Complete Classification of Finite Dimensional Estimation Algebras With State Dimension n, Linear Rank n-1, and Constant Wong Matrix

Hongyu Yu, Xiaopei Jiao, and Stephen S.-T. Yau, *Fellow, IEEE*

***Abstract*—Ever since the Lie algebra method was introduced to construct finite dimensional nonlinear filters by Brockett and Mitter independently, there has been an intense interest in classifying all finite dimensional estimation algebras and finding new classes of finite dimensional recursive filters. The estimation algebra method has been proven to be an invaluable tool in the nonlinear filtering theory. This article considers the finite dimensional estimation algebras derived from a nonlinear filtering system with state dimension n, linear rank n-1, and constant Wong matrix. Related theories of the underdetermined partial differential equations and the Euler operator are applied to classify the estimation algebras. It is proved that the Mitter conjecture holds and the dimension of the finite dimensional estimation algebras must be 2n or 2n+1 with the abovementioned conditions. Therefore, we can construct the explicit solution of filtering systems by Wei–Norman approach. This result is of great significance because it is the first classification of nonmaximal rank finite dimensional estimation algebras with arbitrary state dimension.**

***Index Terms*—Classification, finite dimensional estimation algebras, nonmaximal rank, nonlinear filters, Wong matrix.**

I. Introduction

The filtering problem is a special type of state estimation problem whose goal is to estimate the present state x_t given the observation history $\{y_s : 0 \leq s \leq t\}$. The problem of the linear filtering system with Gaussian initial distributions has already been solved by the well-known Kalman filter proposed by Kalman and Bucy [1], [2]. Thereafter, there have been a lot of interests in studying the nonlinear filtering problem. Different methods have been put forward to solve this problem. In the late 1960s and the early 1970s, "innovations method" was proposed by Kailath and then developed by Fujisaki et al. [3]. However, this method cannot solve the problem explicitly. In the late 1970s, Brockett and Clark [4], Brockett [5], and Mitter [6] established a novel method known as Lie algebra method motivated by the Wei–Norman technique [7], which allows one to express the solutions of time-varying linear differential equations when corresponding Lie algebra is finite dimensional. The method introduces the estimation algebra of a filtering system. The meaning of the Lie algebra method is that if the estimation algebra is finite dimensional, its corresponding filtering system must be a finite dimensional recursive filter. Estimation algebra method can be widely applied in any filtering systems, such as engineering and physical models, which include navigation on aircraft and submarines, radar, tracking problem, trajectory detection, guidance, positioning, orbit determination, etc. The explicit structure of the estimation algebras is also of great significance. The famous Duncan–Mortensen–Zakai (DMZ) equation and its robust version describe the unnormalized probability density function of the state conditioned on the observation history [8]. By Wei–Norman technique, if the explicit basis of the estimations algebras is known, the DMZ equation can be solved. There are also many other ways to solve the DMZ equation. For example, Yau and Yau [9] proposed a novel effective method to solve the "pathwise–robust" DMZ equation. Recently, Chen et al.[10] and Shi et al. [11] put forward another method to solve the robust DMZ equation based on the "direct method" and Gaussian approximation.

Manuscript received 15 October 2021; revised 3 November 2022; accepted 18 March 2023. Date of publication 10 April 2023; date of current version 29 December 2023. This work was supported in part by the National Natural Science Foundation of China (NSFC) under Grant 11961141005, in part by the Tsinghua University Start-up Fund, and in part by the Tsinghua University Education Foundation fund under Grant 042202008. Recommended by Associate Editor Vincent Andrieu.
(Corresponding author: Stephen S.-T. Yau.)

Hongyu Yu is with the Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China (e-mail: 1127900763@qq.com).

Xiaopei Jiao is with the Yanqi Lake Beijing Institute of Mathematical Sciences and Applications, Beijing 101408, China (e-mail: jiaoxiaopei0302@163.com).

Stephen S.-T. Yau is with the Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China, and also with the Yanqi Lake Beijing Institute of Mathematical Sciences and Applications, Beijing 101408, China (e-mail: yau@uic.edu).
Digital Object Identifier 10.1109/TAC.2023.3266012

Estimation algebra plays a critical role in algebraic classification in nonlinear filtering systems and has several merits. First, Lie algebra introduces notion of geometry. Second, it can explain why it is easier to find exact recursive filters for linear system but it is more difficult for some nonlinear systems, such as cubic sensor. Most importantly, once estimation algebra of a nonlinear system is finite dimensional, finite recursive filter can be constructed explicitly and it is universal in the sense of Maurel and Michel [12]. From the aspect of computation, the number of sufficient statistics linearly depends on state dimension n for all known finite dimensional estimation algebra.

For the practical importance of the estimation algebras in solving nonlinear filtering problems, Brockett [13] proposed the problem of classifying all finite dimensional estimation algebras. In 1987, Wong [14] introduced the concept of Wong's Ω matrix. Since the 1990s, Yau and Hu[15] completely classified all finite dimensional estimation algebras with maximal rank with arbitrary state dimension in a series of papers, and the results can be seen in [15]. The result shows that for a finite dimensional estimation algebra with maximal rank, the Wong's Ω matrix must be constant, the dimension of the estimation algebra is $2n + 2$, and Mitter conjecture holds, i.e., any polynomial in the estimation algebra is at most degree 1. Since the 2000s, Yau and his collaborators have started to focus on the finite dimensional estimation algebras with nonmaximal rank. Wu and Yau [16] classified all finite dimensional estimation algebras with state dimension 2. Shi and Yau [17], [18] have proved the linear structure of Wong's Ω matrix and the Mitter conjecture with state dimension 3 and linear rank 2. However, for the general classification of estimation algebra with nonmaximal rank on arbitrary state dimension, it is still an open and critical problem.

In the abovementioned studies of estimation algebras, the structure of Wong's Ω matrix plays an important role. However, a series of studies have shown that Wong's Ω matrix may not have simple structure. Shi et al. [19] constructed a novel class of finite dimensional estimation algebras with state dimension 3 and linear rank 1 whose Wong's Ω matrix is not constant. Dong et al. [20] constructed a new class of finite dimensional estimation algebras with state dimension 4 and linear rank 1 whose Wong's Ω matrix can be polynomials of any degree. Recently, Jiao and Yau [21] proved that for finite dimensional estimation algebras with arbitrary state dimension n and linear rank $n-2$, the entries of

Wong's Ω matrix are not necessary to be polynomials. The complexity of Wong's Ω matrix have obstructed the classification of nonmaximal finite dimensional estimation algebras.

Therefore, in this article, we consider the structure of estimation algebras given the conditions that Wong's Ω matrix is constant. We will prove that if the Wong's Ω matrix is constant and the linear rank $r = n - 1$, where n is arbitrary state dimension, then Mitter conjecture holds and the classification can be finished. It is the first classification of nonmaximal rank finite dimensional estimation algebras with arbitrary state dimension.

The following are three main theorems of this article.

Theorem 1.1: Let E be a finite dimensional estimation algebra with arbitrary state dimension n, linear rank $n-1$, and constant Wong's Ω matrix, then any polynomial in E is at most degree 1, i.e., the Mitter conjecture holds in this case.

Theorem 1.2: Let E be a finite dimensional estimation algebra with arbitrary state dimension n, linear rank $n-1$, and constant Wong's Ω matrix, then dim$E = 2n$ or dim$E = 2n+1$. Moreover, we have $E =< L_0, 1, x_1, \ldots, x_{n-1}, D_1, \ldots, D_{n-1} >_{L.A.}$ or $E =< L_0, 1, x_1, \ldots, x_{n-1}, D_1, \ldots, D_{n-1}, D_n + cx_n >_{L.A.}$ in the sense of isomorphism.

Moreover, we can construct the solution of the robust DMZ equation (9) with these conditions.

