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FINITE-DIMENSIONAL FILTERS WITH NONLINEAR DRIFT IV: CLASSIFICATION OF FINITE-DIMENSIONAL ESTIMATION ALGEBRAS OF MAXIMAL RANK WITH STATE–SPACE DIMENSION 3*

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Abstract. The idea of using estimation algebras to construct finite-dimensional nonlinear filters was first proposed by Brockett and Mitter independently. It turns out that the concept of estimation algebra plays a crucial role in the investigation of finite-dimensional nonlinear filters. In his talk at the International Congress of Mathematics in 1983, Brockett proposed a classification of all finite-dimensional estimation algebras. Chiou and Yau classify all finite-dimensional estimation algebras of maximal rank with dimension of the state space less than or equal to two. In this paper we succeed in classifying all finite-dimensional estimation algebras of maximal rank with state–space dimension equal to three. Thus from the Lie algebraic point of view, we have now understood generically all finite dimensional filters with state–space dimension less than four.

Key words. finite-dimensional filter, estimation algebra of maximal rank, nonlinear drift

AMS subject classifications. 17B30, 35J15, 60G35, 93E11

1. Introduction.

In the sixties and early seventies, the basic approach to nonlinear filtering theory was via the "innovation methods" originally proposed by Kailath and subsequently rigorously developed by Fujisaki, Kallianpur, and Kunita [FKK] in 1972. As pointed out by Mitter [Mi], the difficulty with this approach is that the innovations process is not, in general, explicitly computable (except in the well-known Kalman–Bucy case). In the late seventies, Brockett and Clark [BrCl], Brockett [Br], and Mitter [Mi] proposed the idea of using estimation algebras to construct finite-dimensional nonlinear filters. In a previous paper [Ya], Yau has studied the general class of nonlinear filtering systems which included both Kalman–Bucy and Benes filtering systems as special cases. He gives necessary and sufficient conditions for an estimation algebra of such filtering systems to be finite dimensional. Using the Wei–Norman approach, he constructed explicitly finite-dimensional recursive filters for such nonlinear filtering systems.

In his talk at the International Congress of Mathematics in 1983, Brockett proposed classification of all finite-dimensional estimation algebras. Since then, the concept of estimation algebras has proved to be an invaluable tool in the study of nonlinear filtering problems. In [ChYa], Chiou and Yau introduced the concept of an estimation algebra of maximal rank. They were able to classify all finite-dimensional estimation algebras of maximal rank with state–space dimension less than or equal to two. The novelty of their theorem is that there is no assumption on the drift term of the nonlinear filtering system. On the other hand, if the drift term has a potential function (i.e., drift term is a gradient vector field), then the corresponding estimation algebra is called exact. In [TWY], Tam, Wong, and Yau classified all finite-dimensional ex-

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act estimation algebras of maximal rank with arbitrary state–space dimension. This paper is a natural continuation of [ChYa]. We shall classify all finite-dimensional estimation algebras of maximal rank with state–space dimension equal to 3 (without any assumption on the drift term). The following is our main theorem.

THEOREM 1 (main theorem). *Suppose that the state space of the filtering system* (2.0) *is of dimension three. If E is the finite-dimensional estimation algebra of maximal rank, then the drift term f must be a linear vector field (i.e., each component is a polynomial of degree one) plus a gradient vector field, and E is a real vector space of dimension eight with bases given by 1, x_1, x_2, x_3, D_1, D_2, D_3, and L_0.*

This kind of nonlinear filtering system was studied by Yau [Ya]. Therefore, from the Lie algebraic point of view, we have shown that the finite-dimensional filters considered in [Ya] are the most general.

