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009

ON A NECESSARY AND SUFFICIENT CONDITION FOR FINITE DIMENSIONALITY OF ESTIMATION ALGEBRAS*

LUEN-FAI TAM†, WING SHING WONG‡, AND STEPHEN S.-T. YAU§

Abstract. Ever since the technique of the Kalman-Bucy filter was popularized, there has been an intense interest in finding new classes of finite dimensional recursive filters. In the late seventies, the concept of the estimation algebra of a filtering system was introduced. It has proven to be an invaluable tool in the study of nonlinear filtering problems. In this paper, a simple algebraic necessary and sufficient condition is established for an estimation algebra of a special class of filtering systems to be finite-dimensional. Also presented is a rigorous proof of the Wei-Norman program which allows one to construct finite-dimensional recursive filters from finite dimensional estimation algebras.

Key words. nonlinear filters, solvable Lie algebra, estimation algebra

AMS(MOS) subject classifications. 17B30, 35K15, 60G35, 93E11

1. Introduction. The idea of using estimation algebras to construct finite-dimensional nonlinear filters was first proposed in Brockett and Clark [1] and Brockett [2]. The motivation came from the following Wei-Norman approach [3] of using Lie algebraic ideas to solve time varying linear differential equations. Consider the equation

$$\frac{d}{dt}X(t)=A(t)X(t)\equiv\sum_{i=1}^{m}a_i(t)A_iX(t),\qquad X(0)=X_0, \tag{1.0}$$

where X and A_i's are n by n matrices and a_i's are scalar-valued functions. Let $B_1,\cdots,B_l$ be a basis of the Lie algebra generated by $A_1,\cdots,A_m$. Then the Wei-Norman Theorem states that locally in t, $X(t)$ has a representation of the form,

$$X(t)=\exp(b_1(t)B_1)\cdots\exp(b_l(t)B_l)X_0, \tag{1.1}$$

where b_i's satisfy an ordinary differential equation of the form

$$\frac{db_i}{dt}=c_i(b_1,\cdots,b_l),\qquad b_i(0)=0$$

for all i. The function c_i's in the above equation are determined by the structure constants of the Lie algebra generated by the A_i's.

The extension of Wei-Norman's approach to the nonlinear filtering problem is much more complicated. Instead of an ordinary differential equation, we have to solve the Duncan-Mortensen-Zakai (DMZ) equation, which is a stochastic partial differential equation. By working on the robust form of the DMZ equation we can reduce the complexity of the problem to that of solving a time varying partial differential equation. Working independently, Steinberg [4] applied the Wei-Norman approach to solve some partial differential equations that are roughly related to the linear filtering problem. Wong in [5] constructed some new finite-dimensional estimation algebras and used the Wei-Norman approach to synthesize finite-dimensional filters. However, the systems considered in [5] are quite specific and the question whether the Wei-Norman approach works for a general system with finite-dimensional estimation algebra remains open.

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† Department of Mathematics, The Chinese University of Hong Kong, Hong Kong.
‡ Room 3M-329, AT&T Bell Laboratories, Holmdel, New Jersey 07733.
§ Department of Mathematics, University of Illinois at Chicago, Box 4348, Chicago, Illinois 60680.

In this paper we examine the properties of finite-dimensional estimation algebras and the Wei–Norman approach in detail. We consider here a class of filtering systems having the property that the drift-term f of the state evolution equation is a gradient vector field. In [6], the concept of Ω is introduced, which is defined as the matrix whose i, j-element is $(\partial f_j/\partial x_i) - (\partial f_i/\partial x_j)$. For this class of filtering systems, Ω is zero. Conversely, if $\Omega = 0$, then by the Poincaré Lemma, f is a gradient vector field. So, the class of filtering systems considered here is characterized by the fact that $\Omega = 0$.

Motivated by the results in Wong [6] and [7], we investigate the algebraic problem of characterizing and classifying finite-dimensional exact estimation algebras. In [6], a sufficient condition of finite dimensionality is derived for certain filtering systems. In [7], a necessary condition and some theorems of the structure of the estimation algebra are demonstrated. In this paper, we derive a simple necessary and sufficient condition for an exact estimation algebra to be finite-dimensional. As an important consequence of these algebraic results, we prove that for a system with finite-dimensional exact estimation algebras, the Wei–Norman approach always leads to finite dimensional filters. The proof will be presented in § 4. The necessary and sufficient theorem presented here also leads us to prove some classification theorems of finite-dimensional exact estimation algebras, which will be presented in a forthcoming paper.

2. Basic concepts. The filtering problem considered here is based on the following signal observation model:

$$(2.0) \qquad \begin{cases} dx(t) = f(x(t))\, dt + g(x(t))\, dv(t) & x(0) = x_0, \\ dy(t) = h(x(t))\, dt + dw(t) & y(0) = 0, \end{cases}$$

in which x, v, y, and w, are, respectively, $\mathbb{R}^n$, $\mathbb{R}^p$, $\mathbb{R}^m$, and $\mathbb{R}^m$ valued processes, and v and w have components which are independent, standard Brownian processes. We further assume that $n = p$, f, h are C^∞ smooth, and that g is an orthogonal matrix. We will refer to $x(t)$ as the state of the system at time t and $y(t)$ as the observation at time t.

Let $\rho(t, x)$ denote the conditional probability density of the state given the observation $\{y(s): 0 \leqq s \leqq t\}$. It is well known (see [8], for example) that $\rho(t, x)$ is given by normalizing a function, $\sigma(t, x)$, which satisfies the following Duncan–Mortensen–Zakai equation:

$$(2.1) \qquad d\sigma(t, x) = L_0\sigma(t, x)\, dt + \sum_{i=1}^{m} L_i\sigma(t, x)\, dy_i(t), \qquad \sigma(0, x) = \sigma_0,$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2$$

and for $i = 1, \cdots, m$, L_i is the zero degree differential operator of multiplication by h_i.[1] σ_0 is the probability density of the initial point, x_0.

