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FINITE-DIMENSIONAL FILTERS WITH NONLINEAR DRIFT II: BROCKETT'S PROBLEM ON CLASSIFICATION OF FINITE-DIMENSIONAL ESTIMATION ALGEBRAS*

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Abstract. The idea of using estimation algebras to construct finite-dimensional nonlinear filters was first proposed by Brockett and Mitter independently. It turns out that the concept of estimation algebra plays a crucial role in the investigation of finite-dimensional nonlinear filters. In his talk at the International Congress of Mathematics in 1983, Brockett proposed classifying all finite-dimensional estimation algebras. In this paper, all finite-dimensional algebras with maximal rank are classified if the dimension of the state space is less than or equal to two. Therefore, from the Lie algebraic point of view, all finite-dimensional filters are understood generically in the case where the dimension of state space is less than three.

Key words. nonlinear filters, estimation algebra, Wei–Norman approach

AMS subject classifications. 17B30, 35J15, 60G35, 93E11

1. Introduction. In a previous paper [Ya], Yau has studied the general class of nonlinear filtering systems that include both Kalman–Bucy and Benes filtering systems as special cases. Simple algebraic necessary and sufficient conditions were proved for an estimation algebra of such filtering system to be finite-dimensional. Using the Wei–Norman approach, he constructed explicitly finite-dimensional recursive filters for such nonlinear filtering systems. This paper is, in essence, a continuation of [Ya] and we strongly recommend that readers familiarize themselves with the results in [Ya]. However, every effort will be made to make this paper as self-contained as possible without too much duplication of the previous paper.

The idea of using estimation algebras to construct finite-dimensional nonlinear filters was first proposed in Brockett and Clark [Br-Cl], Brockett [Br1], and Mitter [Mi]. The concept of estimation algebras has proved to be an invaluable tool in the study of nonlinear filtering problems. In his famous talk at the International Congress of Mathematics in 1983, Brockett proposed classifying all finite-dimensional estimation algebras. There were some interesting results in 1987 due to Wong [Wo] under the assumptions that the observation $h(x)$ and drift term $f(x)$ are real analytic functions on $\mathbf{R}^n$, and f satisfies the following growth conditions: for any i, all the first-, second-, and third-order partial derivatives of f_i are bounded functions. Under all these conditions, Wong provides partial information toward the classification of finite-dimensional estimation algebra. Namely, he showed that if the estimation algebra is finite-dimensional, then the degree of h in x is at most one, and the estimation algebra has a basis consisting of one second-degree differential operator, L_0 (see (2.1)), first-

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degree differential operators of the form

$$\sum_{i=1}^{n} \alpha_i \left(\frac{\partial}{\partial x_i} - f_i \right) + \sum_{i=1}^{n} \beta_i \frac{\partial \eta}{\partial x_i},$$

where α_i and β_i are constants and

$$\eta = -\frac{1}{2} \left(\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{m} h_i^2 \right),$$

and zero-degree differential operators affine in x. In [T-W-Y], Tam, Wong, and Yau have introduced the concept of an estimation algebra with maximal rank. This is one of the most important general subclass of estimation algebras. Let n be the dimension of the state space. It turns out that all nontrivial finite-dimensional estimation algebras are automatically exact with maximal rank if $n = 1$. It follows from the works of Ocone [Oc], Tam, Wong, and Yau [T-W-Y], and Dong et al. [D-T-W-Y] that the finite-dimensional estimation algebras are completely classified if $n = 1$. In fact, Tam, Wong, and Yau have classified all finite-dimensional exact estimation algebras with maximal rank of arbitrary dimension. In this paper, we classify all finite-dimensional estimation algebras with maximal rank if $n = 2$. The novelty of the problem is that there is no assumption on the drift term of the nonlinear filtering system. The following is our main theorem.

MAIN THEOREM. *Suppose that the state space of the filtering system* (2.0) *below is of dimension two. If E is the finite-dimensional estimation algebra with maximal rank, then the drift term f must be linear vector field plus gradient vector field, and E is a real vector space of dimension 6 with basis given by* 1, x_1, x_2, D_1, D_2, *and* L_0.

This kind of nonlinear filtering systems was studied by Yau [Ya]. Therefore, from the Lie algebraic point of view, we have shown that the finite-dimensional filters considered in [Ya] are the most general finite-dimensional filters.

2. Basic concepts. In this section, we will recall some basic concepts and results from [Ya]. Consider a filtering problem based on the following signal observation model:

$$\begin{aligned} dx(t) &= f(x(t))dt + g(x(t))dv(t), \qquad x(0) = x_0, \\ dy(t) &= h(x(t))dt + dw(t), \qquad y(0) = 0 \end{aligned} \tag{2.0}$$

in which x, v, y, and w are, respectively, $\mathbf{R}^n$, $\mathbf{R}^p$, $\mathbf{R}^m$, and $\mathbf{R}^m$ valued processes, and v and w have components that are independent, standard Brownian processes. We further assume that $n = p$, f, h are C^∞ smooth and that g is an orthogonal matrix. We will refer to $x(t)$ as the state of the system at time t and to $y(t)$ as the observation at time t.

Let $\rho(t, x)$ denote the conditional density of the state given the observation $\{y(s) : 0 \le s \le t\}$. It is well known (see [Da-Ma], for example) that $\rho(t, x)$ is given by normalizing a function, $\sigma(t, x)$, which satisfies the following Duncan–Mortensen–Zakai equation:

$$d\sigma(t, x) = L_0 \sigma(t, x) dt + \sum_{i=1}^{m} L_i \sigma(t, x) dy_i(t), \qquad \sigma(0, x) = \sigma_0, \tag{2.1}$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2$$

and for $i = 1, \dots, m$, L_i is the zero-degree differential operator of multiplication by h_i. σ_0 is the probability density of the initial point x_0. In this paper, we will assume σ_0 is a C^∞ function.

