \phi_2}{\partial x_2} + \frac{\partial \phi_3}{\partial x_3} + \frac{\partial \phi_4}{\partial x_4} + \sum_{i=1}^{m} h_i^2.
\end{aligned} \tag{23}$$

Since the estimation algebra has linear rank 1, we assume that h_i, $1 \le i \le m$ are degree one polynomials of x_1. Now condition (ii) implies:

(a) $(\frac{\partial \phi_1}{\partial x_2})^2 + (\frac{\partial \phi_1}{\partial x_3})^2 + (\frac{\partial \phi_1}{\partial x_4})^2$ is a constant;
(b) h_i, $1 \le i \le m$ are degree one polynomials of x_1;
(c) the following expression:

$$2a_1\phi_1 + 2\phi_2\frac{\partial \phi_1}{\partial x_2} + 2\phi_3\frac{\partial \phi_1}{\partial x_3} + 2\phi_4\frac{\partial \phi_1}{\partial x_4} + \frac{\partial^2 \phi_1}{\partial x_2^2} + \frac{\partial^2 \phi_1}{\partial x_3^2} + \frac{\partial^2 \phi_1}{\partial x_4^2} \tag{24}$$

is a constant.

To summarise, if the following conditions are satisfied,

(C.1) $f_i = a_ix_1 + \phi_i$, $1 \le i \le 4$;
(C.2) $\frac{\partial \phi_1}{\partial x_2} = a_2$, $\frac{\partial \phi_1}{\partial x_3} = a_3$, $\frac{\partial \phi_1}{\partial x_4} = a_4$;
(C.3) a_1 is a constant, and $\sum_{i=1}^{4} a_i^2$ is a positive constant;
(C.4) $2a_1\phi_1 + 2\sum_{i=2}^{4}\phi_i\frac{\partial \phi_1}{\partial x_i} + \sum_{i=2}^{4}\frac{\partial^2 \phi_1}{\partial x_i^2}$ is a constant;
(C.5) h_i, $1 \le i \le m$ are degree one polynomials of x_1;

then the conditions (i) and (ii) are satisfied, and the corresponding estimation algebra E is finite dimensional with basis $\{L_0, D_1, x_1, 1\}$. Moreover, the system (1) satisfying these conditions shall possess FDFs, which will be concretely illustrated by Section 4.

3.2 Special classes of polynomial NLF systems with FDFs

Special classes of polynomial systems with FDF are constructed which satisfy these conditions above. Without loss of generality, we may assume that the observation term is valued in $\mathbb{R}^1$, namely $h(x) = cx_1$, c is a constant and $G(x)$ is an identity matrix. These assumptions are satisfied by all special classes of polynomial NLF systems constructed in the following sequel.

Example 3.4: Let all the a_i's be constants, for example, we take $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1$, then condition (C.3) is satisfied and from (C.2) we can assume that ϕ_1 is a degree at most 1 polynomial of x_2, x_3, x_4. Thus we can take $\phi_1 = x_2 + x_3 + x_4$.

Now the condition (C.4) which says

$$2a_1\phi_1 + 2\sum_{i=2}^{4}\phi_i\frac{\partial\phi_1}{\partial x_i} + \sum_{i=2}^{4}\frac{\partial^2\phi_1}{\partial x_i^2} = 2\sum_{i=1}^{4}\phi_i + \sum_{j=2}^{4}\frac{\partial^2\phi_1}{\partial x_j^2}$$

is a constant can be easily satisfied. If we take $\phi_2 = -x_2 - 6x_3 + 3x_4 - x_3^3 + x_4^3 + x_2x_3$, $\phi_3 = -x_3 - 3x_4 - x_2^3 - x_4^3 - x_2x_3 + x_3x_4$, $\phi_4 = 6x_3 - x_4 + x_2^3 + x_3^3 - x_3x_4$, then the condition (C.4) is satisfied. Condition (C.5) is easily satisfied by letting the observation term $h_j(x) = b_jx_1 + c_j$, $1 \le j \le m$, where b_j, c_j's are constants. Now the Wong's Ω-matrix is given by

