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STRUCTURE AND CLASSIFICATION THEOREMS OF FINITE-DIMENSIONAL EXACT ESTIMATION ALGEBRAS*

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Abstract. Estimation algebra turns out to be a crucial concept in the investigation of finite-dimensional nonlinear filters. In an earlier paper by the authors, a necessary and sufficient algebraic condition was derived for an exact estimation algebra to be finite-dimensional. In this paper, the investigation of the properties of finite-dimensional exact estimation algebras is continued, and some structure and partial classification theorems for such algebras are proved.

Key words. nonlinear filters, solvable Lie algebra, estimation algebra, elliptic partial differential equation

AMS(MOS) subject classifications. 17B30, 35J15, 60G35, 93E11

1. Introduction. In a previous paper [1], we introduced the concept of an exact estimation algebra. A simple algebraic necessary and sufficient condition was proved for an exact estimation algebra to be finite-dimensional. We also provided a detailed examination of the relationship between finite-dimensional exact estimation algebras and finite-dimensional nonlinear filters. This paper is in essence a continuation of our earlier study of exact estimation algebra, and we strongly recommend that the readers familiarize themselves with the results in [1]. However, every effort will be made to make this paper as self-contained as possible without too much duplication of the previous paper.

In this paper, we will prove some structure and partial classification theorems of exact finite-dimensional estimation algebras. The class of nonlinear filtering systems with an exact estimation algebra can be characterized by the solutions of some family of Riccati partial differential equations. These equations are the focal point of this study. We will provide two alternative existence proofs of these equations and examine their uniqueness properties.

2. Basic concepts. In this section, we will recall some basic concepts and results from [1]. The idea of using estimation algebras to construct finite-dimensional nonlinear filters was first proposed in Brockett and Clark [2], Brockett [3], and Mitter [4].[1] The motivation came from the Wei–Norman approach [5] of using Lie algebraic ideas to solve linear time-varying differential equations.

Consider a filtering problem based on the following signal observation model:

$$\begin{aligned} &dx(t)=f(x(t))\,dt+g(x(t))\,dv(t), \qquad x(0)=x_0,\\ &dy(t)=h(x(t))\,dt+dw(t), \qquad y(0)=0, \end{aligned} \tag{2.0}$$

in which x, v, y, and w, are respectively, $\mathbb{R}^n$, $\mathbb{R}^p$, $\mathbb{R}^m$, and $\mathbb{R}^m$-valued processes, and v and w have components that are independent, standard Brownian processes. We further

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[1] This reference was inadvertently omitted in [1].

assume that $n=p$, f, h are C^∞ smooth, and that g is an orthogonal matrix. We will refer to $x(t)$ as the state of the system at time t and to $y(t)$ as the observation at time t.

Let $\rho(t,x)$ denote the conditional probability density of the state given the observation $\{y(s): 0 \leqq s \leqq t\}$. It is well known (see [6], for example) that $\rho(t,x)$ is given by normalizing a function $\sigma(t,x)$, which satisfies the following Duncan–Mortensen–Zakai equation:

$$d\sigma(t,x)=L_0\sigma(t,x)\,dt+\sum_{i=1}^{m}L_i\sigma(t,x)\,dy_i(t), \qquad \sigma(0,x)=\sigma_0, \tag{2.1}$$

where

$$L_0=\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}-\sum_{i=1}^{n}f_i\frac{\partial}{\partial x_i}-\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}-\frac{1}{2}\sum_{i=1}^{m}h_i^2$$

and for $i=1,\cdots,m$, L_i is the zero-degree differential operator of multiplication by h_i.[2] σ_0 is the probability density of the initial point x_0. In this paper, we will assume that σ_0 is a C^∞ function.

Equation (2.1) is a stochastic partial differential equation. In real applications, we are interested in constructing state estimators from observed sample paths with some property of robustness. In [7] Davis studied this problem and proposed some robust algorithms. In our case, his basic idea reduces to defining a new unnormalized density

$$\xi(t,x)=\exp\left(-\sum_{i=1}^{m}h(x)_i y_i(t)\right)\sigma(t,x).$$

It is easy to show that $\xi(t,x)$ satisfies the following time-varying partial differential equation

$$\begin{aligned}\frac{d\xi(t,x)}{dt}&=L_0\xi(t,x)+\sum_{i=1}^{m}y_i(t)[L_0,L_i]\xi(t,x)\\&\quad+\frac{1}{2}\sum_{i=1}^{m}y_i^2(t)[[L_0,L_i],L_i]\xi(t,x), \qquad \xi(0,x)=\sigma_0,\end{aligned} \tag{2.2}$$

where $[\cdot,\cdot]$ is the Lie bracket defined as follows.