Equation (2.1) is a stochastic partial differential equation. In real applications, we are interested in constructing state estimators from observed sample paths with some property of robustness. Davis in [Da] studied this problem and proposed some robust algorithms. In our case, his basic idea reduces to defining a new unnormalized density

$$\xi(t,x) = \exp\left(-\sum_{i=1}^{m} h_i(x) y_i(t)\right)\sigma(t,x).$$

It is easy to show that $\xi(t,x)$ satisfies the following time-varying partial differential equation

$$\begin{aligned} \frac{\partial \xi}{\partial t}(t,x) &= L_0\xi(t,x) + \sum_{j=1}^{m} y_i(t)[L_0, L_i]\xi(t,x) \\ &\quad + \frac{1}{2}\sum_{i,j=1}^{m} y_i(t)y_j(t)[[L_0,L_i],L_j]\xi(t,x), \\ \xi(0,x) &= \sigma_0, \end{aligned} \tag{2.2}$$

where $[\cdot\,,\cdot]$ is the Lie bracket defined as follows.

DEFINITION. If X and Y are differential operators, the Lie bracket of X and Y, $[X,Y]$, is defined by $[X,Y]\varphi = X(Y\varphi) - Y(X\varphi)$ for any C^∞ function φ.

Recall that a real vector space $\mathcal{F}$, with an operation $\mathcal{F}\times\mathcal{F} \to \mathcal{F}$ denoted $(x,y) \mapsto [x,y]$ and called the Lie bracket of x and y, is called a Lie algebra if the following axioms are satisfied:

(1) The Lie bracket operation is bilinear;
(2) $[x,y] = 0$ for all $x \in \mathcal{F}$;
(3) $\big[x,[y,z]\big] + \big[y,[z,x]\big] + \big[z,[x,y]\big] = 0 \quad (x,y,z \in \mathcal{F})$.

DEFINITION. The estimation algebra E of a filtering problem (2.0) is defined to be the Lie algebra generated by $\{L_0, L_1, \dots, L_m\}$ or $E = \langle L_0, L_1, \dots, L_m\rangle_{L.A.}$. If, in addition, there exists a potential function φ such that $f_i = \partial\varphi/\partial x_i$ for all $1 \le i \le n$, then the estimation algebra is called exact.

In [Ya], the following proposition is proven.

PROPOSITION 1. *$\partial f_j/\partial x_i - \partial f_i/\partial x_j = c_{ij}$ are constants for all i and j if and only if $(f_1, \dots, f_n) = (\ell_1, \dots, \ell_n) + (\partial\varphi/\partial x_1, \dots, \partial\varphi/\partial x_n)$, where $\ell_1, \dots, \ell_n$ are polynomials of degree one and φ is a C^∞ function.*

Define

$$D_i = \frac{\partial}{\partial x_i} - f_i$$

and

$$\eta = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{m} h_i^2.$$

Then

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^{n} D_i^2 - \eta\right).$$

We need the following basic results for later discussion.

THEOREM 2 (Ocone). *Let E be a finite-dimensional estimation algebra. If a function ξ is in E, then ξ is a polynomial of degree less than or equal to 2.*

Ocone's theorem ([Oc], see [Co] for an extension) says that $h_1, \ldots, h_m$ in a finite-dimensional estimation algebra are polynomials of degree less than or equal to 2.

The following theorem proved in [Ya] plays a fundamental role in the classification of finite-dimensional estimation algebra.

THEOREM 3. *Let E be a finite-dimensional estimation algebra of (2.0) satisfying $\partial f_j/\partial x_i - \partial f_i/\partial x_j = c_{ij}$, where c_{ij} are constants for all $1 \leq i, j \leq n$. Then $h_1, \ldots, h_m$ are polynomials of degree at most one.*

In view of the above theorem, we introduce the following definition.

DEFINITION. The estimation algebra E of a filtering problem (2.0) is said to be the estimation algebra with maximal rank if $x_i + c_i$ is in E for all $1 \leq i \leq n$ where c_i is a constant.

In [Ya], the following theorem was also proved.

THEOREM 4. *Let $F(x_1, \ldots, x_n)$ be a polynomial on $\mathbf{R}^n$. Suppose that there exists a polynomial path $c : \mathbf{R} \to \mathbf{R}^n$ such that $\lim_{t\to\infty} \|c(t)\| = \infty$ and $\lim_{t\to\infty} F \circ c(t) = -\infty$. Then there is no C^∞ functions $f_1, f_2, \ldots, f_n$ on $\mathbf{R}^n$ satisfying the equation*

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = F.$$

We recall the following simple lemma proved in [Ya].

LEMMA 5. (i) *$[XY, Z] = X[Y, Z] + [X, Z]Y$, where X, Y and Z are differential operators.*

(ii) *$[gD_i, h] = g\ (\partial h/\partial x_i)$, where $D_i = \partial/\partial x_i - f_i$, g and h are functions defined on $\mathbf{R}^n$.*

(iii) *$[gD_i, hD_j] = -gh\omega_{ij} + g(\partial h/\partial x_i)D_j - h(\partial g/\partial x_j)\ D_i$, where $\omega_{ji} = [D_i, D_j] = (\partial f_i/\partial x_j) - (\partial f_j/\partial x_i)$.*

(iv) *$[gD_i^2, h] = 2g(\partial h/\partial x_i)\ D_i + g(\partial^2 h/\partial x_i^2)$.*

(v) *$[D_i^2, hD_j] = 2(\partial h/\partial x_i)\ D_iD_j - 2h\omega_{ij}D_i + (\partial^2 h/\partial x_i^2)\ D_j - h(\partial \omega_{ij}/\partial x_i)$.*

LEMMA 6.

$$\text{(i)} \qquad [D_i^2, D_j^2] = 4\omega_{ji}D_jD_i + 2(\partial\omega_{ji}/\partial x_j)D_i + (\partial\omega_{ji}/\partial x_i)D_j + (\partial^2\omega_{ji}/\partial x_i\partial x_j) + 2\omega_{ji}^2.$$

$$\begin{aligned}\text{(ii)} \qquad [D_k^2, hD_iD_j] = {} & 2\ (\partial h/\partial x_k)D_kD_iD_j + 2h\omega_{jk}D_iD_k + 2h\omega_{ik}D_kD_j \\ & + (\partial^2 h/\partial x_k^2)D_iD_j + 2h(\partial\omega_{jk}/\partial x_i)D_k + h(\partial\omega_{jk}/\partial x_k)D_i \\ & + h(\partial\omega_{ik}/\partial x_k)D_j + h(\partial^2\omega_{jk}/\partial x_i\partial x_k).\end{aligned}$$

$$\begin{aligned}\text{(iii)} \qquad [D_iD_j, hD_k] = {} & (\partial h/\partial x_j)D_iD_k + (\partial h/\partial x_i)D_jD_k + h\omega_{kj}D_i + h\omega_{ki}D_j \\ & + (\partial^2 h/\partial x_i\partial x_j)D_k + h(\partial\omega_{kj}/\partial x_i).\end{aligned}$$

Proof.