Equation (2.1) is a stochastic partial differential equation. In real applications, we are interested in constructing robust state estimators from observed sample paths with some property of robustness. Davis in [9] studied this problem and proposed some robust algorithms. In our case, his basic idea reduces to defining a new unnormalized density

$$\xi(t, x) = \exp\left(-\sum_{i=1}^{m} h_i(x) y_i(t)\right) \sigma(t, x).$$

[1] If p is a vector, we use the notation p_i to represent the ith component of p.

It is easy to show that $\xi(t, x)$ satisfies the following time varying partial differential equation

$$\frac{d\xi(t,x)}{dt} = L_0\xi(t,x) + \sum_{i=1}^{m} y_i(t)[L_0, L_i]\xi(t,x) + \frac{1}{2}\sum_{i=1}^{m} y_i^2(t)[[L_0, L_i], L_i]\xi(t,x), \tag{2.2}$$

$$\xi(0, x) = \sigma_0$$

where $[\cdot, \cdot]$ is the Lie bracket as described by the following definition.

DEFINITION. If X and Y are differential operators, the Lie bracket of X and Y, $[X, Y]$, is defined by

$$[X, Y]\zeta = X(Y\zeta) - Y(X\zeta),$$

for any C^∞ function ζ.

The objective of constructing a robust finite-dimensional filter to (2.0) is equivalent to finding a smooth manifold M and complete C^∞ vector fields μ_i on M and C^∞ functions ν on $M \times \mathbb{R} \times \mathbb{R}^n$ and ω_i's on $\mathbb{R}^m$, such that $\xi(t, x)$ can be represented in the form:

$$\begin{cases} \dfrac{dz(t)}{dt} = \sum_{i=1}^{k} \mu_i(z(t))\omega_i(y(t)), \qquad z(0) \in M, & (2.3a) \\ \xi(t, x) = \nu(z(t), t, x). & (2.3b) \end{cases}$$

Following [10], we say that system (2.0) has a robust universal finite-dimensional filter if for each initial probability density σ_0 there exists a z_0, such that (2.3a) and (2.3b) hold if $z(0) = z_0$, and μ_i, ω_i are independent of σ_0.

In § 5, we will use the Wei–Norman approach to construct a finite-dimensional filter for (2.0). Before we can achieve that, we need to introduce the concept of the estimation algebra of (2.0) and examine its algebraic structure.

DEFINITION. The estimation algebra $\mathbf{E}$ of a filtering problem (2.0), is defined to be the Lie algebra generated by $\{L_0, L_1, \cdots, L_m\}$, or, $\mathbf{E} = \langle L_0, L_1, \cdots, L_m\rangle_{L.A.}$. If in addition there exists a potential function ϕ such that $f_i = (\partial\phi)/(\partial x_i)$ for all $1 \leqq i \leqq n$, then the estimation algebra is called exact.

From now on, unless stated otherwise, we assume the estimation algebra of (2.0) is exact. We use ∇p to denote the column vector

$$\left(\frac{\partial p}{\partial x_1}, \cdots, \frac{\partial p}{\partial x_n}\right)^T.$$

Hence, $\nabla\phi = f$.

In the case where $n = 1$, all estimation algebras are automatically exact. Note also, all exact estimation algebras are characterized by the fact that $\Omega = 0$.

Define

$$D_i = \frac{\partial}{\partial x_i} - f_i,$$

and

$$\eta = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{m} h_i^2.$$

Then,

$$L_0=\frac{1}{2}\left(\sum_{i=1}^n D_i^2-\eta\right).$$

Recall that $f_i=(\partial\phi)/(\partial x_i)$. Hence,

$$\eta=\Delta\phi+|\nabla\phi|^2+\sum_{i=1}^m h_i^2. \tag{2.4}$$

We need the following basic results for later discussion.

THEOREM 1. (Ocone). *Let* $\mathbf{E}$ *be a finite-dimensional estimation algebra. If a function* ζ *is in* $\mathbf{E}$, *then* ζ *is a polynomial of degree* $\leqq 2$.

Ocone's theorem ([11], see [12] for an extension) says that $h_1,\cdots,h_m$ in a finite-dimensional estimation algebra are polynomials of degree $\leqq 2$.

LEMMA 1. *Let* ζ *be a* C^∞ *function on* $\mathbb{R}^n$. *Suppose* $E_l(\zeta)$ *is a polynomial of degree at most k where* $E_l=\sum_{i=1}^l x_i\,\partial/(\partial x_i)$. *Then* $\zeta=p_k(x_1,\cdots,x_n)+\zeta(0,\cdots,0,x_{l+1},\cdots,x_n)$ *where* p_k *is a polynomial of degree k in* $x_1,\cdots,x_n$.

Proof.

$$\begin{aligned}
&\zeta(x_1,x_2,\cdots,x_n)-\zeta(0,\cdots,0,x_{l+1},\cdots,x_n)\\
&=\int_0^1\frac{d}{dt}\zeta(tx_1,\cdots,tx_l,x_{l+1},\cdots,x_n)\,dt\\
&=\int_0^1\left[x_1\frac{\partial\zeta}{\partial x_1}(tx_1,\cdots,tx_l,x_{l+1},\cdots,x_n)+\cdots\right.\\
&\qquad\left.+x_l\frac{\partial\zeta}{\partial x_l}(tx_1,\cdots,tx_l,x_{l+1},\cdots,x_n)\right]dt\\
&=\int_0^1 E_l(\zeta)(tx_1,\cdots,tx_l,x_{l+1},\cdots,x_n)\,dt.
\end{aligned}$$

Since $E_l(\zeta)$ is a polynomial of degree k, we see that $\int_0^1 E_l(\zeta)\times(tx_1,\cdots,tx_l,x_{l+1},\cdots,x_n)\,dt$ is also a polynomial of degree k. □