Theorem 1.3: Let E be the estimation algebra of system (3). Suppose E is finite dimensional with linear rank $n-1$ and constant Wong matrix. If dim$E = 2n+1$, then the robust DMZ (9) has a solution for all t of the form.

$$u(t,x) = e^{T(t)} \left(\prod_{i=1}^{n} e^{r_i(t)x_i}\right) \left(\prod_{i=1}^{n} e^{s_i(t)D_i}\right) e^{tL_0}\sigma_0. \quad (1)$$

If dim$E = 2n$, then the robust DMZ (9) has a solution for all t of the form

$$u(t,x) = e^{T(t)} \left(\prod_{i=1}^{n-1} e^{r_i(t)x_i}\right) \left(\prod_{i=1}^{n-1} e^{s_i(t)D_i}\right) e^{tL_0}\sigma_0. \quad (2)$$

T, r_i, and s_i in the formula can be determined by the observation history y_i.

The rest of this article is organized as follows. In Section II, we will first recall some basic concepts and the derivation of this problem. Then, some important preliminary results and fundamental tools will be introduced. In Section III, we will first prove Theorem 1.1 and then derive Theorem 1.2 with the help of Theorem 1.1. In Section IV, we will prove Theorem 1.3 by Wei–Norman technique. Finally, Section V concludes this article.

II. Basic Concepts and Preliminary Results

A. Basic Concepts

In this article, the set of real numbers is denoted by $\mathbb{R}$. $\mathbb{R}^k$ refers to k-dimensional Euclidean space. $A = (a_{ij})$ denotes a matrix A with i, j-entry a_{ij}. rank(A) denotes rank of matrix A. δ_{ij} denotes Kronecker symbol, which means $\delta_{ij} = 1$ if $i = j$; otherwise, $\delta_{ij} = 0$. Let $C^\infty(U)$ be the set of smooth function defined on U. span$\{v_1, \ldots, v_k\}$ refers to a linear space spanned by vectors $\{v_1, v_2, \ldots, v_k\}$.

Our study is based on the following continuous signal observation model:

$$\begin{cases} dx(t) = f(x(t))dt + g(x(t))dv(t),\ x(0) = x_0 \\ dy(t) = h(x(t))dt + dw(t),\ y(0) = 0 \end{cases} \quad (3)$$

where $x(t)$ is the state of the system at time t in $\mathbb{R}^n$(n is known as state dimension), $y(t)$ is the observation at time t in $\mathbb{R}^m$, and v and w are independent standard Brownian motions that take values in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Assume f and h are C^∞ smooth functions and $g(x(t))$ is an orthogonal matrix for any t.

Define $\rho(t,x) = p(x(t)|y(s), 0 \le s \le t)$ and let $\sigma(t,x)$ to be the unnormalized version of $\rho(t,x)$. We have the well-known DMZ equation [8] as follows:

$$\begin{cases} d\sigma(t,x) = L_0\sigma(t,x)dt + \sum_{i=1}^m L_i\sigma(t,x) \circ dy_i(t) \\ \sigma(0,x) = \sigma_0 \end{cases} \quad (4)$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^n f_i \frac{\partial}{\partial x_i} - \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^m h_i^2 \quad (5)$$

and L_i is the operator of multiplication by h_i. The term σ_0 is the probability density of the initial state x_0. It is worth noticing that the DMZ equation is written in the Stratonovich sense in the Lie algebra method while the previous filtering system is in the sense of Ito integral.

Filtering system is considered as Ito sense and DMZ equation is written in the Stratonovich sense.

In order to represent the operator L_0 in a more compact form, we introduce

$$D_i = \frac{\partial}{\partial x_i} - f_i$$
$$\eta = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^n f_i^2 + \sum_{i=1}^m h_i^2. \quad (6)$$

Then,

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^n D_i^2 - \eta\right). \quad (7)$$

In real applications, we focus more on the robust state estimator from the observation history with some properties of robustness. Davis [22] considered this problem and proposed some robust algorithms. His basic idea is reduced to consider the following unnormalized density in our case:

$$u(t,x) = \exp\left(-\sum_{i=1}^m h_i(x)y_i(t)\right)\sigma(t,x). \quad (8)$$

Then, we have the following robust DMZ equation:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = L_0 u + \sum_{i=1}^m y_i(t)[L_0, L_i]u \\ \qquad + \frac{1}{2}\sum_{i,j=1}^m y_i(t)y_j(t)[[L_0, L_i], L_j]u \\ u(0,x) = \sigma_0(x) \end{cases} \quad (9)$$

where $[,]$ is the Lie bracket defined as follows.

It is obvious to see that $\sigma(t,x)$ can be solved if we can construct the solution of the robust DMZ (9).

Definition 2.1: Let V be a real vector space with a dyadic operation $(x,y) \to [x,y]$. Then, V can be seen as a Lie algebra and the operation can be called as Lie bracket if the operation satisfies the following three properties:

1) the operation is bilinear;
2) $[x,x] = 0$ for all $x \in V$;
3) $[x,[y,z]] + [y,[x,z]] + [z,[x,y]] = 0$ for all $x, y, z \in V$.

Definition 2.2: Let $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ be two Lie algebras. An isomorphism $f: \mathfrak{g} \to \tilde{\mathfrak{g}}$ is a linear map and satisfies the following.

1) f is a bijection.
2) f is a homomorphism of Lie algebras, i.e., $f([g_1, g_2]) = [f(g_1), f(g_2)]$ for any $g_1, g_2 \in \mathfrak{g}$.

If there exists an isomorphism, we denote $\mathfrak{g}$ is isomorphic to $\tilde{\mathfrak{g}}$, i.e., $\mathfrak{g} \cong \tilde{\mathfrak{g}}$.

Remark 2.3: If two Lie algebras are isomorphic, then they have the same Lie algebra structure.

Definition 2.4: If X and Y are differential operators, then the Lie bracket can be defined by $[X, Y]\phi = X(Y\phi) - Y(X\phi)$ for any C^∞ function ϕ.

Definition 2.5: The estimation algebra E of a filtering system (3) is defined as the Lie algebra generated by $L_0, L_1, \ldots, L_m$. Equivalently, we denote $E =< L_0, L_1, \ldots, L_m >_{L.A.}$.

Definition 2.6: The Wong matrix $\Omega = (\omega_{ij})$ is defined by $\omega_{ij} = [D_j, D_i] = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$. It is obvious to see $\omega_{ij} = -\omega_{ji}$.

Definition 2.7: Let $L(E) \subset E$ be the vector space consisting of all the homogeneous degree 1 polynomials in E. Then the linear rank of estimation algebra E is defined by $r := \dim L(E)$.

Definition 2.8: Let $I = \{i_1, \ldots, i_l\}$ be a subset of $\{1, \ldots, n\}$. The Euler operator $E_{i_1,\ldots,i_l}(\phi) = \sum_{i\in I} x_i \frac{\partial \phi}{\partial x_i}$.

Definition 2.9: Let U_l be the set of differential operators in the form

$$\sum_{i_1+\cdots+i_n \le l} a_{i_1,\ldots,i_n} D_1^{i_1} \ldots D_n^{i_n} \tag{10}$$

where $a_{i_1,\ldots,i_n} \in C^\infty(\mathbb{R}^n)$. Define $E_l := E \cap U_l$.

The following notations are used in this article.

1) V is a subspace of E, $A, B \in E$, then we say $A = B (\mathrm{mod}\ V)$ if $A - B \in V$.
2) $\mathrm{pol}_k(x_{i_1}, \ldots x_{i_m})$ denotes a polynomial of degree at most k in $x_{i_1}, \ldots, x_{i_m}$. $\overline{\mathrm{pol}}_k(x_{i_1}, \ldots x_{i_m})$ denotes a polynomial of degree k in $x_{i_1}, \ldots, x_{i_m}$. const denotes a constant and $\overline{\mathrm{const}}$ denotes a nonzero constant.
3) If $A, B \in E$, define $Ad_A^0 B = B$, $Ad_A^l B = [A, Ad_A^{l-1} B]$.

B. Preliminary Results

In this section, we will introduce some important preliminary results of estimation algebras and fundamental tools, including the properties of the Euler operator and underdetermined partial differential equations.

Theorem 2.10 (See [23]): Let E be a finite dimensional estimation algebra. If a function ϕ is in E, then ϕ is a polynomial of degree at most 2.

Theorem 2.11 (See [24]): Let E be an estimation algebra of system (3). Suppose Ω is a constant matrix.