(i)
$$
\begin{aligned}
[D_i^2, D_j^2] &= D_i[D_i, D_j^2] + [D_i, D_j^2]D_i \\
&= -D_i\left[2\omega_{ij}D_j + \frac{\partial \omega_{ij}}{\partial x_j}\right] - \left[2\omega_{ij}D_j + \frac{\partial \omega_{ij}}{\partial x_j}\right]D_i \\
&= -2\frac{\partial \omega_{ij}}{\partial x_i}D_j - 2\omega_{ij}D_iD_j - \frac{\partial^2 \omega_{ij}}{\partial x_i \partial x_j} - \frac{\partial \omega_{ij}}{\partial x_j}D_i \\
&\quad - 2\omega_{ij}D_jD_i - \frac{\partial \omega_{ij}}{\partial x_j}D_i \\
&= 4\omega_{ji}D_jD_i + 2\frac{\partial \omega_{ji}}{\partial x_j}D_i + 2\frac{\partial \omega_{ji}}{\partial x_i}D_j + \frac{\partial^2 \omega_{ji}}{\partial x_i \partial x_j} + 2\omega_{ji}^2.
\end{aligned}
$$

(ii)
$$
\begin{aligned}
[D_k^2, hD_iD_j] &= -[hD_iD_j, D_k^2] \\
&= -h[D_iD_j, D_k^2] - [h, D_k^2]D_iD_j \\
&= -h\{D_i[D_j, D_k^2] + [D_i, D_k^2]D_j\} + \left(\frac{\partial^2 h}{\partial x_k^2} + 2\frac{\partial h}{\partial x_k}D_k\right)D_iD_j \\
&= -h\left\{D_i\left(-2\omega_{jk}D_k - \frac{\partial \omega_{jk}}{\partial x_k}\right) + \left(-2\omega_{ik}D_k - \frac{\partial \omega_{ik}}{\partial x_k}\right)D_j\right\} \\
&\quad + \left(\frac{\partial^2 h}{\partial x_k^2} + 2\frac{\partial h}{\partial x_k}D_k\right)D_iD_j \\
&= -h\left\{-2\frac{\partial \omega_{jk}}{\partial x_i}D_k - 2\omega_{jk}D_iD_k - \frac{\partial^2 \omega_{jk}}{\partial x_i \partial x_k} - \frac{\partial \omega_{jk}}{\partial x_k}D_i\right. \\
&\qquad \left. - 2\omega_{ik}D_kD_j - \frac{\partial \omega_{ik}}{\partial x_k}D_j\right\} + \left(\frac{\partial^2 h}{\partial x_k^2} + 2\frac{\partial h}{\partial x_k}D_k\right)D_iD_j \\
&= 2\frac{\partial h}{\partial x_k}D_kD_iD_j + 2h\omega_{jk}D_iD_k + 2h\omega_{ik}D_kD_j + \frac{\partial^2 h}{\partial x_k^2}D_iD_j \\
&\quad + 2h\frac{\partial \omega_{jk}}{\partial x_i}D_k + h\frac{\partial \omega_{jk}}{\partial x_k}D_i \\
&\quad + h\frac{\partial \omega_{ik}}{\partial x_k}D_j + h\frac{\partial^2 \omega_{jk}}{\partial x_i \partial x_k}.
\end{aligned}
$$

(iii)
$$
\begin{aligned}
[D_iD_j, hD_k] &= -h[D_k, D_iD_j] - [h, D_iD_j]D_k \\
&= h[D_iD_j, D_k] + [D_iD_j, h]D_k \\
&= h\{D_i[D_j, D_k] + [D_i, D_k]D_j\} \\
&\quad + D_i[D_j, h]D_k + [D_i, h]D_jD_k \\
&= h\{D_i\omega_{kj} + \omega_{ki}D_j\} \\
&\quad + D_i\frac{\partial h}{\partial x_j}\ D_k + \frac{\partial h}{\partial x_i}\ D_jD_k \\
&= h\ \omega_{kj}D_i + h\ \frac{\partial \omega_{kj}}{\partial x_i} \\
&\quad + h\omega_{ki}D_j + \frac{\partial h}{\partial x_j}D_iD_k
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\partial^2 h}{\partial x_i \partial x_j} D_k + \frac{\partial h}{\partial x_i} D_j D_k \\
&= \frac{\partial h}{\partial x_j}\, D_i D_k + \frac{\partial h}{\partial x_i}\, D_j D_k \\
&+ h\, \omega_{kj} D_i + h\, \omega_{ki} D_j + \frac{\partial^2 h}{\partial x_i \partial x_j} D_k \\
&+ h\, \frac{\partial \omega_{kj}}{\partial x_i}. \qquad \square
\end{aligned}
$$

LEMMA 7. *Let $\widetilde{x} = Rx$ be an orthogonal change of coordinate, i.e., R is an orthogonal matrix. Then*

(1) $\widetilde{f}(\widetilde{x}) = Rf(x)$;

(2) $\widetilde{L}_0 = L_0$;

(3) $(\widetilde{\omega}_{ji}) = R(\omega_{\ell k})R^T$ *where* $\widetilde{L}_0 = \frac{1}{2}\left(\sum_{i=1}^n \widetilde{D}_i^2 - \widetilde{\eta}(\widetilde{x})\right)$, $\widetilde{D}_i = \partial/\partial\widetilde{x}_i - \widetilde{f}_i$, $\widetilde{h}(\widetilde{x}) = h(x)$, $\widetilde{\eta}(\widetilde{x}) = \sum_{i=1}^n (\partial \widetilde{f}_i(\widetilde{x})/\partial\widetilde{x}_i) + \widetilde{f}(\widetilde{x}) \cdot \widetilde{f}(\widetilde{x}) + \sum_{i=1}^m \widetilde{h}_i^2(\widetilde{x})$, *and* $\widetilde{\omega}_{ji} = (\partial\widetilde{f}_i/\partial\widetilde{x}_j) - (\partial\widetilde{f}_j/\partial\widetilde{x}_i)$;

(4) $\widetilde{E}$ *is isomorphic to E as Lie algebra, where $\widetilde{E}$ is the Lie algebra generated by* $\widetilde{L}_0, \widetilde{h}_1, \ldots, \widetilde{h}_m$.