$$\Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -3x_2^2 + 3x_3^2 - x_2 - x_3 + 6 & 3x_2^2 - 3x_4^2 - 3 \\ 0 & 3x_2^2 - 3x_3^2 + x_2 + x_3 - 6 & 0 & 3x_3^2 + 3x_4^2 - x_3 - x_4 + 9 \\ 0 & -3x_2^2 + 3x_4^2 + 3 & -3x_3^2 - 3x_4^2 + x_3 + x_4 - 9 & 0 \end{pmatrix} \tag{25}$$

and $\eta = 4x_1^2 + \gamma(x_2, x_3, x_4)$, where $\gamma(x_2, x_3, x_4)$ is the $\mathbb{C}^\infty$ function of x_2, x_3, x_4, then the estimation algebra corresponding to this class of NLF system is finite dimensional with basis given by $\{L_0, D_1, x_1, 1\}$. It is apparent that the entries of Wong's Ω-matrix (25) are polynomials of degree two at most.

Then we can get the signal observation system (1) shown as:

$$\begin{cases} \mathrm{d}x_1(t) = (x_1 + x_2 + x_3 + x_4)\,\mathrm{d}t + \mathrm{d}v_1(t), \\ \mathrm{d}x_2(t) = (x_1 - x_2 - 6x_3 + 3x_4 + x_2x_3 - x_3^3 \\ \qquad\qquad + x_4^3)\,\mathrm{d}t + \mathrm{d}v_2(t), \\ \mathrm{d}x_3(t) = (x_1 - x_3 - 3x_4 - x_2x_3 + x_3x_4 \\ \qquad\qquad - x_2^3 - x_4^3)\,\mathrm{d}t + \mathrm{d}v_3(t), \\ \mathrm{d}x_4(t) = (x_1 + 6x_3 - x_4 - x_3x_4 + x_2^3 + x_3^3)\,\mathrm{d}t + \mathrm{d}v_4(t), \\ \mathrm{d}y(t) = cx_1\,\mathrm{d}t + \mathrm{d}w(t), \end{cases} \tag{26}$$

where $\mathbf{v}(t) = (v_1(t), \ldots, v_4(t))^\top$ and $w(t)$ are mutally independent, standard Brownian motions.

Let us give another example to explain that the Wong's Ω-matrix can be higher degree polynomials, not just as Example 3.4 showed that the Wong's Ω-matrix is polynomial of degree two.

Example 3.5: If

(1) let all the a_i's be constants, for instance, we take $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1$, then condition (C.3) is satisfied;
(2) for condition (C.2), we may assume that ϕ_1 is a degree at most 1 polynomial of x_2, x_3, x_4. Thus we take $\phi_1 = x_2 + x_3 + x_4$;
(3) set $\phi_2 = -x_2 - x_3^7 + x_4^7 + x_2^8x_3^5$, $\phi_3 = -x_3 - x_2^8 - x_4^7 + x_3^6x_4^3 - x_2^8x_3^5$, $\phi_4 = -x_4 + x_2^8 + x_3^7 - x_3^6x_4^3$, then the condition (C.4) is easily satisfied;
(4) condition (C.5) is easily satisfied by letting the observation term $h_j(x) = b_jx_1 + c_j$, $1 \le j \le m$, where b_j, c_j's are constants;

Now the Wong's Ω-matrix is given by

$$\Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5x_2^8x_3^4 - 8x_2^7x_3^5 - 8x_2^7 + 7x_3^6 & 8x_2^7 - 7x_4^6 \\ 0 & 5x_2^8x_3^4 + 8x_2^7x_3^5 + 8x_2^7 - 7x_3^6 & 0 & -6x_3^5x_4^3 - 3x_3^6x_4^2 + 7x_3^6 + 7x_4^6 \\ 0 & -8x_2^7 + 7x_4^6 & 6x_3^5x_4^3 + 3x_3^6x_4^2 - 7x_3^6 - 7x_4^6 & 0 \end{pmatrix} \tag{27}$$

and $\eta = 4x_1^2 + \gamma(x_2, x_3, x_4)$, where $\gamma(x_2, x_3, x_4)$ is the $\mathbb{C}^\infty$ function of x_2, x_3, x_4, then the estimation algebra corresponding to this class of NLF system is finite dimensional with basis given by $\{L_0, D_1, x_1, 1\}$. It can be noted that the Wong's Ω-matrix (27) is polynomial of degree higher than two.