LEMMA 2. *Let* ζ *be a* C^∞ *function on* $\mathbb{R}^n$. *Suppose* $E_l\zeta+2\zeta$ *is a sum of polynomials of degree two and a* C^∞ *function on* $\mathbb{R}^n$ *which depends only on* $x_{l+1},\cdots,x_n$ *variables. Then for any* $(a_{l+1},\cdots,a_n)\in\mathbb{R}^{n-l}$, $\zeta(x_1,\cdots,x_l,a_{l+1},\cdots,a_n)$ *is a polynomial of degree two in* $x_1,\cdots,x_l$ *variables.*

Proof. Let $\tilde{\zeta}(x_1,\cdots,x_l)=\zeta(x_1,\cdots,x_l,a_{l+1},\cdots,a_n)$. Then

$$\begin{aligned}
E_l(\tilde{\zeta})(x_1,\cdots,x_l)+2\tilde{\zeta}(x_1,\cdots,x_l)&=E_l(\zeta)(x_1,\cdots,x_l,a_{l+1},\cdots,a_n)\\
&\quad+2\zeta(x_1,\cdots,x_l,a_{l+1},\cdots,a_n)
\end{aligned}$$

is a polynomial of degree two in $x_1,\cdots,x_l$ variables. It is well known that $\tilde{\zeta}$ can be written in the following form

$$\tilde{\zeta}(x_1,\cdots,x_l)=\text{polynomial of degree two}+\sum_{i\leqq j\leqq k}a_{ijk}x_ix_jx_k$$

where a_{ijk} are C^∞ functions on $\mathbb{R}^l$. Clearly $E_l(\tilde{\zeta})=$ polynomial of degree two $+\sum_{i\leqq j\leqq k}(E_l(a_{ijk})+3a_{ijk})x_ix_jx_k$ and $E_l(\tilde{\zeta})+2\tilde{\zeta}=$ polynomial of degree two $+\sum_{i\leqq j\leqq k}(E_l(a_{ijk})+5a_{ijk})x_ix_jx_k$. This implies $\sum_{i\leqq j\leqq k}(E_l(a_{ijk})+5a_{ijk})x_ix_jx_k$ is a polynomial of degree two. It follows that for each $i\leqq j\leqq k$, we have $E_l(a_{ijk})+5a_{ijk}=0$. Observe that $E_l(x_1^5a_{ijk})=5x_1^5a_{ijk}+x_1^5E_l(a_{ijk})=0$. In view of Lemma 1, we know that $x_1^5a_{ijk}$ is a polynomial of degree zero, i.e., $x_1^5 a_{ijk} =$ constant. Since a_{ijk} is a C^∞ function on $\mathbb{R}^l$, we conclude that the constant is actually zero. So $\zeta(x_1, \cdots, x_l, a_{l+1}, \cdots, a_n) = \tilde{\zeta}(x_1, \cdots, x_l)$ is a polynomial of degree two in $x_1, \cdots, x_l$ variables. □

3. Structure theorems. The following theorem plays a fundamental role in the classification of exact estimation algebra. It is similar to Theorem 1 in [7], although assuming the estimation algebra is exact allows us to drop certain technical requirements on *f*, *g* and *h*.

THEOREM 2. *Let* **E** *be a finite-dimensional exact estimation algebra. Then* $h_1, \cdots, h_m$ *are polynomials of degree at most one.*

Proof. By Theorem 1, each h_i is a polynomial of degree at most two. Suppose h_1 is of degree two, then by using the affine transformation $\tilde{x} = Ax + b$, where A is orthogonal, we may assume h_1 is of the form

$$\sum_{i=1}^{l} c_i \tilde{x}_i^2 + \sum_{i=l+1}^{n} c_i \tilde{x}_i + c_0,$$

where $c_1, \cdots, c_l$ are nonzero real numbers, and $l \leqq n$. (If $l = n$, the second summation vanishes.) Define $\tilde{f}(\tilde{x}) = Af(x)$ and $\tilde{D}_i = \partial/\partial\tilde{x}_i - \tilde{f}_i$. If $\tilde{\phi}(\tilde{x}) = \phi(x)$, it is easy to see that

$$\tilde{f}(\tilde{x}) = \left(\frac{\partial\tilde{\phi}}{\partial\tilde{x}_1}, \cdots, \frac{\partial\tilde{\phi}}{\partial\tilde{x}_n}\right)^T.$$

Under the transformation, L_0 is mapped into:

$$\tilde{L}_0 = \frac{1}{2}\left(\sum_{i=1}^{n} \tilde{D}_i^2 - \tilde{\eta}(\tilde{x})\right),$$

where

$$\tilde{\eta}(\tilde{x}) = \left[\sum_{i=1}^{n} \frac{\partial\tilde{f}_i(\tilde{x})}{\partial\tilde{x}_i} + \tilde{f}(\tilde{x})^T\tilde{f}(\tilde{x}) + \sum_{i=1}^{m} \tilde{h}_i(\tilde{x})^2\right],$$

and h is transformed into

$$\tilde{h}(\tilde{x}) = h(x).$$

E is isomorphic to the Lie algebra generated by $\tilde{L}_0$ and $\tilde{h}_i$. Note that the degree of h_i in x is the same as the degree of $\tilde{h}_i$ in $\tilde{x}$. Without causing any confusion, from now on, we drop the tilde notation.