Proof. Statement (1) is obvious. For (2), observe that

$$
\widetilde{L}_0 = \frac{1}{2}\sum_{i=1}^{2} \frac{\partial^2}{\partial \widetilde{x}_i} - \sum_{i=1}^{2} \widetilde{f}_i(\widetilde{x}) \frac{\partial}{\partial \widetilde{x}_i} - \sum_{i=1}^{2} \frac{\partial \widetilde{f}_i}{\partial \widetilde{x}_i} - \frac{1}{2}\sum_{i=1}^{m} \widetilde{h}_i^2.
$$

Let S be the inverse matrix of R. Then $x = S\widetilde{x}$ and $S = (s_{ij}) = R^T = (r_{ji})$.

$$
\begin{aligned}
\widetilde{L}_0 &= \frac{1}{2}\sum_{i=1}^{n}\left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial \widetilde{x}_i}\frac{\partial}{\partial x_j}\right)^2 - \sum_{i=1}^{n}\left(\sum_{j=1}^{n} r_{ij} f_j(x)\right)\left(\sum_{j=1}^{n} \frac{\partial x_j}{\partial \widetilde{x}_i}\frac{\partial}{\partial x_j}\right) \\
&\quad - \sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij} \frac{\partial f_j}{\partial \widetilde{x}_i}(x) - \frac{1}{2}\sum_{i=1}^{m} h_i^2(x) \\
&= \frac{1}{2}\sum_{i=1}^{n}\left(\sum_{j=1}^{n} s_{ji}\frac{\partial}{\partial x_j}\right)^2 - \sum_{i=1}^{n}\left(\sum_{j=1}^{n} r_{ij} f_j(x)\right)\left(\sum_{j=1}^{n} s_{ji}\frac{\partial}{\partial x_j}\right) \\
&\quad - \sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij} \sum_{k=1}^{n} \frac{\partial f_j}{\partial x_k}\frac{\partial x_k}{\partial \widetilde{x}_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2(x) \\
&= \frac{1}{2}\sum_{i,j,k=1}^{n} s_{ji}s_{ki}\frac{\partial^2}{\partial x_j \partial x_k} - \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} r_{ik} f_k(x) s_{ji}\frac{\partial}{\partial x_j} \\
&\quad - \sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij} \sum_{k=1}^{n} s_{ki}\frac{\partial f_j}{\partial x_k} - \frac{1}{2}\sum_{i=1}^{m} h_i^2(x) \\
&= \frac{1}{2}\sum_{j,k=1}^{n} \delta_{jk}\frac{\partial^2}{\partial x_j \partial x_k} - \sum_{j,k=1}^{n} \delta_{jk} f_k(x)\frac{\partial}{\partial x_j} - \sum_{j,k=1}^{n} \delta_{jk}\frac{\partial f_j}{\partial x_k} - \frac{1}{2}\sum_{i=1}^{m} h_i^2(x) \\
&= \frac{1}{2}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} - \sum_{j=1}^{n} f_j(x)\frac{\partial}{\partial x_j} - \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j} - \frac{1}{2}\sum_{i=1}^{m} h_i^2(x)
\end{aligned}
$$

$= L_0.$

Statement (3) follows from the following computation:

$$\begin{aligned}
\widetilde{\omega}_{ji} = \frac{\partial \widetilde{f}_i}{\partial \widetilde{x}_j} - \frac{\partial \widetilde{f}_j}{\partial \widetilde{x}_i} &= \sum_{k=1}^{n} r_{ik} \frac{\partial f_k}{\partial \widetilde{x}_j} - \sum_{k=1}^{n} r_{jk} \frac{\partial f_k}{\partial \widetilde{x}_i} \\
&= \sum_{k=1}^{n} r_{ik} \sum_{\ell=1}^{n} \frac{\partial x_\ell}{\partial \widetilde{x}_j} \frac{\partial f_k}{\partial x_\ell} - \sum_{k=1}^{n} r_{jk} \sum_{\ell=1}^{n} \frac{\partial x_\ell}{\partial \widetilde{x}_i} \frac{\partial f_k}{\partial x_\ell} \\
&= \sum_{k,\ell=1}^{n} r_{ik} \frac{\partial f_k}{\partial x_\ell} s_{\ell j} - \sum_{k,\ell=1}^{n} r_{jk} \frac{\partial f_k}{\partial x_\ell} s_{\ell i} \\
&= \sum_{k,\ell=1}^{n} s_{ki} \frac{\partial f_k}{\partial x_\ell} r_{j\ell} - \sum_{k,\ell=1}^{n} r_{j\ell} \frac{\partial f_\ell}{\partial x_k} s_{ki} \\
&= \sum_{k,\ell=1}^{n} r_{ik} r_{j\ell} \left(\frac{\partial f_k}{\partial x_\ell} - \frac{\partial f_\ell}{\partial x_k} \right) \\
&= \sum_{k,\ell=1}^{n} r_{ik} r_{j\ell} \omega_{\ell k}
\end{aligned}$$

Statement (4) is a particular case of Brockett's result in [Br3].

3. Classification theorems. Let us first recall that the following two theorems were stated in Ocone [Oc].

THEOREM 8 (Ocone). *With the notation in §2, let $n = m = p = 1$, $g = 1$. Then $\dim E$ is finite only if* (i)

$$h(x) = \alpha x, \quad \text{and} \quad f' + f^2 = ax^2 + bx + c \tag{*}$$

or

(ii) $h(x) = \alpha x^2 + \beta x$, $\alpha \neq 0$ *and*

$$f' + f^2 = -h^2 + a(2\alpha x + \beta)^2 + b + c(2\alpha x + \beta)^{-2} \tag{**}$$

$$\text{or} \quad f' + f^2 = -h^2 + ax^2 + bx + c. \tag{***}$$

THEOREM 9 (Ocone). *If f satisfies $(*)$, f must have a singularity in any unbounded interval.*

The following theorem follows easily from Ocone's Theorem 8 and Theorem 9 in the case where $m = 1$. Since Theorem 8 was stated without proof in [Oc], it is interesting to know that Theorem 9 follows from the proof of Theorem A as well. In fact we do not need to assume $m = 1$.