DEFINITION. If X and Y are differential operators, the Lie bracket of X and Y, $[X,Y]$, is defined by

$$[X,Y]\zeta=X(Y\zeta)-Y(X\zeta)$$

for any C^∞ function ζ.

DEFINITION. The estimation algebra $\mathbf{E}$ of a filtering problem (2.0), is defined to be the Lie algebra generated by $\{L_0,L_1,\cdots,L_m\}$, or $\mathbf{E}=\langle L_0,L_1,\cdots,L_m\rangle_{L.A.}$. If in addition there exists a potential function ϕ such that $f_i=\partial\phi/\partial x_i$, for all $1\leqq i\leqq n$, then the estimation algebra is called exact.

From now on, unless stated otherwise, we assume the estimation algebra of (2.0) is exact. We use ∇p to denote the column vector $(\partial p/\partial x_1,\cdots,\partial p/\partial x_n)^T$. Hence, $\nabla\phi=f$.

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[2] If p is a vector, we use the notation p_i to represent the ith component of p.

Define $D_i = \partial/\partial x_i - f_i$, and $\eta = \sum_{i=1}^n \partial f_i/\partial x_i + \sum_{i=1}^n f_i^2 + \sum_{i=1}^m h_i^2$. Then,

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^n D_i^2 - \eta\right). \tag{2.3}$$

Recall that $f_i = \partial\phi/\partial x_i$. Hence,

$$\eta = \Delta\phi + |\nabla\phi|^2 + \sum_{i=1}^m h_i^2. \tag{2.4}$$

In [8] the two matrices Ω and J_η were introduced. Ω is the matrix whose i, j element is $\partial f_j/\partial x_i - \partial f_i/\partial x_j$. Note that all exact estimation algebras are characterized by the fact that $\Omega = 0$. $J_\eta = [\partial^2\eta/\partial x_i\,\partial x_j]$ is the Hessian matrix of η.

In [1], we proved the following structure theorems.

THEOREM 1. *Let* **E** *be a finite-dimensional exact estimation algebra. Then* $h_1, \cdots, h_m$ *are polynomials of degree at most one.*

THEOREM 2. *Let* $F(x_1, \cdots, x_n)$ *be a* C^∞ *function on* $\mathbb{R}^n$. *Suppose that there exists a path* $C: \mathbb{R} \to \mathbb{R}^n$ *and* $\delta > 0$ *such that* $\lim_{t\to\infty} \|C(t)\| = \infty$ *and* $\lim_{t\to\infty} \sup_{B_\delta(C(t))} F = -\infty$, *where* $B_\delta(C(t)) = \{x \in \mathbb{R}^n \mid \|x - C(t)\| < \delta\}$. *Then there is no* C^∞ *function* ψ *on* $\mathbb{R}^n$ *satisfying*

$$\Delta\psi + |\nabla\psi|^2 = F.$$

COROLLARY 1. *Let* $F(x_1, \cdots, x_n)$ *be a polynomial on* $\mathbb{R}^n$. *Suppose that there exists a polynomial path* $C: \mathbb{R} \to \mathbb{R}^n$ *such that* $\lim_{t\to\infty} \|C(t)\| = \infty$ *and* $\lim_{t\to\infty} F \circ C(t) = -\infty$. *Then there is no* C^∞ *function* ψ *on* $\mathbb{R}^n$ *satisfying*

$$\Delta\psi + |\nabla\psi|^2 = F. \tag{2.5}$$

THEOREM 3. *Suppose* **E** *is an exact estimation algebra. Then,* **E** *is finite-dimensional if and only if* $\nabla h_i^T J_\eta^j$ *is a constant for* $1 \leqq i \leqq m$ *and all* $j = 0, 1, \cdots$.

THEOREM 4. *Suppose* **E** *is an exact finite-dimensional estimation algebra. Then it has a basis consisting of one second-degree differential operator* L_0, *first-degree differential operator(s) with constant coefficients for the* $\partial/\partial x_i$ *terms,*[3] *and zero-degree differential operator(s) affine in* x. *Moreover, if* X *and* Y *are in* **E** *with degree less than or equal to one, then* $[X, Y]$ *is a constant.*

Theorem 5 follows from Theorem 4.

THEOREM 5. *An exact finite-dimensional estimation algebra is solvable.*

To show the relevancy of studying finite-dimensional exact estimation algebra in nonlinear filtering problems, we proved in [1] that a system defined by (2.0) with a finite-dimensional exact estimation algebra admits a universal finite-dimensional filter and provided an explicit Lie-algebraic method to construct such a filter.

