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General convergence result for continuous-discrete feedback particle filter

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Dedicate to Professor Peter Caines on the occasion of his 75th birthday

ABSTRACT

In this paper, we shall discuss the convergence of the continuous-discrete feedback particle filter (FPF) proposed in Yang et al. (2014). The FPF is an interacting system of *N* particles where the interaction is designed such that the empirical distribution of the particles approximates the posterior distribution by an innovation error-based feedback control structure. Under some assumptions, it is proved that, for a class of functions ϕ and $\forall p \geq 2$, the estimate of $\phi(X_{t_n})$ by FPF converges to its optimal estimate $\mathbb{E}[\phi(X_{t_n}) \mid \mathscr{F}_{t_n}]$ in L^p sense, as the number of particles goes to infinity and the numerical approximation error of computing the control input U goes to zero. Furthermore, the bound of the estimation error is also delicately analyzed.

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1. Introduction

The aim of the stochastic filtering is to obtain a good estimate of the state in the stochastic dynamic system recursively in time, based on the noisy observations of the state. There are three properties that the estimate is required to have (Bain & Dan, 2009):

- *Causal:* The state at time t to be estimated should use the observations up to time t.
- *Optimal:* The estimate should minimize the mean squared error (MSE).
- *Real-time:* At arbitrary time t, the estimate should be obtained on the spot, while the observation data keep coming in.

It is well known that the conditional expectation of the state at time t based on the observations till t is the optimal estimate (Jazwinski, 1970).

The study of the stochastic filtering problems has a long history which can be dated back to 1940s, when Wiener and Kolmogorov firstly investigated in the pioneering work (Kolmogorov, 1941; Wiener, 1950). The next major development in stochastic filtering was the introduction of the Kalman filter (KF) for linear Gaussian systems proposed in 1960 Kalman (1960). Subsequently, Kalman and Bucy proposed the continuous version of the KF, the so-called Kalman-Bucy filter (Kalman & Bucy, 1961). The KF is optimal and can be easily computed, enabling it to gain widespread success since 1960s. For example, it was used by NASA to get the Apollo missions off the ground and to the moon (Cipra, 1993). However, the KF only works for linear Gaussian systems, which urges mathematicians and engineers to pursue a computationally efficient, recursive optimal solution applicable to the general nonlinear filtering (NLF) problems.

There have been a lot of works that aim to solve the NLF problems, such as the extended Kalman filter (EKF) (Jazwinski, 1970), ensemble Kalman filter (Evensen, 2003), unscented Kalman filter (Julier et al., 2000), particle filter (PF) (Gordon et al., 1993), Yau-Yau algorithm (Luo & Yau, 2013a, 2013b; Yau & Yau, 2008), estimation algebra method (Shi & Yau, 2017), direct method (Chen et al., 2019) and so on. As for the systems with unknown nonlinear functions arising in the control and filtering problems, we can first estimate the unknown functions (Li et al., 2020; Li, Yang et al., 2020), and then estimate the state (Parlos et al., 2001). In PF, one use a large number of independent random particles to approximate the apriori probability, and update the posterior by the latest observation. The particles are properly located, weighted and propagated recursively by the Bayes' rule. However, the PF suffers from several drawbacks, such as 'particle degeneracy' and 'curse of dimensionality'. Therefore, there have been various variants of PF, and the interested readers can refer to the survey paper (Chen, 2003).

Recently, Yang et al. proposed a novel PF, named Feedback Particle Filter (FPF) (Yang et al., 2014, 2013). The importance sampling step and resampling step in traditional PF are avoided by a feedback control-based approach. The numerical efficiency

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of FPF has been examined by many works, such as Berntorp (2015), Radhakrishnan and Meyn (2019) and Yang (2014). Despite the success of the experiments, the convergence result of FPF for general nonlinear systems remains open. As far as we know, there are only some convergence results of FPF for linear Gaussian systems (Taghvaei & Mehta, 2018a, 2018b). To fill in the blank and take the consideration that, in most real applications, the observations often arrive and are processed at discrete time instant, we shall investigate the convergence of FPF for systems with continuous state and discrete observation, i.e. continuous-discrete FPF.

The central problem considered in this paper is that, under what conditions and for which functions ϕ, does the estimate of $\phi(X_t)$ by FPF converge to its optimal estimate $\mathbb{E}[\phi(X_{t_n}) \mid \mathscr{F}_{t_n}]$? The main contribution of this work is that we prove the convergence of the FPF in the L^p-sense for $\forall p \geq 2$, under certain conditions, and give an explicit error bound. More specifically, we study the L^p-error between the estimate of $\phi(X_{t_n})$ by FPF and its optimal estimate $\mathbb{E}[\phi(X_{t_n}) \mid \mathscr{F}_t]$ for certain class of function ϕ. The estimate error of FPF are introduced in two aspects. The one is the finite number of particles and the other one is the numerical approximation error of computing the optimal control input U (or $\{K, u\}$) in (6). We prove that, the estimate error converges to zero as the number of the particle tends to infinity and the numerical approximation error of computing the optimal control input U vanishes. It needs to be stressed that there are two key mechanisms which help us to complete the proof.

(1) In the updating step, by the use of Markov transition kernel κ_{2n} which is dependent on the observations, we give the key evolution equation (38) for conditional distribution from $\pi_{n\,|\,n-1}$ to $\pi_{n\,|\,n}$, and this equation plays an important role in the proof of the main theorem.
(2) We make a slight modification to the standard FPF, and call it modified FPF in this paper. The modification is that, in prediction step, the particles are sampled from $\frac{1}{N}\sum_{i=1}^{N} \kappa_1(dx_{t_n} \mid \tilde{X}^i_{t_{n-1}})$ instead of $\kappa_1(dx_{t_n} \mid X^i_{t_{n-1}})$ in the standard FPF, which can be seen in Figure 3. This is standard and is used by nearly all existing theoretical analysis for PF, see Crisan and Doucet (2002), Hu et al. (2008) and Hu et al. (2011).

Notations: $|\star|$ is the Euclidean norm of $\star$. Some important notations are summarized in Table 1, after all of them have been introduced in the subsequent sections.

The organization of the paper is as follows. We introduce some preliminary results of FPF in Section 2. The theoretical FPF with the optimal control input U and the standard FPF with the numerical approximation U have also been stated there. In order to complete the proof of the convergence of the standard FPF, we have made a slight modification to the standard FPF, motivated by the theoretical analysis of PF in the literature. The modified FPF and its convergence result have been stated in Section 3, while the detailed proof is in Section 4. A benchmark numerical simulation has been included to show the efficiencies of FPF and modified FPF in Section 5. In addition, we also investigate the MSE versus the number of particles of this particular numerical experiment. The conclusion have been drawn in the end.

2. Preliminary

The continuous-discrete time NLF system considered in this paper is

$$\mathrm{d}X_t = a(X_t)\,\mathrm{d}t + \mathrm{d}B_t, \tag{1}$$

$$Y_n = h(X_{t_n}) + W_n, \tag{2}$$

where $X_t \in \mathbb{R}^{d\times 1}$ is the state at time t, B_t is an d-dimensional standard Brownian motion process independent of X_0, $\pi_{0\,|\,0}$ is the distribution of the initial state X_0, $Y_n \in \mathbb{R}^{m\times 1}$ is the observation arriving at discrete time $t = t_n = n\Delta t$ ($\Delta t > 0$), and W_n is the white noise independent of $\{X_t\}$. Here, we assume that $a(x)$ is a global Lipschitz function such that (1) has a pathwise unique solution for each initial X_0 (Bain & Dan, 2009), and $h(x) \in C^2$. The probability space we considered is denoted as $(\Theta, \mathscr{F}, P)$.

Let us denote the observation history as $\mathscr{F}_t \triangleq \sigma(\{Y_n : t_n \leq t\})$, and denote p_X as the conditional density function of the state X_t based on $\mathscr{F}_t$. More specifically, for any measurable set $A \in \mathscr{F}$,

$$\int_A p_X(x,t)\,\mathrm{d}x = P(X_t \in A \mid \mathscr{F}_t), \tag{3}$$

where $p_X(x,0) = \pi_{0\,|\,0}$ is the density function of the initial state X_0. And p_X can be approximated by the empirical distribution of the particles generated by the FPF proposed in Yang et al. (2014).

2.1 Theoretical feedback particle filter

The continuous-discrete FPF proposed in Yang et al. (2014) is a controlled system where the state evolves in two alternating steps for $n = 1, 2, 3, \ldots$.

(1) Prediction: given N particles $X^i_{t_{n-1}} \in \mathbb{R}^d$, $i = 1, 2, \ldots, N$ (they are sampled i.i.d. from $p_X(x,0)$ at time $t = 0$), the particles evolve according to (1) in the time interval $t \in [t_{n-1}, t_n)$:

$$\mathrm{d}X^i_t = a(X^i_t)\,\mathrm{d}t + \mathrm{d}B^i_t, \tag{4}$$

with initial value $X^i_{t_{n-1}}$, where $X^i_t \in \mathbb{R}^d$ is the state for the i-th particle at time t and $\{B^i_t\}$ are mutually independent standard Wiener processes. We denote the left limit as:

$$X^i_{t_n^-} := \lim_{t \nearrow t_n} X^i_t. \tag{5}$$

(2) Updating: let $S^i_n(0) := X^i_{t_n^-}, i = 1, \ldots, N$, $S^i_n(\lambda)$ evolves according to the following equation

$$\frac{\mathrm{d}S^i_n}{\mathrm{d}\lambda}(\lambda) = \underbrace{K(S^i_n(\lambda), \lambda)Y_n + u(S^i_n(\lambda), \lambda)}_{\text{optimal } U^i_n(\lambda)}, \tag{6}$$

with initial condition $S^i_n(0)$ for $i = 1, 2, \ldots, N$, and the pseudo-time $\lambda \in [0, 1]$. The control input $U^i_n(\lambda)$ (or $\{K, u\}$) is designed such that the empirical distribution of the ensemble $\{S^i_n(1)\}_{i=1}^N$ approximates the posterior distribution. The initial condition for the next interval is assigned as $X^i_{t_n} = S^i_n(1)$ for $i = 1, 2, \ldots, N$.

And the evolution structure of the continuous-discrete FPF is shown in Figure 1 (Yang et al., 2014).

Let us denote the conditional distribution of the particle X_t^i given $\mathscr{F}_t$ as p_{X^i}, i.e.

$$\int_A p_{X^i}(x,t)\,\mathrm{d}x = P(X_t^i \in A \mid \mathscr{F}_t), \quad \forall A \in \mathscr{F}. \tag{7}$$

Similarly, we denote the conditional distribution of the particle $S_n^i(\lambda)$ given $\mathscr{F}_{t_n}$ as $p_{S_n^i}$, i.e.

$$\int_A p_{S_n^i}(x,\lambda)\,\mathrm{d}x = P(S_n^i(\lambda) \in A \mid \mathscr{F}_{t_n}), \quad \forall A \in \mathscr{F}. \tag{8}$$

The goal of FPF is to choose the optimal control input $U_n^i(\lambda)$ in (6) such that p_{X^i} can well approximate p_X in (3). Now we need to review the evolution of the conditional density p_X before we give the optimal control.

Given the initial density $p_X(x,0)$ and the increasing filtration $\mathscr{F}_t$, the evolution of the posterior $p_X(x,t)$ is obtained by two alternative steps: prediction and updating, which is shown in the following proposition.

Proposition 2.1 (Proposition 4.2.1 in Yang (2014)): *Consider the filtering problem (1)–(2) over time interval $[t_{n-1}, t_n]$. For $t \in [t_{n-1}, t_n)$, $p_X(x,t)$ satisfies the following Fokker-Planck equation (Jazwinski, 1970):*

$$\frac{\partial p_X}{\partial t}(x,t) = \mathscr{L}^\dagger p_X(x,t), \tag{9}$$

where $\mathscr{L}^\dagger p_X = -\nabla\cdot(p_X a) + \frac{1}{2}\Delta p_X$ is the forward generator of the diffusion process X_t, Δ denotes the Laplacian in $\mathbb{R}^d$, and $\nabla\cdot$ is the divergence operator. Then we have

$$p_X(x,t_n^-) := \lim_{t \nearrow t_n} p_X(x,t).$$

Note $p_X(x,t_n^-)$ is the apriori distribution of X_{t_n} given $\mathscr{F}_{t_{n-1}}$.

At the discrete time instant $t = t_n$ when the observation is made, the posterior density is updated using Bayes' rule:

$$p_X(x,t_n) = p_X(x,t_n^-) \cdot \exp\left[-\frac{1}{2}(Y_n - h(x))^T(Y_n - h(x))\right]/C_n, \tag{10}$$

where C_n is the normalization constant.

The two Equations (9)–(10) define the mapping of p_X from t_{n-1} to t_n.

Apparently, $p_{X^i}(x,t)$ satisfies the same evolution equation (9) for $t \in [t_{n-1}, t_n)$ according to (1) and (4), and we list the evolution equations of $p_X(x,t)$ and $p_{X^i}(x,t)$ in Figure 2.

Recall that our aim is to approximate p_X by p_{X^i}. Now it is known that they have the same initial value at $t=0$ and they satisfy the same evolution equation in the prediction step. Obviously, if they satisfy the same evolution equation in the updating step, then we have $p_X = p_{X^i}$. According to Figure 2, we know that the evolution equation of p_{X^i} in the updating step is equivalent to the evolution equation of $p_{S_n^i}(x,\lambda)$ which is determined by the control input $\{K, u\}$. Therefore the functions $\{K(x,\lambda), u(x,\lambda)\}$ (or say the control input $U_n^i(\lambda)$) in (6) are said to be *optimal* if $p_X = p_{X^i}$. That is, given $p_X(x,t_n^-) = p_{S_n^i}(x,0)\ (= p_{X^i}(x,t_n^-))$, our goal is to choose $\{K,u\}$ in (6) such that $p_X(x,t_n) = p_{S_n^i}(x,1)\ (= p_{X^i}(x,t_n))$. The optimal $\{K,u\}$ are given in Yang (2014).