THEOREM A. *Suppose that the state space of the filtering system* (2.0) *is of dimension one. If the estimation algebra E is finite-dimensional, then one of the following holds:* (i) *E is a real vector space of dimension* 4 *with basis given by* 1, x, $D = (\partial/\partial x) - f$ *and* $L_0 = \frac{1}{2}(D^2 - \eta)$ *or* (ii) *E is a real vector space of dimension* 2 *with basis given by* 1, *and* $L_0 = \frac{1}{2}(D^2 - \eta)$ *or* (iii) *E is a real vector space of dimension* 1 *with basis given by* $L_0 = \frac{1}{2}(D^2 - \eta)$.

Proof. In view of Theorem 3, all the observation terms h_i $1 \leq i \leq m$ are necessarily affine polynomials. So we have only three cases.

If all the h_i for $1 \leq i \leq m$ are actually zero, then obviously we are in case (iii) above.

If all the h_i for $1 \leq i \leq m$ are at most constants and one of them is nonzero, then $1 \in E$. By Lemma 5 (iv), we have

$$[L_0, 1] = \frac{1}{2}[D^2 - \eta, 1] = 0.$$

Therefore we are in case (ii) above.

Finally we may assume that there is a constant c such that $x + c$ is in E. In view of Lemma 5, we have

$$[L_0, x + c] = \tfrac{1}{2}[D^2 - \eta, x + c] = D, \tag{3.1}$$

$$[D, x + c] = 1, \tag{3.2}$$

$$[L_0, D] = \frac{1}{2}[D^2 - \eta, D] = \frac{1}{2}\frac{d\eta}{dx}. \tag{3.3}$$

$d\eta/dx \in E$ implies η is a polynomial of degree at most 3 by Theorem 2. Recall that

$$\frac{df}{dx} + f^2 = \eta - \sum_{i=1}^{m} h_i^2. \tag{3.4}$$

If η is a polynomial of degree 3, then $\eta - \sum_{i=1}^{m} h_i^2$ is also a polynomial of degree 3. According to Theorem 4, (3.4) has no C^∞ solution f since

$$\lim_{x \to +\infty} \left(\eta - \sum_{i=1}^{m} h_i^2\right) = -\infty \quad \text{or} \quad \lim_{x \to -\infty} \left(\eta - \sum_{i=1}^{m} h_i^2\right) = -\infty.$$

This leads to a contradiction. Therefore, we have shown that η is a polynomial of degree 2. In view of (3.1)–(3.3), E is four-dimensional real vector space with basis 1, x, $D = (d/dx) - f$ and $L_0 = \frac{1}{2}(D^2 - \eta)$. □

THEOREM B. *Suppose that the state space of the filtering system* (2.0) *is of dimension two. If E is the finite-dimensional estimation algebra with maximal rank, then E is a real vector space of dimension* 6 *with basis given by* 1, x_1, x_2, D_1, D_2, *and* L_0.

Proof. Since E is a finite-dimensional estimation algebra with maximal rank, there are constants c_i's such that $x_i + c_i$ is in E for $i = 1, 2$. In view of Lemma 5, we have the following

$$[L_0, x_j + c_j] = \frac{1}{2}\left[\sum_{i=1}^{2} D_i^2 - \eta, x_j\right] = \frac{1}{2}\sum_{i=1}^{2}[D_i^2, x_j] = D_j \in E, \tag{3.5}$$

$$\omega_{ji} = [D_i, D_j] \in E \tag{3.6}$$

(3.7)

$$\begin{aligned} Y_j = [L_0, D_j] &= \frac{1}{2}\left[\sum_{i=1}^{2} D_i^2 - \eta, D_j\right] = -\sum_{i=1}^{2}\left(\omega_{ij} D_i + \frac{1}{2}\frac{\partial \omega_{ij}}{\partial x_i}\right) + \frac{1}{2}\frac{\partial \eta}{\partial x_j} \\ &= \sum_{i=1}^{2}\left(\omega_{ji} D_i + \frac{1}{2}\frac{\partial \omega_{ji}}{\partial x_i}\right) + \frac{1}{2}\frac{\partial \eta}{\partial x_j} \in E, \end{aligned}$$

$$[Y_j, \omega_{k\ell}] = \left[\sum_{i=1}^{2} \omega_{ji} D_i + \frac{1}{2}\sum_{i=1}^{2} \frac{\partial \omega_{ji}}{\partial x_i} + \frac{1}{2}\frac{\partial \eta}{\partial x_j},\ \omega_{k\ell}\right] = \sum_{i=1}^{2} \omega_{ji} \frac{\partial \omega_{k\ell}}{\partial x_i} \in E, \tag{3.8}$$

$$[Y_j, D_k] = \left[\sum_{i=1}^{2} \omega_{ji} D_i + \frac{1}{2}\sum_{i=1}^{2} \frac{\partial \omega_{ji}}{\partial x_i} + \frac{1}{2}\frac{\partial \eta}{\partial x_j},\ D_k\right] = \sum_{i=1}^{2}\left(\omega_{ji}\omega_{ki} - \frac{\partial \omega_{ji}}{\partial x_k} D_i\right) - \frac{1}{2}\sum_{i=1}^{2} \frac{\partial^2 \omega_{ji}}{\partial x_k \partial x_i} - \frac{1}{2}\frac{\partial^2 \eta}{\partial x_k \partial x_j}. \tag{3.9}$$

By Theorem 2 and (3.6), ω_{ij}'s are polynomials of degree less than or equal to 2. Recall that $\omega_{11} = 0 = \omega_{22}$. By (3.8), we have

$$\omega_{12}\frac{\partial \omega_{12}}{\partial x_1} \in E \quad \text{and} \quad \omega_{12}\frac{\partial \omega_{12}}{\partial x_2} \in E,$$

which implies that

$$\frac{\partial \omega_{12}^2}{\partial x_1} \in E \quad \text{and} \quad \frac{\partial \omega_{12}^2}{\partial x_2} \in E.$$