Therefore, the signal observation system (1) is obtained as follows:

$$\begin{cases} \mathrm{d}x_1(t) = (x_1 + x_2 + x_3 + x_4)\,\mathrm{d}t + \mathrm{d}v_1(t), \\ \mathrm{d}x_2(t) = (x_1 - x_2 - x_3^7 + x_4^7 + x_2^8x_3^5)\,\mathrm{d}t + \mathrm{d}v_2(t), \\ \mathrm{d}x_3(t) = (x_1 - x_3 - x_2^8 - x_4^7 + x_3^6x_4^3 - x_2^8x_3^5)\,\mathrm{d}t + \mathrm{d}v_3(t), \\ \mathrm{d}x_4(t) = (x_1 - x_4 + x_2^8 + x_3^7 - x_3^6x_4^3)\,\mathrm{d}t + \mathrm{d}v_4(t), \\ \mathrm{d}y(t) = cx_1\,\mathrm{d}t + \mathrm{d}w(t), \end{cases} \tag{28}$$

where $\mathbf{v}(t) = (v_1(t), \ldots, v_4(t))^\top$ and $w(t)$ are independent, standard Brownian motions.

Generally, we can construct an arbitrary degree polynomial NLF system, which satisfies all conditions above, while the Wong's Ω-matrix can be any degree polynomials. For $\forall\, k \ge 1$, a NLF system can be established as follows.

Example 3.6: We give the following NLF system which satisfies all conditions above, and its Wong's Ω-matrix can be any degree polynomials for any given positive integer $k \ge 1$.

The signal observation system (1) shall be specifically expressed as:

$$\begin{cases} \mathrm{d}x_1(t) = (x_1 + x_2 + x_3 + x_4)\,\mathrm{d}t + \mathrm{d}v_1(t), \\ \mathrm{d}x_2(t) = (x_1 - x_2 - x_3^k + x_4^k + x_2^{k+1}x_3^{k+3})\,\mathrm{d}t + \mathrm{d}v_2(t), \\ \mathrm{d}x_3(t) = (x_1 - x_3 - x_2^{k+2} - x_4^k + x_3^{k+5}x_4^k \\ \qquad\qquad - x_2^{k+1}x_3^{k+3})\,\mathrm{d}t + \mathrm{d}v_3(t), \\ \mathrm{d}x_4(t) = (x_1 - x_4 + x_2^{k+2} + x_3^k - x_3^{k+5}x_4^k)\,\mathrm{d}t + \mathrm{d}v_4(t), \\ \mathrm{d}y(t) = cx_1\,\mathrm{d}t + \mathrm{d}w(t), \end{cases} \tag{29}$$

where $\mathbf{v}(t) = (v_1(t), \ldots, v_4(t))^\top$ and $w(t)$ are independent, standard Brownian motions.

It is easily to verify that the Wong's Ω-matrix is as follows:

$$\Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_{23} & \omega_{24} \\ 0 & -\omega_{23} & 0 & \omega_{34} \\ 0 & -\omega_{24} & -\omega_{34} & 0 \end{pmatrix}, \tag{30}$$

where $\omega_{23} = -(k+3)x_2^{k+1}x_3^{k+2} - (k+1)x_2^k x_3^{k+3} - (k+2)x_2^{k+1} + kx_3^{k-1}$, $\omega_{24} = (k+2)x_2^{k+1} - kx_4^{k-1}$, $\omega_{34} = -kx_3^{k+5}x_4^{k-1} - (k+5)x_3^{k+4}x_4^k + kx_3^{k-1} + kx_4^{k-1}$.

Remark 3.2: For the Example 3.6 above, it is easily illustrated that it satisfies all the conditions (C.1)–(C.5). Without loss of generality, let all the a_i's be constants, for instance, we take $a_1 = 1,\ a_2 = 1,\ a_3 = 1,\ a_4 = 1$, then condition (C.3) is satisfied and from (C.2) we can assume that ϕ_1 is a degree at most 1 polynomial of x_2, x_3, x_4. Thus we can still take $\phi_1 = x_2 + x_3 + x_4$. Now the condition (C.4) which says

$$2a_1\phi_1 + 2\sum_{i=2}^{4}\phi_i\frac{\partial\phi_1}{\partial x_i} + \sum_{i=2}^{4}\frac{\partial^2\phi_1}{\partial x_i^2} = 2\sum_{i=1}^{4}\phi_i + \sum_{j=2}^{4}\frac{\partial^2\phi_1}{\partial x_j^2}$$