Given the importance of the estimation algebra, a natural question arises as to whether we can classify all finite-dimensional exact estimation algebras up to Lie-algebraic isomorphism. Theorems 4 and 5 provide a starting point for solving this problem. In Theorem 6, we provide a more explicit structure theorem for an important subclass of finite-dimensional exact estimation algebras. A second question that arises naturally is whether we can classify all filtering systems with finite-dimensional exact estimation algebras up to state-space diffeomorphism. This is apparently a very difficult problem and requires a careful study of partial differential equations of the type (2.4).

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[3] This clarifies the original statement of Theorem 5 of [1].

The connection between these types of equations and the nonlinear filtering problem was first noted by Benes (see [9]). The properties of these equations, however, are not well known. In Theorems 9 and 12, we provide some answers in regard to the existence and uniqueness of the solutions of these types of equations. Our result here is far from providing a reasonable classification theory of systems with finite-dimensional exact estimation algebras, but it may be viewed as a necessary first step.

3. Classification theorems. If **E** is finite-dimensional, then the matrix

$$M=[\nabla h_1,\cdots,\nabla h_m, J_\eta \nabla h_1,\cdots, J_\eta\nabla h_m, J_\eta^2\nabla h_1,\cdots, J_\eta^2\nabla h_m,\cdots] \tag{3.0}$$

is a constant matrix and

$$h_1,\cdots,h_m, \nabla h_1^T\nabla\eta,\cdots,\nabla h_m^T\nabla \eta, \nabla h_1^T J_\eta\nabla\eta,\cdots,\nabla h_m^T J_\eta \nabla\eta,\cdots$$

are all linear functions in **E**. If the rank of M is n, we say that the corresponding estimation algebra has full rank. In this case, it is easy to describe the Lie algebra structure of the estimation algebra.

THEOREM 6. *Suppose* **E** *is of maximal rank. Then it is a real vector space of dimension* $2n+2$ *with basis given by* $1, x_1, x_2, \cdots, x_n, D_1, \cdots, D_n$, *and* L_0. *Moreover,* η *is a polynomial of degree at most two and the quadratic part of* $\eta-\sum_{i=1}^m h_i^2$ *is positive semidefinite.*

Proof. Since the columns of M represent gradient vectors of functions in **E** and M is a constant matrix with rank n, there are constants c_i's such that x_i+c_i is in **E** for $i=1,\cdots,m$. It is easy to show the following relations:

$$[L_0, x_i+c_i]=\frac{1}{2}\left[\sum_{j=1}^n D_j^2, x_i+c_i\right]=D_i,$$

$$[D_i, x_j+c_j]=\begin{cases}1 & \text{if } i=j,\\ 0 & \text{if } i\neq j,\end{cases}$$

$$[L_0, D_i]=\frac{1}{2}\left[\sum_{j=1}^n D_j^2, D_i\right]+\frac{1}{2}[D_i,\eta]=\frac{1}{2}\frac{\partial\eta}{\partial x_i}.$$

$\partial\eta/\partial x_i$ is a polynomial of degree at most one, for all $1\leqq i\leqq n$. Hence **E** is a real vector space spanned by $1, x_1,\cdots,x_n, D_1,\cdots,D_n$ and L_0. The fact that the quadratic part of $\eta-\sum_{i=1}^m h_i^2$ is positive semidefinite again follows from Theorem 2. □

Theorem 6 implies that all exact finite-dimensional estimation algebras with maximal rank come from Benes filters (see [9] for details concerning Benes filters).

For any filtering system defined in (2.0) with an exact estimation algebra, (2.4) assigns a characteristic η. Theorem 6 implies that if the estimation algebra is finite-dimensional with maximal rank, then this mapping maps the given system to a quadratic polynomial. In order to develop a classification of systems with finite-dimensional estimation algebras, we need to know the range of this mapping restricted to such systems. We also need to understand the properties of the inverse of this mapping. In the following, we will provide some partial results to these questions. The key to these questions is a complete understanding of the existence and uniqueness properties of (2.5).

Let q be a C^∞ function defined on $\mathbb{R}^n$. Extend $-\Delta+q$ in the standard way to act on a closed subspace of $L^2(\mathbb{R}^n)$. It follows from the definition that the first eigenvalue λ_1 of the operator $-\Delta+q$ is equal to

$$\lambda_1=\inf_\phi\frac{\int|\nabla\phi|^2\,dx+\int q\phi^2\,dx}{\int\phi^2\,dx}, \tag{3.1}$$

where infimum is taken on all nonzero C^∞ functions with compact support. The following theorem by Fischer-Colbrie and Schoen is well known (see [10]).