If ω_{12} were polynomial of degree 2, then there would be a nonzero polynomial of degree 3 in E, which contradicts Theorem 2. Therefore, we conclude that ω_{12} is a polynomial of degree at most 1. We will prove that ω_{12} is actually a constant. From (3.9) and (3.5), we have

$$\sum_{i=1}^{2} \omega_{ji}\omega_{ki} - \frac{1}{2}\frac{\partial^2 \eta}{\partial x_k \partial x_j} \in E. \tag{3.10}$$

Since

$$\sum_{i=1}^{2} \omega_{ji}\omega_{ki} - \frac{1}{2}\frac{\partial^2 \eta}{\partial x_k \partial x_j}$$

is a polynomial of degree at most 2 for all $1 \le j,\ k \le 2$ we deduce easily that η is a polynomial of degree at most 4. Assume that $\eta = a_{40}x_1^4 + a_{31}x_1^3x_2 + a_{22}x_1^2x_2^2 + a_{13}x_1x_2^3 + a_{04}x_2^4 +$ degree 3 polynomial and $\omega_{12} = ax_1 + bx_2 + c$. Equation (3.10) implies that

$$\omega_{12}^2 - \frac{1}{2}\frac{\partial^2 \eta}{\partial x_1^2}, \quad \frac{\partial^2 \eta}{\partial x_1 \partial x_2}, \quad \text{and} \quad \omega_{12}^2 - \frac{1}{2}\frac{\partial^2 \eta}{\partial x_2^2}$$

are in E. Hence we have

$$\begin{aligned} E \ni \omega_{12}^2 - \frac{1}{2}\frac{\partial^2 \eta}{\partial x_1^2} &= a^2x_1^2 + 2abx_1x_2 + b^2x_2^2 \\ &\quad - (6a_{40}x_1^2 + 3a_{31}x_1x_2 + a_{22}x_2^2) \\ &\quad + \text{polynomial of degree one} \\ &= (a^2 - 6a_{40})x_1^2 + (2ab - 3a_{31})x_1x_2 + (b^2 - a_{22})x_2^2 \\ &\quad + \text{polynomial of degree one.} \end{aligned} \tag{3.11}$$

$$E \ni \frac{\partial^2 \eta}{\partial x_1 \partial x_2} = 3a_{31}x_1^2 + 4a_{22}x_1x_2 + 3a_{13}x_2^2 + \text{ polynomial of degree one.} \tag{3.12}$$

$$\begin{aligned} E \ni \omega_{12}^2 - \frac{1}{2}\frac{\partial^2 \eta}{\partial x_1^2} &= a^2x_1^2 + 2abx_1x_2 + b^2x_2^2 - (a_{22}x_1^2 + 3a_{13}x_1x_2 + 6a_{04}x_2^2) \\ &\quad + \text{ polynomial of degree one} \\ &= (a^2 - a_{22})x_1^2 + (2ab - 3a_{13})x_1x_2 + (b^2 - 6a_{04})x_2^2 \\ &\quad + \text{ polynomial of degree one.} \end{aligned} \tag{3.13}$$

Since $1 = [D_1, x_1 + c_1] \in E$, we have $1, x_1, x_2 \in E$. It follows from (3.11)–(3.13) that

$$(a^2 - 6a_{40})x_1^2 + (2ab - 3a_{31})x_1x_2 + (b^2 - a_{22})x_2^2 \in E, \tag{3.14}$$

$$3a_{31}x_1^2 + 4a_{22}x_1x_2 + 3a_{13}x_2^2 \in E, \tag{3.15}$$

$$(a^2 - a_{22})x_1^2 + (2ab - 3a_{13})x_1x_2 + (b^2 - 6a_{04})x_2^2 \in E. \tag{3.16}$$

We will prove that ω_{12} is a constant. If there is no polynomial of degree 2 in E, then we have $a = b = a_{22} = 0$. This implies that ω_{12} is a constant.

Suppose that there is a polynomial of degree 2 in E. Then, by using the affine transformation $\widetilde{x} = Rx$, where R is an orthogonal matrix, we may assume that there exists a degree 2 polynomial in E of the form $k_1x_1^2 + k_2x_2^2+$ polynomial of degree one, where either $k_1 \neq 0$ or $k_2 \neq 0$. This can be seen by using Lemma 7 because $\widetilde{\omega}_{12} = \sum_{k,\ell=1}^2 r_{2k}r_{1\ell}\omega_{\ell k}$ is still a polynomial in x_i of degree at most one. As $1, x_1, x_2 \in E$, we deduce that there exists a polynomial in E of the form $k_1x_1^2 + k_2x_2^2$, where either $k_1 \neq 0$ or $k_2 \neq 0$. Without loss of generality we may assume that $k_1 \neq 0$. So we have $p(x)$

$$p(x) = x_1^2 + kx_2^2 \in E \text{ where } k = k_2/k_1. \tag{3.17}$$

Case 1. $k \neq 0$.
We observe that

$$\begin{aligned} [Y_1, p(x)] &= \left[\omega_{12}D_2 + \frac{1}{2}\frac{\partial \omega_{12}}{\partial x_2} + \frac{1}{2}\frac{\partial \eta}{\partial x_1}\ ,\ x_1^2 + kx_2^2\right] \\ &= 2k\omega_{12}x_2 = 2k(ax_1x_2 + bx_2^2 + cx_2) \in E, \end{aligned} \tag{3.18}$$

$$\begin{aligned} [Y_2, p(x)] &= \left[\omega_{21}D_1 + \frac{1}{2}\frac{\partial \omega_{21}}{\partial x_1} + \frac{1}{2}\frac{\partial \eta}{\partial x_2}\ ,\ x_1^2 + kx_2^2\right] \\ &= -2\omega_{12}x_1 = -2(ax_1^2 + bx_1x_2 + cx_1) \in E. \end{aligned} \tag{3.19}$$