is a constant can be easily satisfied. For example, if we take $\phi_2 = -x_2 - x_3^k + x_4^k + x_2^{k+1}x_3^{k+3}, \phi_3 = -x_3 - x_2^{k+2} - x_4^k + x_3^{k+5}x_4^k - x_2^{k+1}x_3^{k+3}, \phi_4 = -x_4 + x_2^{k+2} + x_3^k - x_3^{k+5}x_4^k$, then the condition (C.4) is satisfied. Condition (C.5) is easily satisfied by letting the observation term $h_j(x) = b_j x_1 + c_j, 1 \le j \le m$, where b_j, c_j's are constants.

And further let $\eta = 4x_1^2 + \gamma(x_2, x_3, x_4)$, where $\gamma(x_2, x_3, x_4)$ is the $\mathbb{C}^\infty$ function of x_2, x_3, x_4, then the estimation algebra corresponding to this class of NLF system is finite dimensional with basis given by $\{L_0, D_1, x_1, 1\}$. It can be noted that the Wong's Ω-matrix (30) can be polynomial of degree higher than k.

In this section, we obtain a new class of NLF system with FDF, especially a set of polynomial filtering system, when we consider the FDEA on state space dimension of four with non-maximal linear rank of one, whose basis is formed by $\{L_0, D_1, x_1, 1\}$. Moreover, as presented in the polynomial systems above, it is notable that the coefficients in Wong's Ω-matrix are shown not to be constant, even not necessary to be a polynomial of degree one.

In some sense, the coefficients in Wong's Ω-matrix which are polynomials of two, or higher can be seen as higher nonlinearity of the filtering systems. The previous results for non-maximal rank cases are shown that the coefficients in Wong's Ω-matrix are polynomials of degree at most one. However, we find a class of polynomial NLF system with FDF, but the coefficients in Wong's Ω-matrix are polynomials of two, or higher. Therefore, we find a class of highly nonlinear FDF system in this paper.

4. The construction of finite dimensional filters by Wei–Norman approach

4.1 Wei–Norman approach

Before we proceed, we briefly introduce the Wei–Norman approach (Wei & Norman, 1964) which was first introduced to solve the linear PDEs by Lie algebra. Suppose that the linear operator $A(t)$ can be expressed in the form

$$A(t) = \sum_{i=1}^{m} a_i(t)X_i, \quad \text{m is finite}, \tag{31}$$

where the $a_i(t)$ are scalar functions of time, and $X_1, X_2, \ldots, X_m$ are time-independent operators. We shall denote the $\mathcal{G}$ as the Lie algebra generated by $X_1, X_2, \ldots, X_m$ under the operations of the Lie bracket, and further assume $\mathcal{G}$ is of the finite dimension l, thus without loss of generality, we may assume that $X_1, X_2, \ldots, X_l$ form the basis of $\mathcal{G}$, then there exists a neighbourhood of $t=0$ in which the solution of the differential equation

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = A(t)Z, \quad U(0) = I \tag{32}$$

can be expressed in the form

$$Z(t) = \exp(s_1(t)X_1)\exp(s_2(t)X_2)\cdots\exp(s_l(t)X_l), \tag{33}$$

where the $s_i(t)$ are scalar functions of time t, and I is identity matrix. Moreover, the $s_i(t)$'s satisfy a set of differential equations which depend only on the Lie algebra $\mathcal{G}$ (i.e. the basis of $\mathcal{G}$), and the $a_i(t)$'s.

4.2 The construction of FDF for the new class of NLF system

In this section, we use the structure results of Section 3 to derive FDFs explicitly for the robust-DMZ equation by the Wei–Norman approach, if FDEA is considered with the state space dimension $n=4$ and with non-maximal linear rank $r=1$. The main theorem is presented as follows.