THEOREM 7 [10]. *Let q be a C^∞ function defined on $\mathbb{R}^n$. Then there exists a positive function ζ satisfying the equation $\Delta\zeta - q\zeta = 0$ on $\mathbb{R}^n$ if and only if the first eigenvalue λ_1 of $-\Delta + q$ on $\mathbb{R}^n$ is nonnegative.*

Assume now that the estimation algebra is finite-dimensional and has maximal rank. Then by Theorem 6, we know that

$$\eta - \sum_{i=1}^{m} h_i^2 = q, \tag{3.2}$$

where q is a polynomial of degree two with quadratic part positive semidefinite. Recall that

$$\begin{aligned}\eta &= \sum_{i=1}^{m} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{m} h_i^2 \\ &= \Delta\phi + |\nabla\phi|^2 + \sum_{i=1}^{m} h_i^2.\end{aligned}$$

Putting this into (3.2), we have

$$\Delta\phi + |\nabla\phi|^2 = q. \tag{3.3}$$

Let $u = e^\phi$. Then $\partial u/\partial x_i = (\partial\phi/\partial x_i)\, e^\phi$ and $\partial^2 u/\partial x_i^2 = (\partial^2\phi/\partial x_i^2 + (\partial\phi/\partial x_i)^2)\, e^\phi$, hence

$$\Delta u - qu = 0. \tag{3.4}$$

We observe that (3.3) has a C^∞-solution ϕ if and only if (3.4) has a C^∞ positive solution u.

THEOREM 8. *Let q be a quadratic polynomial in $x_1, \cdots, x_n$. Let λ_1 be the first eigenvalue of the operator $-\Delta + q$. Then λ_1 is nonnegative if and only if under an orthogonal transformation and a translation, q can be written in the form*

$$\sum_{i=1}^{n} a_i x_i^2 - c,$$

where a_i and c are constants, $a_i \geqq 0$, and $c \leqq \sum_{i=1}^{n} \sqrt{a_i}$.

Proof. Suppose that $x = (x_1, \cdots, x_n)^T = Ay - y_0$, where A is an orthogonal matrix and y_0 is a constant vector. Then $\Delta_y = \Delta_x$, and the first eigenvalue of the operator $-\Delta_x + q$ is nonnegative if and only if the first eigenvalue of $-\Delta_y + \tilde{\eta}$ is nonnegative where $\tilde{\eta}(y) = \eta(x(y))$. Hence after an orthogonal transformation and a translation, we may assume that

$$q(x) = \sum_{i=1}^{l} a_i x_i^2 + \sum_{i=l+1}^{n} b_i x_i - c,$$

where a_i, b_i, and c are constants, $a_i \neq 0$, for $i = 1, \cdots, l$.

By Theorem 7, we know that $\lambda_1 \geqq 0$ if and only if (3.4) has C^∞ positive solution if and only if (3.3) has C^∞ solution. In view of Theorem 2, this implies that $b_i = 0$ and we have that $a_i \geqq 0$ for $i = 1, \cdots, n$. Hence it remains to prove that the first eigenvalue of the operator $-\Delta + r - c$ is nonnegative if and only if $c \leqq \sum_{i=1}^{n} \sqrt{a_i}$, where $r = \sum_{i=1}^{n} a_i x_i^2$. This is equivalent to proving that the first eigenvalue λ_1' of the operator $-\Delta + r$ is $\sum_{i=1}^{n} \sqrt{a_i}$.

Denote $c_0=\sum_{i=1}^n \sqrt{a_i}$. Let ξ be a C^∞ function with compact support. Then

$$\begin{aligned}\int |\nabla\xi|^2\,dx+\int r\xi^2\,dx &= \int \sum_{i=1}^n \left(\left(\frac{\partial \xi}{\partial x_i}\right)^2 + a_i x_i^2 \xi^2\right) dx \\ &= \int \sum_{i=1}^n \left(\frac{\partial \xi}{\partial x_i}+\sqrt{a_i}\, x_i\xi\right)^2 dx - 2\int \left(\sum_{i=1}^n \sqrt{a_i}\, x_i \xi \frac{\partial \xi}{\partial x_i}\right) dx \\ &= \int \sum_{i=1}^n \left(\frac{\partial \xi}{\partial x_i}+\sqrt{a_i}\, x_i\xi\right)^2 dx \\ &\qquad - \int \sum_{i=1}^n \frac{\partial}{\partial x_i}(\sqrt{a_i}\, x_i \xi^2)\,dx + \int \left(\sum_{i=1}^n \sqrt{a_i}\right)\xi^2\,dx \\ &= \int \sum_{i=1}^n \left(\frac{\partial \xi}{\partial x_i}+\sqrt{a_i}\, x_i\xi\right)^2 dx + c_0 \int \xi^2\,dx \\ &\geqq c_0 \int \xi^2\,dx.\end{aligned}$$