Theorem 2.1 (Theorem 4.2.3, Yang (2014)): *Let ρ_n be the solution of the following equation:*

$$\frac{\partial \rho_n}{\partial \lambda}(x,\lambda) = \rho_n(x,\lambda)\left[(h(x) - \hat{h}(\lambda))^T Y_n - \frac{1}{2}|h(x)|^2 + \frac{1}{2}\widehat{|h|^2}\right], \tag{11}$$

where $\rho_n(x,0) = p_X(x,t_n^-)$, $\hat{h}(\lambda) := \int_{R^d}\rho_n(x,\lambda)h(x)\,dx$, and $\widehat{|h|^2} := \int_{R^d}\rho_n(x,\lambda)\,|h(x)|^2\,dx$.

For each fixed $\lambda \in [0,1]$, let η_j be the solution of:

$$\nabla\cdot(\rho_n\nabla\eta_j) = -(h_j - \hat{h}_j)\rho_n, \quad \int_{R^d}\eta_j(x,\lambda)\rho_n(x,\lambda)\,\mathrm{d}x = 0, \tag{12}$$

for $j = 1,\ldots,m$. Then the optimal gain function is

$$K = [\nabla\eta_1^T, \nabla\eta_2^T, \ldots, \nabla\eta_m^T]. \tag{13}$$

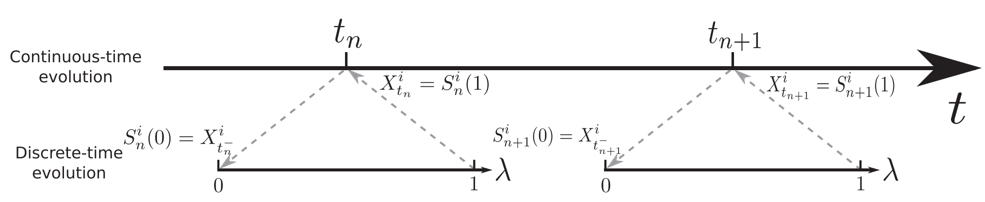

Figure 1. An illustration of the two-step evolution structure in the continuous-discrete time FPF.

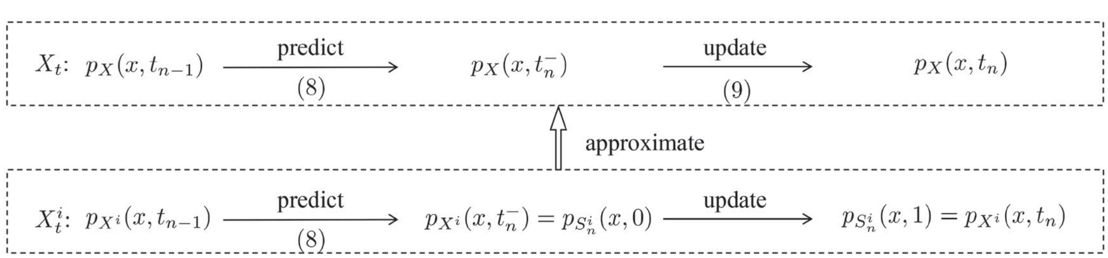

Figure 2. Evolutions of the conditional density functions $p_X(x,t)$ and $p_{X^i}(x,t)$.

The optimal function u is obtained as

$$u(x,\lambda) = -\frac{1}{2}K(x,\lambda)(h(x)+\hat{h}) + \frac{1}{2}\Omega(x,\lambda), \qquad (14)$$

where $\Omega = \nabla\varphi$, φ is a scalar function, and it is a solution to

$$\nabla\cdot(\rho_n\nabla\varphi) = -(g-\hat{g})\rho_n, \qquad \int_{\mathbb{R}^d}\varphi(x,\lambda)\rho_n(x,\lambda)\,\mathrm{d}x = 0, \qquad (15)$$

where $g := \sum_{j=1}^m \nabla\eta_j\cdot\nabla h_j^T$ *and* $\hat{g} := \int_{\mathbb{R}^d}\rho_n(x,\lambda)g(x)\,\mathrm{d}x = \left|\hat{h}\right|^2 - \widehat{|h|^2}$. *Substituting the optimal $\{K,u\}$ in (13) and (14) into (6), we have*

$$\frac{\mathrm{d}S_n^i}{\mathrm{d}\lambda}(\lambda) = K(S_n^i(\lambda),\lambda)\left[Y_n - \frac{h(S_n^i(\lambda))+\hat{h}}{2}\right] + \frac{1}{2}\Omega(S_n^i(\lambda),\lambda). \qquad (16)$$

If $S_n^i(\lambda)$ evolves according to (16) with the optimal control input $\{K,u\}$ obtained by (13)–(15), then $p_X(x,t_n) = p_{S_n^i}(x,1)$ ($= p_{X^i}(x,t_n)$), provided that $p_X(x,t_n^-) = p_{S_n^i}(x,0)$ ($= p_{X^i}(x,t_n^-)$).

Proof: The complete proof can be found in Yang (2014) and the idea of the proof can be found in Appendix 1. ■

In summary, the particles in the theoretical FPF evolve according to (4) in prediction step and (16) in updating step, with $\{K,u\}$ obtained from (13)–(15).

Apparently, we have $p_X(x,t) = p_{X^i}(x,t)$ with the optimal control input. Since we are concerned about the apriori and posterior distributions, and for the conciseness of the notations in the proof, let

$$\pi_{n\,|\,n-1}(x) := p_X(x,t_n^-), \quad \pi_{n\,|\,n}(x) := p_X(x,t_n), \quad \forall\, n\geq 1. \qquad (17)$$

Then we have

$$p_{S_n^i}(x,0) = \pi_{n\,|\,n-1}(x), \quad p_{S_n^i}(x,1) = \pi_{n\,|\,n}(x), \qquad (18)$$

since $p_X(x,t) = p_{X^i}(x,t)$.

We shall use $\pi^N_{n\,|\,n-1}$ and $\pi^N_{n\,|\,n}$ to denote the empirical distributions formed by the N particles:

$$\pi^N_{n\,|\,n-1} := \frac{1}{N}\sum_{i=1}^N \delta_{X^i_{t_n^-}} = \frac{1}{N}\sum_{i=1}^N \delta_{S_n^i(0)},$$
$$\pi^N_{n\,|\,n} := \frac{1}{N}\sum_{i=1}^N \delta_{X^i_{t_n}} = \frac{1}{N}\sum_{i=1}^N \delta_{S_n^i(1)}, \qquad (19)$$

where δ is the Dirac delta measure.

2.2 Standard feedback particle filter

In this subsection, we shall analyze the FPF with the numerical errors introduced by the approximations of (12)–(15). In the theoretical FPF, the optimal control in (12)–(15) are assumed to be obtained exactly, without any approximation. However, it is known that the approximations of the solutions are inevitable, since the closed-form solutions of the boundary value problems (12) and (15) can only be obtained in certain special cases, say the linear Gaussian system in Proposition A.2. For general nonlinear system, various numerical approximations of U can be found in the paper (Berntorp, 2018).

The FPF with numerical approximations of $\{K,u\}$ (or $\{K,\Omega,\hat{h}\}$) is named standard FPF which is described in Yang (2014). Let us use $\check{X}^i_{t_n}$ to denote the particles obtained in standard FPF, which is listed as follows: For $n = 1,2,3,\ldots$.

(1) Prediction: given N particles $\check{X}^i_{t_{n-1}}\in\mathbb{R}^d$, $i=1,2,\ldots,N$ (they are sampled i.i.d. from $\pi_{0\,|\,0}$ at time $t=0$), the particles evolve according to (1) at time interval $t\in[t_{n-1},t_n)$:

$$\mathrm{d}\check{X}^i_t = a(\check{X}^i_t)\,\mathrm{d}t + \mathrm{d}B^i_t \qquad (20)$$

with initial value $\check{X}^i_{t_{n-1}}$, where $\check{X}^i_t\in\mathbb{R}^d$ is the state for the i-th particle at time t and $\{B^i_t\}$ are mutually independent standard Wiener processes. We denote the left limit as:

$$\check{X}^i_{t_n^-} := \lim_{t\nearrow t_n}\check{X}^i_t. \qquad (21)$$

(2) Updating: let $\check{S}^i_n(0) := \check{X}^i_{t_n^-}, i=1,\ldots,N$, $\check{S}^i_n(\lambda)$ evolve according to the following equation

$$\frac{\mathrm{d}\check{S}^i_n}{\mathrm{d}\lambda}(\lambda) = \tilde{K}(\check{S}^i_n(\lambda),\lambda)\left[Y_n - \frac{h(\check{S}^i_n(\lambda))+\bar{\hat{h}}}{2}\right] + \frac{1}{2}\tilde{\Omega}(\check{S}^i_n(\lambda),\lambda), \qquad (22)$$

with initial condition $\check{S}^i_n(0)$ for $i=1,2,\ldots,N$ when $\lambda\in[0,1]$, where $\tilde{K}$, $\tilde{\Omega}$, $\bar{\hat{h}}$ are numerical approximations of K, Ω, $\hat{h}$, and they are computed as follows:

$$K\approx\tilde{K}(\check{S}_n) := \frac{1}{N}\sum_{i=1}^N \check{S}^i_n(\lambda)\left(h\left(\check{S}^i_n(\lambda)\right) - \bar{\hat{h}}\left(\check{S}^i_n(\lambda)\right)\right)^T, \qquad (23)$$

$$\Omega\approx\tilde{\Omega}(\check{S}_n) := \frac{1}{N}\sum_{i=1}^N \check{S}^i_n(\lambda)\left(g\left(\check{S}^i_n(\lambda)\right) - \bar{\hat{g}}\left(\check{S}^i_n(\lambda)\right)\right)^T, \qquad (24)$$

where $\bar{\hat{g}} = \left|\bar{\hat{h}}\right|^2 - \widehat{|h|^2}$, and

$$\hat{h}\approx\bar{\hat{h}}(\check{S}_n) := \frac{1}{N}\sum_{i=1}^N h(\check{S}^i_n(\lambda)),$$
$$\widehat{|h|^2}\approx\overline{\widehat{|h|^2}}(\check{S}_n) := \frac{1}{N}\sum_{i=1}^N h(\check{S}^i_n(\lambda))^T h(\check{S}^i_n(\lambda)). \qquad (25)$$

Then N particles are updated according to $\check{X}^i_{t_n} = \check{S}^i_n(1)$ for $i=1,2,\ldots,N$.

Remark 2.1: The difference of $\check{S}^i_n$ and S_n are mainly due to the numerical approximations of $\{K,\Omega,\hat{h}\}$ from (23)–(25). S_n is the exact solution of (16), while $\check{S}_n$ is obtained by (22), with $\{K,\Omega,\hat{h}\}$ from (23)–(25).

2.3 Markov transition Kernel

Before we start the error analysis, we need to introduce the Markov kernel, which is a main tool in our proof. The system (1)–(2) will be represented in a sightly different framework. The d-dimensional state X_t is a Markov process with initial state X_0 obeying the distribution $\pi_{0\,|\,0}(\mathrm{d}x_0)$. The dynamic Equation (1) describing the state evolution over time can be modeled by a Markov transition kernel [1] $\kappa_1(\mathrm{d}x_{t_{n+1}}\,|\,x_{t_n})$ (Hu et al., 2008), so that

$$P(X_{t_{n+1}} \in A\,|\,X_{t_n} = x_{t_n}) = \int_A \kappa_1(\mathrm{d}x_{t_{n+1}}\,|\,x_{t_n}) \tag{26}$$

for all $A \in \mathscr{B}(\mathbb{R}^d)$, where $\mathscr{B}(\mathbb{R}^d)$ is the Borel algebra on $\mathbb{R}^d$.

Similarly, for $\check{S}^i_n(\lambda)$, which evolves according to (22) with approximated $\{\tilde{K},\ \tilde{\Omega},\ \bar{\hat{h}}\}$, we can define the following Markov transition kernel $\tilde{\kappa}_{2n}$:

$$P(\check{S}^i_n(1) \in A\,|\,\check{S}^i_n(0) = s_{n0}, Y_n) = \int_A \tilde{\kappa}_{2n}(\mathrm{d}\check{s}_{n1}\,|\,s_{n0}, Y_n). \tag{27}$$

Meanwhile, we need to define the Markov transition kernel for $S^i_n(\lambda)$ which evolves according to (16) with exact $\{K,\ \Omega,\ \hat{h}\}$:

$$P(S^i_n(1) \in A\,|\,S^i_n(0) = s_{n0}, Y_n) = \int_A \kappa_{2n}(\mathrm{d}s_{n1}\,|\,s_{n0}, Y_n). \tag{28}$$

It is obvious that $\tilde{\kappa}_{2n}$ and κ_{2n} are Markov transition kernels depending on Y_n.

Based on the Markov transition kernels defined above, we list the standard FPF in Algorithm 1.