Let $\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$, which was first introduced by Wong [Wo2]. Our strategy is to prove ω_{ij} constant for all i, j. Then we can apply the result of [Ya] to finish the proof. This involves two steps. The first step is to prove that ω_{ij} is a degree-one polynomial. The second step is to prove that ω_{ij} is a constant. Let n be the dimension of the state space. Unlike the case $n = 2$, where there is only one unknown, ω_{12}, the case $n = 3$ for the treatment of the first step is more difficult because there are three unknowns: ω_{12}, ω_{13}, and ω_{23}, and they cannot be separated and thus they cannot be treated individually. For the second step, which is the hard part of the paper, we have to introduce a new concept and technique in addition to the method used in [ChYa] to overcome the difficulties.

The paper is in essence a continuation of [Ya], [ChYa], and we strongly recommend that readers familiarize themselves with the results in [Ya], [ChYa]. However, every effort will be made to make this paper as self-contained as possible with minimal duplication of the previous papers.

2. Basic concepts. In this section, we shall recall some basic concepts and results from [Ya]. Consider a filtering problem based on the following signal observation model:

$$\begin{cases} dx(t) = f(x(t))dt + g(x(t))dv(t), & x(0) = x_0, \\ dy(t) = h(x(t))dt + dw(t), & y(0) = 0, \end{cases} \tag{2.0}$$

in which x, v, y, and w are, respectively, $\mathbf{R}^n$-, $\mathbf{R}^p$-, $\mathbf{R}^m$-, and $\mathbf{R}^m$-valued processes, and v and w have components which are independent, standard Brownian processes. We further assume that $n = p$, f, h are C^∞ smooth, and that g is an orthogonal matrix. We shall refer to $x(t)$ as the state of the system at time t and to $y(t)$ as the observation at time t.

Let $\rho(t,x)$ denote the conditional density of the state given the observation $\{y(s) : 0 \le s \le t\}$. It is well known (see [DaMa], for example) that $\rho(t,x)$ is given by normalizing a function, $\sigma(t,x)$, which satisfies the Duncan–Mortensen–Zakai equation.

$$d\sigma(t,x) = L_0\sigma(t,x)dt + \sum_{i=1}^m L_i\sigma(t,x)dy_i(t), \qquad \sigma(0,x) = \sigma_0, \tag{2.1}$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^n f_i \frac{\partial}{\partial x_i} - \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^m h_i^2$$

and for $i = 1, \dots, m$, L_i is the zero-degree differential operator of multiplication by h_i and σ_0 is the probability density of the initial point x_0. In this paper, we will assume σ_0 is a C^∞ function.

Equation (2.1) is a stochastic partial differential equation. The stochastic differential is a Stratonovich one, not an Ito one. In real applications, we are interested in constructing state estimators from observed sample paths with some property of robustness. Davis [Da] studied this problem and proposed some robust algorithms. In our case, his basic idea reduces to defining a new, unnormalized density

$$\xi(t,x) = \exp\Big(-\sum_{i=1}^{m} h_i(x) y_i(t)\Big)\sigma(t,x).$$

It is easy to show that $\xi(t,x)$ satisfies the following time-varying partial differential equation:

$$\begin{aligned} \frac{\partial \xi}{\partial t}(t,x) &= L_0\xi(t,x) + \sum_{i=1}^{m} y_i(t)[L_0, L_i]\xi(t,x) \\ &\cdot \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} y_i(t)y_j(t)[[L_0,L_i],L_j]\xi(t,x) \end{aligned} \tag{2.2}$$

where $[\cdot,\cdot]$ is the Lie bracket defined as follows.

DEFINITION. *If X and Y are differential operators, the Lie bracket of X and Y, $[X,Y]$, is defined by $[X,Y]\phi = X(Y\phi) - Y(X\phi)$ for any C^∞ function ϕ.*

DEFINITION. *The estimation algebra E of a filtering problem* (2.0) *is defined as the Lie algebra generated by $\{L_0, L_1, \dots, L_m\}$. E is said to be an estimation algebra of maximal rank if, for any $1 \le i \le n$, there exists a constant c_i such that $x_i + c_i$ is in E.*

Most of the known finite-dimensional estimation algebras are maximal. For example, if the equation (2.0) is linear, i.e., $f(x) = Ax$, $g(x) = B$, and $h(x) = Cx$, and if also (A,B,C) is minimal, then the corresponding estimation algebra is maximal [Ha].