Hence $\lambda_1' \geqq c_0$. On the other hand,

$$\chi(x)=\prod_{i=1}^n \exp\left(-\frac{\sqrt{a_i}\, x_i^2}{2}\right)$$

is an eigenfunction of $-\Delta+r$ with eigenvalue c_0, so $c_0 \geqq \lambda_1'$. Hence $c_0=\lambda_1'$. □

THEOREM 9. *Suppose* **E** *is a finite estimation algebra of maximal rank. Then under an orthogonal transformation and a translation,* η *can be written in the form*

$$\sum_{i=1}^m h_i^2+\sum_{i=1}^n a_i x_i^2 - c,$$

where a_i *and* c *are constants,* $a_i \geqq 0$, *and* $c \leqq \sum_{i=1}^n \sqrt{a_i}$.

Proof. This result follows from Theorems 7 and 8. □

4. Alternative proof. Theorem 9 provides a constraint that the coefficients of (2.0) must satisfy so that the system has a finite-dimensional estimation algebra of maximal rank. It is a first step in providing some classification results of all finite-dimensional exact estimation algebras. In the following, we provide an alternative proof of these results by applying a technique pioneered by Li and Yau [11]. In fact, Theorem 12 sharpens the results stated in Theorem 9.

THEOREM 10. *Consider the following equation*:

$$\Delta\xi+|\nabla\xi|^2=\sum_{ij=1}^n a_{ij}x_i x_j - c, \tag{4.0}$$

where $(x_1,\cdots,x_n)\in\mathbb{R}^n$, $c\in\mathbb{R}$ *and the constant matrix* $A=(a_{ij})$ *is positive semidefinite. Then for any smooth solution* ξ *of* (4.0) *defined on* $\mathbb{R}^n$, $|\nabla\xi|$ *has at most linear growth, namely,*

$$|\nabla\xi(x)|\leqq C(1+|x|), \qquad x\in\mathbb{R}^n$$

for some constant C.

Proof. Let $u=-\xi$. After orthogonal change of coordinates, (4.0) becomes

$$-\Delta u+|\nabla u|^2=\sum_{i=1}^n \lambda_i x_i^2 - c. \tag{4.1}$$

Let $v=\sum_{i=1}^n \frac{1}{2}\sqrt{\lambda_i}\, x_i^2$. It is easy to see that

$$-\Delta v+|\nabla v|^2=\sum_{i=1}^n \lambda_i x_i^2-c_0, \tag{4.2}$$

where $c_0=\sum_{i=1}^n \sqrt{\lambda_i}$. Let $w=u-v$. Then subtracting (4.2) from (4.1), we get

$$\begin{aligned} c_0-c&=-\Delta u+\Delta v+|\nabla u|^2-|\nabla v|^2\\ &=-\Delta(u-v)+\nabla(u-v)\cdot\nabla(u-v)+2\nabla(u-v)\cdot\nabla v\\ &=-\Delta w+|\nabla w|^2+2\nabla w\cdot\nabla v, \end{aligned} \tag{4.3}$$

where $x\cdot y$ represents the standard inner product between vectors x and y. Denote $F=|\nabla w|^2$. Direct computation yields

$$\begin{aligned} \Delta F&=\Delta(\nabla w\cdot\nabla w)\\ &=2\sum_{i,j=1}^n\left(\frac{\partial^2 w}{\partial x_i\,\partial x_j}\right)^2+2(\nabla\Delta w)\cdot\nabla w\\ &\geqq\frac{2}{n}|\Delta w|^2+2\nabla(F+2\nabla v\cdot\nabla w+c-c_0)\cdot\nabla w. \end{aligned} \tag{4.4}$$

If $\nabla^2 v$ and $\nabla^2 w$ represent the Hessian of v and w, respectively, then

$$\begin{aligned} 4\nabla(\nabla v\cdot\nabla w)\cdot\nabla w&=4[\nabla^2 v\nabla w+\nabla^2 w\nabla v]\cdot\nabla w\\ &\geqq 4[\nabla^2 w\nabla v]\cdot\nabla w=4[\nabla^2 w\nabla w]\cdot\nabla v\\ &=2\nabla(\nabla w\cdot\nabla w)\cdot\nabla v=2\nabla F\cdot\nabla v. \end{aligned} \tag{4.5}$$