Since h_1 is not of degree one, then $l \geqq 1$. We shall produce a contradiction. Let $X_0 = h_1$, and define X_i for $i \geqq 1$ recursively by $X_i = [[L_0, X_{i-1}], X_0]$. Since $L_0 = \frac{1}{2}(\sum_{i=1}^n D_i^2 - \eta)$, it is easy to see that

$$X_1 = 4\sum_{i=1}^{l} c_i^2 x_i^2 + \sum_{i=l+1}^{n} c_i^2,$$

and for $j > 1$

$$X_j = 4^j \sum_{i=1}^{l} c_i^{j+1} x_i^2.$$

By the invertibility of the Vandermonde matrix, it follows after some relabeling, if necessary, that

$$p \equiv \frac{1}{2}\sum_{i=1}^{l} x_i^2$$

is an element in **E**. Let Y_0 be the zero degree differential operator defined by multiplication by p. Define

$$Y_1=[L_0, Y_0]=\sum_{i=1}^{l} x_iD_i+l/2,$$

$$Y_2=[L_0, Y_1]=\sum_{i=1}^{l} D_i^2+\frac{1}{2}\sum_{i=1}^{l} x_i\frac{\partial\eta}{\partial x_i}=\sum_{i=1}^{l} D_i^2+\frac{1}{2}E_l(\eta),$$

and

$$Y_3=[Y_2, Y_1]=2\sum_{i=1}^{l} D_i^2-\frac{1}{2}E_l^2(\eta).$$

Then,

$$2Y_2-Y_3=\tfrac{1}{2}E_l^2(\eta)+E_l(\eta)=\tfrac{1}{2}E_l(E_l\eta+2\eta).$$

By Lemma 1, we know that $E_l(\eta)+2\eta$ is a sum of polynomial of degree two and a C^∞ function which depends on $x_{l+1},\cdots,x_n$ variables. By Lemma 2, it follows that η is a polynomial of degree two in $x_1,\cdots,x_l$, with coefficients which are C^∞ functions in $x_{l+1},\cdots,x_n$ only. Recall that

$$\Delta\phi+|\nabla\phi|^2=-\sum_{i=1}^{m} h_i^2+\eta. \tag{3.0}$$

Let $\psi\in C_0^\infty$ be any C^∞ function with compact support. Multiply (3.0) with ψ^2 and integrate the equation over $\mathbb{R}^n$.

$$\begin{aligned}-\int_{\mathbb{R}^n}\left(\sum_{i=1}^{m} h_i^2-\eta\right)\psi^2&=\int_{\mathbb{R}^n}\psi^2\Delta\phi+\int_{\mathbb{R}^n}\psi^2|\nabla\phi|^2\\&=-\int_{\mathbb{R}^n}2\psi\langle\nabla\psi,\nabla\phi\rangle+\int_{\mathbb{R}^n}\psi^2|\nabla\phi|^2.\end{aligned}\tag{3.1}$$

By the Schwartz inequality

$$2\int_{\mathbb{R}^n}\psi\langle\nabla\psi,\nabla\phi\rangle\leqq\int_{\mathbb{R}_n}|\nabla\psi|^2+\int_{\mathbb{R}^n}\psi^2|\nabla\phi|^2. \tag{3.2}$$

Putting (3.2) into (3.1), we get

$$\int_{\mathbb{R}^n}|\nabla\psi|^2-\int_{\mathbb{R}^n}\left(\sum_{i=1}^{m} h_i^2-\eta\right)\psi^2\geqq 0, \tag{3.3}$$

which is true for all $\psi\in C_0^\infty$. Take any nonzero C^∞ function θ with compact support. Define ψ to be θ followed by a translation in $x_1,\cdots,x_l$ variables direction. Observe that $\int_{\mathbb{R}^n}|\nabla\psi|^2$ is independent of the translation selected. On the other hand, since η is quadratic in $x_1,\cdots,x_l$ variables and h_1 is of degree four in $x_1,\cdots,x_l$, $\sum_{l=1}^m h_i^2-\eta$ becomes very positive when one of the $x_1,\cdots,x_l$ tends to infinity while the other variables remain fixed. We can choose translation in directions, $x_1,\cdots,x_l$, in such a way that

$$\int_{\mathbb{R}^n}\left(\sum_{i=1}^{m} h_i^2-\eta\right)\psi^2$$

is arbitrarily large while $\int_{\mathbb{R}^n} |\nabla \psi|^2$ is bounded. This of course contradicts the inequality (3.3). □

The argument above actually proves the following theorem.

THEOREM 3. *Let $F(x_1, \cdots, x_n)$ be a C^∞ function on $\mathbb{R}^n$. Suppose that there exists a path $C:\mathbb{R}\to\mathbb{R}^n$ and $\delta > 0$ such that $\lim_{t\to\infty} \|C(t)\| = \infty$ and $\lim_{t\to\infty} \sup_{B_\delta(C(t))} F = -\infty$, where $B_\delta(C(t)) = \{x \in \mathbb{R}^n \mid \|x - C(t)\| < \delta\}$. Then there is no C^∞ function ψ on $\mathbb{R}^n$ satisfying the equation*

$$\Delta\psi + |\nabla\psi|^2 = F.$$

COROLLARY. *Let $F(x_1, \cdots, x_n)$ be a polynomial on $\mathbb{R}^n$. Suppose that there exists a polynomial path $C:\mathbb{R}\to\mathbb{R}^n$ such that $\lim_{t\to\infty} \|C(t)\| = \infty$ and $\lim_{t\to\infty} F\circ C(t) = -\infty$. Then there is no C^∞ function ψ on $\mathbb{R}^n$ satisfying the equation*

$$\Delta\psi + |\nabla\psi|^2 = F.$$

Proof. It suffices to prove that $\lim_{t\to\infty} \sup_{B_\delta(C(t))} F(x_1, \cdots, x_n) = -\infty$, where $B_\delta(C(t)) = \{x \in \mathbb{R}^n \mid \|x - C(t)\| < \delta\}$ for some $\delta > 0$. Let $C(t) = (C_1(t), \cdots, C_n(t))$, where

$$\begin{aligned}
C_1(t) &= a_{11}t^k + a_{12}t^{k-1} + \cdots + a_{1k}t + b_1\\
C_2(t) &= a_{21}t^k + a_{22}t^{k-1} + \cdots + a_{2k}t + b_2\\
&\vdots\\
C_n(t) &= a_{n1}t^k + a_{n2}t^{k-1} + \cdots + a_{nk}t + b_n.
\end{aligned}$$