Algorithm 1 Standard Continuous-Discrete Time FPF Yang (2014)

INITIALIZATION
Set $n = 0$
for $i = 1$ to N **do**
  Sample $\check{X}^i_0 \sim \pi_{0\,|\,0}$
$n = n + 1$
ITERATION 1: Prediction
for $i = 1$ to N **do**
  Sample $\check{X}^i_{t_n^-} \sim \kappa_1(\mathrm{d}x_{t_n}\,|\,\check{X}^i_{t_{n-1}})$ by (20)
ITERATION 2: Updating
Set $\lambda = 0$
for $i = 1$ to N **do**
  Set $\check{S}^i_n(0) = \check{X}^i_{t_n^-}$
for $i = 1$ to N **do**
  Calculate the gain function $\tilde{K}$ by (23)
  Calculate the function $\tilde{\Omega}$ by (24)
  Calculate $\hat{h}$ by (25)
  Sample $\check{S}^i_n(1) \sim \tilde{\kappa}_{2n}(d\check{s}_{n1}\,|\,\check{S}^i_n(0), Y_n)$ by (22)
for $i = 1$ to N **do**
  Set $\check{X}^i_{t_n} = \check{S}^i_n(1)$
$n = n + 1$
goto ITERATION 1

For convenience, we assume that $\kappa_1, \tilde{\kappa}_{2n}$ and κ_{2n} have densities with respect to a Lebesgue measure (Hu et al., 2008), i.e.

$$\begin{aligned}
\kappa_1(\mathrm{d}x_{t_{n+1}}\,|\,x_{t_n}) &= \kappa_1(x_{t_{n+1}}\,|\,x_{t_n})\,\mathrm{d}x_{t_{n+1}},\\
\tilde{\kappa}_{2n}(\mathrm{d}s_1\,|\,s_0, Y_n) &= \tilde{\kappa}_{2n}(s_1\,|\,s_0, Y_n)\,\mathrm{d}s_1,\\
\kappa_{2n}(\mathrm{d}s_1\,|\,s_0, Y_n) &= \kappa_{2n}(s_1\,|\,s_0, Y_n)\,\mathrm{d}s_1.
\end{aligned} \tag{29}$$

According to the law of total probability, we have the following evolution of the real conditional density (Hu et al., 2008):

$$\pi_{n\,|\,n-1}(\mathrm{d}x_{t_n}) = \int_{R^d} \pi_{n-1\,|\,n-1}(\mathrm{d}x_{t_{n-1}})\kappa_1(\mathrm{d}x_{t_n}\,|\,x_{t_{n-1}}). \tag{30}$$

Now we use the Markov transition kernel to give the evolution equation between $\pi_{n\,|\,n}$ and $\pi_{n\,|\,n-1}$. Recalling that $X_{t_n^-}$ and $S^i_n(0)$ have conditional density $\pi_{n\,|\,n-1}$ and $\rho_n(x,0)$, respectively. If $\rho_n(x,0) = \pi_{n\,|\,n-1}$, then we have $\rho_n(x,1) = \pi_{n\,|\,n}$ for $S_n(1)$ and X_{t_n}, by Theorem 2.1. It follows that

$$\begin{aligned}
\pi_{n\,|\,n}(\mathrm{d}x_{t_n}) &= \int_{R^d} P(S^i_n(0) \in \mathrm{d}z\,|\,\mathscr{F}_{t_n})\\
&\quad \times P(S^i_n(1) \in \mathrm{d}x_{t_n}\,|\,S^i_n(0) = z, Y_n)\\
&= \int_{R^d} P(S^i_n(0) \in \mathrm{d}z\,|\,\mathscr{F}_{t_{n-1}})\\
&\quad \times P(S^i_n(1) \in \mathrm{d}x_{t_n}\,|\,S^i_n(0) = z, Y_n)\\
&= \int_{R^d} \pi_{n\,|\,n-1}(\mathrm{d}z)\kappa_{2n}(\mathrm{d}x_{t_n}\,|\,z, Y_n),
\end{aligned} \tag{31}$$

where the second equality in (31) follows from the fact that

$$P(S^i_n(0) \in \mathrm{d}z\,|\,\mathscr{F}_{t_n}) = P(S^i_n(0) \in \mathrm{d}z\,|\,\mathscr{F}_{t_{n-1}}). \tag{32}$$

This is because, by the construction of $S^i_n(0)$, it can be known that $S^i_n(0)$ depends on $\{Y_k, 1 \le k \le n-1\}$ and $\{B^i_s, s \le t_n\}$. Therefore, we have (32) by (2), since W_n is independent of $\{Y_k, 1 \le k \le n-1\}$ and $\{B^i_s, s \le t_n\}$, i.e. W_n is independent of $S^i_n(0)$.

3. The convergence analysis of modified FPF

In this section, we shall analyze the error between the conditional density function π and its numerical approximation formed by FPF.

3.1 Modified FPF

Similar to the treatments in Crisan and Doucet (2002) and Hu et al. (2011), we need to make a slight modification to the standard FPF in Algorithm 1 for the technical treatment in the proof in Section 4.

Instead of sampling from $\check{X}^i_{t_n^-} \sim \kappa_1(\mathrm{d}x_{t_n}\,|\,\check{X}^i_{t_{n-1}})$ in the standard FPF, we sample

$$\tilde{X}^i_{t_n^-} \sim \frac{1}{N}\sum_{i=1}^{N} \kappa_1\left(dx_{t_n}\,|\,\tilde{X}^i_{t_{n-1}}\right), \tag{33}$$

where $\tilde{X}^i_t$ denotes the i-th particle in modified FPF. The procedures of modified FPF are described in Algorithm 2.

Algorithm 2 Modified Continuous-Discrete Time FPF

INITIALIZATION
Set $n = 0$
for $i = 1$ to N **do**
  Sample $\tilde{X}_0^i \sim \pi_{0\,|\,0}$
$n = n + 1$
ITERATION 1: Prediction
for $i = 1$ to N **do**
  Sample $\tilde{X}^i_{t_n^-} \sim \frac{1}{N}\sum_{i=1}^{N} \kappa_1(dx_{t_n}|\tilde{X}^i_{t_{n-1}})$
ITERATION 2: Updating
Set $\lambda = 0$
for $i = 1$ to N **do**
  Set $\tilde{S}^i_n(0) = \tilde{X}^i_{t_n^-}$
for $i = 1$ to N **do**
  Calculate the gain function $\tilde{K}$ by (23)
  Calculate the function $\tilde{\Omega}$ by (24)
  Calculate $\hat{h}$ by (25)
  Sample

$$\tilde{S}^i_n(1) \sim \tilde{\kappa}_{2n}(d\tilde{s}_{n1}|\tilde{S}^i_n(0), Y_n) \tag{34}$$

for $i = 1$ to N **do**
  Set $\tilde{X}^i_{t_n} = \tilde{S}^i_n(1)$
$n = n + 1$
goto ITERATION 1

We use $\tilde{\pi}^N$ to denote the empirical distribution formed by the N particles of the modified FPF, i.e.

$$\tilde{\pi}^N_{n\,|\,n} := \frac{1}{N}\sum_{i=1}^{N}\delta_{\tilde{X}^i_{t_n}} = \frac{1}{N}\sum_{i=1}^{N}\delta_{\tilde{S}^i_n(1)},$$
$$\tilde{\pi}^N_{n\,|\,n-1} := \frac{1}{N}\sum_{i=1}^{N}\delta_{\tilde{X}^i_{t_n^-}} = \frac{1}{N}\sum_{i=1}^{N}\delta_{\tilde{S}^i_n(0)}. \tag{35}$$

To help the readers understand the paper better, we list theoretical FPF, standard FPF and modified FPF in Figure 3. It is shown that the difference between theoretical FPF and standard FPF is that there are numerical approximations of $\{K, u\}$ in updating step of FPF. The only difference between standard FPF and modified FPF is that they have different Markov transition kernel in prediction step.

Notations: Given a measure ν, a function ϕ and a Markov transition kernel κ, denote

$$(\nu, \phi) \triangleq \int_{R^d} \phi(x)\nu(\mathrm{d}x), \quad \kappa\phi(x) \triangleq \int_{R^d} \kappa(\mathrm{d}z\,|\,x)\phi(z). \tag{36}$$

Hence, we have $\mathbb{E}[\phi(X_{t_n})\,|\,\mathscr{F}_{t_n}] = (\pi_{n\,|\,n}, \phi)$, and

$$(\pi_{n\,|\,n-1}, \phi) = (\pi_{n-1\,|\,n-1}, \kappa_1\phi), \tag{37}$$
$$(\pi_{n\,|\,n}, \phi) = (\pi_{n\,|\,n-1}, \kappa_{2n}\phi), \tag{38}$$

by (30) and (31), respectively.

Besides, we clarify and restate the three types of particles considered in this paper:

(1) $\{X_t^i, S_n^i\}$ denote the particles in theoretical FPF proposed in Yang et al. (2014), and the optimal control input U is exact without any approximation in the updating step. Their empirical distributions are $\pi^N_{n\,|\,n-1}$ and $\pi^N_{n\,|\,n}$;
(2) $\{\check{X}_t^i, \check{S}_n^i\}$ denote the particles in standard FPF (Algorithm 1) proposed in Yang et al. (2014), and there are numerical approximations of the optimal control input U in the updating step;
(3) $\{\tilde{X}_t^i, \tilde{S}_n^i\}$ denote the particles in modified FPF (Algorithm 2), and it differs from the standard FPF in the prediction step.

The notations are listed in Table 1 and the connections between these three types of particles are clearly shown in Figure 3.

3.2 Error analysis

In this part, we shall analyze the estimation error of the modified FPF. For a class of functions ϕ and arbitrary $p \geq 2$, the L^p-error between the estimate of $\phi(X_{t_n})$ by modified FPF and its optimal estimate $\mathbb{E}[\phi(X_{t_n})\,|\,\mathscr{F}_{t_n}]$ will be analyzed. More explicitly, we shall give an error bound of $\mathbb{E}[|(\tilde{\pi}^N_{n\,|\,n}, \phi) - (\pi_{n\,|\,n}, \phi)|^p]$, which is composed of two parts: the one is caused by finite N particles, and the other one is from the numerical approximations of $\{K, u\}$ (or $\{K, \Omega, \hat{h}\}$) in updating step. We declare that, the error caused by Euler scheme used in Algorithm 2 is not taken into consideration in this paper. We shall put some mild restrictions on the function $\phi(x)$:

Assumption 3.1: The function $\phi(x)$ has bounded p-th moment for $p \geq 2$, i.e. $\forall n \geq 1$,

$$\mathbb{E}\left[\int |\phi(x)|^p \pi_{k\,|\,k}(\mathrm{d}x)\right] < \infty, \quad 0 \leq k \leq n, \tag{39}$$

i.e.

$$\mathbb{E}\left[\left|\phi(X_{t_k})\right|^p\right] < \infty, \quad 0 \leq k \leq n. \tag{40}$$

The class of functions ϕ satisfying Assumption 3.1 will be denoted by L^p_n. For any $\phi(x) \in L^p_n$, we define

$$\|\phi\|_{n,p} \triangleq \max_{k=0,1,2,\ldots,n} \{1, \mathbb{E}^{1/p}\left[(\pi_{k\,|\,k}, |\phi|^p)\right]\}$$

Table 1. Notations.

Variable	Notation	Equation
State of the hidden process	X_t	(1)
Particles in theoretical FPF	$X^i_{t_n^-} = S^i_n(0)$	(4)
	$X^i_{t_n} = S^i_n(1)$	(6)
Particles in standard FPF	$\check{X}^i_{t_n^-} = \check{S}^i_n(0)$	(20)
	$\check{X}^i_{t_n} = \check{S}^i_n(1)$	(22)
Particles in modified FPF	$\tilde{X}^i_{t_n^-} = \tilde{S}^i_n(0)$	(33)
	$\tilde{X}^i_{t_n} = \tilde{S}^i_n(1)$	(34)
Conditional densities	$\pi_{n\,|\,n-1}(\cdot) = \rho_n(\cdot, 0)$	(A6),(17)
	$\pi_{n\,|\,n}(\cdot) = \rho_n(\cdot, 1)$	(A7),(17)
Empirical distribution of $X^i_{t_n}$	$\pi^N_{n\,|\,n-1}, \pi^N_{n\,|\,n}$	(19)
Empirical distribution of $\tilde{X}^i_{t_n}$	$\tilde{\pi}^N_{n\,|\,n-1}, \tilde{\pi}^N_{n\,|\,n}$	(35)
Markov transition kernel	κ_1, κ_{2n}	(26),(28)
	$\tilde{\kappa}_{2n}$	(27)

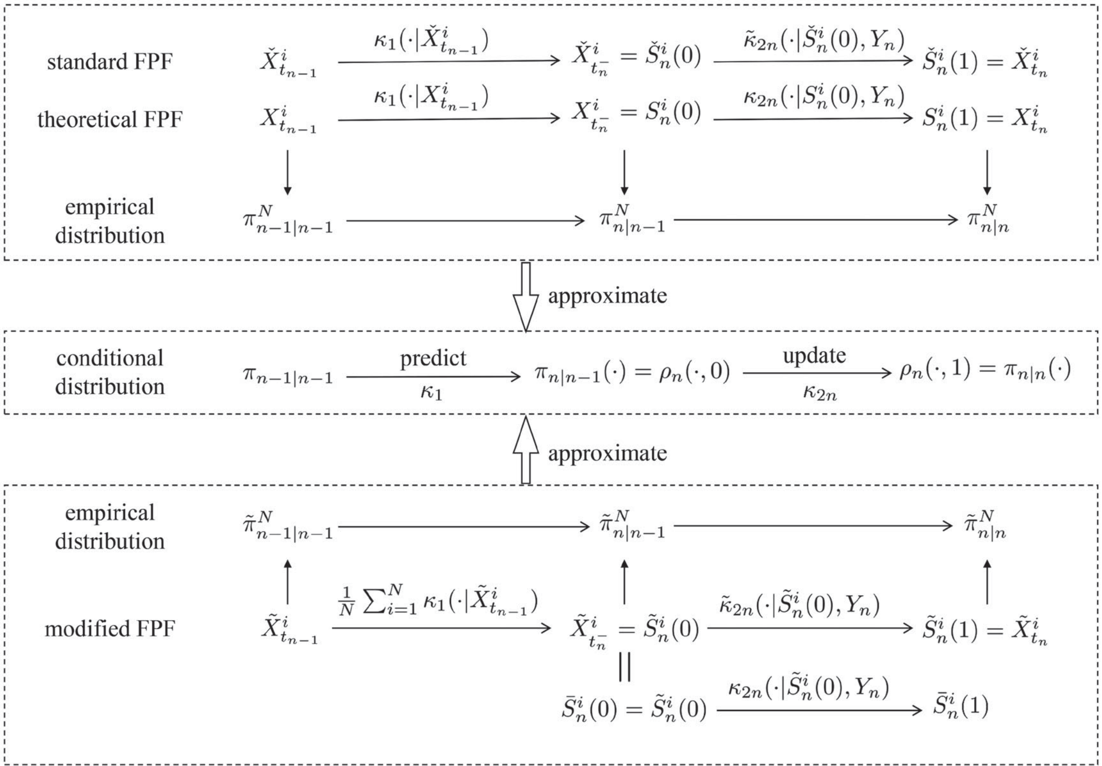

Figure 3. Framework of the FPF.

$$= \max_{k=0,1,2,\ldots,n} \{1, \mathbb{E}^{1/p}\left[\left|\phi(X_{t_k})\right|^p\right]\}, \tag{41}$$

which is introduced in Hu et al. (2008).