It follows from (3.18) and (3.19) that we have

$$ax_1x_2 + bx_2^2 \in E, \tag{3.20}$$

$$ax_1^2 + bx_1x_2 \in E. \tag{3.21}$$

Equations (3.20) and (3.21) imply that

$$a^2x_1^2 - b^2x_2^2 \in E. \tag{3.22}$$

On the other hand, we have

$$\begin{aligned}[L_0, p(x)] &= \frac{1}{2}\left[\sum_{i=1}^{2} D_i^2 - \eta,\ x_1^2 + kx_2^2\right] \\ &= \frac{1}{2}[D_1^2,\ x_1^2 + kx_2^2] + \frac{1}{2}[D_2^2,\ x_1^2 + kx_2^2] \\ &= \frac{1}{2}(2 + 4x_1D_1) + \frac{1}{2}(2k + 4kx_2D_2) \\ &= 2\left(x_1D_1 + kx_2D_2 + \frac{k+1}{2}\right) \in E,\end{aligned}$$

$$\left[x_1D_1 + kx_2D_2 + \frac{k+1}{2}\ ,\ x_1^2 + kx_2^2\right] = 2x_1^2 + 2k^2x_2^2 \in E.$$

So we have

$$x_1^2 + k^2x_2^2 \in E. \tag{3.23}$$

Equations (3.17) and (3.23) imply $(k^2 - k)x_2^2 \in E$. So if $k \neq 1$, then both x_1^2 and x_2^2 are in E. If $k = 1$, then it follows from (3.17) and (3.22) that $(a^2 + b^2)x_2^2 \in E$. If $a^2 + b^2 = 0$, then ω_{12} is constant as claimed. On the other hand, if $a^2 + b^2 \neq 0$, then we conclude that x_1^2, x_2^2 are in E. Therefore in view of Lemma 5, we have

$$\left[L_0,\ \frac{1}{2}x_1^2\right] = \frac{1}{4}[D_1^2 + D_2^2 - \eta,\ x_1^2] = \frac{1}{4}[D_1^2,\ x_1^2] = x_1D_1 + \frac{1}{2} \in E, \tag{3.24}$$

$$\left[L_0,\ \frac{1}{2}x_2^2\right] = \frac{1}{4}[D_1^2 + D_2^2 - \eta,\ x_2^2] = \frac{1}{4}[D_2^2,\ x_2^2] = x_2D_2 + \frac{1}{2} \in E, \tag{3.25}$$

$$\left[x_1D_1 + \frac{1}{2},\ x_2D_2 + \frac{1}{2}\right] = -x_1x_2\omega_{12} \in E. \tag{3.26}$$

By Theorem 2, $x_1x_2\omega_{12}$ is a polynomial of degree 2. So ω_{12} is a constant.

Case 2. $k = 0$.

By (3.19) we have $ax_1^2 + bx_1x_2 \in E$ which implies $bx_1x_2 \in E$. If $b \neq 0$, then $x_1x_2 \in E$. It follows that

(3.27)

$$\begin{aligned}[L_0, x_1x_2] &= \frac{1}{2}\left[\sum_{i=1}^{2} D_i^2 - \eta, x_1x_2\right] = \frac{1}{2}[D_1^2, x_1x_2] + \frac{1}{2}[D_2^2, x_1x_2] \\ &= x_2D_1 + x_1D_2 \in E,\end{aligned}$$

$$[x_2D_1 + x_1D_2,\ x_1x_2] = x_1^2 + x_2^2 \in E. \tag{3.28}$$

We deduce from (3.28) that x_1^2 and x_2^2 are in E. Hence (3.24)–(3.26) imply that ω_{12} must be a constant as claimed.

From now on, we assume that $b = 0$, i.e., $\omega_{12} = ax_1 + c$, and $p(x) = x_1^2$.

Let $Z_0 = \frac{1}{2}p(x) = \frac{1}{2}x_1^2 \in E$ and $Z_k = [L_0, Z_{k-1}]$. Then by (3.24) $Z_1 = [L_0, Z_0] = x_1 D_1 + \frac{1}{2}$. In view of Lemma 5, we have

$$\begin{aligned} Z_2 = [L_0, Z_1] &= \frac{1}{2}\left[D_1^2 + D_2^2 - \eta,\ x_1 D_1 + \frac{1}{2}\right] \\ &= \frac{1}{2}[D_1^2,\ x_1 D_1] + \frac{1}{2}[D_2^2, x_1 D_1] + \frac{1}{2}[x_1 D_1,\ \eta] \\ &= D_1^2 + x_1\omega_{12}D_2 + \frac{1}{2}E_1(\eta) \text{ where } E_1 = x_1\frac{\partial}{\partial x_1}. \end{aligned}$$

Let U^k be the space of differential operators of order up to and including k. Then

$$\begin{aligned} Z_3 = [L_0, Z_2] &= \frac{1}{2}\left[D_1^2 + D_2^2 - \eta,\ D_1^2 + x_1\omega_{12}D_2 + \frac{1}{2}E_1(\eta)\right] \\ &= \frac{1}{2}[D_1^2, x_1\omega_{12}D_2] + \frac{1}{2}[D_2^2, D_1^2] + \frac{1}{2}[D_2^2, x_1\omega_{12}D_2] \bmod U^1 \\ &= \frac{\partial(x_1\omega_{12})}{\partial x_1}D_1D_2 + 2\omega_{12}D_1D_2 + \frac{\partial(x_1\omega_{12})}{\partial x_2}D_2^2 \bmod U^1 \\ &= (4ax_1 + 3c)D_1D_2 \bmod U^1. \end{aligned}$$

Here $(\ \cdot\ )$ mode U^k signifies a member of the affine class of operators obtained by adding members of U^k to the argument. Suppose $a \neq 0$. Then $A = Z_3/4a = (x_1 + 3c/4a)D_1D_2 \bmod U^1$ is an element in E. We claim that $(-1)^{k+1}Ad_A^k Z_2 = 2^k D_1^2 D_2^k \bmod U^{k+1}$. For $k = 1$,

$$\begin{aligned} (-1)Ad_A Z_2 = [Z_2, A] &= \left[D_1^2 \bmod U^1, \left(x_1 + \frac{3c}{4a}\right)D_1D_2 \bmod U^1\right] \\ &= \left[D_1^2, \left(x_1 + \frac{3c}{4a}\right)D_1D_2\right] \bmod U^2 \\ &= 2D_1^2D_2 \bmod U^2. \end{aligned}$$