Since F is a polynomial, we have

$$(F\circ C)(t) = \gamma_1 t^d + \gamma_2 t^{d-1} + \cdots + \gamma_{d+1},$$

where $\gamma_1, \cdots, \gamma_{d+1}$ are polynomials in a_{ij} and b_i for $i \leqq n$ and $1 \leqq j \leqq k$. γ_1 must be negative since $\lim_{t\to\infty} (F\circ C)(t) = -\infty$. By continuity, we know that there exists a $\delta > 0$, and a sphere center at $(b_1, \cdots, b_n)$ with radius δ, $B_\delta(b)$, such that for any point $(b_1', \cdots, b_n')$ in it, the following bounds hold:

$$\begin{aligned}
&\gamma_1(a_{ij}; b_1', \cdots, b_n') \leqq \tfrac{1}{2}\gamma_1(a_{ij}; b_1, \cdots, b_n) < 0\\
&|\gamma_2(a_{ij}; b_1', \cdots, b_n') - \gamma_2(a_{ij}; b_1, \cdots, b_n)| \leqq 1\\
&\vdots\\
&|\gamma_{d+1}(a_{ij}; b_1', \cdots, b_n') - \gamma_{d+1}(a_{ij}; b_1, \cdots, b_n)| \leqq 1.
\end{aligned}$$

It follows that for $t > 0$,

$$\begin{aligned}
&\sup_{B_\delta(C(t))} F(x_1, \cdots, x_n)\\
&\quad= \sup_{b'\in B_\delta(b)} F(a_{11}t^k + \cdots + a_{1k} + b_1', \cdots, a_{n1}t^k + \cdots + a_{nk} + b_n')\\
&\quad= \sup_{b'\in B_\delta(b)} \{\gamma_1(a_{ij}; b_1', \cdots, b_n')t^d + \gamma_2(a_{ij}; b_1', \cdots, b_n')t^{d-1} + \cdots\\
&\qquad\qquad + \gamma_{d+1}(a_{ij}; b_1', \cdots, b_n')\}\\
&\quad\leqq \tfrac{1}{2}\gamma_1(a_{ij}; b_1, \cdots, b_n)t^d + (1 + \gamma_2(a_{ij}; b_1, \cdots, b_n))t^{d-1} + \cdots\\
&\qquad\qquad + (1 + \gamma_{d+1}(a_{ij}; b_1, \cdots, b_n)).
\end{aligned}$$

As $\gamma_1(a_{ij}; b_1, \cdots, b_n)$ is negative, the right-hand side tends to $-\infty$ as t tends to ∞. The assertion follows immediately. □

The following result provides a simple characterization of when the dimension of an estimation algebra is finite.

THEOREM 4. *Suppose* **E** *is an exact estimation algebra. Then,* **E** *is finite-dimensional if and only if* $\nabla h_i^T J_\eta^j$ *is a constant for* $1 \leqq i \leqq m$ *and all* $j = 0, 1, \cdots$, *where* $J_\eta = (\partial^2 \eta)/(\partial x_i \, \partial x_j)$, *denote the Hessian matrix of* η.

Proof. The sufficiency of the condition follows from the main theorem of [2]. For completeness reason, we provide the proof here. Assume the condition in Theorem 4 holds. Note that **E** is generated by $L_0, L_1, \cdots, L_m$. Recall that for $i = 1, \cdots, m$ we define

$$L_{m+i} = [L_0, L_i] = \nabla h_i^T D,$$

where D denotes the vector

$$(D_1, \cdots, D_n)^T.$$

Define **F** to be the linear space generated by first and zero degree differential operators of the form $\nabla h_i^T J_\eta^j D$ and $\nabla h_i^T J_\eta^j \nabla \eta$, for $i = 1, \cdots, m$, $j = 0, 1, \cdots$. Clearly, $L_{m+1}, \cdots, L_{2m}$ are elements in **F**. Using our stated assumption, it is also straightforward to check that

(i) $[X, Y] =$ constant if $X, Y \in \mathbf{F}$,

(ii) $[L_0, X] \in \mathbf{F}$ if $X \in \mathbf{F}$,

(iii) $[h_i, X] =$ constant for $i = 1, \cdots, m$ and X in **F**.

Conditions (i), (ii), and (iii) imply that

$$\dim \mathbf{E} \leqq \dim \operatorname{span}\{L_0, h_1, \cdots, h_m, 1\} + \dim \mathbf{F}.$$

By our stated assumption,

$$\mathbf{F} \subset \operatorname{span}\{\partial\eta/\partial x_1, \cdots, \partial\eta/\partial x_n, \partial/\partial x_1, \cdots, \partial/\partial x_n\}.$$

It follows that dimension of **E** is finite.

To prove the necessary condition, assume **E** is finite-dimensional and the condition in Theorem 4 does not hold. Without loss of generality, we may assume there is a $k \geqq 0$, such that $\nabla h_1^T, \nabla h_1^T J_\eta, \cdots, \nabla h_1^T J_\eta^k$ are constant vectors, but $\nabla h_1^T J_\eta^{k+1}$ is not. (Notice that ∇h_1 is a constant vector by Theorem 2.) Let $c = \nabla h_1$. Hence, $c^T x$, $c^T \nabla \eta$, $c^T J_\eta \nabla \eta, \cdots, c^T J_\eta^{k-1} \nabla \eta$ all have degrees at most 1. (If $k = 0$, only the first term is present.) It follows that

$$Ad_{L_0}^{2i+1} h_1 = \frac{1}{2^i} c^T J_\eta^i D \qquad i = 0, \cdots, k+1, \tag{3.4a}$$

$$Ad_{L_0}^{2i} h_1 = \frac{1}{2^i} c^T J_\eta^{i-1} \nabla \eta \qquad i = 1, \cdots, k+1. \tag{3.4b}$$