1) If η is a polynomial of degree at most 2, then E is finite dimensional and has a basis consisting of L_0 and operators of the form

$$\sum_{j=1}^{n} \alpha_j D_j + \beta_j \tag{11}$$

where α_j are constant and β_j are affine in x. Moreover, the quadratic part of $\eta - \sum_{i=1}^m h_i^2$ is positive semidefinite.

2) Conversely, if E is finite dimensional, then h_i are affine in x for all i. Furthermore, if the linear rank of E is n, then η is a polynomial of degree at most 2.

Theorem 2.12 (See [24]): Let $\tilde{x} = Ax + b$ where A is orthogonal. Consider the estimation algebra $\tilde{E}$ with

$$\tilde{f}(\tilde{x}) = Af(x)$$

$$\tilde{h}(\tilde{x}) = h(x)$$

$$\tilde{D}_i = \frac{\partial}{\partial \tilde{x}_i} - \tilde{f}_i$$

$$\tilde{\eta} = \sum_{i=1}^{n} \frac{\partial \tilde{f}_i}{\partial \tilde{x}_i} + \sum_{i=1}^{n} \tilde{f}_i^2 + \sum_{i=1}^{m} \tilde{h}_i^2$$

$$\tilde{L}_0 = \frac{1}{2}\left(\sum_{i=1}^{n} \tilde{D}_i^2 - \tilde{\eta}\right). \tag{12}$$

Then $\tilde{E}$ is isomorphic to E, i.e., the transformation $\tilde{x} = Ax + b$ maintains the Lie structure.

Theorem 2.13 (See [15]): Let E be a finite dimensional estimation algebra. If $l \ge 0$ and

$$A = \sum_{|i|=l+1} a_{i_1,\ldots,i_n} D_1^{i_1} \ldots D_n^{i_n} (\mathrm{mod}\ U_l) \in E \tag{13}$$

then $a_{i_1,\ldots,i_n}$ are polynomials.

Theorem 2.14 (See [16]): Let m be an integer, $I = \{i_1, \ldots, i_l\}$, and $\xi \in C^\infty(\mathbb{R}^n)$ such that $E_{i_1,\ldots,i_l}(\xi) + m\xi$ is a polynomial of degree k in $x_{i_1}, \ldots, x_{i_l}$ with coefficients in C^∞ functions of $x_j (j \notin I)$.

1) If $m + k + 1 > 0$, then ξ is a degree k polynomial in $x_{i_1}, \ldots, x_{i_l}$ with coefficients in C^∞ functions of $x_j (j \notin I)$.
2) If $m + k + 1 \le 0$, then ξ is a degree k or degree $-m$ polynomial in $x_{i_1}, \ldots, x_{i_l}$ with coefficients in C^∞ functions of $x_j (j \notin I)$.

Theorem 2.15 (See [16]): Let $F(x_1, \ldots, x_n)$ be a C^∞ function on $\mathbb{R}^n$. Suppose that there exists a path $c : \mathbb{R} \to \mathbb{R}^n$ and $\delta > 0$ such that $\lim_{t\to\infty} ||c(t)|| = \infty$ and $\lim_{t\to\infty} \sup_{B_\delta(c(t))} F = -\infty$, where $B_\delta(c(t)) = \{x \in \mathbb{R}^n : ||x - c(t)|| < \delta\}$. Then, there are no C^∞ functions $f_1, \ldots, f_n$ on $\mathbb{R}^n$ satisfying

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = F. \tag{14}$$

Corollary 2.16 (See [16]): $I = \{i_1, \ldots, i_l\}$. If $F(x_1, \ldots, x_n)$ is a polynomial in $x_{i_1}, \ldots, x_{i_l}$ with coefficients in C^∞ functions of $x_j (j \notin I)$ satisfying

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = F. \tag{15}$$

Then, the degree of F with respect to $x_{i_1}, \ldots, x_{i_l}$ must be even.

Remark 2.17: We consider the finite dimensional estimation algebras with constant Wong's Ω matrix. By the definition of η and Corollary 2.16, if $\eta - \sum_{i=1}^m h_i^2$ is a polynomial with respect to some variables from $x_1, \ldots, x_n$, then the degree must be even in those variables. By Theorem 2.11, h_i $(i = 1, \ldots, m)$ are affine in x. Therefore, if η is a polynomial with respect to some variables from $x_1, \ldots, x_n$, then the degree must be even in those variables.

Finally, we provide some practical calculation results.

Lemma 2.18 (See [15]):

1) $[gD_i, h] = g\frac{\partial h}{\partial x_i}$.
2) $[gD_i, hD_j] = gh\omega_{ji} + g\frac{\partial h}{\partial x_i} D_j - h\frac{\partial g}{\partial x_j} D_i$.
3) $[gD_i^2, h] = 2g\frac{\partial h}{\partial x_i} D_i + g\frac{\partial^2 h}{\partial x_i^2}$.
4) $[D_i^2, hD_j] = 2\frac{\partial h}{\partial x_i} D_i D_j + 2h\omega_{ji} D_i + \frac{\partial^2 h}{\partial x_i^2} D_j + h\frac{\partial \omega_{ji}}{\partial x_i}$.
5) $[D_i^2, D_j^2] = 4\omega_{ji} D_j D_i + 2\frac{\partial \omega_{ji}}{\partial x_j} D_i + 2\frac{\partial \omega_{ji}}{\partial x_i} D_j + \frac{\partial^2 \omega_{ji}}{\partial x_i \partial x_j} + 2\omega_{ji}^2$.

Lemma 2.19 (See [16]): Let $g, h \in C^\infty(\mathbb{R}^n)$ and let $i_1, \ldots, i_n, j_1, \ldots, j_n$ be nonnegative integers with $\sum_{l=1}^n i_l = r$, $\sum_{l=1}^n j_l = s$, and $r + s \ge 2$. Let δ_{ij} be the Kronecker symbol. Then

$$\begin{aligned}&[gD_1^{i_1} \ldots D_n^{i_n}, hD_1^{j_1} \ldots D_n^{j_n}] \\ &\quad = \sum_{l=1}^{n} \left(i_l g \frac{\partial h}{\partial x_l} - j_l h \frac{\partial g}{\partial x_l}\right) D_1^{i_1+j_1-\delta_{1l}} \ldots D_n^{i_n+j_n-\delta_{nl}} \\ &\quad (\mathrm{mod}\ U_{r+s-2}).\end{aligned} \tag{16}$$

Lemma 2.20 (See [17]): Suppose $i = (i_1, \ldots, i_n)$ and $|i| = \sum_{l=1}^{n} i_l \geq 2$; then,

$$gD_1^{i_1} \ldots D_n^{i_n} = gD_{k_1}^{i_{k_1}} \ldots D_{k_n}^{i_{k_n}} \pmod{U_{|i|-2}} \tag{17}$$

where g is a C^∞ function of $x_1, \ldots, x_n$ and $k = (k_1, \ldots, k_n)$ is a permutation of $(1, \ldots, n)$.

Finally, we recall the classical Baker–Campbell–Hausdorff type relation.

Lemma 2.21: (See [25]) Let F_i and F_j be two elements in a Lie algebra V, then

$$e^{r(t)F_i} F_j = \left(\sum_{l=0}^{+\infty} \frac{r(t)^l}{l!} Ad_{F_i}^l F_j \right) e^{r(t)F_i}. \tag{18}$$

III. Classification

In this section, we consider the finite dimensional estimation algebra E arising from system (3) with state dimension n, linear rank $n-1$, and constant Wong's Ω matrix. By Theorem 2.12, without the loss of generality, we can assume $x_1, \ldots, x_{n-1} \in E$, $x_n \notin E$.

By Lemma 2.18, for $i, j = 1, \ldots, n-1$

$$[L_0, x_i] = D_i \in E$$
$$[D_i, x_j] = \delta_{ij} \in E$$
$$[D_i, D_j] = \omega_{ji} \in E$$
$$Y_i := [L_0, D_i] - \sum_{j \neq n} \omega_{ij} D_j = \omega_{in} D_n + \frac{1}{2} \frac{\partial \eta}{\partial x_i} \in E. \tag{19}$$

So we have $L_0, 1, x_1, \ldots, x_{n-1}, D_1, \ldots, D_{n-1}, Y_1, \ldots, Y_{n-1} \in E$.