Theorem 4.1 (Main Theorem): *Suppose that the state space of the NLF system (1) is of state dimension four and the FDEA is denoted as E with non-maximal linear rank 1. Without loss of generality, the basis of E is given by $\{L_0, D_1, x_1, 1\}$. Furthermore, we assume that the NLF system (1) satisfies the conditions (C.1)–(C.5), G is an identity matrix, $\eta = a_2x_1^2 + a_1x_1 + a_0(x_2, x_3, x_4)$, and $h_i = c_{i1}x_1 + c_{i0},\ 1 \le i \le m$, where c_{i1}, c_{i0}, a_2, a_1 are constants and $a_0(x_2, x_3, x_4)$ is a $\mathbb{C}^\infty$ function of x_2, x_3, x_4. Then its robust-DMZ equation (7) has a solution of the form*

$$u(t,x) = \mathrm{e}^{r_0(t)}\,\mathrm{e}^{r_1(t)x_1}\,\mathrm{e}^{r_2(t)D_1}\,\mathrm{e}^{tL_0}\sigma_0,$$

where r_i's satisfy the following ODEs for all $t \ge 0$,

$$\begin{aligned}
\dot r_1(t) &= a_2 r_2(t),\\
\dot r_2(t) &= r_1(t) + \sum_{i=1}^{m} c_{i1}y_i(t),\\
\dot r_0(t) &= \frac{r_1(t)^2}{2} + \frac{a_2}{2}r_2(t)^2 + \frac{1}{2}a_1 r_2(t)\\
&\quad + \sum_{i=1}^{m} c_{i1}y_i(t)r_1(t) + \frac{1}{2}\sum_{i,j=1}^{m} c_{i1}c_{j1}y_i(t)y_j(t).
\end{aligned} \tag{34}$$

Table 1. Lie bracket multiplication of E.

	L_0	D_1	x_1	1
L_0	0	$a_2x_1 + \frac{a_1}{2}$	D_1	0
D_1	$-a_2x_1 - \frac{a_1}{2}$	0	1	0
x_1	$-D_1$	-1	0	0
1	0	0	0	0

Proof: As described in Section 3, the estimation algebra E of (4) satisfies conditions (C.1)–(C.5) with basis of $\{L_0, D_1, x_1, 1\}$. First, we give the basis calculation of estimation algebra E in Table 1.

By differentiating $u(t,x)$ with respect to t, we have

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x) &= e^{r_0(t)}\, e^{r_1(t)x_1}\, e^{r_2(t)D_1} L_0\, e^{tL_0}\sigma_0 \\ &\quad + \dot r_2(t)\cdot e^{r_0(t)}\, e^{r_1(t)x_1} D_1\, e^{r_2(t)D_1}\, e^{tL_0}\sigma_0 \\ &\quad + (\dot r_0(t) + \dot r_1(t)x_1)\, u(t,x) \\ &= I + II + (r_0(t) + \dot r_1(t)x_1)\, u(t,x), \end{aligned} \tag{35}$$

where we denote

$$\begin{aligned} I &:= e^{r_0(t)}\, e^{r_1(t)x_1}\, e^{r_2(t)D_1} L_0\, e^{tL_0}\sigma_0, \\ II &:= \dot r_2(t)\cdot e^{r_0(t)}\, e^{r_1(t)x_1} D_1\, e^{r_2(t)D_1}\, e^{tL_0}\sigma_0. \end{aligned} \tag{36}$$

Recall the classical Baker–Campbell–Hausdorff type relation, i.e.

$$\begin{aligned} e^{r(t)E_i} E_k\, e^{s(t)E_j} &= \left(E_k + r(t)[E_i,E_k] + \frac{r(t)^2}{2!}[E_i,[E_i,E_k]] + \cdots\right) \\ &\quad e^{r(t)E_i}\, e^{s(t)E_j}, \end{aligned} \tag{37}$$

where E_i, E_k, E_j are elements of a Lie algebra. The following calculations basically come from (37), thus we have

$$\begin{aligned} e^{r_0(t)} L_0\, e^{r_1(t)x_1}\, e^{r_2(t)D_1}\, e^{tL_0}\sigma_0 &= L_0 u(t,x), \\ e^{r_0(t)} r_1(t) D_1\, e^{r_1(t)x_1}\, e^{r_2(t)D_1}\, e^{tL_0}\sigma_0 &= r_1(t) D_1 u(t,x), \end{aligned} \tag{38}$$

and

$$\begin{aligned} & e^{r_0(t)}\, e^{r_1(t)x_1} L_0\, e^{r_2(t)D_1}\, e^{tL_0}\sigma_0 \\ &= e^{r_0(t)}\left(L_0 + r_1(t)[x_1,L_0] + \frac{r_1(t)^2}{2}[x_1,[x_1,L_0]]\right) \\ &\quad \cdot e^{r_1(t)x_1}\, e^{r_2(t)D_1}\, e^{tL_0}\sigma_0 \\ &= e^{r_0(t)}\left(L_0 - r_1(t)D_1 + \frac{r_1(t)^2}{2}\right) e^{r_1(t)x_1}\, e^{r_2(t)D_1}\, e^{tL_0}\sigma_0. \end{aligned} \tag{39}$$