Let $b^T = c^T J_\eta^k$. There exists an orthogonal matrix, Q, such that

$$b^T Q = (d_1, 0, 0, \cdots, 0) \equiv d^T.$$

Define an orthogonal transformation on the state space by $\tilde{x} = Q^T x$. Under this new coordinate, $c^T x$ is mapped to $c^T Q \tilde{x}$, $\eta(x)$ is mapped to $\eta(Q\tilde{x})$, and $c^T J_\eta^k$ is mapped to d^T. So we may assume $b = d$. Equation (3.4) implies that D_1 and $b^T \nabla \eta = d_1(\partial \eta/\partial x_1)$ are both in **E**. By Ocone's Theorem, $(\partial\eta)/(\partial x_1)$ is a polynomial with degree at most 2. By the assumption that $\nabla h_1^T J_\eta^{k+1}$ is not a constant vector, it follows that the degree of $(\partial \eta)/(\partial x_1)$ is exactly 2. So,

$$\eta = x_1 q + r,$$

where q is a polynomial with degree 2, r is independent of x_1. Depending on the degree of q in x_1, we have three possible cases.

(i) *Degree* 2 *case.* Clearly, $\eta-\sum_{i=1}^m h_i^2$ can be arbitrarily negative on some polynomial path as the path tends to infinity.

(ii) *Degree* 1 *case.* It follows that $\eta=\sum_{i=2}^n \alpha_i x_i x_1^2+\beta x_1+r$, where α_i's are constants, at least one of them nonzero, β and r are independent of x_1. Clearly, $\eta-\sum_{i=1}^m h_i^2$ can be arbitrarily negative on some polynomial path as the path tends to infinity.

(iii) *Degree* 0 *case.* Since q is independent of x_1, $\eta=sx_1+t$, where s and t are independent of x_1. If $\sum_{i=1}^m h_i^2$ is independent of x_1, then $\eta-\sum_{i=1}^m h_i^2$ can be arbitrarily negative. If $\sum_{i=1}^m h_i^2$ is dependent on x_1, it must be of degree 2 in x_1. Again, $\eta-\sum_{i=1}^m h_i^2$ can be arbitrarily negative on some polynomial path as the path tends to infinity.

In all three cases, there is a contradiction to the Corollary of Theorem 3. □

If $\mathbf{E}$ is finite-dimensional, then $\nabla h_i^T J_\eta^j$ is a constant for $1 \leqq i \leqq n$ and all $j=0, 1, \cdots$. It is easy to show by inductive argument that the following theorem holds.

THEOREM 5. *Suppose* $\mathbf{E}$ *is an exact finite-dimensional estimation algebra. Then it has a basis consisting of one second degree differential operator* L_0, *first degree differential operator(s) with constant coefficients, and zero degree differential operator(s) affine in* x. *Moreover, if* X *and* Y *are in* $\mathbf{E}$ *with degree less than or equal to* 1, *then* $[X, Y]$ *is a constant.*

Theorem 6 follows from Theorem 5.

THEOREM 6. *An exact finite-dimensional estimation algebra is solvable.*

4. The Wei–Norman approach. In this section we will use the structural results of previous sections to derive finite-dimensional filters by the Wei–Norman–Brockett approach. To do this, the first step we have to establish is a representation analogous to (1.1).

Consider the filtering system as defined by (2.0). In the following discussion it is not necessary to assume that the estimation algebra of (2.0) is exact. However, we will retain all the notation introduced earlier. In particular, notice that (2.2) still holds. We assume that the estimation algebra is finite dimensional and has a basis consisting of $E_0=L_0$, differential operators, $E_1, \cdots, E_p$, (for some p) of the form

$$\sum_{j=1}^n \alpha_{ij} D_j+\beta_i,$$

where α_{ij}'s are constants and β_i's are polynomial in x, and zero degree differential operators, $E_{p+1}, \cdots, E_q$, (for some $q>p$) affine in x. Moreover, we assume for $1 \leqq i, j \leqq p$, $[E_i, E_j]$ is a constant and that all zero degree differential operators in the estimation algebra are spanned by $E_{p+1}, \cdots, E_q$.[2]

It follows from Theorem 5 that if the estimation algebra of (2.0) is exact and finite-dimensional then it possesses such a basis. However, the exactness is not always necessary. For example, in [6] sufficient conditions are provided for nonexact systems to possess finite-dimensional estimation algebras.

It is clear that by the assumption on the basis that for $1 \leqq i, j \leqq q$,

$$[E_i, E_j]=\text{constant}.$$

For $p+1 \leqq i, j \leqq q$,

$$[E_i, E_j]=0,$$

and for $1 \leqq i \leqq q$ the degree of $[E_0, E_i]$ as a differential operator is not greater than one.

[2] Our earlier definition of L_i still holds. Notice that the L_i's may not form a basis of the estimation algebra.

Since $[[L_0, \sum_{i=1}^{l} c_i x_i + d], \sum_{i=1}^{l} c_i x_i + d] = \sum_{i=1}^{l} c_i^2$, if c_i's and d are constants, the constant function is in the estimation algebra. Without loss of generality, we assume that E_q is the constant function 1.

For a filtering system with such a basis, $[[L_0, L_i], L_i] =$ constant for all $i = 1, \cdots, m$. Hence, $\frac{1}{2}\sum_{i=1}^{m} [[L_0, L_i], L_i] y_i^2(t)$, denoted by $u(t)$, is a function of t independent of x. Equation (2.2) becomes

$$\frac{d\xi(t,x)}{dt} = L_0 \xi(t,x) + \sum_{i=1}^{m} [L_0, L_i] \xi(t,x) y_i(t) + u(t)\xi(t,x). \tag{4.0}$$

DEFINITION. Suppose X is a differential operator, ζ_0 is in the domain of X, r is a continuous function, and $R(t) = \int_0^t r(s)\, ds$. We denote by $e^{R(t)X}\zeta_0$ the solution at time t of the following equation:

$$\frac{d\zeta(t)}{dt} = r(t) X \zeta(t), \qquad \zeta(0) = \zeta_0,$$

if it is well defined.