Suppose that it is true for $k-1$, i.e., $(-1)^k Ad_A^{k-1}Z_2 = 2^{k-1}D_1^2D_2^{k-1} \bmod U^k$. Then

$$\begin{aligned} (-1)^{k+1}Ad_A^k Z_2 &= (-1)Ad_A[(-1)^k Ad_A^{k-1}Z_2] \\ &= \left[2^{k-1}D_1^2D_2^{k-1} \bmod U^k,\ \left(x_1 + \frac{3c}{4a}\right)D_1D_2 \bmod U^1\right] \\ &= 2^{k-1}\left[D_1^2D_2^{k-1},\ \left(x_1 + \frac{3c}{4a}\right)D_1D_2\right] \bmod U^{k+1} \\ &= -\left(x_1 + \frac{3c}{4a}\right)[D_1D_2,\ 2^{k-1}D_1^2D_2^{k-1}] \\ &\quad - \left[x_1 + \frac{3c}{4a},\ 2^{k-1}D_1^2D_2^{k-1}\right]D_1D_2 \bmod U^{k+1}. \end{aligned} \tag{3.29}$$

We show that $[D_1D_2, D_1^2D_2^{k-1}] \equiv 0 \bmod U^{k+1}$. This can be seen easily by induction as follows. For $k = 1$, this follows from Lemma 6 (ii):

$$\begin{aligned}[D_1D_2, D_1^2D_2^{k-1}] &= -[D_1^2D_2^{k-2}D_2, D_1D_2] \\ &= -D_1^2D_2^{k-1}[D_2, D_1D_2] \\ &\quad - [D_1^2D_2^{k-2}, D_1D_2]D_2 = 0 \bmod U^{k+1}\end{aligned}$$

in view of Lemma 6 (iii) and induction hypothesis. Put this into (3.29), and we obtain

$$\begin{aligned}(-1)^{k+1}Ad_A^kZ_2 &= 2^{k-1}\left[D_1^2D_2^{k-1},\ x_1 + \frac{3c}{4a}\right]D_1D_2 \bmod U^{k+1} \\ &= 2^{k-1}D_1^2\left[D_2^{k-1},\ x_1 + \frac{3c}{4a}\right]D_1D_2 + 2^{k-1}\left[D_1^2,\ x_1 + \frac{3c}{4a}\right]D_2^{k-1}D_1D_2 \\ &= 2^{k-1}D_1^2\left\{D_2\left[D_2^{k-2}, x_1 + \frac{3c}{4a}\right] + \left[D_2,\ x_1 + \frac{3c}{4a}\right]D_2^{k-2}\right\}D_1D_2 \\ &\quad + 2^{k-1}\cdot 2D_1D_2^{k-1}D_1D_2 \bmod U^{k+1} \\ &= 2^kD_1D_2^{k-1}D_1D_2 \bmod U^{k+1} \\ &= 2^kD_1^2D_2^k \bmod U^{k+1}.\end{aligned}$$

This proves our claim. We have shown that if $a \neq 0$, then E is infinite-dimensional. Hence the finite-dimensionality of E implies that $a = 0$, i.e., ω_{12} is a constant. We can apply Theorem 6 of [Ya] to deduce our result. □

REFERENCES

[Br-Cl] R. W. Brockett and J. M. C. Clark, *The geometry of the conditional density functions*, in Analysis and Optimization of Stochastic Systems, O. L. R. Jacobs, et. al., eds., Academic Press, New York, 1980, pp. 299–309.

[Br1] R. W. Brockett, *Nonlinear systems and nonlinear estimation theory*, in The Mathematics of Filtering and Identification and Applications, M. Hazewinkel and J. S. Willems, eds., Reidel, Dordrecht, 1981.

[Br2] ———, *Nonlinear Control Theory and Differential Geometry*, Proceedings of the International Congress of Mathematics, 1983, pp. 1357–1368.

[Br3] ———, *Classification and Equivalence in Estimation Theory*, Proceedings, 18th IEEE Conference on Decision and Control, Fort Lauderdale, FL, December, 1979.

[Co] P. C. Collingwood, *Some remarks on estimation algebras*, Systems Control Lett., 7 (1986), pp. 217–224.

[Da] M. H. A. Davis, *On a multiplicative functional transformation arising in nonlinear filtering theory*, Z. Wahrsch. Verw. Gebiete, 54 (1980), pp. 125–139.

[Da-Ma] M. H. A. Davis and S. I. Marcus, *An introduction to nonlinear filtering*, in The Mathematics of Filtering and Identification and Applications, M. Hazewinkel and J. S. Willems, eds.,, eds., Reidel, Dordrecht, 1981.

[D-T-W-Y] R. T. Dong, L. F. Tam, W. S. Wong and S. S.-T. Yau,, *Structure and classification theorems of finite dimensional exact estimation algebras*, SIAM J. Control Optim., to appear.

[Oc] D. L. Ocone, *Finite dimensional estimation algebras in nonlinear filtering*, in The Mathematics of Filtering and Identification and Applications, M. Hazewinkel and J. S. Willems, eds., Reidel, Dorrecht, 1981.

[Mi] S. K. Mitter, *On the analogy between mathematical problems of nonlinear filtering and quantum physics*, Ricerche di Automatica, 10 (1979), pp. 163–216.

[T-W-Y] L. F. TAM, W. S. WONG, AND S. S.-T. YAU, *On a necessary and sufficient condition for finite dimensionality of estimation algebras*, SIAM J. Control Optim., 28 (1990), pp. 173–185.

[Wo] W. S. WONG, *Theorems on the structure of finite dimensional estimation algebras*, Systems Control Lett., 9 (1987), pp. 117–124.

[Ya] S. S.-T. YAU, *Finite Dimensional Filters with Nonlinear Drift I: A Class of Filters Including both Kalman-Bucy Filters and Benes Filters*, J. Math. Systems, Estimation Control,, to appear.

[Ya-Ch] S. S.-T. YAU AND W.-L. CHIOU, *Recent results on classification of finite dimensional estimation algebras: dimension of state space ≤ 2*, Proceedings of 30th Conf. on Decision and Control, Brighton, England, Dec. 11-13, 1991, pp. 2758–2760.