Putting (4.5) into (4.4), we have

$$\begin{aligned} \Delta F&\geqq\frac{2}{n}(F+2\nabla v\cdot\nabla w+c-c_0)^2+2\nabla F\cdot\nabla w+2\nabla F\cdot\nabla v\\ &\geqq\frac{2}{n}F^2+\frac{4}{n}F(2\nabla v\cdot\nabla w+c-c_0)+2\nabla F\cdot\nabla(v+w)\\ &\geqq\frac{2}{n}F^2-\frac{8}{n}F^{3/2}|\nabla v|+2\nabla F\cdot\nabla(v+w)+\frac{4(c-c_0)}{n}F. \end{aligned} \tag{4.6}$$

Denote $r^2=\sum_{i=1}^n x_i^2$. For $a>0$, the function $(a^2-r^2)^2F$ achieves its maximum at $x_0\in B_a(0)=\{x\in\mathbb{R}^n:|x|<a\}$. At that point,

$$\nabla[(a^2-r^2)^2F]=0,$$

which implies

$$4rF\nabla r=(a^2-r^2)\nabla F. \tag{4.7}$$

Also at the point x_0,

$$\begin{aligned} 0&\geqq\Delta[(a^2-r^2)^2F]\\ &=(a^2-r^2)^2\Delta F+2\nabla(a^2-r^2)^2\cdot\nabla F+F\Delta(a^2-r^2)^2\\ &=(a^2-r^2)^2\Delta F-8(a^2-r^2)r\nabla r\cdot\nabla F+[8r^2-4n(a^2-r^2)]F. \end{aligned} \tag{4.8}$$

Using (4.7), we get

$$(a^2-r^2)^2\Delta F-24r^2F-4n(a^2-r^2)F\leqq 0. \tag{4.9}$$

Using (4.6), we have

$$\begin{aligned}(a^2-r^2)^2\left[\frac{2}{n}F^2-\frac{8}{n}F^{3/2}|\nabla v|+2\nabla F\cdot\nabla(v+w)+\frac{4(c-c_0)}{n}F\right]&\\ -[24r^2+4n(a^2-r^2)]F\leqq 0.&\end{aligned} \tag{4.10}$$

Dotting (4.7) with $\nabla(v+w)$, we get

$$\begin{aligned}(a^2-r^2)\nabla(v+w)\cdot\nabla F&=4rF\nabla r\cdot(\nabla v+\nabla w)\\ &\geqq -4rF|\nabla v|-4rF^{3/2}.\end{aligned} \tag{4.11}$$

Putting (4.11) back into (4.10) and dividing it by F, we have

$$\begin{aligned}\frac{2}{n}(a^2-r^2)^2F-\frac{8}{n}(a^2-r^2)^2F^{1/2}|\nabla v|-(a^2-r^2)[8r|\nabla v|+8rF^{1/2}]&\\ +4(a^2-r^2)^2\frac{c-c_0}{n}-[24r^2+4n(a^2-r^2)]\leqq 0.&\end{aligned} \tag{4.12}$$

By denoting $M=(a^2-r^2)F^{1/2}$, (4.12) becomes

$$\begin{aligned}&\frac{2}{n}M^2-\left[\frac{8}{n}(a^2-r^2)|\nabla v|+8r\right]M\\ &\quad+\left[4(a^2-r^2)^2\frac{c-c_0}{n}-4n(a^2-r^2)-8r(a^2-r^2)|\nabla v|-24r^2\right]\leqq 0.\end{aligned}$$

Noting the fact that $|\nabla v|\leqq c_1r$ and $r\leqq a$, we can see that

$$M\leqq c_2a^3,$$

where c_1 and c_2 are constants. The inequality

$$\begin{aligned}M&=\max_{|x|\leqq a}(a^2-r^2(x))F^{1/2}(x)\\ &\geqq \max_{|x|\leqq a/2}(a^2-|x|^2)F^{1/2}(x)\\ &\geqq \max_{|x|\leqq a/2}\left[a^2-\left(\frac{a}{2}\right)^2\right]F^{1/2}(x)\\ &=\frac{3}{4}a^2\max_{|x|\leqq a/2}|\nabla w|\end{aligned}$$

yields the estimate

$$\max_{|x|\leqq a/2}|\nabla w|\leqq C_3a. \tag{4.13}$$

Combining (4.13) with the relation $w=u-v$, we can conclude that $|\nabla u|$ has at most linear growth. □

Remark. We can also deduce the above theorem by making use of Theorem 1.3 of [11].

THEOREM 11. *If* $c<\sqrt{\lambda}$, *and* $\lambda>0$, *then*

$$-u''+(u')^2=\lambda x^2-c,\quad u(0)=a,\quad u'(0)=b, \tag{4.14}$$

has a global solution for any a and small $|b|$.