Lemma 3.1: Any polynomial $p \in E$ does not contain $x_i x_n$ terms where $i = 1, \ldots, n-1$.

Proof: By Theorem 2.10 we know that any polynomial in E has at most degree 2. If $p \in E$ contain $x_i x_n$ terms, then $[D_i, p]$ is a degree 1 polynomial that contains x_n. Since $1, x_1, \ldots, x_{n-1} \in E$, we have $x_n \in E$, which contradicts with the assumption that $x_n \notin E$. ∎

Lemma III.2: If there exists a degree 2 polynomial in E, then we only need to consider the following two cases.

Case (A): There exists $p = x_n^2 + (x_{i_1}^2 + \cdots + x_{i_k}^2) \in E$.

Case (B): There exists $p = x_{i_1}^2 + \cdots + x_{i_k}^2 \in E$, where n is not in $i_1, \ldots, i_k$.

Proof: By Lemma 3.1, we have all degree 2 polynomials in E should be of the form

$$p_0 = \sum_{i=1}^{n} \text{const } x_i^2 + \sum_{1 \leq j < k \leq n-1} \text{const } x_j x_k + \sum_{i=1}^{n} \text{const } x_i + \text{const}. \tag{20}$$

By using orthogonal transformation of $x_1, .., x_{n-1}$, we can assume

$$p_0 = \sum_{i=1}^{n} \text{const } x_i^2 + \text{const } x_n \in E. \tag{21}$$

We still have $L(E) = \text{span}\{x_1, \ldots, x_{n-1}\}$ since the transformation is independent of x_n. If the coefficient of x_n^2 is not zero, similarly by using a translation $\tilde{x}_n = x_n + \text{const}$, we can assume that $p_0 = \sum_{i=1}^{n} \text{const } x_i^2 \in E$ and $L(E)$ does not change. If the coefficient is zero, consider $[L_0, p_0], p_0] - \text{const} = \sum_{i=1}^{n-1} \text{const } x_i^2$, so we can assume there is a polynomial only consisting of items x_i^2 if Mitter conjecture does not hold. We assume $p_0 = \sum_{i=1}^{n} a_i x_i^2 \in E$.

Consider

$$p_1 = \frac{1}{4}[[L_0, p_0], p_0] = \sum_{i=1}^{n} a_i^2 x_i^2 \in E$$
$$p_2 = \frac{1}{4}[[L_0, p_1], p_0] = \sum_{i=1}^{n} a_i^3 x_i^2 \in E$$
$$\ldots$$
$$p_l = \frac{1}{4}[[L_0, p_{n-1}], p_0] = \sum_{i=1}^{n} a_i^{l+1} x_i^2 \in E. \tag{22}$$

By the invertibility of Vandermonde matrix and similar arguments of Wu and Yau [16], we can assume $a_i \in \{0, 1\}$, i.e., Lemma 3.2 holds. ∎

Therefore, in order to prove Mitter conjecture, it suffices to show that Case (A) and Case (B) are both impossible. In the following two lemmas, we provide two useful techniques for subsequent main theorem.

Lemma 3.3: If $T = D_n + \phi(x_n) \in E$ where $\phi(x_n)$ is a C^∞ function, then $\phi(x_n)$ is a polynomial with at most degree 2.

Proof: Consider

$$T_m = Ad_{L_0}^m T = \frac{\partial^m \phi}{\partial x_n^m} D_n^m (\text{mod } U_{m-1}) \in E \ (m > 1). \tag{23}$$

If ϕ is not a polynomial, then $\frac{\partial^m \phi}{\partial x_n^m} \neq 0$ so $\dim E = \infty$, a contradiction. Therefore, ϕ must be a polynomial.

We assume $\phi(x_n)$ is a degree l polynomial and $\frac{\partial^{l-1} \phi}{\partial x_n^{l-1}} = e_0 x_n + e_1$, where e_i are constant and e_0 is not zero. If $l > 2$, consider

$$R_0 = T_{l-1} = (e_0 x_n + e_1) D_n^{l-1} \pmod{U_{l-2}} \in E$$
$$R_1 = T_l = e_0 D_n^l \pmod{U_{l-1}} \in E. \tag{24}$$

By induction and Lemma 2.19, we can create an infinite sequence in E

$$R_2 = [R_1, R_0] = l e_0^2 D_n^{2l-2} \pmod{U_{2l-3}} \in E$$
$$\ldots$$
$$R_{n+1} = [R_n, R_0] = \overline{\text{const}} D_n^{(l-2)n+l} \pmod{U_{(l-2)n+l-1}} \in E. \tag{25}$$

It contradicts with $\dim E < \infty$, so ϕ is a polynomial of degree at most 2. ∎

Lemma 3.4: Assume there exists an operator $T = D_n + \text{pol}_2(x_n) \in E$. If η can be of the form $\text{pol}_3(x_1, \ldots, x_n) + f(x_n)$ where f is a C^∞ smooth function, then $f(x_n)$ must be a polynomial of degree less than or equal to 3.

Proof: If $T = D_n + \text{pol}_2(x_n) \in E$ and $\eta = \text{pol}_3(x_1, \ldots, x_n) + f(x_n)$, consider

$$A_1 = [L_0, T] - \sum_{i=1}^{n-1} \omega_{ni} D_i = \frac{1}{2} f'(x_n) + \text{pol}_2(x_1, \ldots, x_n) + \text{pol}_1(x_n) D_n \in E$$
$$A_2 = [L_0, A_1] = \frac{1}{2} f''(x_n) D_n + \sum_{i=1}^{n} \text{pol}_1(x_1, \ldots, x_n) D_i + \text{const } D_n^2 \pmod{U_0} \in E$$
$$A_3 = [L_0, A_2] = \frac{1}{2} f^{(3)}(x_n) D_n^2 + \sum_{i \leq j} \text{const } D_i D_j \pmod{U_1} \in E$$
$$A_4 = [L_0, A_3] = \frac{1}{2} f^{(4)}(x_n) D_n^3 \pmod{U_2} \in E$$
$$\ldots$$
$$A_l = [L_0, A_{l-1}] \in E. \tag{26}$$

So f must be a polynomial; otherwise, A_i is an infinite sequence in E. Assume f is a degree k polynomial. If $k > 3$, then

$$B_1 := A_k = [L_0, A_{l-1}] = \overline{\text{const}} D_n^{k-1} \pmod{U_{k-2}} \in E$$
$$B_0 := A_{k-1} = (\overline{\text{const}}\, x_n + \text{const}) D_n^{k-2} \pmod{U_{k-3}} \in E. \tag{27}$$

Therefore, we can create an infinite sequence in E

$$B_{l+1} = [B_l, B_0] = \overline{\text{const}} D_n^{(k-3)l+k-1} \pmod{U_{(k-3)l+k-2}} \in E \tag{28}$$

a contradiction. ■

In the following, we will prove all cases of Lemma 3.2 that do not occur.

Lemma 3.5: $p = x_n^2 \notin E$.

Proof: If $p \in E$, let $Z_0 = p$

$$Z_1 = [L_0, Z_0] - \text{const} = x_n D_n \in E. \tag{29}$$

For $i = 1, \ldots, n-1$

$$[D_i, Z_1] = \omega_{ni} x_n \in E. \tag{30}$$

Therefore, $\omega_{in} = 0$ since $L(E) = \text{span}\{x_1, \ldots, x_{n-1}\}$. So

$$Y_i = \frac{1}{2}\frac{\partial \eta}{\partial x_i} \in E. \tag{31}$$

Consider

$$Z_2 = [L_0, Z_1] = D_n^2 + \frac{1}{2}E_n(\eta) \in E$$

$$Z_3 = [Z_2, Z_1] = 2D_n^2 - \frac{1}{2}E_n^2(\eta) \in E$$

$$Z_4 = 2Z_2 - Z_3 = \frac{1}{2}E_n^2(\eta) + E_n(\eta) = E_n\left(\frac{1}{2}E_n(\eta) + \eta\right) \in E. \tag{32}$$

By Theorem 2.14, we can conclude that $\frac{1}{2}E_n(\eta) + \eta$ is at most degree 2 in x_n. Apply Theorem 2.14 again and we have η is at most degree 2 in x_n.