Consequently,

$$\begin{aligned} I &= e^{r_0(t)}\, e^{r_1(t)x_1}\, (L_0 + r_2(t)[D_1, L_0] \\ &\quad + \frac{r_2(t)^2}{2}[D_1,[D_1,L_0]] + \cdots\Big)\, e^{r_2(t)D_1}\, e^{tL_0}\sigma_0 \\ &= e^{r_0(t)}\, e^{r_1(t)x_1}\left(L_0 - r_2(t)\left(a_2x_1 + \frac{1}{2}a_1\right)\right. \\ &\quad \left. - \frac{r_2(t)^2}{2}a_2\right) e^{r_2(t)D_1}\, e^{tL_0}\sigma_0 \\ &= e^{r_0(t)}\, e^{r_1(t)x_1} L_0\, e^{r_2(t)D_1}\, e^{tL_0}\sigma_0 - \left(r_2(t)\left(a_2x_1 + \frac{1}{2}a_1\right)\right. \\ &\quad \left. + \frac{r_2(t)^2}{2}a_2\right) u(t,x). \end{aligned} \tag{40}$$

Putting (38) and (39) into (40), we have

$$\begin{aligned} I &= L_0 u(t,x) - r_1(t) D_1 u(t,x) \\ &\quad + \left(\frac{r_1(t)^2}{2} - \frac{r_2(t)^2}{2}a_2 - r_2(t)\left(a_2x_1 + \frac{1}{2}a_1\right)\right) u(t,x). \end{aligned} \tag{41}$$

Similarly, we have

$$\begin{aligned} II &:= \dot r_2(t)\, e^{r_0(t)}\, e^{r_1(t)x_1} D_1\, e^{r_2(t)D_1}\, e^{tL_0}\sigma_0 \\ &= \dot r_2(t)\, e^{r_0(t)}(D_1 - r_1(t))\, e^{r_1(t)x_1}\, e^{r_2(t)D_1}\, e^{tL_0}\sigma_0 \\ &= \dot r_2(t) D_1 u(t,x) - \dot r_2(t) r_1(t) u(t,x). \end{aligned} \tag{42}$$

Putting (41) and (42) into (35), we have

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= L_0 u(t,x) + (\dot r_2(t) - r_1(t))\, D_1 u(t,x) \\ &\quad + \left(\frac{r_1(t)^2}{2} - r_2(t)\left(a_2x_1 + \frac{1}{2}a_1\right) - \frac{r_2(t)^2}{2}a_2\right. \\ &\quad \left. + \dot r_0(t) + \dot r_1(t)x_1 - \dot r_2(t) r_1(t)\right) u(t,x). \end{aligned} \tag{43}$$

Notice that L_i is the zero degree differential operator of multiplication by h_i, then (7) becomes

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= L_0 u(t,x) + \left(\sum_{i=1}^m c_{i1} y_i(t)\right) D_1 u(t,x) \\ &\quad + \left(\frac{1}{2}\sum_{i,j=1}^m c_{i1}c_{j1} y_i(t) y_j(t)\right) u(t,x). \end{aligned} \tag{44}$$

Comparing (43) and (44), we have

$$\dot r_2(t) - r_1(t) = \sum_{i=1}^m c_{i1} y_i(t), \tag{45}$$

and

$$\begin{aligned} & (\dot r_1(t) - a_2 r_2(t))\, x_1 + \dot r_0(t) - \frac{a_1}{2} r_2(t) - \frac{a_2}{2} r_2(t)^2 \\ &\quad + \frac{r_1(t)^2}{2} - \dot r_2(t) r_1(t) = \frac{1}{2}\sum_{i,j=1}^m c_{i1}c_{j1} y_i(t) y_j(t). \end{aligned} \tag{46}$$