Proof. Let $v=u'$. We have

$$v'=v^2-\lambda x^2+c,\qquad v(0)=b.$$

Suppose (A, B) is the maximum open interval containing zero, such that v exists. Define two auxiliary functions:

$$v_+(x)=\varepsilon x+k,\qquad v_-(x)=-\delta x-l.$$

We have that

$$\begin{aligned} v_+'-v_+^2+\lambda x^2-c&=\varepsilon-(\varepsilon x+k)^2+\lambda x^2-c\\ &=(\lambda-\varepsilon^2)x^2-2k\varepsilon x+(\varepsilon-c-k^2). \end{aligned}$$

Choose $\varepsilon>0$, such that

$$\lambda-\varepsilon^2>0\quad\text{and}\quad \varepsilon-c>0.$$

This is possible because $\sqrt{\lambda}>c$. Choose $k>0$ small enough, so that

$$(\lambda-\varepsilon^2)x^2-2k\varepsilon x+(\varepsilon-c-k^2)>0,\qquad x\in[0, B].$$

By the standard comparison theorem [12], we have

$$v(x)<v_+(x)\quad\text{for } x\in[0, B),$$

as long as $v(0)=b<k=v_+(0)$.

Similarly, we can show that if δ is sufficiently large, so that

$$\lambda-\delta^2<0\quad\text{and}\quad \delta+c>0,$$

then

$$v_-'-v_-^2+\lambda x^2-c=(\lambda-\delta^2)x^2-2l\delta x-(\delta+c+l^2)<0$$

for all $x\in[0, B]$ and $l\geqq 0$.

The comparison theorem again implies that

$$v(x)>v_-(x)\quad\text{for } x\in[0, B),$$

if $v(0)=b>-l=v_-(0)$.

This implies that when $-l<b<k$, $B=\infty$. Otherwise, as $v(B)$ is bounded, we can extend v beyond B, a contradiction to the hypothesis that (A, B) is the maximal interval on which v is defined.

Similar arguments show that $A=-\infty$ when $|b|$ is sufficiently small. □

Remark. If $\lambda=0$, then we can prove by direct integration that there is a global solution to (4.14) if $|b|$ is small enough.

THEOREM 12. *Consider the following equation*:

$$\Delta\xi+|\nabla\xi|^2=\sum_{i,j=1}^{n} a_{ij}x_ix_j-c, \tag{4.15}$$

where $(x_1,\cdots,x_n)\in\mathbb{R}^n$, $c\in\mathbb{R}$ *and the constant matrix* $A=(a_{ij})$ *is positive semidefinite. Let* $\{\lambda_1,\cdots,\lambda_n\}$ *be the eigenvalues of A and* $c_0=\sum_{i=1}^n\sqrt{\lambda_i}$. *Then we have the following*:

(I) (*Existence*). *When* $c < c_0$, *there is a family of* C^∞ *solution of* (4.15) *with* $2n$ *parameters such that* $|\nabla \xi|$ *has at most linear growth at* ∞, *namely,*

$$|\nabla \xi(x)| \leqq C(1+|x|),$$

for some constant C.

(II) (*Uniqueness*). *When* $c = c_0$, *there is a quadratic polynomial, uniquely determined up to a constant, which satisfies* (4.15). *Moreover, this is the unique solution up to a constant if either one of the following conditions holds*:

(i) rank $A = 0$ (*namely,* $A = 0$), *or*

(ii) rank $A \geqq n-2$.

(III) (*Nonexistence*). *When* $c > c_0$, *there is no smooth solution to* (4.15).

Proof. Let $u = -\xi$. After an orthogonal change of coordinates, (4.15) becomes

$$-\Delta u + |\nabla u|^2 = \sum_{i=1}^{n} \lambda_i x_i^2 - c. \tag{4.16}$$

For part (I), let $c = \sum_{i=1}^n c_i$ with $c_i < \sqrt{\lambda_i}$. By Theorem 11, there is a 2-parameter family of solution of the

$$-u_i'' + (u_i')^2 = \lambda_i x_i^2 - c_i. \tag{4.17}$$

It is easy to see that $u(x) = \sum_{i=1}^n u_i(x_i)$ satisfies (4.15) and that $|\nabla u|$ has at most linear growth at ∞. To see that there is a $2n$-parameter family of such solutions to (4.16), note that $n-1$ parameters come from the different ways of decomposing c into c_i's so that $c = \sum_{i=1}^n c_i$, n parameters come from $u_i'(0)$, and the last parameter comes from the arbitrary constant added to the whole solution.

For part (II), it is clear that there exists a uniquely determined quadratic polynomial solution. If in addition the first rank condition is satisfied, we need only to prove that the only solutions of

$$-\Delta u + |\nabla u|^2 = 0 \tag{4.18}$$

are constants. Taking $\Phi = e^{-u}$, (4.18) can be written as

$$\Delta \Phi = 0. \tag{4.19}$$

It is equivalent to prove that (4.19) has no positive solutions other than constants. However, this is well known to be the case.