By Theorem 2.10, we have $\frac{\partial \eta}{\partial x_i}$ that are at most degree 2 polynomial for $i = 1, \ldots, n-1$. So η is at most degree 3 in $x_1, .., x_{n-1}$ with coefficient in C^∞ functions of x_n. From Remark 2.17, the degree of η in $x_1, .., x_{n-1}$ is actually no more than 2.

In other words, we have the following:

$$\eta = \text{pol}_2(x_1, \ldots, x_{n-1}) + \sum_{i=1}^{n-1} \phi_i(x_n) x_i + g(x_n) \tag{33}$$

where ϕ_i and g are at most degree 2.

So η is at most degree 3 in $x_1, .., x_n$ and, therefore, is at most degree 2 by Remark 2.17. We can easily check that $E = < L_0, 1, x_1, \ldots, x_{n-1}, D_1, \ldots, D_{n-1} >_{L.A.}$, which contradicts with $p \in E$. ■

Lemma 3.6: Let $\{i_1, \ldots, i_k\}$ be a subset of $\{1, .., n-1\}$, then $p = x_n^2 + (x_{i_1}^2 + \cdots + x_{i_k}^2) \notin E$, i.e., Case (A) in Lemma 3.2 will not happen.

Proof: If $p \in E$, let $Z_0 = p$

$$Z_1 = [L_0, Z_0] - \text{const} = x_n D_n + (x_{i_1} D_{i_1} + \cdots + x_{i_k} D_{i_k}) \in E. \tag{34}$$

For $i = 1, \ldots, n-1$

$$[D_i, Z_1] = \omega_{ni} x_n \pmod{E_1} \in E. \tag{35}$$

Therefore, $\omega_{in} = 0$ since $L(E) = \text{span}\{x_1, \ldots, x_{n-1}\}$.

$$2Y_i = \frac{\partial \eta}{\partial x_i} \in E \; (i = 1, \ldots, n-1).$$

Similar to the process in Lemma 3.5, we have

$$\eta = \text{pol}_2(x_1, \ldots, x_{n-1}) + \sum_{i=1}^{n-1} \phi_i(x_n) x_i + g(x_n) \tag{36}$$

$$\frac{\partial \eta}{\partial x_i} = \text{pol}_1(x_1, \ldots, x_{n-1}) + \phi_i(x_n). \tag{37}$$

Therefore, the degree of ϕ_i is less than 3.

If all $\phi_i(x_n)$ are not degree 2 for $i = 1, \ldots, n-1$, then $Y_i = \text{pol}_1(x_1, \ldots, x_{n-1})$. It is not hard to see that $E = < L_0, 1, x_1, \ldots, x_{n-1}, D_1, \ldots, D_{n-1} >_{L.A}, p \notin E$.

If there exists $\phi_{i_0}(x_n)$ that is degree 2 polynomial, $\phi_{i_0}(x_n) + \text{const } x_n \in E$. By Theorem 2.12, we can assume $x_n^2 \in E$, which contradicts with Lemma 3.5.

Therefore, Case (A) in Lemma 3.2 will not happen. ■

Lemma 3.7: Let $I = \{i_1, \ldots, i_k\}$ be a subset of $\{1, \ldots, n-1\}$, then $p = x_{i_1}^2 + \cdots + x_{i_k}^2 \notin E$, i.e., Case (B) in Lemma 3.2 will not happen.

Proof:

If the lemma is false, then there are degree 2 polynomials in E and all of them are independent of x_n by Lemmas 3.1 and 3.6.

Consider the quadratic part of those degree 2 polynomials and each of them corresponds to a symmetric matrix. For example, given a degree 2 polynomial q, its quadratic part $q^{(2)} = x^T A_{q^{(2)}} x$, where $A_{q^{(2)}}$ is symmetric. Define $r_q = \max_{q \in E} \text{rank}(A_{q^{(2)}})$.

Let p be a quadratic polynomial with maximal rank of quadratic part. By Theorem 2.12, we can assume that $p = x_1^2 + \cdots + x_k^2 \in E$, where $k = r_q$ by doing orthogonal transformation of $x_1, .., x_{n-1}$. Then, $x_{t_1} x_{t_2} \notin E$ unless $t_1 \le k$ and $t_2 \le k$; otherwise, we can create a polynomial corresponding to a symmetric matrix that has rank more than $k = r_q$. Moreover, for any degree 2 polynomial $q \in E$, $A_{q^{(2)}} = (a_{t_1 t_2})$, then $a_{t_1 t_2} = 0$ unless $t_1 \le k$ and $t_2 \le k$.

Let $Z_0 = p$

$$Z_1 = [L_0, Z_0] = x_1 D_1 + \cdots + x_k D_k \in E. \tag{38}$$

For $j = 1, \ldots, n-1$

$$Y_j = \omega_{jn} D_n + \frac{1}{2}\frac{\partial \eta}{\partial x_j} \in E \tag{39}$$

$$P_j = [Z_1, Y_j] = \frac{1}{2}E_{1,\ldots,k}\left(\frac{\partial \eta}{\partial x_j}\right) \pmod{L(E)} \in E. \tag{40}$$

Consider P_j where $j = 1, .., k$, we know that $\frac{\partial \eta}{\partial x_j}$ is at most degree 2 in $x_1, \ldots, x_k$. Therefore, η is at most degree 2 in $x_1, \ldots, x_k$ since the degree must be even by Remark 2.17.

In the previous analysis, we conclude that the quadratic part of any degree 2 polynomial in E is independent of x_s where $s > k$. So the coefficient of quadratic part of η in $x_1, \ldots, x_k$ is actually constant since $E_{1,\ldots,k}(\frac{\partial \eta}{\partial x_i}) \in E$ for $i = 1, \ldots, k$.

Therefore,

$$\eta = pol_2(x_1, \ldots, x_k) + \sum_{i=1}^{k} a_i x_i + u_1 \tag{41}$$

where a_i and u_1 are functions that are independent with $x_1, \ldots, x_k$. Consider P_j when $k < j < n$, we have

$$\sum_{i=1}^{k} \frac{\partial a_i}{\partial x_j} x_i \in E. \tag{42}$$

By the fact that a_i are independent with $x_1, \ldots, x_k$ and the quadratic part of any degree 2 polynomial in E is independent of x_s where $s > k$, we have $\frac{\partial a_i}{\partial x_j} = \text{const}$.

Therefore,

$$a_i = \sum_{j>k} \text{const } x_j + \phi_i(x_n) \tag{43}$$

$$\eta = \text{pol}_2(x_1, \ldots, x_{n-1}) + \sum_{i=1}^{k} \phi_i(x_n) x_i + u_1. \tag{44}$$

For $i, j = 1, \ldots, n-1$, consider

$$G_{ij} := 2[D_i, Y_j] - \text{const} = \frac{\partial^2 \eta}{\partial x_i \partial x_j} - \text{const} = \frac{\partial^2 u_1}{\partial x_i \partial x_j} \in E. \tag{45}$$

We know u_1 is independent with $x_1, \ldots, x_k$, so $\frac{\partial^2 u_1}{\partial x_i \partial x_j}$ cannot contain the quadratic part and, therefore, is independent with x_n.

As a result, u_1 is at most degree 3 in $x_1, \ldots, x_{n-1}$, so η is actually at most degree 2 in $x_1, \ldots, x_{n-1}$ by Remark 2.17 and the degree 2 part has constant coefficient. In other words, we have

$$\eta = \text{pol}_2(x_1, \ldots, x_{n-1}) + \sum_{i=1}^{n} \phi_i(x_n) x_i + u_2(x_n) \tag{46}$$

$$Y_i = \omega_{in} D_n + \frac{1}{2}(pol_1(x_1, \ldots, x_{n-1}) + \phi_i(x_n)) \in E \tag{47}$$

$$\omega_{in} D_n + \phi_i(x_n) \in E. \tag{48}$$

If $\omega_{in} = 0$, then $\phi_i = \text{const}$. If $\omega_{in} \neq 0$, then ϕ_i is at most degree 2 by Lemma 3.3. Therefore, no matter whether $\omega_{in} = 0$ or not, ϕ_i is at most degree 2.