From (45) and (46) we have

$$\dot{r}_1(t) = a_2 r_2(t),$$

$$\dot{r}_2(t) = r_1(t) + \sum_{i=1}^{m} c_{i1} y_i(t),$$

$$\dot{r}_0(t) = \frac{1}{2} r_1(t)^2 + \sum_{i=1}^{m} c_{i1} y_i(t) r_1(t) + \frac{a_2}{2} r_2(t)^2 + \frac{1}{2} a_1 r_2(t) + \frac{1}{2} \sum_{i,j=1}^{m} c_{i1} c_{j1} y_i(t) y_j(t). \qquad (47)$$

It is clear that (47) have solutions for all $t \geq 0$. Note that $r_1(t), r_2(t)$ is uniquely determined by the first two equations of (47), then by the last equation of (47), we know that $r_0(t)$ is unique up to a constant. ■

The algorithm in Table 2 illustrates how to combine the FDEA method with Wei–Norman approach to solve the NLF problems proposed in this paper.

4.3 The construction of FDFs for a novel class of polynomial filtering system

As we know, it is of much hardness to solve the DMZ equation (2) or even robust-DMZ equation (7) explicitly. However, we find a novel class of FDF system by establishing FDEA, shown in section 3, and its corresponding robust-DMZ equation has an explicit solution by means of Wei–Norman approach, which is presented in the main theorem of this paper completely and further illustrated by the algorithm shown in Table 2. Now we show how to compute the explicit solution of robust-DMZ equation with respect to the system constructed in Example 3.4.

Table 2. Finite dimensional filtering algorithm for a class of nonlinear system.

Algorithm	
1:	For NLF system (1) which satisfies the conditions (C.1)–(C.5), one can write out its corresponding DMZ equation (4) (or (2));
2:	By taking transformation (6), we get the robust-DMZ equation (7) of the related NLF system (26);
3:	By the construction of FEDA in Section 3, one can obtain the basis $\{L_0, D_1, x_1, 1\}$ of robust-DMZ equation of NLF system;
4:	By the Wei–Norman approach shown as the main theorem in Section 4, the solution of robust-DMZ equation (7) is $u(t,x) = e^{r_0(t)} e^{r_1(t)x_1} e^{r_2(t)D_1} e^{tL_0} \sigma_0(x)$, where r_i's satisfy the following ODEs for all $t \geq 0$, $\dot{r}_1(t) = a_2 r_2(t)$, $\dot{r}_2(t) = r_1(t) + \sum_{i=1}^{m} c_{i1} y_i(t)$, $\dot{r}_0(t) = \frac{r_1(t)^2}{2} + \frac{a_2}{2} r_2(t)^2 + \frac{1}{2} a_1 r_2(t) + \sum_{i=1}^{m} c_{i1} y_i(t) r_1(t) + \frac{1}{2} \sum_{i,j=1}^{m} c_{i1} c_{j1} y_i(t) y_j(t)$;
5:	Then by taking inverse transformation $\sigma(t,x) = \exp[h^\top(x) y_t] u(t,x)$, the solution of DMZ equation (4) is $\sigma(t,x) = \exp[h^\top(x) y_t] e^{r_0(t)} e^{r_1(t)x_1} e^{r_2(t)D_1} e^{tL_0} \sigma_0(x)$;
6:	Finally, the normalised conditional density function of state $\rho(x_t \mid \mathcal{Y}_t)$ is given by $\rho(x_t \mid \mathcal{Y}_t) = \frac{\sigma(t,x)}{\int \sigma(t,x)}$.

$$\begin{cases} dx_1(t) = f_1(x)\,dt + dv_1(t), \\ dx_2(t) = f_2(x)\,dt + dv_2(t), \\ dx_3(t) = f_3(x)\,dt + dv_3(t), \\ dx_4(t) = f_4(x)\,dt + dv_4(t), \\ dy(t) = h(x)\,dt + dw(t), \end{cases} \qquad (48)$$