Next, assume the second rank condition is satisfied, that is, rank $A \geqq n-2$. Let $v(x) = \sum_{i=1}^n \frac{1}{2}\sqrt{\lambda_i}\, x_i^2$. Note that $v(x)$ satisfies

$$-\Delta v + |\nabla v|^2 = \sum_{i=1}^{n} \lambda_i x_i^2 - c_0. \tag{4.20}$$

Subtracting (4.20) from (4.16) and letting $w = u - v$, we get

$$-\Delta w + 2\nabla v \cdot \nabla w + |\nabla w|^2 = 0. \tag{4.21}$$

Define $B(r) = \{x : |x_i| \leqq r, i = 1, 2, \cdots, n\}$. Multiplying by e^{-2v} and integrating on both sides of (4.21), we get

$$\begin{aligned} 0 &= \int_{B(r)} e^{-2v}(-\Delta w + 2\nabla v \cdot \nabla w) + \int_{B(r)} e^{-2v}|\nabla w|^2 \\ &= \int_{B(r)} -\nabla \cdot (e^{-2v}\nabla w) + \int_{B(r)} e^{-2v}|\nabla w|^2 \\ &= -\int_{\partial B(r)} e^{-2v}\nabla w \cdot \vec{n} + \int_{B(r)} e^{-2v}|\nabla w|^2, \end{aligned}$$

where $\vec{n}$ is the unit outward normal of $\partial B(r)$. By the Schwartz inequality, we get

$$\begin{aligned}\int_{B(r)} e^{-2v}|\nabla w|^2 &= \int_{\partial B(r)} e^{-2v}\nabla w\cdot\vec{n} \leqq \int_{\partial B(r)} e^{-2v}|\nabla w| \\ &\leqq \sqrt{\left(\int_{\partial B(r)} e^{-2v}|\nabla w|^2\right)\left(\int_{\partial B(r)} e^{-2v}\right)}.\end{aligned} \tag{4.22}$$

Denoting $f(r)=\int_{B(r)} e^{-2v}|\nabla w|^2$, $g(r)=\int_{B(r)} e^{-2v}$, (4.22) becomes

$$(f(r))^2 \leqq f'(r)\cdot g'(r). \tag{4.23}$$

Supposing $f(r_0)>0$ for some r_0, we have

$$\frac{f'}{f^2} \geqq \frac{1}{g'}. \tag{4.24}$$

Integrating (4.24) over $(r_0, +\infty)$, we have

$$\infty > \frac{1}{f(r_0)} - \frac{1}{f(\infty)} = \int_{r_0}^{\infty} \frac{f'}{f^2}\,dr \geqq \int_{r_0}^{\infty} \frac{1}{g'}\,dr. \tag{4.25}$$

Other the other hand,

$$g(r) = \int_{B(r)} e^{-2v} = \prod_{i=1}^{n}\left(\int_{-r}^{r} \exp\left(-\sqrt{\lambda_i}\,x_i^2\right)dx_i\right).$$

It is easy to see that

(i) If rank $A=n$, $g'(r)\to 0$ as $r\to\infty$.
(ii) If rank $A=n-1$, $g'(r)\to c>0$ as $r\to\infty$.
(iii) If rank $A=n-2$, $g'(r)\approx cr$ as $r\to\infty$.

In all of the above three cases, the right-hand side of (4.25) is divergent. The contradiction says that $f(r)\equiv 0$, namely, w is a constant. So $u=v+\text{const}$.

For part (III), the statement holds even if A is degenerate. Since $c_0=\sum_{i=1}^{n}\sqrt{\lambda_i}<c$, we can find δ small such that $\sum_{i=1}^{n}(\sqrt{\lambda_i}+\delta)<c$. Let $v=\sum_{i=1}^{n}\frac{1}{2}(\sqrt{\lambda_i}+\delta)x_i^2$. v satisfies

$$-\Delta v+|\nabla v|^2 = \sum_{i=1}^{n}(\sqrt{\lambda_i}+\delta)^2 x_i^2 - \sum_{i=1}^{n}(\sqrt{\lambda_i}+\delta). \tag{4.26}$$

Subtracting (4.26) from (4.16) and letting $w=u-v$, we get

$$-\Delta w + 2\nabla v\cdot\nabla w \leqq 0.$$

Using the same argument as before, but without assuming the rank condition on A, we can show that $u=v+\text{const}$. So u cannot be a solution to (4.16). A contradiction and no smooth solution to (4.16) exists. □

Remark. Equation (4.15) may have other solutions in addition to those listed in part (I). Some examples are given on page 86 of [9].

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