If $\omega_{in} = 0$ for $i = 1, \ldots, n-1$, then it is obvious to see that

$$E = < L_0, 1, x_1, \ldots, x_{n-1}, D_1, \ldots, D_{n-1} >_{L.A.}. \tag{49}$$

If there exists $\omega_{i_0 n} \neq 0$, then by Lemma 3.4, we have u_2 is at most degree 3. As a result, η must be a degree 2 polynomial in $x_1, \ldots, x_n$. By Theorem 2.11, we know that there are no degree 2 polynomial in E, a contradiction. ■

By Lemma 3.6 and Lemma 3.7, we can conclude that Mitter conjecture holds. It shows that any polynomial in E is at most degree 1.

Finally, we will finish the complete classification of finite dimensional estimation algebras with arbitrary state dimension n, linear rank $n-1$, and constant Wong's Ω matrix.

Theorem 3.8: Let E be a finite dimensional estimation algebra with arbitrary state dimension n, linear rank $n-1$, and constant Wong's Ω matrix, then $\dim E = 2n$ or $\dim E = 2n+1$. Moreover, we have $E = < L_0, 1, x_1, \ldots, x_{n-1}, D_1, \ldots, D_{n-1} >_{L.A.}$ or $E = < L_0, 1, x_1, \ldots, x_{n-1}, D_1, \ldots, D_{n-1}, D_n + cx_n >_{L.A.}$ in the sense of isomorphism.

Proof: We have $1, L_0, x_1, \ldots, x_{n-1}, D_1, \ldots, D_{n-1}, Y_1, \ldots, Y_{n-1} \in E$, where

$$Y_j = \omega_{jn} D_n + \frac{1}{2} \frac{\partial \eta}{\partial x_j} \in E.$$

If $\omega_{jn} = 0$ for $j = 1, .., n-1$, it is obvious to see that

$$E = < L_0, 1, x_1, \ldots, x_{n-1}, D_1, \ldots, D_{n-1} >_{L.A.} \tag{50}$$

$\dim E = 2n$, so we only consider the case when ω_{in} are not all 0. Without the loss of generality, we assume $\omega_{1n} \neq 0$.

For $i, j = 1, \ldots, n-1$, we have

$$2[D_i, Y_j] - \text{const} = \frac{\partial^2 \eta}{\partial x_i \partial x_j} \in E. \tag{51}$$

By the Mitter conjecture, any polynomial in E must be no more than degree 1, which concludes that η is at most degree 2 in $x_1, \ldots, x_{n-1}$ since the degree needs to be even.

Therefore, we can assume that η has the following form:

$$\eta = \text{pol}_2(x_1, \ldots, x_{n-1}) + \sum_{i=1}^{n-1} \phi_i(x_n) x_i + u_1(x_n) \tag{52}$$

$$Y_j = t\omega_{jn} D_n + \frac{1}{2}(\text{pol}_1(x_1, \ldots, x_{n-1}) + \phi_j(x_n)) \in E \tag{53}$$

$$\omega_{jn} D_n + \phi_j(x_n) \in E. \tag{54}$$

It is obvious to see that if $\omega_{jn} = 0$, then $\phi_j(x_n) = 0$, and if $\omega_{j_0 n} \neq 0$, we have $\phi_j(x_n)$ is at most a degree 2 polynomial by Lemma 3.3.

By Lemma 3.4, $u_1(x_n)$ is at most degree 3, so we have $\eta = \text{pol}_2(x_1, \ldots, x_n)$ since the degree should be even. By Theorem 2.11, $E \subset \text{span}\{L_0, 1, x_1, \ldots, x_n, D_1, \ldots, D_n\}$. $x_n \notin E$ so $\dim E \leq 2n + 1$.

Consider Y_{j_0}, we have $T = D_n + cx_n \in E$, so $\dim E \geq 2n+1$. Therefore, if there exists $\omega_{j_0 n} \neq 0$, $\dim E = 2n+1$ and

$$E = < L_0, 1, x_1, \ldots, x_{n-1}, D_1, \ldots, D_{n-1}, D_n + cx_n >_{L.A.}. \tag{55}$$

The complete classification is finished. ■

Remark 3.9: If $\omega_{in} = 0$ for $i = 1, \ldots, n-1$, then $\dim E = 2n$ and $\eta = \text{pol}_2(x_1, \ldots, x_{n-1}) + \phi(x_n)$. If there exists i_0, s.t. $\omega_{i_0 n} \neq 0$, then $\dim E = 2n+1$ and $\eta = \text{pol}_2(x_1, \ldots, x_n)$.

IV. Finite Dimensional Filter

The structure of estimation algebras can help us solve the nonlinear filtering system (3). Actually, if the estimation algebra is finite dimensional and its basis is known, we can construct a solution of the robust DMZ (9), and therefore, we can solve system (3). In detail, we have the following two results.

Theorem 4.1: Let E be the estimation algebra of system (3). Suppose E is finite dimensional with linear rank $n-1$ and constant Wong matrix. If $\dim E = 2n+1$, $\eta = \sum_{i,j=1}^n a_{ij} x_i x_j + \sum_{i=1}^n b_i x_i + d$, where a_{ij}, b_i, and d are constant, then the robust DMZ equation (9) has a solution for all t of the form

$$u(t,x) = e^{T(t)} \left(\prod_{i=1}^{n} e^{r_i(t) x_i} \right) \left(\prod_{i=1}^{n} e^{s_i(t) D_i} \right) e^{t L_0} \sigma_0 \tag{56}$$

where r_i, s_i, and T satisfy the following ordinary differential equation:

$$\begin{cases} s_i'(t) = \sum_{j=1}^m h_{ji} y_j(t) + r_i(t) + \sum_{j=1}^n s_j(t) \omega_{ji} \ (1 \leq i \leq n-1) \\ s_n'(t) = r_n(t) + \sum_{j=1}^n s_j(t) \omega_{jn} \\ r_i'(t) = \frac{1}{2} \sum_{j=1}^n s_j(t)(a_{ij} + a_{ji}) \ (1 \leq i \leq n) \\ T'(t) = -G_1(s(t), r(t)) + \frac{1}{2} \sum_{i,j=1}^m y_i(t) y_j(t) \left(\sum_{k=1}^{n-1} h_{ik} h_{jk} \right) \end{cases} \tag{57}$$

where

$$\begin{aligned} G_1(s(t), r(t)) = & \\ \sum_{j=1}^{n} s_j'(t) & \left(-r_j(t) + \sum_{i=1}^{j-1} \omega_{ji} s_i(t) \right) \\ & + \frac{1}{2} \sum_{i=1}^n r_i^2(t) - \frac{1}{2} \sum_{i=1}^n s_i(t) b_i + \frac{1}{2} \sum_{i=1}^n s_i^2(t) \left(\sum_{j=1}^n \omega_{ij}^2 - a_{ii} \right) \\ & - \sum_{1 \leq i < k \leq n} s_i(t) s_k(t) \left(\sum_{j=1}^n \omega_{ij} \omega_{jk} + \frac{1}{2}(a_{ik} + a_{ki}) \right) \\ & + \sum_{j,k=1}^n s_k(t) r_j(t) \omega_{kj}. \end{aligned} \tag{58}$$

(We define the order of the product to be $\prod_{i=1}^n A_i = A_1 \ldots A_n$.)