Assumption 3.2: Given $p \geq 2$, ϕ and ϕ^p are Lipschitz functions, i.e. for all $x,\ y \in \mathbb{R}^d$,

$$\begin{aligned} \left|\phi(x) - \phi(y)\right| &\leq C_\phi \left|x - y\right|, \\ \left|\phi^p(x) - \phi^p(y)\right| &\leq C_{\phi,p} \left|x - y\right|, \end{aligned} \tag{42}$$

where C_ϕ and $C_{\phi,p}$ are Lipschitz constants.

Remark 3.1: One sufficient condition for Assumption 3.2 to hold is that the function ϕ is bounded and Lipschitz, say the trigonometry functions $\sin(x)$, $\cos(x)$, the inverse trigonometric functions $\arctan(x)$, $\text{arccot}(x)$, and $1/(x^2+1)$.

Now we need to analyze the error caused by the approximations of K, Ω and $\hat{h}$ in (16). It has been declared that $\tilde{S}_n^i(\lambda)$ is the solution of (22) with the initial value $\tilde{S}_n^i(0)$, and we denote the solution of (16) with initial value $\tilde{S}_n^i(0)$ as $\bar{S}_n^i(\lambda)$, i.e.

$$\begin{aligned} \frac{\mathrm{d}\bar{S}_n^i}{\mathrm{d}\lambda}(\lambda) &= K(\bar{S}_n^i(\lambda),\lambda)\left[Y_n - \frac{h(\bar{S}_n^i(\lambda)) + \hat{h}}{2}\right] \\ &\quad + \frac{1}{2}\Omega(\bar{S}_n^i(\lambda),\lambda) \\ &\triangleq f_n(\bar{S}_n^i(\lambda),\lambda), \\ \bar{S}_n^i(0) &= \tilde{S}_n^i(0), \end{aligned} \tag{43}$$

and

$$\begin{aligned} \frac{\mathrm{d}\tilde{S}_n^i}{\mathrm{d}\lambda}(\lambda) &= \tilde{K}(\tilde{S}_n^i(\lambda),\lambda)\left[Y_n - \frac{h(\tilde{S}_n^i(\lambda)) + \bar{\hat{h}}}{2}\right] \\ &\quad + \frac{1}{2}\tilde{\Omega}(\tilde{S}_n^i(\lambda),\lambda) \\ &\triangleq \tilde{f}_n(\tilde{S}_n^i(\lambda),\lambda), \end{aligned} \tag{44}$$

with $\tilde{S}_n^i(0) = \bar{S}_n^i(0)$, where $\tilde{K}$, $\tilde{\Omega}$, and $\bar{\hat{h}}$ are the approximations of K, Ω and $\hat{h}$, respectively. It is clear that the error between $\tilde{S}_n^i(1)$ and $\bar{S}_n^i(1)$ is caused by the numerical approximations of $\{K, \Omega, \hat{h}\}$ from Figure 3. The bound of the error is given in Theorem 3.1 under the following assumption.

Assumption 3.3: $f_n(S,\lambda)$ is Lipschitz with respect to (w.r.t.) S, and the error between $f_n(\tilde{S}_n^i(\lambda),\lambda)$ and its numerical approximation $\tilde{f}_n(\tilde{S}_n^i(\lambda),\lambda)$ is bounded, i.e. for $\lambda \in [0,1]$, there exist $\tilde{\gamma}_{1n}$ and $\tilde{\gamma}_{2n}$, which are functions w.r.t. Y_n, such that

- $\left|f_n(S_1,\lambda) - f_n(S_2,\lambda)\right| \leq \tilde{\gamma}_{1n}\left|S_1 - S_2\right|$, for all $S_1, S_2 \in \mathbb{R}^d$.
- $\left|f_n(\tilde{S}_n^i(\lambda),\lambda) - \tilde{f}_n(\tilde{S}_n^i(\lambda),\lambda)\right| \leq \tilde{\gamma}_{2n}$.

Remark 3.2: (i) The first inequality in Assumption 3.3 holds, if K, Kh and Ω are Lipschitz functions, and $\hat{h}$ is bounded, since

$$\begin{aligned} &f_n(\bar{S}_n^i(\lambda),\lambda) - f_n(\tilde{S}_n^i(\lambda),\lambda) \\ &\quad = \left[K(\bar{S}_n^i(\lambda),\lambda) - K(\tilde{S}_n^i(\lambda),\lambda)\right] Y_n \end{aligned}$$

$$- \frac{1}{2}\left[K(\bar{S}_n^i(\lambda),\lambda)h(\bar{S}_n^i(\lambda)) - K(\tilde{S}_n^i(\lambda),\lambda)h(\tilde{S}_n^i(\lambda))\right]$$
$$- \frac{1}{2}\left[K(\bar{S}_n^i(\lambda),\lambda) - K(\tilde{S}_n^i(\lambda),\lambda)\right]\hat{h}$$
$$+ \frac{1}{2}\left[\Omega(\bar{S}_n^i(\lambda),\lambda) - \Omega(\tilde{S}_n^i(\lambda),\lambda)\right].$$

(ii) The second inequality is satisfied, if K, Ω, $\hat{h}$ can be well numerically approximated by $\tilde{K}$, $\tilde{\Omega}$ and $\bar{\hat{h}}$, since

$$\begin{aligned} & f_n(\tilde{S}_n^i(\lambda),\lambda) - \tilde{f}_n(\tilde{S}_n^i(\lambda),\lambda) \\ &= \left[K(\tilde{S}_n^i(\lambda),\lambda) - \tilde{K}(\tilde{S}_n^i(\lambda),\lambda)\right] \\ &\quad \cdot \left[Y_n - \frac{1}{2}\left(h(\tilde{S}_n^i(\lambda)) + \hat{h}\right)\right] \\ &\quad - \frac{1}{2}\tilde{K}(\tilde{S}_n^i(\lambda),\lambda)\left[\hat{h} - \bar{\hat{h}}\right] \\ &\quad + \frac{1}{2}\left[\Omega(\tilde{S}_n^i(\lambda),\lambda) - \tilde{\Omega}(\tilde{S}_n^i(\lambda),\lambda)\right]. \end{aligned}$$

Now we can give an error bound between $\bar{S}_n^i(1)$ and $\tilde{S}_n^i(1)$.

Theorem 3.1: *If Assumption 3.3 is satisfied, then for any $p \geq 2$, we have*

$$\left|\bar{S}_n^i(1) - \tilde{S}_n^i(1)\right|^p \leq \tilde{\gamma}_{n,p}, \tag{45}$$

where $\bar{S}_n^i(1)$ and $\tilde{S}_n^i(1)$ are the solutions to (43) and (44), respectively,

$$\tilde{\gamma}_{n,p} := \left[\frac{\tilde{\gamma}_{2n}}{\tilde{\gamma}_{1n}}\left(e^{\tilde{\gamma}_{1n}} - 1\right)\right]^p, \tag{46}$$

with $\tilde{\gamma}_{1n}$ and $\tilde{\gamma}_{2n}$ are defined in Assumption 3.3.

Proof: According to (43)–(44), we have

$$\frac{\mathrm{d}(\bar{S}_n^i - \tilde{S}_n^i)}{\mathrm{d}\lambda} = f_n(\bar{S}_n^i(\lambda),\lambda) - f_n(\tilde{S}_n^i(\lambda),\lambda) + f_n(\tilde{S}_n^i(\lambda),\lambda) - \tilde{f}_n(\tilde{S}_n^i(\lambda),\lambda). \tag{47}$$

Multiplying both sides of (47) by $2(\bar{S}_n^i - \tilde{S}_n^i)^T$, we get

$$\begin{aligned} \frac{\mathrm{d}\left(\left|\bar{S}_n^i - \tilde{S}_n^i\right|^2\right)}{\mathrm{d}\lambda} &= 2(\bar{S}_n^i - \tilde{S}_n^i)^T\frac{\mathrm{d}(\bar{S}_n^i - \tilde{S}_n^i)}{\mathrm{d}\lambda} \\ &= 2(\bar{S}_n^i - \tilde{S}_n^i)^T\left(f_n(\bar{S}_n^i(\lambda),\lambda) - f_n(\tilde{S}_n^i(\lambda),\lambda)\right) \\ &\quad + 2(\bar{S}_n^i - \tilde{S}_n^i)^T\left(f_n(\tilde{S}_n^i(\lambda),\lambda) - \tilde{f}_n(\tilde{S}_n^i(\lambda),\lambda)\right) \\ &\triangleq I_1 + I_2. \end{aligned} \tag{48}$$

The first term on the right-hand side of (48) can be easily estimated as

$$\begin{aligned} |I_1| &\leq 2\left|\bar{S}_n^i - \tilde{S}_n^i\right| \cdot \left|f_n(\bar{S}_n^i(\lambda),\lambda) - f_n(\tilde{S}_n^i(\lambda),\lambda)\right| \\ &= 2\tilde{\gamma}_{1n}\left|\bar{S}_n^i - \tilde{S}_n^i\right|^2 \end{aligned} \tag{49}$$

by Cauchy–Schwarz inequality and Assumption 3.3, respectively. Meanwhile, the second term can be controlled by

$$\begin{aligned} |I_2| &\leq 2\left|\bar{S}_n^i - \tilde{S}_n^i\right| \cdot \left|f_n(\tilde{S}_n^i(\lambda),\lambda) - \tilde{f}_n(\tilde{S}_n^i(\lambda),\lambda)\right| \\ &\leq 2\left|\bar{S}_n^i - \tilde{S}_n^i\right| \cdot \tilde{\gamma}_{2n} \end{aligned} \tag{50}$$

by Assumption 3.3. Substituting (49) and (50) back to (48), then we can estimate the norm's p-th power as

$$\begin{aligned} \frac{\mathrm{d}\left(\left|\bar{S}_n^i - \tilde{S}_n^i\right|^p\right)}{\mathrm{d}\lambda} &= \frac{\mathrm{d}\left(\left|\bar{S}_n^i - \tilde{S}_n^i\right|^2\right)^{p/2}}{\mathrm{d}\lambda} \\ &= \frac{p}{2}\left(\left|\bar{S}_n^i - \tilde{S}_n^i\right|^2\right)^{p/2-1} \cdot \frac{\mathrm{d}\left(\left|\bar{S}_n^i - \tilde{S}_n^i\right|^2\right)}{\mathrm{d}\lambda} \\ &\leq p\tilde{\gamma}_{1n}\left|\bar{S}_n^i - \tilde{S}_n^i\right|^p + p\tilde{\gamma}_{2n}\left|\bar{S}_n^i - \tilde{S}_n^i\right|^{p-1}. \end{aligned} \tag{51}$$

Integrating (51) with respect to λ from 0 to 1, one has

$$\left|\bar{S}_n^i(1) - \tilde{S}_n^i(1)\right|^p \leq \int_0^1 \left[p\tilde{\gamma}_{1n}\left|\bar{S}_n^i(\lambda) - \tilde{S}_n^i(\lambda)\right|^p + p\tilde{\gamma}_{2n}\left|\bar{S}_n^i(\lambda) - \tilde{S}_n^i(\lambda)\right|^{p-1}\right]\mathrm{d}\lambda,$$

with $\bar{S}_n^i(0) = \tilde{S}_n^i(0)$. By Lemma A.1 in Appendix 3, it yields that

$$\begin{aligned} \left|\bar{S}_n^i(1) - \tilde{S}_n^i(1)\right|^p &\leq \left\{(1/p)\int_0^1 p\tilde{\gamma}_{2n} \times \exp\left[\frac{1}{p}\int_s^t p\tilde{\gamma}_{1n}\,\mathrm{d}r\right]\mathrm{d}s\right\}^{\frac{1}{1-(p-1)/p}} \\ &= \left[\frac{\tilde{\gamma}_{2n}}{\tilde{\gamma}_{1n}}\left(e^{\tilde{\gamma}_{1n}} - 1\right)\right]^p. \end{aligned}$$

■

According to Assumption 3.3, it is known that $\tilde{\gamma}_{n,p}$ is a function w.r.t. Y_n. Therefore we define

$$\gamma_{n,p} \triangleq \max\left\{\mathbb{E}\left[\tilde{\gamma}_{k,p}\right],\ k = 0, 1, 2, \ldots, n\right\}, \tag{52}$$

where $\tilde{\gamma}_{k,p}$ is defined in (46). Furthermore, we assume that $\gamma_{n,p}$ is bounded.