For $1 \leqq i \leqq q$, $e^{tE_i}\zeta(x)$ can be expressed in the form $\int k(t,x,r)\zeta(r)\, dr$, for some integrable kernel k. Hence, we can extend the definition of $e^{tE_i}\zeta(x)$ to $e^{tE_i}\zeta(t,x)$, where ζ is also a function of t.

PROPOSITION 1. *If ζ is a C^∞ function in x, then for all $0 \leqq s$, the following Baker-Campbell-Hausdorff type relations hold:*

(1) *For $1 \leqq i < q$,*

$$e^{sE_i} E_0 \zeta = \left(E_0 + s \sum_{i=1}^{q} a_{ij} E_j + s^2 \delta_i \right) e^{sE_i} \zeta,$$

where a_{ij}'s and δ_i's are constants.

(2) *For $1 \leqq i \leqq p$, $1 \leqq j < q$, or $1 \leqq i < q$, $1 \leqq j \leqq p$,*

$$e^{sE_i} E_j \zeta = (E_j + s\gamma_{ji}) e^{sE_i} \zeta,$$

where γ_{ji}'s are constants in x.

(3) *For $p+1 \leqq i, j \leqq q$, or $i = q$, $1 \leqq j \leqq q$, or $j = p$, $1 \leqq i \leqq q$,*

$$e^{sE_i} E_j \zeta = E_j e^{sE_i} \zeta.$$

Proof. If E_i is a zero degree differential operator, e^{sE_i} is simply $\exp(sE_i)$. If it is a first degree differential operator, we may assume it is of the form: $\sum_{j=1}^{n} \alpha_{ij} D_j + \beta_i$. Define α_i to be column nth-dimensional vector whose jth component is α_{ij}. Then, it is well known that

$$\begin{aligned} e^{sE_i}\zeta(x) &= \exp\left(\phi(x) - \phi(x + s\alpha_i) + \int_0^s \beta_i(x + \alpha_i(s-r))\, dr \right) \zeta(x + s\alpha_i) \\ &= \exp\left(\phi(x) - \phi(x + s\alpha_i) + \int_0^s \beta_i(x + \alpha_i r)\, dr \right) \zeta(x + s\alpha_i). \end{aligned} \tag{4.1}$$

Assume first that ϕ and ζ are analytic functions. Let $\tilde{\zeta}$ be an arbitrary analytic function in x. From our discussion, it is clear that $e^{sE_i}\tilde{\zeta}$ is well defined for all real s and $1 \leqq i \leqq q$. Moreover, for any fixed x, $e^{sE_i} E_0 e^{-sE_i} \tilde{\zeta}$ is analytic in s. Hence, the classical Baker-Campbell-Hausdorff formula holds from the Taylor series expansion. That is:

$$e^{sE_i} E_0 e^{-sE_i} \tilde{\zeta} = \left(E_0 + s[E_i, E_0] + \frac{s^2}{2} [E_i, [E_i, E_0]] \right) \tilde{\zeta}.$$

Now let $\tilde{\zeta}=e^{sE_i}\zeta$. By using the previously stated properties of the basis, it is easy to see that (1) holds under the analytic assumption.

Next, we relax the condition that ϕ is analytic to that it is C^∞. If E_i is a zero degree differential operator, then clearly $e^{sE_i}E_0e^{-sE_i}\tilde{\zeta}$ is still analytic in s and (1) holds as proven before. Hence, we assume that $E_i=\sum_{j=1}^n \alpha_{ij}D_j+\beta_i$. (Recall that β_i is a polynomial in x.) We can find a polynomial sequence, $\{\tilde{\phi}_i\}$, so that $\tilde{\phi}_i$ converges to ϕ and the first and second order derivatives of $\tilde{\phi}_i$ converge to the respective first and second order derivatives of ϕ. Define $\tilde{f}_{j,i}$ to be $(\partial\tilde{\phi}_j)/(\partial x_i)$ and $\tilde{D}_{j,i}$ to be $\partial/(\partial x_i)-\tilde{f}_{j,i}$. Define $\tilde{E}_{j,i}$ to be $\sum_{k=1}^n \alpha_{ik}\tilde{D}_{j,k}+\beta_i$. Finally, define

$$\tilde{E}_{j,0}=\frac{1}{2}\sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}-\sum_{k=1}^n \tilde{f}_{j,k}\frac{\partial}{\partial x_k}-\sum_{k=1}^n \frac{\partial \tilde{f}_{j,k}}{\partial x_k}-\frac{1}{2}\sum_{k=1}^m h_k^2.$$

It is easy to show by (4.1) that there exist functions u and v such that:

$$\begin{aligned} &e^{s\tilde{E}_{j,i}}\tilde{E}_{j,0}e^{-s\tilde{E}_{j,i}}\tilde{\zeta} \\ &\quad=\left(-\frac{1}{2}\sum_{k=1}^n \frac{\partial \tilde{f}_{j,k}(x)}{\partial x_k}+\frac{1}{2}\sum_{k=1}^n \tilde{f}_{j,k}^2(x)-\frac{1}{2}\sum_{k=1}^n \frac{\partial \tilde{f}_{j,k}(x+s\alpha_i)}{\partial x_k}\right. \\ &\qquad\left.-\frac{1}{2}\sum_{k=1}^n \tilde{f}_{j,k}^2(x+s\alpha_i)\right)\tilde{\zeta}(x)+\sum_{k=1}^n \tilde{f}_{j,k}(x)u(s,x)+v(s,x), \end{aligned}$$

and

$$\begin{aligned} &e^{sE_i}E_0e^{-sE_i}\tilde{\zeta} \\ &\quad=\left(-\frac{1}{2}\sum_{k=1}^n \frac{\partial f_k(x)}{\partial x_k}+\frac{1}{2}\sum_{k=1}^n f_k^2(x)-\frac{1}{2}\sum_{k=1}^n \frac{\partial f_k(x+s\alpha_i)}{\partial x_k}\right. \\ &\qquad\left.-\frac{1}{2}\sum_{k=1}^n f_k^2(x+s\alpha_i)\right)\tilde{\zeta}(x)+\sum_{k=1}^n f_k(x)u(s,x)+v(s,x). \end{aligned}$$