Proof: By Lemma 2.21, we have

$$e^{r(t) F_i} F_j = \left(\sum_{l=0}^{+\infty} \frac{r(t)^l}{l!} Ad_{F_i}^l F_j \right) e^{r(t) F_i}. \tag{59}$$

Therefore,

$$\begin{aligned} e^{s_i(t) D_i} L_0 = & \left(L_0 - s_i(t) \sum_{j=1}^n \omega_{ij} D_j \right. \\ & \left. - \frac{s_i(t)}{2} \left(\sum_{j=1}^n (a_{ij} + a_{ji}) x_j + b_i \right. \right. \end{aligned}$$

$$+\frac{s_i^2(t)}{2}\left(\sum_{j=1}^{n}\omega_{ij}^2-a_{ii}\right)e^{s_i(t)D_i} \tag{60}$$

$$e^{s_i(t)D_i}D_j=(D_j+\omega_{ji}s_i(t))e^{s_i(t)D_i} \tag{61}$$

$$e^{r_i(t)x_i}L_0=\left(L_0-r_i(t)D_i+\frac{r_i^2(t)}{2}\right)e^{r_i(t)x_i} \tag{62}$$

$$e^{r_i(t)x_i}D_j=(D_j-r_i(t)\delta_{ij})e^{r_i(t)x_i}. \tag{63}$$

Then,

$$\frac{\partial u}{\partial t}=\left(T'(t)+\sum_{i=1}^{n}r_i'(t)x_i\right)u+A+\sum_{j=1}^{n}B_j \tag{64}$$

where

$$A=e^{T(t)}\left(\prod_{i=1}^{n}e^{r_i(t)x_i}\right)\left(\prod_{i=1}^{n}e^{s_i(t)D_i}\right)L_0e^{tL_0}\sigma_0$$

$$B_j=e^{T(t)}\left(\prod_{i=1}^{n}e^{r_i(t)x_i}\right)\left(\prod_{i=1}^{j-1}e^{s_i(t)D_i}\right)$$

$$(s_j'(t)D_je^{s_j(t)D_j})\left(\prod_{i=j+1}^{n}e^{s_i(t)D_i}\right)e^{tL_0}\sigma_0. \tag{65}$$

By (60)–(63), we have

$$\begin{aligned}A=&\left(L_0-\sum_{i=1}^{n}r_i(t)D_i+\frac{1}{2}\sum_{i=1}^{n}r_i^2(t)\right.\\&-\sum_{i,j=1}^{n}s_i(t)\omega_{ij}D_j\\&-\frac{1}{2}\sum_{i=1}^{n}s_i(t)\left(\sum_{j=1}^{n}(a_{ij}+a_{ji})x_j+b_i\right)\\&+\frac{1}{2}\sum_{i=1}^{n}s_i^2(t)\left(\sum_{j=1}^{n}\omega_{ij}^2-a_{ii}\right)\\&-\sum_{1\le i<k\le n}s_i(t)s_k(t)\left(\sum_{j=1}^{n}\omega_{ij}\omega_{jk}+\frac{1}{2}(a_{ik}+a_{ki})\right)\\&\left.+\sum_{j,k=1}^{n}s_k(t)r_j(t)\omega_{kj}\right)u\end{aligned} \tag{66}$$

$$B_j=s_j'(t)\left(D_j-r_j(t)+\sum_{i=1}^{j-1}\omega_{ji}s_i(t)\right)u. \tag{67}$$

Therefore,

$$\begin{aligned}\frac{\partial u}{\partial t}=&\left(L_0+\sum_{i=1}^{n}\left(-r_i(t)-\sum_{j=1}^{n}s_j(t)\omega_{ji}+s_i'(t)\right)D_i\right.\\&+\sum_{j=1}^{n}\left(-\frac{1}{2}\sum_{i=1}^{n}s_i(t)(a_{ij}+a_{ji})+r_j'(t)\right)x_j\\&+T'(t)+G_1(s(t),r(t))u\end{aligned} \tag{68}$$

where

$$\begin{aligned}G_1(s(t),r(t))=&\\&\sum_{j=1}^{n}s_j'(t)\left(-r_j(t)+\sum_{i=1}^{j-1}\omega_{ji}s_i(t)\right)\\&+\frac{1}{2}\sum_{i=1}^{n}r_i^2(t)-\frac{1}{2}\sum_{i=1}^{n}s_i(t)b_i+\frac{1}{2}\sum_{i=1}^{n}s_i^2(t)\left(\sum_{j=1}^{n}\omega_{ij}^2-a_{ii}\right)\\&-\sum_{1\le i<k\le n}s_i(t)s_k(t)\left(\sum_{j=1}^{n}\omega_{ij}\omega_{jk}+\frac{1}{2}(a_{ik}+a_{ki})\right)\\&+\sum_{j,k=1}^{n}s_k(t)r_j(t)\omega_{kj}.\end{aligned} \tag{69}$$

We know that h_i are linear from Theorem 2.11. Suppose $h_i=\sum_{j=1}^{n-1}h_{ij}x_j+e_i$. By the robust DMZ (9), we have

$$\begin{aligned}\frac{\partial u}{\partial t}=&\left(L_0+\sum_{i=1}^{n-1}\left(\sum_{j=1}^{m}h_{ji}y_j(t)\right)D_i\right.\\&\left.+\frac{1}{2}\sum_{i,j=1}^{m}y_i(t)y_j(t)\left(\sum_{k=1}^{n-1}h_{ik}h_{jk}\right)\right)u.\end{aligned} \tag{70}$$

Comparing (68) and (70), we know that u is the solution of the robust DMZ equation if

$$\begin{cases}s_i'(t)=\sum_{j=1}^{m}h_{ji}y_j(t)+r_i(t)+\sum_{j=1}^{n}s_j(t)\omega_{ji}\ (i\le n-1)\\ s_n'(t)=r_n(t)+\sum_{j=1}^{n}s_j(t)\omega_{jn}\\ r_i'(t)=\frac{1}{2}\sum_{j=1}^{n}s_j(t)(a_{ij}+a_{ji})\ (i\le n)\end{cases} \tag{71}$$

and

$$T'(t)=-G_1(s(t),r(t))+\frac{1}{2}\sum_{i,j=1}^{m}y_i(t)y_j(t)\left(\sum_{k=1}^{n-1}h_{ik}h_{jk}\right). \tag{72}$$

Equation (71) is a proper determined ordinary linear equation so there exists a unique solution. By (72), we know that T is determined by $s(t), r(t)$, and $y(t)$. As a result, we can construct a solution for the robust DMZ (9).

Similarly, for the case when $\dim E=2n$, we have the following theorem.

Theorem 4.2: Let E be the estimation algebra of system (3). Suppose E is finite dimensional with linear rank $n-1$ and constant Wong matrix. If $\dim E=2n$, $\eta=\sum_{i,j=1}^{n-1}a_{ij}x_ix_j+\sum_{i=1}^{n-1}b_ix_i+d+\phi(x_n)$, where a_{ij}, b_i, and d are constant, then the robust DMZ (9) has a solution for all t of the form

$$u(t,x)=e^{T(t)}\left(\prod_{i=1}^{n-1}e^{r_i(t)x_i}\right)\left(\prod_{i=1}^{n-1}e^{s_i(t)D_i}\right)e^{tL_0}\sigma_0 \tag{73}$$

where r_i, s_i, and T satisfy the following equation:

$$\begin{cases}s_i'(t)=\sum_{j=1}^{m}h_{ji}y_j(t)+r_i(t)+\sum_{j=1}^{n-1}s_j(t)\omega_{ji}\ (1\le i\le n-1)\\ r_i'(t)=\frac{1}{2}\sum_{j=1}^{n-1}s_j(t)(a_{ij}+a_{ji})\ (1\le i\le n-1)\\ T'(t)=-G_2(s(t),r(t))+\frac{1}{2}\sum_{i,j=1}^{m}y_i(t)y_j(t)(\sum_{k=1}^{n-1}h_{ik}h_{jk})\end{cases} \tag{74}$$

where

$$\begin{aligned}G_2(s(t),r(t))=&\\&\sum_{j=1}^{n-1}s_j'(t)\left(-r_j(t)+\sum_{i=1}^{j-1}\omega_{ji}s_i(t)\right)\\&+\frac{1}{2}\sum_{i=1}^{n-1}r_i^2(t)-\frac{1}{2}\sum_{i=1}^{n-1}s_i(t)b_i+\frac{1}{2}\sum_{i=1}^{n-1}s_i^2(t)\left(\sum_{j=1}^{n-1}\omega_{ij}^2-a_{ii}\right)\\&-\sum_{1\le i<k\le n-1}s_i(t)s_k(t)\left(\sum_{j=1}^{n-1}\omega_{ij}\omega_{jk}+\frac{1}{2}(a_{ik}+a_{ki})\right)\end{aligned}$$

$$+\sum_{j,k=1}^{n-1} s_k(t) r_j(t)\omega_{kj}. \tag{75}$$

Proof: The proof is similar to Theorem 4.1. The only slight difference is that in this case η may not be degree 2 polynomial. However, ϕ will not appear in calculation by (60)–(63). So the whole process of the proof can be transferred to this theorem. ∎