Assumption 3.4: $\gamma_{n,p}$ is bounded by a positive constant, i.e.

$$\gamma_{n,p} \leq \Gamma, \quad \text{for} \quad \forall\, n \geq 0, \tag{53}$$

where Γ is a positive constant.

Remark 3.3: It is known from Remark 3.2 that $\gamma_{n,p}$ indicates how well $\{K, \Omega, \hat{h}\}$ can be approximated by $\{\tilde{K}, \tilde{\Omega}, \bar{\hat{h}}\}$. If $\{K, \Omega, \hat{h}\}$

in (16) can be exactly computed without any approximation, then $\gamma_{n,p} = 0$.

Since $\|\phi\|_{n,p} \geq 1$ according to (41), we can easily see that

$$\gamma_{n,p} \leq \Gamma \|\phi\|_{n,p}. \tag{54}$$

Under the assumptions above, we state the main result of this paper here.

Theorem 3.2 (Main Theorem): *Under Assumptions 3.1–3.4, for any $\phi \in L_n^p$, for $p \geq 2$, there exist constants $A_{n\,|\,n}$, $B_{n\,|\,n}$ and $C_{n\,|\,n}$, independent of N and $\gamma_{n,p}$, such that*

$$\mathbb{E}\Big[\Big|(\tilde{\pi}^N_{n\,|\,n}, \phi) - (\pi_{n\,|\,n}, \phi)\Big|^p\Big] \leq A_{n\,|\,n}\frac{\|\phi\|^p_{n,p}}{N^{p/2}} + B_{n\,|\,n}\gamma_{n,p} + C_{n\,|\,n}(\gamma_{n,p})^{1/p}, \tag{55}$$

where $\tilde{\pi}^N_{n\,|\,n}$ defined in (35) is the empirical conditional density formed by the N particles of modified FPF in Algorithm 2, $\pi_{n\,|\,n}$ defined in (17) is the conditional posterior density of the state, $\|\phi\|_{n,p}$ is defined in (41), and $\gamma_{n,p}$ is defined in (52).

Theorem 3.2 tells us that the error between the optimal estimate $(\pi_{n\,|\,n}, \phi)$ and the numerical estimate $(\tilde{\pi}^N_{n\,|\,n}, \phi)$ by FPF can be divided into two parts. The first part is due to the fact that we can only use finite N particles in FPF, and this part of error will tend to zero as the number of particles N goes to infinity at the rate of $O(N^{-1/2})$. The second part is due to the numerical approximations of K, Ω, $\hat{h}$ in (23)–(25) in the updating step. It is clear that if $\gamma_{n,p}$ goes to zero, i.e. the approximation errors of K, Ω, $\hat{h}$ go to zero, then this part of error will tend to zero.

Corollary 3.1: *If* (1)–(2) *is a linear Gaussian system, under Assumptions 3.1–3.4, then for any $\phi \in L_n^p$, for $p \geq 2$, there exists constant $A_{n\,|\,n}$, independent of N and $\gamma_{n,p}$, such that*

$$\mathbb{E}\Big[\Big|(\tilde{\pi}^N_{n\,|\,n}, \phi) - (\pi_{n\,|\,n}, \phi)\Big|^p\Big] \leq A_{n\,|\,n}\frac{\|\phi\|^p_{n,p}}{N^{p/2}}. \tag{56}$$

We refer the interested readers to see the details in Appendix 2.

Furthermore, by the Borel-Cantelli lemma, we have a corollary as follows.

Corollary 3.2: *Under Assumptions 3.1–3.4, for any $\phi \in L_n^p$,*

$$\lim_{N\to\infty, \gamma_{n,p}\to 0} (\tilde{\pi}^N_{n\,|\,n}, \phi) = (\pi_{n\,|\,n}, \phi), \text{ a.s.} \tag{57}$$

4. Proof of the main Theorem

In this section we shall give the proof of Theorem 3.2. The proof is by induction as in Crisan and Doucet (2002). Some technical lemmas are listed in Appendix 3.

1. Initialization

Let $\{\tilde{X}^i_0\}_{i=1}^N$ be independent random variables with the same distribution $\pi_{0\,|\,0}$ (see Algorithm 2), then by (35) and (36) we have:

$$\tilde{\pi}^N_{0\,|\,0} = \frac{1}{N}\sum_{i=1}^N \delta_{\tilde{X}^i_0} \quad \Longrightarrow \quad (\tilde{\pi}^N_{0\,|\,0}, \phi) = \frac{1}{N}\sum_{i=1}^N \phi(\tilde{X}^i_0).$$

Consequently, we have

$$\begin{aligned}
&\mathbb{E}\Big[\Big|(\tilde{\pi}^N_{0\,|\,0}, \phi) - (\pi_{0\,|\,0}, \phi)\Big|^p\Big] \\
&= \frac{1}{N^p}\mathbb{E}\left[\left|\sum_{i=1}^N \left(\phi(\tilde{X}^i_0) - \mathbb{E}[\phi(\tilde{X}^i_0)]\right)\right|^p\right] \\
&\overset{(A12)}{\leq} \frac{C(p)}{N^p}\sum_{i=1}^N \mathbb{E}\left[\left|\phi(\tilde{X}^i_0) - \mathbb{E}[\phi(\tilde{X}^i_0)]\right|^p\right] \\
&\quad + \frac{C(p)}{N^p}\left(\sum_{i=1}^N \mathbb{E}\left[\left|\phi(\tilde{X}^i_0) - \mathbb{E}[\phi(\tilde{X}^i_0)]\right|^2\right]\right)^{p/2} \\
&\overset{(A13)}{\leq} \frac{2^p C(p)}{N^{p-1}}\mathbb{E}\left[|\phi(\tilde{X}^i_0)|^p\right] \\
&\quad + \frac{2^p C(p)}{N^{p/2}}\mathbb{E}^{p/2}\left[|\phi(\tilde{X}^i_0)|^2\right] \\
&\overset{(A14)}{\leq} \frac{2^{p+1} C(p)}{N^{p/2}}\mathbb{E}\left[|\phi(\tilde{X}^i_0)|^p\right] \\
&\triangleq A_{0\,|\,0}\frac{\|\phi\|^p_{0,p}}{N^{p/2}} + B_{0\,|\,0}\gamma_{0,p} + C_{0\,|\,0}(\gamma_{0,p})^{1/p},
\end{aligned} \tag{58}$$

where $A_{0\,|\,0} = 2^{p+1}C(p), B_{0\,|\,0} = C_{0\,|\,0} = 0$, and $C(p)$ is a constant that depends only on p.

With the similar argument as in (58), we deduce that

$$\begin{aligned}
&\mathbb{E}\Big[\Big|(\tilde{\pi}^N_{0\,|\,0}, |\phi|^p) - (\pi_{0\,|\,0}, |\phi|^p)\Big|\Big] \\
&= \frac{1}{N}\mathbb{E}\left[\left|\sum_{i=1}^N \left(|\phi(\tilde{X}^i_0)|^p - \mathbb{E}[|\phi(\tilde{X}^i_0)|^p]\right)\right|\right] \leq 2\mathbb{E}[|\phi(\tilde{X}^i_0)|^p].
\end{aligned}$$

Note that $\tilde{X}^i_0$ have the same distribution for all i, so the expectation does not depend on i. Hence, one has

$$\begin{aligned}
\mathbb{E}\Big[\Big|(\tilde{\pi}^N_{0\,|\,0}, |\phi|^p)\Big|\Big] &\leq \mathbb{E}\Big[\Big|(\tilde{\pi}^N_{0\,|\,0}, |\phi|^p) - (\pi_{0\,|\,0}, |\phi|^p)\Big|\Big] \\
&\quad + \mathbb{E}\Big[\Big|(\pi_{0\,|\,0}, |\phi|^p)\Big|\Big] \\
&\leq 3\mathbb{E}[|\phi(\tilde{X}^i_0)|^p] \triangleq M_{0\,|\,0}\|\phi\|^p_{0,p},
\end{aligned}$$

where $M_{0\,|\,0} = 3$, i.e.

$$\mathbb{E}\Big[\Big|(\tilde{\pi}^N_{0\,|\,0}, |\phi|^p)\Big|\Big] \leq M_{0\,|\,0}\|\phi\|^p_{0,p}. \tag{59}$$

2. Prediction

Based on (58) and (59), by induction, we assume that for $n-1$, for $n \geq 1$, there exist the generic nonnegative constants $A_{n-1\,|\,n-1}$, $B_{n-1\,|\,n-1}$, $C_{n-1\,|\,n-1}$, and $M_{n-1\,|\,n-1}$, independent of N and $\gamma_{n-1,p}$, such that

$$\mathbb{E}\Big[\Big|(\tilde{\pi}^N_{n-1\,|\,n-1}, \phi) - (\pi_{n-1\,|\,n-1}, \phi)\Big|^p\Big]$$

$$\leq A_{n-1\,|\,n-1}\frac{\|\phi\|_{n-1,p}^p}{N^{p/2}} + \left(B_{n-1\,|\,n-1}\gamma_{n-1,p} + C_{n-1\,|\,n-1}(\gamma_{n-1,p})^{1/p}\right) \tag{60}$$

and

$$\mathbb{E}\left[\left|(\tilde{\pi}^N_{n-1\,|\,n-1}, |\phi|^p)\right|\right] \leq M_{n-1\,|\,n-1}\|\phi\|_{n-1,p}^p \tag{61}$$

hold for all $\phi \in L^p_n$.

2.1 We look at $\mathbb{E}[|(\tilde{\pi}^N_{n\,|\,n-1}, \phi) - (\pi_{n\,|\,n-1}, \phi)|^p]$.

Let $\mathcal{G}_{n-1}$ be the σ-algebra generated by $\{\tilde{X}^i_{t_{n-1}},\ i = 1, \ldots N\}$, i.e.

$$\mathcal{G}_{n-1} \triangleq \sigma\{\tilde{X}^i_{t_{n-1}},\ i = 1, \ldots N\}. \tag{62}$$

Notice that

$$\begin{aligned} I &\triangleq (\tilde{\pi}^N_{n\,|\,n-1}, \phi) - (\pi_{n\,|\,n-1}, \phi) \\ &= (\tilde{\pi}^N_{n\,|\,n-1}, \phi) - (\tilde{\pi}^N_{n-1\,|\,n-1}, \kappa_1\phi) \\ &\quad + (\tilde{\pi}^N_{n-1\,|\,n-1}, \kappa_1\phi) - (\pi_{n\,|\,n-1}, \phi) \\ &\triangleq I_1 + I_2. \end{aligned} \tag{63}$$

In the sequel, we shall estimate $\mathbb{E}[|I_1|^p]$ and $\mathbb{E}[|I_2|^p]$, respectively.

As for I_1, according to Algorithm 2, we know that $\tilde{X}^i_{t_n^-} \sim (1/N)\sum_{i=1}^N \kappa_1(\mathrm{d}x_{t_n}\,|\,\tilde{X}^i_{t_{n-1}}) = \tilde{\pi}^N_{n-1\,|\,n-1}\kappa_1(\mathrm{d}x_{t_n})$.[2] It can be easily checked that

$$\begin{aligned} (\tilde{\pi}^N_{n-1\,|\,n-1}, \kappa_1\phi) &= \int_{R^d} \kappa_1\phi(x)\tilde{\pi}^N_{n-1\,|\,n-1}\,\mathrm{d}x \\ &= \int_{R^d}\int_{R^d} \kappa_1(\mathrm{d}z\,|\,x)\phi(z)\tilde{\pi}^N_{n-1\,|\,n-1}(\mathrm{d}x) \\ &= \frac{1}{N}\sum_{i=1}^N \int_{R^d} \kappa_1(\mathrm{d}z\,|\,\tilde{X}^i_{t_{n-1}})\phi(z), \end{aligned}$$

and

$$\mathbb{E}\left[\phi(\tilde{X}^i_{t_n^-})\,\middle|\,\mathcal{G}_{n-1}\right] = \frac{1}{N}\sum_{i=1}^N \int_{R^d} \kappa_1(\mathrm{d}z\,|\,\tilde{X}^i_{t_{n-1}})\phi(z).$$

Then we have

$$\mathbb{E}\left[\phi(\tilde{X}^i_{t_n^-})\,\middle|\,\mathcal{G}_{n-1}\right] = (\tilde{\pi}^N_{n-1\,|\,n-1}, \kappa_1\phi). \tag{64}$$

It follows that

$$I_1 = \frac{1}{N}\sum_{i=1}^N \left[\phi(\tilde{X}^i_{t_n^-}) - \mathbb{E}[\phi(\tilde{X}^i_{t_n^-})\,|\,\mathcal{G}_{n-1}]\right]. \tag{65}$$

Then similarly as argued in (58), we have

$$\begin{aligned} &\mathbb{E}\left[|I_1|^p\,\middle|\,\mathcal{G}_{n-1}\right] \\ &= \frac{1}{N^p}\mathbb{E}\left[\left|\sum_{i=1}^N\left[\phi(\tilde{X}^i_{t_n^-}) - \mathbb{E}\left[\phi(\tilde{X}^i_{t_n^-})\,\middle|\,\mathcal{G}_{n-1}\right]\right]\right|^p\,\middle|\,\mathcal{G}_{n-1}\right] \\ &\overset{(A12)--(A14)}{\leq} \frac{2^pC(p)}{N^{p-1}}(\tilde{\pi}^N_{n-1\,|\,n-1}, \kappa_1|\phi|^p) \\ &\quad + \frac{2^pC(p)}{N^{p/2}}(\tilde{\pi}^N_{n-1\,|\,n-1}, \kappa_1|\phi|^p) \\ &\leq \frac{2^{p+1}C(p)}{N^{p/2}}(\tilde{\pi}^N_{n-1\,|\,n-1}, \kappa_1|\phi|^p), \end{aligned} \tag{66}$$

where $C(p)$ is a constant that depends only on p.