where $\mathbf{v}(t) = (v_1(t), \ldots, v_4(t))^{\mathrm{T}}$ and $w(t)$ are mutually independent, standard Brownian motions, $x = (x_1, x_2, x_3, x_4)$. $f_1(x) = x_1 + x_2 + x_3 + x_4$, $f_2(x) = x_1 - x_2 - 6x_3 + 3x_4 + x_2x_3 - x_3^3 + x_4^3$, $f_3(x) = x_1 - x_3 - 3x_4 - x_2x_3 + x_3x_4 - x_2^3 - x_4^3$, and $f_4(x) = x_1 + 6x_3 - x_4 - x_3x_4 + x_2^3 + x_3^3$. In addition, $h = cx_1$, where c is constant. It is worth noted that this example satisfies all conditions of Theorem 4.1, then we can write out its corresponding DMZ equation according to (4) as,

$$\begin{cases} d\sigma(t,x) = L_0 \sigma(t,x)\,dt + h\sigma(t,x)\,dy(t), \\ \sigma(0,x) = \sigma_0(x), \end{cases} \qquad (49)$$

where L_0 in this system is defined by

$$L_0 = \frac{1}{2} \sum_{i=1}^{4} \frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{4} f_i \frac{\partial}{\partial x_i} - \sum_{i=1}^{4} \frac{\partial f_i}{\partial x_i} - \frac{1}{2} h^2. \qquad (50)$$

After taking the transformation of $u(t,x) = exp(-h(x)y(t))\sigma(t,x)$ according to (6), one can obtain the corresponding robust-DMZ equation according to (7) as follows,

$$\begin{cases} \frac{d}{dt} u(t,x) &= L_0 u(t,x) + y(t)[L_0, h] u(t,x) \\ &\quad + \frac{1}{2}(y(t))^2 [[L_0, h], h] u(t,x), \\ u(0,x) &= \sigma_0(x), \end{cases} \qquad (51)$$

Where L_0 is defined in (50). From the compact form of L_0 (see (9)), and the assumption of η in Example 3.4, we can write out

$$L_0 = \frac{1}{2} \left(\sum_{i=1}^{4} D_i^2 - \eta \right).$$

Thus after simple computations, the robust-DMZ equation of system (48) can be more clearly written as

$$\begin{cases} \frac{d}{dt} u(t,x) = \frac{1}{2} \left(\sum_{i=1}^{4} D_i^2 - \eta \right) u(t,x) + cy(t) D_1 u(t,x) \\ \qquad + \frac{1}{2}(cy(t))^2 u(t,x), \\ u(0,x) = \sigma_0(x), \end{cases} \qquad (52)$$

with $\eta = 4x_1^2 + \gamma(x_2, x_3, x_4)$, and $h = cx_1$, in which $\gamma(x_2, x_3, x_4)$ is the $\mathbb{C}^\infty$ function of x_2, x_3, x_4, and c is a constant.

According to the Theorem 4.1, we can get the solution of (52) as

$$u(t,x) = e^{r_0(t)} e^{r_1(t)x_1} e^{r_2(t)D_1} e^{tL_0} \sigma_0,$$

where r_i's satisfy the following ODEs for all $t \geq 0$,

$$\dot{r}_1(t) = 4r_2(t),$$

$$\dot{r}_2(t) = r_1(t) + cy(t), \qquad (53)$$

$$\dot{r}_0(t) = \frac{r_1(t)^2}{2} + 2r_2(t)^2 + cy(t) r_1(t) + \frac{1}{2} c^2 (y(t))^2.$$

It can be seen that the robust-DMZ equation of system (26) is solved by a finite number of ODEs and SDEs. Therefore, the FDFs for the robust-DMZ equation of Example 3.4 in Section 4 can be successfully constructed by Theorem 4.1.

5. Conclusion

The idea of using estimation algebra to construct FDF for NLF problem has been proven to be invaluable in the study of NLF problems. Once we obtain the FDEAs, we can construct FDFs by Wei–Norman approach for a class of NLF problems. By interpreting the DMZ equation or its robust form as a partial differential equation with time varying parameters, one derives an approach to filtering based on Lie algebra as well as the theory of linear differential operators. In this paper, we find a novel class of FDF system for NLF problems by estimation algebra method. When we consider FDEAs with state dimension 4 and linear rank equal to 1, we establish several conditions for FDEA which can be used to construct special classes of filtering systems of finite dimension, especially constructing a class of polynomial NLF systems, where the Wong's Ω-matrix is shown not necessary to be a polynomial at most degree one, and further, it can be polynomial of degree two, or higher. Furthermore, by using the Wei–Norman approach, we derive the explicit solution for the robust-DMZ equation of the proposed finite dimensional NLF systems.

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