It follows then, that

$$\lim_{j\to\infty} e^{s\tilde{E}_{j,i}}\tilde{E}_{j,0}\, e^{-s\tilde{E}_{j,i}}\tilde{\zeta}=e^{sE_i}E_0\, e^{-sE_i}\tilde{\zeta}.$$

Similarly,

$$\lim_{j\to\infty}\left(\tilde{E}_{j,0}+s[\tilde{E}_{j,i},\tilde{E}_{j,0}]+\frac{s^2}{2}[\tilde{E}_{j,i},[\tilde{E}_{j,i},\tilde{E}_{j,0}]]\right)\tilde{\zeta}=\left(E_0+s[E_i,E_0]+\frac{s^2}{2}[E_i,[E_i,E_0]]\right)\tilde{\zeta}.$$

Hence, (1) holds in this case also. For the general case, for any given x, construct sequences of analytic functions $\{\tilde{\zeta}_i\}$, so that they converge to ζ. It follows that (1) holds in the general case as well. Statements (2), (3), and (4) can be proved similarly. □

THEOREM 7. *If the estimation algebra of* (2.0) *has a basis as described earlier, then its robust* DMZ *equation* (4.0) *has a solution for all* $t\geqq 0$ *of the form*:

$$\xi(t,x)=e^{r_q(t)E_q}\cdots e^{r_1(t)E_1}\,e^{tE_0}\sigma_0, \tag{4.2}$$

where r_i's satisfy an ordinary differential equation for all t. It follows then that a universal finite-dimensional filter exists for (2.0).

Proof. Since E_0 is elliptic, for any $t>0$, $e^{tE_0}\sigma_0$ is C^∞. By differentiating $\xi(t,x)$ we have

$$\frac{d\xi(t,x)}{dt}=e^{r_qE_q}\cdots e^{r_1E_1}E_0\,e^{tE_0}\sigma_0$$

$$+\frac{dr_1}{dt}e^{r_qE_q}\cdots e^{r_2E_2}E_1\,e^{r_1E_1}\,e^{tE_0}\sigma_0+\cdots+\frac{dr_q}{dt}E_q\,e^{r_qE_q}\cdots e^{r_1E_1}\,e^{tE_0}\sigma_0.$$

By applying Proposition 1,

$$e^{r_qE_q}\cdots e^{r_1E_1}E_0\,e^{tE_0}\sigma_0=e^{r_qE_q}\cdots e^{r_2E_2}\left(E_0+r_1\sum_{j=1}^{q}a_{1j}E_j+r_1^2\delta_1\right)e^{r_1E_1}\,e^{tE_0}\sigma_0$$

$$=\left(E_0+\sum_{i=1}^{q-1}\sum_{j=1}^{q}r_ia_{ij}E_j+\kappa_0\right)\xi(t,x),$$

where κ_0 is a polynomial in $r_1,\cdots,r_{q-1}$ and constant in x and r_q.

For $1\leqq i\leqq p$,

$$\frac{dr_i}{dt}e^{r_qE_q}\cdots e^{r_{i+1}E_{i+1}}E_i\,e^{r_iE_i}\,e^{r_{i-1}E_{i-1}}\cdots e^{tE_0}\sigma_0$$

$$=\frac{dr_i}{dt}e^{r_qE_q}\cdots e^{r_{i+2}E_{i+2}}(E_i+r_{i+1}\gamma_{i+1,i})\,e^{r_{i+1}E_{i+1}}\cdots e^{tE_0}\sigma_0$$

$$=\frac{dr_i}{dt}(E_i+\kappa_i)\xi(t,x),$$

where κ_i is a polynomial with degree 1 in r_j for $i+1\leqq j<q$ and constant in the remaining r_j's and x.

For $p+1\leqq i\leqq q$,

$$\frac{dr_i}{dt}e^{r_qE_q}\cdots e^{r_{i+1}E_{i+1}}E_i\,e^{r_iE_i}\,e^{r_{i-1}E_{i-1}}\cdots e^{tE_0}\sigma_0=\frac{dr_i}{dt}E_i\xi(t,x).$$

Hence,

$$\frac{d\xi(t,x)}{dt}=\left(E_0+\sum_{i=1}^{q-1}\sum_{j=1}^{q}r_ia_{ij}E_j+\sum_{i=1}^{q}\frac{dr_i}{dt}E_i+\sum_{i=1}^{p}\frac{dr_i}{dt}\kappa_i+\kappa_0\right)\xi(t,x). \tag{4.3}$$

By substituting (4.3) into (4.0), it is clear that ξ_t is a solution to (4.0), if for $1\leqq j<q$,

$$\frac{dr_j}{dt}=\sum_{i=1}^{m}y_i(t)e_{ij}-\sum_{i=1}^{q-1}r_ia_{ij}, \tag{4.4}$$

and

$$\frac{dr_q}{dt}=u_t+\sum_{i=1}^{m}y_i(t)e_{iq}-\sum_{i=1}^{q-1}r_ia_{iq}-\sum_{i=1}^{p}\frac{dr_i}{dt}\kappa_i-\kappa_0, \tag{4.5}$$

where we represent $[L_0,L_i]$ as $\sum_{j=1}^{q}e_{ij}E_j$.

By the aforementioned property of κ_i, it is clear that (4.4) and (4.5) have solutions for all t.

To see that these results lead to a finite-dimensional filter for (2.0), notice that if we let the r_i's play the role of the z_i's in (2.3a), then (4.4) and (4.5) are of the form (2.3a). By using (4.1), it is easy to check that (4.2) is of the form (2.3b). □

Remark. For the Benes systems, the β_i's are all linear. It is well known that finite-dimensional filters exist in those cases [13].

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