It can be easily checked that

$$\mathbb{E}\left[(\pi_{k\,|\,k}, \kappa_1|\phi|^p)\right] = \mathbb{E}\left[\kappa_1|\phi|^p(X_{t_k})\right] = \mathbb{E}\left[|\phi|^p(X_{t_{k+1}})\right], \tag{67}$$

then it follows that

$$\begin{aligned} &\|(\kappa_1|\phi|^p)^{1/p}\|_{n-1,p}^p \\ &\overset{(41)}{=} \max_{k=0,1,2,\ldots,n-1}\{1, \mathbb{E}\left[(\pi_{k\,|\,k}, \kappa_1|\phi|^p)\right]\} \\ &= \max_{k=1,2,\ldots,n}\{1, \mathbb{E}\left[(\pi_{k\,|\,k}, |\phi|^p)\right]\} \leq \|\phi\|_{n,p}^p. \end{aligned} \tag{68}$$

Now we obtain the estimate of $\mathbb{E}[|I_1|^p]$ by the tower property of the conditional expectation

$$\begin{aligned} \mathbb{E}\left[|I_1|^p\right] &= \mathbb{E}\left[\mathbb{E}[\,|I_1|^p\,|\,\mathcal{G}_{n-1}]\right] \\ &\overset{(66)(61)}{\leq} \frac{2^{p+1}C(p)}{N^{p/2}}M_{n-1\,|\,n-1}\|(\kappa_1|\phi|^p)^{1/p}\|_{n-1,p}^p \\ &\overset{(68)}{\leq} 2^{p+1}C(p)M_{n-1\,|\,n-1}\frac{\|\phi\|_{n,p}^p}{N^{p/2}}. \end{aligned} \tag{69}$$

As for I_2,

$$\begin{aligned} \mathbb{E}\left[|I_2|^p\right] &\overset{(37)}{=} \mathbb{E}\left[\left|(\tilde{\pi}^N_{n-1\,|\,n-1}, \kappa_1\phi) - (\pi_{n-1\,|\,n-1}, \kappa_1\phi)\right|^p\right] \\ &\overset{(60)}{\leq} A_{n-1\,|\,n-1}\frac{\|\kappa_1\phi\|_{n-1,p}^p}{N^{p/2}} \\ &\quad + \left(B_{n-1\,|\,n-1}\gamma_{n-1,p} + C_{n-1\,|\,n-1}(\gamma_{n-1,p})^{1/p}\right) \\ &\leq A_{n-1\,|\,n-1}\frac{\|\phi\|_{n,p}^p}{N^{p/2}} + B_{n-1\,|\,n-1}\gamma_{n-1,p} \\ &\quad + C_{n-1\,|\,n-1}(\gamma_{n-1,p})^{1/p}, \end{aligned} \tag{70}$$

where the last inequality is due to the fact that

$$\begin{aligned} \|\kappa_1\phi\|_{n-1,p}^p &\overset{(41)}{=} \max_{k=0,1,2,\ldots,n-1}\{1, \mathbb{E}\left[(\pi_{k\,|\,k}, |\kappa_1\phi|^p)\right]\} \\ &\leq \max_{k=1,2,\ldots,n}\{1, \mathbb{E}\left[|\phi(X_{t_{k+1}})|^p\right]\} \leq \|\phi\|_{n,p}^p, \end{aligned} \tag{71}$$

since

$$\begin{aligned} \mathbb{E}\left[(\pi_{k\,|\,k}, |\kappa_1\phi|^p)\right] &= \int_{R^d} P(X_{t_k} \in \mathrm{d}x)\left|\int_{R^d}\kappa_1(\,\mathrm{d}z\,|\,x)\phi(z)\right|^p \\ &\leq \int_{R^d} P(X_{t_k} \in \mathrm{d}x)\int_{R^d}\kappa_1(\mathrm{d}z\,|\,x)|\phi(z)|^p \\ &= \mathbb{E}\left[|\phi(X_{t_{k+1}})|^p\right], \end{aligned}$$

where the first equality is due to the tower property of the conditional expectation.

Combining (63), (69) and (70), we have

$$\mathbb{E}\left[\left|(\tilde{\pi}^N_{n\,|\,n-1}, \phi) - (\pi_{n\,|\,n-1}, \phi)\right|^p\right]$$

$$
\begin{aligned}
&\le 2^{p-1}\left(\mathbb{E}\left[|I_1|^p\right]+\mathbb{E}\left[|I_2|^p\right]\right)\\
&\overset{(69),(70)}{\le} 2^{p-1}(2^{p+1}C(p)M_{n-1\,|\,n-1}+A_{n-1\,|\,n-1})\frac{\|\phi\|_{n,p}^p}{N^{p/2}}\\
&\quad + 2^{p-1}B_{n-1\,|\,n-1}\gamma_{n-1,p}+2^{p-1}C_{n-1\,|\,n-1}(\gamma_{n-1,p})^{1/p}\\
&:= A_{n\,|\,n-1}\frac{\|\phi\|_{n,p}^p}{N^{p/2}}+B_{n\,|\,n-1}\gamma_{n-1,p}+C_{n\,|\,n-1}(\gamma_{n-1,p})^{1/p},
\end{aligned}
\tag{72}
$$

by Minkowski inequality and Jensen's inequality, where

$$
A_{n\,|\,n-1}=2^{p-1}(2^{p+1}C(p)M_{n-1\,|\,n-1}+A_{n-1\,|\,n-1}),
$$
$$
B_{n\,|\,n-1}=2^{p-1}B_{n-1\,|\,n-1},\ C_{n\,|\,n-1}=2^{p-1}C_{n-1\,|\,n-1}.
$$

2.2 We analyze $\mathbb{E}[|(\tilde{\pi}^N_{n\,|\,n-1},|\phi|^p)|]$.
Similarly as in (63), we have

$$
\begin{aligned}
&(\tilde{\pi}^N_{n\,|\,n-1},|\phi|^p)-(\pi_{n\,|\,n-1},|\phi|^p)\\
&\quad=(\tilde{\pi}^N_{n\,|\,n-1},|\phi|^p)-(\tilde{\pi}^N_{n-1\,|\,n-1},\kappa_1|\phi|^p)\\
&\qquad+(\tilde{\pi}^N_{n-1\,|\,n-1},\kappa_1|\phi|^p)-(\pi_{n\,|\,n-1},|\phi|^p)\triangleq \tilde{I}_1+\tilde{I}_2.
\end{aligned}
\tag{73}
$$

As for $\tilde{I}_1$, we have

$$
\begin{aligned}
\tilde{I}_1&=(\tilde{\pi}^N_{n\,|\,n-1},|\phi|^p)-(\tilde{\pi}^N_{n-1\,|\,n-1},\kappa_1|\phi|^p)\\
&=\frac{1}{N}\sum_{i=1}^{N}\left[|\phi(\tilde{X}^i_{t_n^-})|^p-\mathbb{E}\left[|\phi(\tilde{X}^i_{t_n^-})|^p\,\Big|\,\mathcal{G}_{n-1}\right]\right].
\end{aligned}
$$

Taking conditional expectation of $|\tilde{I}_1|$ with respect to $\mathcal{G}_{n-1}$, one has

$$
\begin{aligned}
\mathbb{E}\left[|\tilde{I}_1|\,\big|\,\mathcal{G}_{n-1}\right]&\le\frac{2}{N}\sum_{i=1}^{N}\mathbb{E}\left[|\phi(\tilde{X}^i_{t_n^-})|^p\,\Big|\,\mathcal{G}_{n-1}\right]\\
&\overset{(64)}{=}2(\tilde{\pi}^N_{n-1\,|\,n-1},\kappa_1|\phi|^p).
\end{aligned}
$$

Consequently, we have

$$
\mathbb{E}\left[|\tilde{I}_1|\right]=\mathbb{E}\left[\mathbb{E}\left[|\tilde{I}_1|\,\big|\,\mathcal{G}_{n-1}\right]\right]\overset{(61),(68)}{\le}2M_{n-1\,|\,n-1}\|\phi\|^p_{n,p},
\tag{74}
$$

by tower property.

As for $\tilde{I}_2$, one obtain that

$$
\begin{aligned}
\mathbb{E}\left[|\tilde{I}_2|\right]&\overset{(37),(73)}{=}\mathbb{E}\left[\left|(\tilde{\pi}^N_{n-1\,|\,n-1},\kappa_1|\phi|^p)-(\pi_{n-1\,|\,n-1},\kappa_1|\phi|^p)\right|\right]\\
&\overset{(41),(61)}{\le}\left(M_{n-1\,|\,n-1}+1\right)\|(\kappa_1|\phi|^p)^{1/p}\|^p_{n-1,p}\\
&\overset{(68)}{\le}\left(M_{n-1\,|\,n-1}+1\right)\|\phi\|^p_{n,p}.
\end{aligned}
\tag{75}
$$

Therefore, we have

$$
\begin{aligned}
&\mathbb{E}\left[\left|(\tilde{\pi}^N_{n\,|\,n-1},|\phi|^p)\right|\right]\\
&\overset{(37)}{\le}\mathbb{E}\left[\left|(\tilde{\pi}^N_{n\,|\,n-1},|\phi|^p)-(\pi_{n\,|\,n-1},|\phi|^p)\right|\right]\\
&\quad+\mathbb{E}\left[\left|(\pi_{n-1\,|\,n-1},\kappa_1|\phi|^p)\right|\right]\le M_{n\,|\,n-1}\|\phi\|^p_{n,p},
\end{aligned}
\tag{76}
$$

where $M_{n\,|\,n-1}:=3M_{n-1\,|\,n-1}+2$, and the last inequality is due to (61), (71), and (73)–(75).

3. Updating

In this step we analyze $\mathbb{E}[|(\tilde{\pi}^N_{n\,|\,n},\phi)-(\pi_{n\,|\,n},\phi)|^p]$ and $\mathbb{E}[|(\tilde{\pi}^N_{n\,|\,n},|\phi|^p)|]$ from the apriori estimation (72) and (76).

3.1 We analyze $\mathbb{E}[|(\tilde{\pi}^N_{n\,|\,n},\phi)-(\pi_{n\,|\,n},\phi)|^p]$.

By (38) we have

$$
\begin{aligned}
\Pi&\triangleq(\tilde{\pi}^N_{n\,|\,n},\phi)-(\pi_{n\,|\,n},\phi)\\
&=(\tilde{\pi}^N_{n\,|\,n},\phi)-\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[\phi(\tilde{S}^i_n(1))\,\Big|\,\mathcal{G}_{n0}\right]\\
&\quad+\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[\phi(\tilde{S}^i_n(1))\,\Big|\,\mathcal{G}_{n0}\right]-\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[\phi(\bar{S}^i_n(1))\,\big|\,\mathcal{G}_{n0}\right]\\
&\quad+\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[\phi(\bar{S}^i_n(1))\,\big|\,\mathcal{G}_{n0}\right]-(\tilde{\pi}^N_{n\,|\,n-1},\kappa_{2n}\phi)\\
&\quad+(\tilde{\pi}^N_{n\,|\,n-1},\kappa_{2n}\phi)-(\pi_{n\,|\,n-1},\kappa_{2n}\phi)\\
&\triangleq\Pi_1+\Pi_2+\Pi_3+\Pi_4,
\end{aligned}
\tag{77}
$$

where $\mathcal{G}_{n0}:=\sigma(\{\tilde{S}^i_n(0)\},Y_n)$, apparently $\mathcal{G}_n\subset\mathcal{G}_{n0}$ with $\mathcal{G}_n$ defined in (62). $\bar{S}^i_n(1)$ denotes the solution of (43) at $\lambda=1$ with initial value $\tilde{S}^i_n(0)$ for every i. It can be seen that both $\tilde{S}^i_n(1)$ and $\bar{S}^i_n(1)$ are measurable w.r.t. $\mathcal{G}_{n0}$ by (43) and (44).

As for Π_1, we have

$$
\Pi_1=\frac{1}{N}\sum_{i=1}^{N}\left(\phi(\tilde{S}^i_n(1))-\mathbb{E}\left[\phi(\tilde{S}^i_n(1))\,\Big|\,\mathcal{G}_{n0}\right]\right)=0,
\tag{78}
$$

where the last equation is due to the fact that $\tilde{S}^i_n(1), 1\le i\le N$ are measurable w.r.t. $\mathcal{G}_{n0}$.