Finally, we illustrate a nontrivial finite dimensional recursive filter that satisfies our conditions to show that the case we focus on really happens. Let state dimension $n = 2d+1$ and observation dimension $m = 2\,d$, then consider the filtering system

$$\begin{cases} dx_{2i-1}(t) = \left(x_{2d+1} + \frac{e^{x_{2i-1}}}{e^{x_{2i-1}} + e^{x_{2i}}}\right) dt + dv_{2i-1}(t),\ i=1,\ldots,d \\ dx_{2i}(t) = \left(x_{2d+1} + \frac{e^{x_{2i}}}{e^{x_{2i-1}} + e^{x_{2i}}}\right) dt + dv_{2i}(t),\ i=1,\ldots,d \\ dx_{2d+1}(t) = x_{2d+1} + \frac{\sqrt{n}}{\sqrt{n}-1}(x_1 + \cdots + x_{2\,d}) + dv_{2d+1}(t) \\ dy_j(t) = x_j dt + dw_j(t),\ j=1,\ldots,2d. \end{cases} \tag{76}$$

It is easy to see that the linear rank is $n-1$ and the Wong matrix

$$\Omega = (\omega_{ij}) = \begin{cases} \frac{1}{\sqrt{n}-1},\ i \neq n, j = n \\ \frac{-1}{\sqrt{n}-1},\ i = n, j \neq n \\ 0, \text{otherwise} \end{cases}$$

is constant. We can further check that

$$\eta = (2d+1)x_{2d+1}^2 + \left[2\left(\frac{\sqrt{n}}{\sqrt{n}-1}\right)\left(\sum_{i=1}^{2\,d} x_i\right) + 2d\right] x_{2d+1} + \left(\frac{\sqrt{n}}{\sqrt{n}-1}\right)^2 \left(\sum_{i=1}^{2\,d} x_i\right)^2 + \sum_{i=1}^{2\,d} x_i^2 + d + 1 \tag{77}$$

and its corresponding estimation algebra is finite dimensional with following basis:

$$E = < L_0, 1, x_1, \ldots, x_{n-1}, D_1, \ldots, D_{n-1}, D_n + \sqrt{n}x_n >_{L.A.} \tag{78}$$

Therefore, this filter is a finite dimensional recursive filter. Given the observation history $y_j(t),\ j=1,\ldots,2\,d$, we can explicitly write down the solution of the robust DMZ equation according to Theorem 4.1.

V. Conclusion

In this article, we classify all finite dimensional estimation algebras with linear rank $n-1$ and constant Wong's Ω matrix. The result is that estimation algebras with these conditions can be divided into two types in the sense of isomorphism. It is the first classification of nonmaximal rank estimation algebras. By the classification result, we can construct all finite dimensional recursive filters that correspond to the estimation algebras with linear rank $n-1$ and constant Wong's Ω matrix.

Acknowledgment

The author Stephen Shing-Toung Yau would like to thank the National Center for Theoretical Sciences (NCTS) for providing an excellent research environment while part of this research was done.

References

[1] R. E. Kalman, "A new approach to linear filtering and prediction problem," *J. Fluids Eng.*, vol. 82, pp. 35–45, 1960.

[2] R. E. Kalman and R. S. Bucy, "New results in linear filtering and prediction theory," *J. Fluids Eng.*, vol. 83, pp. 95–108, 1961.

[3] M. Fujisaki, G. Kallianpur, and H. Kunita, "Stochastic differential equations for the nonlinear filtering problems," *Osaka J. Math.*, vol. 9, pp. 19–40, 1972.

[4] R. W. Brockett and J. M. C. Clark, "The geometry of the conditional density function," in*Analysis and Optimization of Stochastic Systems*. New York, NY, USA: Acad. Press, 1980.

[5] R. W. Brockett, *Nonlinear Systems and Nonlinear Estimation Theory*. Berlin, Germany: Springer, 1981.

[6] S. K. Mitter, "On the analogy between mathematical problems of nonlinear filtering and quantum physics," *Ricerche di Automatica*, vol. 10, pp. 163–216, 1979.

[7] J. Wei and E. Norman, "On global representations of the solutions of linear differential equations as a product of exponentials," *Proc. Amer. Math. Soc.*, vol. 15, no. 2, pp. 327–334, 1964.

[8] A. Bensoussan, *Stochastic Control of Partially Observable Systems*. Cambridge, U.K.: Cambridge Univ. Press, 2004.

[9] S.-T. Yau and S. S.-T. Yau, "Real time solution of the nonlinear filtering problem without memory II," *SIAM J. Control Optim.*, vol. 47, no. 1, pp. 163–195, 2008.

[10] X. Chen, J. Shi, and S. S.-T. Yau, "Real-time solution of time-varying Yau filtering problems via direct method and Gaussian approximation," *IEEE Trans. Autom. Control*, vol. 64, no. 4, pp. 1648–1654, Apr. 2018.

[11] J. Shi, Z. Yang, and S. S.-T. Yau, "Direct method for Yau filtering system with nonlinear observations," *Int. J. Control*, vol. 91, no. 3, pp. 678–687, 2018.

[12] M. C. Maurel and D. Michel, "Des resultats de non existence de filtre de dimension finie," *Stochastics*, vol. 13, no. 1/2, pp. 83–102, 1984.

[13] R. W. Brockett, "Nonlinear control theory and differential geometry," in *Proc. Int. Cong. Math.*, 1983, p. 2.

[14] W. S. Wong, "On a new class of finite dimensional estimation algebras," *Syst. Control Lett.*, vol. 9, no. 1, pp. 79–83, 1987.

[15] S. S.-T. Yau and G.-Q. Hu, "Classification of finite-dimensional estimation algebras of maximal rank with arbitrary state–space dimension and Mitter conjecture," *Int J. Control*, vol. 78, no. 10, pp. 689–705, 2005.

[16] X. Wu and S. S.-T. Yau, "Classification of estimation algebras with state dimension 2," *SIAM J. Control Optim.*, vol. 45, no. 3, pp. 1039–1073, 2006.

[17] J. Shi and S. S.-T. Yau, "Finite dimensional estimation algebras with state dimension 3 and rank 2, I: Linear structure of Wong matrix," *SIAM J. Control Optim.*, vol. 55, no. 6, pp. 4227–4246, 2017.

[18] J. Shi and S. S.-T. Yau, "Finite dimensional estimation algebras with state dimension 3 and rank 2, Mitter conjecture," *Int. J. Control*, vol. 93, pp. 2177–2186, 2018.

[19] J. Shi, X. Chen, W. Dong, and S. S.-T. Yau, "New classes of finite dimensional filters with non-maximal rank," *IEEE Control Syst. Lett.*, vol. 1, no. 2, pp. 233–237, Oct. 2017.

[20] W. Dong, X. Chen, and S. S.-T. Yau, "The novel classes of finite dimensional filters with non-maximal rank estimation algebra on state dimension four and rank of one," *Int. J. Control*, vol. 94, pp. 1–10, 2019.

[21] X. Jiao and S. S.-T. Yau, "New classes of finite dimensional filters with nonmaximal rank estimation algebra on state dimension n and linear rank n-2," *SIAM J. Control Optim.*, vol. 58, pp. 3413–3427, 2020.

[22] M. H. A. Davis, "On a multiplicative functional transformation arising in nonlinear filtering theory," *Z. Wahrscheinlichkeitstheorie verw Gebiete*, vol. 54, pp. 125–139, 1980.

[23] D. L. Ocone, "Finite dimensional estimation algebras in nonlinear filtering," *in Proc. Math. Filtering Identification Appl.*, 1981, pp. 629–636.

[24] S. S.-T. Yau, "Finite-dimensional filters with nonlinear drift. I: A class of filters including both Kalman-Bucy and Benes filters," *J. Math. Syst., Estimation Control*, vol. 4, pp. 181–203, 1994.

[25] L. F. Tam, W. S. Wong, and S. S.-T. Yau, "On a necessary and sufficient condition for finite dimensionality of estimation algebras," *SIAM J. Control Optim.*, vol. 28, pp. 173–185, 1990.