As for Π_2, we have

$$
\begin{aligned}
\Pi_2&=\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[\phi(\tilde{S}^i_n(1))\,\Big|\,\mathcal{G}_{n0}\right]-\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[\phi(\bar{S}^i_n(1))\,\big|\,\mathcal{G}_{n0}\right]\\
&=\frac{1}{N}\sum_{i=1}^{N}\left(\phi(\tilde{S}^i_n(1))-\phi(\bar{S}^i_n(1))\right).
\end{aligned}
\tag{79}
$$

To estimate $\mathbb{E}[|\Pi|^p]$ at the end, we take the conditional expectation of $|\Pi_2|^p$ first. Since both $\tilde{S}^i_n(1)$ and $\bar{S}^i_n(1)$ are measurable w.r.t. $\mathcal{G}_{n0}$, we have

$$
\begin{aligned}
\mathbb{E}\left[|\Pi_2|^p\,\big|\,\mathcal{G}_{n0}\right]&=\left|\frac{1}{N}\sum_{i=1}^{N}\left(\phi(\tilde{S}^i_n(1))-\phi(\bar{S}^i_n(1))\right)\right|^p\\
&\le\frac{1}{N}\sum_{i=1}^{N}C^p_\phi\left|\bar{S}^i_n(1)-\tilde{S}^i_n(1)\right|^p,
\end{aligned}
\tag{80}
$$

where the inequality is due to Jensen's inequality and Assumption 3.2. By the tower property of the conditional expectation, we have

$$
\mathbb{E}[|\Pi_2|^p]=\mathbb{E}\left[\mathbb{E}\left[|\Pi_2|^p\,\big|\,\mathcal{G}_{n0}\right]\right]
$$

$$\overset{(45)}{\leq} \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[C_\phi^p\left|\bar{S}_n^i(1)-\tilde{S}_n^i(1)\right|^p\right]$$
$$\overset{(52)}{\leq} C_\phi^p\gamma_{n,p}. \tag{81}$$

As for Π_3, let us look at the second term in Π_3 first,

$$\begin{aligned}(\tilde{\pi}_{n\,|\,n-1}^N,\kappa_{2n}\phi) &= \frac{1}{N}\sum_{i=1}^{N}\kappa_{2n}\phi(\tilde{S}_n^i(0))\\
&= \frac{1}{N}\sum_{i=1}^{N}\int_{R^d}\kappa_{2n}(\mathrm{d}s|\tilde{S}_n^i(0),Y_n)\phi(s)\\
&\overset{(82)}{=} \frac{1}{N}\sum_{i=1}^{N}\int_{R^d}\delta_{s=\bar{S}_n^i(1)}\phi(s)\,\mathrm{d}s\\
&= \frac{1}{N}\sum_{i=1}^{N}\phi(\bar{S}_n^i(1)).\end{aligned} \tag{82}$$

Thus, we have

$$\begin{aligned}\Pi_3 &= \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[\phi(\bar{S}_n^i(1))\middle|\,\mathcal{G}_{n0}\right]-(\tilde{\pi}_{n\,|\,n-1}^N,\kappa_{2n}\phi)\\
&\overset{(82)}{=} \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[\phi(\bar{S}_n^i(1))\middle|\,\mathcal{G}_{n0}\right]-\frac{1}{N}\sum_{i=1}^{N}\phi(\bar{S}_n^i(1))=0,\end{aligned} \tag{83}$$

where the last equality holds due to $\bar{S}_n^i(1)$ is $\mathcal{G}_{n0}$- measurable.

As for Π_4, one has

$$\begin{aligned}\mathbb{E}[|\Pi_4|^p] &= \mathbb{E}[|(\tilde{\pi}_{n\,|\,n-1}^N,\kappa_{2n}\phi)-(\pi_{n\,|\,n-1},\kappa_{2n}\phi)|^p]\\
&\overset{(72)}{\leq} A_{n\,|\,n-1}\bar{C}_n\frac{\|\phi\|_{n,p}^p}{N^{p/2}}+B_{n\,|\,n-1}\gamma_{n-1,p}\\
&\quad + C_{n\,|\,n-1}(\gamma_{n-1,p})^{1/p},\end{aligned} \tag{84}$$

where $\bar{C}_n=\max\{1/C_k,\ 1\le k\le n\}$, and C_k is the normalization constant in (10). This is because

$$\begin{aligned}&\left(\pi_{k\,|\,k},|\kappa_{2n}\phi|^p\right)\\
&\overset{(10)}{\leq}\left(\pi_{k\,|\,k-1},\kappa_{2n}\,|\phi|^p\right)/C_k\\
&\overset{(18)}{=}\int_{R^d}P(S_k^i(0)\in\,\mathrm{d}x\,|\,\mathscr{F}_{t_{k-1}})\\
&\quad\cdot\int_{R^d}P(S_k^i(1)\in\,\mathrm{d}z|S_k^i(0)=x,Y_k)\,|\phi(z)|^p\,/C_k\\
&=\int_{R^d}\int_{R^d}P(S_k^i(0)\in\,\mathrm{d}x\,|\,\mathscr{F}_{t_{k-1}})\\
&\quad\times P(S_k^i(1)\in\,\mathrm{d}z\,|\,S_k^i(0)=x,Y_k)\cdot|\phi(z)|^p\,/C_k\\
&\overset{(32)}{=}\int_{R^d}\int_{R^d}P(S_k^i(0)\in\,\mathrm{d}x\,|\,\mathscr{F}_{t_k})\\
&\quad\times P(S_k^i(1)=\mathrm{d}z\,|\,S_k^i(0)=x,\mathscr{F}_{t_k})\cdot|\phi(z)|^p\,/C_k\\
&=\int_{R^d}P(S_k^i(1)=\mathrm{d}z\,|\,\mathscr{F}_{t_k})\,|\phi(z)|^p\,/C_k=\left(\pi_{k\,|\,k},|\phi|^p\right)/C_k,\end{aligned} \tag{85}$$

where the first and last equalities follow from Theorem 2.1.

Similarly as in (72),

$$\begin{aligned}&\mathbb{E}\left[\left|(\tilde{\pi}_{n\,|\,n}^N,\phi)-(\pi_{n\,|\,n},\phi)\right|^p\right]\\
&\overset{(77),(78),(83)}{=}\mathbb{E}\left[|\Pi_2+\Pi_4|^p\right]\le 2^{p-1}\left(\mathbb{E}\left[|\Pi_2|^p\right]+\mathbb{E}\left[|\Pi_4|^p\right]\right)\\
&\overset{(81),(84)}{\leq} 2^{p-1}\left(A_{n\,|\,n-1}\bar{C}_n\frac{\|\phi\|_{n,p}^p}{N^{p/2}}+(C_\phi^p+B_{n\,|\,n-1})\gamma_{n,p}\right.\\
&\quad\left.+\,C_{n\,|\,n-1}(\gamma_{n,p})^{1/p}\right)\\
&\triangleq A_{n\,|\,n}\frac{\|\phi\|_{n,p}^p}{N^{p/2}}+B_{n\,|\,n}\gamma_{n,p}+C_{n\,|\,n}(\gamma_{n,p})^{1/p},\end{aligned} \tag{86}$$

where $A_{n\,|\,n}=2^{p-1}A_{n\,|\,n-1}\bar{C}_n$, $B_{n\,|\,n}=2^{p-1}(C_\phi^p+B_{n\,|\,n-1})$, $C_{n\,|\,n}=2^{p-1}C_{n\,|\,n-1}$.

3.2 As for the $\mathbb{E}[|(\tilde{\pi}_{n\,|\,n}^N,|\phi|^p)|]$, firstly we can get

$$\begin{aligned}\tilde{\Pi}&\triangleq(\tilde{\pi}_{n\,|\,n}^N,|\phi|^p)-(\pi_{n\,|\,n},|\phi|^p)\\
&=(\tilde{\pi}_{n\,|\,n}^N,|\phi|^p)-\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[|\phi(\tilde{S}_n^i(1))|^p\middle|\,\mathcal{G}_{n0}\right]\\
&\quad+\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[|\phi(\tilde{S}_n^i(1))|^p\middle|\,\mathcal{G}_{n0}\right]\\
&\quad-\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[|\phi(\bar{S}_n^i(1))|^p\middle|\,\mathcal{G}_{n0}\right]\\
&\quad+\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[|\phi(\bar{S}_n^i(1))|^p\middle|\,\mathcal{G}_{n0}\right]-(\tilde{\pi}_{n\,|\,n-1}^N,\kappa_{2n}|\phi|^p)\\
&\quad+(\tilde{\pi}_{n\,|\,n-1}^N,\kappa_{2n}|\phi|^p)-(\pi_{n\,|\,n-1},\kappa_{2n}|\phi|^p)\\
&\triangleq\tilde{\Pi}_1+\tilde{\Pi}_2+\tilde{\Pi}_3+\tilde{\Pi}_4.\end{aligned} \tag{87}$$

Using the similar procedures as for Π_1, Π_3 and Π_4, we can obtain that

$$\tilde{\Pi}_1=0,\quad \tilde{\Pi}_3=0, \tag{88}$$

and

$$\mathbb{E}\left[\left|\tilde{\Pi}_4\right|\right]\overset{(76),(85)}{\leq}M_{n\,|\,n-1}\bar{C}_n\|\phi\|_{n,p}^p. \tag{89}$$

Now we only need to deal with $\tilde{\Pi}_2$.

$$\begin{aligned}\mathbb{E}\left[\left|\tilde{\Pi}_2\right|\right]&\le\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[\left||\phi(\tilde{S}_n^i(1))|^p-|\phi(\bar{S}_n^i(1))|^p\right|\right]\\
&\le\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[C_{\phi,p}\left|\bar{S}_n^i(1)-\tilde{S}_n^i(1)\right|\right]\\
&\le C_{\phi,p}(\gamma_{n,p})^{1/p},\end{aligned} \tag{90}$$

where the last inequality follows from Theorem 3.1.

It follows that

$$\mathbb{E}\left[\left|\tilde{\Pi}\right|\right]\overset{(87),(88)}{\leq}\mathbb{E}\left[\left|\tilde{\Pi}_2\right|\right]+\mathbb{E}\left[\left|\tilde{\Pi}_4\right|\right]$$

$$\overset{(89),(90)}{\leq} M_{n\,|\,n-1}\bar{C}_n\|\phi\|^p_{n,p} + C_{\phi,p}(\gamma_{n,p})^{1/p}$$
$$\leq M_{n\,|\,n-1}\bar{C}_n\|\phi\|^p_{n,p} + C_{\phi,p}(\Gamma_n)^{1/p}\|\phi\|^p_{n,p}, \qquad (91)$$

where the last inequality is due to the fact that $\|\phi\|^p_{n,p} \geq 1$ and $\gamma_{n,p} \leq \Gamma$ by (41) and Assumption 3.4.

Therefore we have

$$\mathbb{E}\left[\left|(\tilde{\pi}^N_{n\,|\,n}, |\phi|^p)\right|\right] \overset{(87),(91)}{\leq} \mathbb{E}\left[\left|(\tilde{\pi}^N_{n\,|\,n}, |\phi|^p) - (\pi_{n\,|\,n}, |\phi|^p)\right|\right] + \mathbb{E}\left[\left|(\pi_{n\,|\,n}, |\phi|^p)\right|\right] \leq M_{n\,|\,n}\|\phi\|^p_{n,p},$$

where $M_{n\,|\,n} = (M_{n\,|\,n-1}\bar{C}_n + C_{\phi,p}(\Gamma)^{1/p}) + 1$, and the last inequality is due to the fact that $\|\phi\|^p_{n,p}$ is non-decreasing w.r.t. n.

In summary, (55) holds for all $n \geq 0$ by induction. ■

5. Simulation

In this section, we verify our theoretic result by a benchmark numerical experiment (Yang et al., 2014). Let us consider the following linear continuous-discrete filtering system:

$$\begin{cases} \mathrm{d}X_t = AX_t\,\mathrm{d}t + \mathrm{d}B_t, \\ Y_n = HX_{t_n} + \sigma_W W_n, \end{cases} \qquad (92)$$

where $A = -0.5$, $H = 3$, $\sigma_B = 1$, $\sigma_W = 2$, $\{B_t\}$ is the standard Brownian motion process, $\{W_n\}$ is the standard Gaussian white noise, and $\{B_t\}$, $\{W_n\}$ are mutually independent. Discrete observations are available at time $t_n = 0.5, 1.0, \ldots, 10$.

To compare the performance, we introduce the MSE based on 500 realizations which is defined as follows:

$$\mathrm{MSE} := \frac{1}{500}\sum_{i=1}^{500}\frac{1}{N_t+1}\sum_{n=0}^{N_t}\left(X^{(i)}_{t_n} - \hat{X}^{(i)}_{t_n}\right)^2, \qquad (93)$$

where $X^{(i)}_{t_n}$ is the real state at instant t_n in the i-th experiment and $\hat{X}^{(i)}_{t_n}$ is the estimation of $X^{(i)}_{t_n}$, with $0 \leq t_n \leq 10$, $0 \leq n \leq N_t = 100$. We use Euler's method in time discretization with the same time step of 0.05 s in both t and λ.

Here N is the number of particles. Firstly, we use $N = 100$ particles in FPF and modified FPF. The estimation results of KF, FPF and modified FPF in one trail is displayed in Figure 4. It can be seen that both FPF and modified FPF have a well performance as that of optimal KF. The average running times of KF, FPF and modified FPF with 500 simulations are 0.0016, 0.0034 and 0.0031, respectively. Apparently, these three algorithms can track the real state very fast and can be implemented in a realtime manner (Luo & Yau, 2013a).

Furthermore, to investigate the performance of modified FPF with different number of particles, we choose the number of particles ranging from 20 to 110. As suggested by our theoretical result (56) with $p = 2$, the MSE of modified FPF is bounded by $O(1/N)$. We plot the MSE of modified FPF with different number of particles in Figure 5. It is shown that, when the number of particles N is relatively small, the error is approximately linear with $1/N$. But we cannot reduce the error further by increasing the number of particles, since the MSE has a lower bound (Jazwinski, 1970).

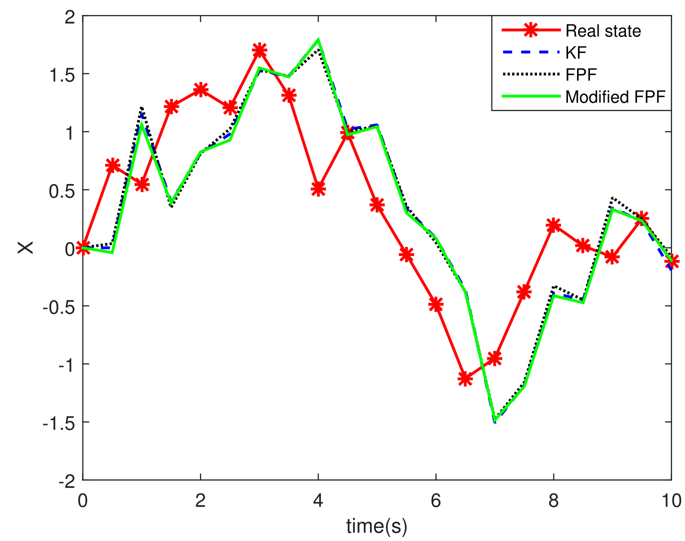

Figure 4. The estimation results of different methods with $N = 100$.

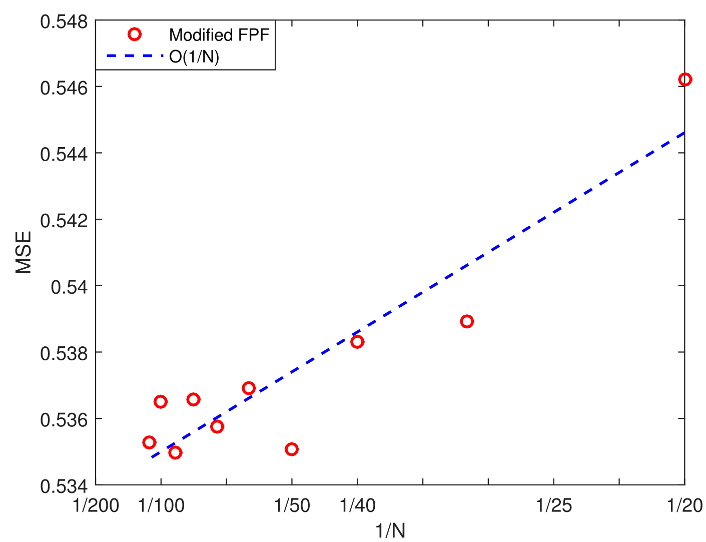

Figure 5. The relationship between MSE and $1/N$.

6. Conclusion

In this paper, we investigate the convergence of FPF. For some technical reason, we modified the standard FPF in predicting step, called modified FPF in our paper. Under some assumptions, we prove that, for a class of function ϕ, the estimate given by the modified FPF converges to its optimal estimate as the number of the particles tends to infinity and the approximation errors of $\{K, \ \Omega, \ \hat{h}\}$ go to zero. The estimate error is controlled by two factors, the one is the number of particles, and the other one is from the numerical error in computing the control input in updating step. The theoretical results have also been verified by a benchmark numerical experiment.

Notes

1. $\kappa(\cdot,\cdot) : (\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d)) \to [0,1]$ is a Markov transition kernel on $\mathscr{B}(\mathbb{R}^d)$ if, for any $x \in \mathbb{R}^d$, $\kappa(\cdot\,|\,x)$ is a probability measure and, for any $A \in \mathscr{B}(\mathbb{R}^d)$, $\kappa(A\,|\,\cdot)$ is a measurable function.

2. According to the standard FPF, one samples from $\check{X}^i_{t_n^-} \sim \kappa_1(dx_{t_n}|\check{X}^i_{t_{n-1}})$. Therefore, $\{\check{X}^i_{t_n^-}\}$ have different distributions, for each $i = 1, 2, \ldots, N$, but $\tilde{X}^i_{t_n^-} \sim \frac{1}{N}\sum_{i=1}^N \kappa_1(dx_{t_n}|\tilde{X}^i_{t_{n-1}})$ in modified FPF are i.i.d..

Disclosure statement

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Appendices

Appendix 1. About Theorem 2.1

Let us take a logarithm of both sides of (10):

$$\ln p^*(x, t_n) = \ln p^*(x, t_n^-) + \left[h(x)^T(Y_n - \frac{1}{2}h(x))\right] - \ln C_n', \tag{A1}$$

where C_n' is a constant that does not depend on x, and this constant can be dropped to obtain the recursion for the unnormalized density $q^*(t, x)$:

$$\ln q^*(x, t_n) = \ln q^*(x, t_n^-) + \left[h(x)^T(Y_n - \frac{1}{2}h(x))\right], \tag{A2}$$

where

$$p^*(x, t_n) = \frac{q^*(x, t_n)}{\int q^*(x', t_n)\,\mathrm{d}x'}, \quad p^*(x, t_n^-) = \frac{q^*(x, t_n^-)}{\int q^*(x', t_n^-)\,\mathrm{d}x'}$$

Let us define two homotopy functions $\zeta_n(x, \lambda)$ and $\rho_n^*(x, \lambda)$ as follows:

$$\begin{aligned} \zeta_n(x, \lambda) &:= \ln q^*(x, t_n^-) + \lambda h(x)^T\left(Y_n - \frac{1}{2}h(x)\right), \\ \rho_n^*(x, \lambda) &:= \frac{\exp(\zeta_n(x, \lambda))}{\int \exp(\zeta_n(x', \lambda))\,\mathrm{d}x'} \end{aligned} \tag{A3}$$

where $\lambda \in [0, 1]$ is the pseudo-time parameter.

By construction, it can be easily checked that, for $\lambda = 0$ and $\lambda = 1$:

$$\begin{aligned} \zeta_n(x,0) &= \ln q^*\left(x, t_n^-\right), \ \zeta_n(x,1) = \ln q^*\left(x, t_n\right) \\ \rho_n^*(x,0) &= p^*\left(x, t_n^-\right), \ \rho_n^*(x,1) = p^*\left(x, t_n\right). \end{aligned} \tag{A4}$$

And the evolution of $\rho_n^*(x,\lambda)$ is described in the following proposition.

Proposition A.1 (Proposition 2 Yang et al. 2014): *Consider the normalized density function $\rho_n^*(x,\lambda)$ as defined in (A3) with $\lambda \in [0,1]$. Then its evolution is given by the following partial differential equation: For $\lambda \in [0,1]$*

$$\frac{\partial \rho_n^*}{\partial \lambda}(x,\lambda) = \rho_n^*(x,\lambda)\left[(h(x) - \hat{h}(\lambda))^T Y_n - \frac{1}{2}|h(x)|^2 + \frac{1}{2}\widehat{|h|^2}\right], \tag{A5}$$

where $\hat{h}(\lambda) := \int \rho_n^*(x,\lambda) h(x)\, dx$, $|h(\cdot)|^2 := h(\cdot)^T h(\cdot)$ and $\widehat{|h|^2} := \int \rho_n^*(x,\lambda)|h(x)|^2\, dx$.

Let us denote $\rho_n(x,\lambda)$ the distribution of $S_n^i(\lambda)$ in (6). More specifically, we have

$$\rho_n(\mathrm{d}x, 0) := P(S_n^i(0) \in \mathrm{d}x \,|\, \mathscr{F}_{t_{n-1}}), \tag{A6}$$

$$\rho_n(\mathrm{d}x, 1) := P(S_n^i(1) \in \mathrm{d}x \,|\, \mathscr{F}_{t_n}). \tag{A7}$$

And the evolution equation for $\rho_n(x,\lambda)$ is given by the following Kolmogorov's forward equation (Jazwinski, 1970):

$$\frac{\partial \rho_n}{\partial \lambda}(x,\lambda) = -\nabla \cdot (\rho_n \mathrm{K})\, Y_n - \nabla \cdot (\rho_n u). \tag{A8}$$

If we can choose control input $U_n^i(\lambda)$ in (6), such that $\rho_n(x,\lambda) = \rho_n^*(x,\lambda)$, then by (A4), we have

$$\rho_n(x,0) = p^*\left(x, t_n^-\right), \quad \rho_n(x,1) = p^*\left(x, t_n\right). \tag{A9}$$

Therefore the functions $\{u(x,\lambda), K(x,\lambda)\}$ (or control input $U_n^i(\lambda)$) in (6) are said to be optimal if $\rho_n = \rho_n^*$. That is, given $\rho_n(\cdot,0) = \rho_n^*(\cdot,0)$, our goal is to choose $\{u, K\}$ in (6) such that the evolution equations of these distributions $\rho_n(x,\lambda)$ and $\rho^*(x,\lambda)$ coincide (see (A5) and (A8) and thus $\rho_n(x,1) = p^*(x,t_n)$. And the optimal $\{u,K\}$ is given in the following Theorem 2.1.

Appendix 2. Linear gaussian case

If the filtering system (1)–(2) is linear and Gaussian, i.e.

$$\begin{aligned} \mathrm{d}X_t &= AX_t\,\mathrm{d}t + \mathrm{d}B_t, \\ Y_n &= HX_{t_n} + W_n, \end{aligned} \tag{A10}$$

where $A \in \mathbb{R}^{d\times d}$, $H \in \mathbb{R}^{m\times d}$, $\{B_t\}$, W_n and X_0 are independent of each other. The initial distribution $p_X(x,0)$ is Gaussian with mean vector μ_0 and covariance matrix Σ_0. The following proposition in Yang et al. (2014) gives the optimal control input $\{K, \Omega\}$ for the system (A10).

Proposition A.2 (Proposition 3 Yang et al. 2014): *Consider the d-dimensional linear system (A10). Suppose the homotopy density function ρ_n in (11) is Gaussian, i.e.*

$$\rho_n(x,\lambda) = \frac{1}{(2\pi)^{\frac{d}{2}}\,|\Sigma_\lambda|^{\frac{1}{2}}}\exp\left[-\frac{1}{2}(x-\mu_\lambda)^T\Sigma_\lambda^{-1}(x-\mu_\lambda)\right],$$

where $x = (x_1,\ldots,x_d)^T$, $\mu_\lambda = (\mu_1(\lambda),\ldots,\mu_d(\lambda))^T$ is the mean, Σ_λ is the covariance matrix, and $|\Sigma_\lambda| > 0$ denotes the determinant. A solution of the boundary value problem (12) and (15) is:

$$\begin{aligned} \eta_j(x,\lambda) &= \sum_{k=1}^d \left[\Sigma_\lambda H^T\right]_{kj}(x_k - \mu_k(\lambda)), \quad j = 1,\ldots,m \\ \Omega(x,\lambda) &= (0,\ldots,0), \end{aligned}$$

where $[\star]_{kj}$ is the (k,j)-th entry of the matrix $\star$. Using $K = [\nabla\eta_1^T,\ldots,\nabla\eta_m^T]$, we obtain $K(x,\lambda) = \Sigma_\lambda H^T$.

Appendix 3. Some Technical Lemmas

Lemma A.1 (Gronwall's inequality): *Let $u(t)$ be a nonlinear function that satisfies the integral inequality*

$$u(t) \le c + \int_{t_0}^t (b_1(s)u(s) + b_2(s)u^\alpha(s))\,\mathrm{d}s, \tag{A11}$$

where $c \ge 0$, $\alpha \ge 0$, $b_1(t)$ and $b_2(t)$ are continuous nonnegative functions for $t \ge t_0$. For $0 \le \alpha < 1$, we have

$$u(t) \le \left\{c^{1-\alpha}\exp\left[(1-\alpha)\int_{t_0}^t b_1(s)\,\mathrm{d}s\right] + (1-\alpha)\int_{t_0}^t b_2(s)\exp\left[(1-\alpha)\int_s^t b_1(r)\,\mathrm{d}r\right]\mathrm{d}s\right\}^{\frac{1}{1-\alpha}};$$

for $\alpha = 1$,

$$u(t) \le c\exp\left\{\int_{t_0}^t [b_1(s) + b_2(s)]\,\mathrm{d}s\right\};$$

and for $\alpha > 1$ with the additional hypothesis

$$c < \left\{\exp\left[(1-\alpha)\int_{t_0}^{t_0+h} b_1(s)\,\mathrm{d}s\right]\right\}^{\frac{1}{\alpha-1}}\left\{(\alpha-1)\int_{t_0}^{t_0+h} b_2(s)\,\mathrm{d}s\right\}^{-\frac{1}{\alpha-1}}$$

we also get for $t_0 \le t \le t_0 + h$, for $h > 0$,

$$u(t) \le c\left\{\exp\left[(1-\alpha)\int_{t_0}^t b_1(s)\,\mathrm{d}s\right] - c^{-1}(\alpha-1)\int_{t_0}^t b_2(s) \times \exp\left[(1-\alpha)\int_s^t b_1(r)\,\mathrm{d}r\right]\mathrm{d}s\right\}^{\frac{1}{\alpha-1}}.$$

Lemma A.2 (Rosenthal type inequality Hu et al., 2011): *Let $p > 0$, and let $\{\xi_i,\ i = 1,\ldots,n\}$ be conditionally independent random variables given $\sigma-$algebra $\mathcal{G}$ such that $\mathbb{E}[\xi_i \,|\, \mathcal{G}] = 0$ and $\mathbb{E}[|\xi|^p|\mathcal{G}] < \infty$. Then*

$$\mathbb{E}\left[\left|\sum_{i=1}^n \xi_i\right|^p \middle| \mathcal{G}\right] \le C(p)\left[\sum_{i=1}^n \mathbb{E}(|\xi|^p|\mathcal{G}) + \left(\sum_{i=1}^n \mathbb{E}(|\xi|^2|\mathcal{G})\right)^{p/2}\right], \tag{A12}$$

where $C(p)$ is a constant that depends only on p. This inequality holds in the almost sure sense.

Lemma A.3 (Hu et al., 2008): *If $\mathbb{E}[|\xi|^p] < \infty$, then*

$$\mathbb{E}\left[|\xi - \mathbb{E}\xi|^p\right] \le 2^p\mathbb{E}\left[|\xi|^p\right], \tag{A13}$$

for any $p \ge 1$.

Lemma A.4 (Hu et al., 2008): *If $1 \le r_1 \le r_2$ and $\mathbb{E}[|\xi|^{r_2}] < \infty$, then*

$$\mathbb{E}^{1/r_1}\left[|\xi|^{r_1}\right] \le \mathbb{E}^{1/r_2}\left[|\xi|^{r_2}\right]. \tag{A14}$$