In [Ya], the following proposition is proven.

PROPOSITION 1 (Yau). $\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$ *are constant functions for all i and j if and only if $(f_1,\dots,f_n) = (l_1,\dots,l_n) + (\frac{\partial \phi}{\partial x_1},\dots,\frac{\partial \phi}{\partial x_n})$, where $l_1,\dots,l_n$ are polynomials of degree one and ϕ is a C^∞ function.*

We need the following basic result for later discussion.

THEOREM 2 (Ocone). *Let E be a finite-dimensional estimation algebra. If a function ξ is in E, then ξ is a polynomial of degree ≤ 2.*

Define

$$D_i = \frac{\partial}{\partial x_i} - f_i,$$

$$\eta = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{m} h_i^2.$$

Then

$$L_0 = \frac{1}{2}\Big(\sum_{i=1}^{n} D_i^2 - \eta\Big).$$

The following theorem proved in [Ya] plays a fundamental role in the classification of finite-dimensional estimation algebras.

THEOREM 3 (Yau). *Let E be a finite-dimensional estimation algebra of (2.0) such that $\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$ are constant functions. If E is of maximal rank, then E is a real vector space of dimension $2n+2$ with bases given by $1, x_1, x_2, \dots, x_n, D_1, D_2, \dots, D_n$, and L_0.*

For the convenience of readers, we also list the following elementary lemmas without proof. The lemmas were proven in [Ya] and [ChYa].

LEMMA 4. (i) $[XY, Z] = X[Y, Z] + [X, Z]Y$ *where X, Y and Z are differential operators.*

(ii) $[gD_i, h] = g\frac{\partial h}{\partial x_i}$, *where $D_i = \frac{\partial}{\partial x_i} - f_i$, g and h are functions defined on $\mathbf{R}^n$.*

(iii) $[gD_i, hD_j] = -gh\omega_{ij} + g\frac{\partial h}{\partial x_i}D_j - h\frac{\partial g}{\partial x_j}D_i$, *where $\omega_{ji} = [D_i, D_j] = \frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i}$.*

(iv) $[gD_i^2, h] = 2g\frac{\partial h}{\partial x_i}D_i + g\frac{\partial^2 h}{\partial x_i^2}$.

(v) $[D_i^2, hD_j] = 2\frac{\partial h}{\partial x_i}D_iD_j - 2h\omega_{ij}D_i + \frac{\partial^2 h}{\partial x_i^2}D_j - h\frac{\partial \omega_{ij}}{\partial x_i}$.

(vi) $[D_i^2, D_j^2] = 4\omega_{ji}D_jD_i + 2\frac{\partial \omega_{ji}}{\partial x_j}D_i + 2\frac{\partial \omega_{ji}}{\partial x_i}D_j + \frac{\partial^2 \omega_{ji}}{\partial x_i \partial x_j} + 2\omega_{ji}^2$.

(vii) $[D_k^2, hD_iD_j] = 2\frac{\partial h}{\partial x_k}D_kD_iD_j + 2h\omega_{jk}D_iD_k + 2h\omega_{ik}D_kD_j + \frac{\partial^2 h}{\partial x_k^2}D_iD_j + 2h\frac{\partial \omega_{jk}}{\partial x_i}D_k + h\frac{\partial \omega_{jk}}{\partial x_k}D_i + h\frac{\partial \omega_{ik}}{\partial x_k}D_j + h\frac{\partial^2 \omega_{jk}}{\partial x_i \partial x_k}$.

(viii) $[gD_iD_j, hD_k] = g\frac{\partial h}{\partial x_j}D_iD_k + g\frac{\partial h}{\partial x_i}D_jD_k + gh\omega_{kj}D_i + gh\omega_{ki}D_j + g\frac{\partial^2 h}{\partial x_i \partial x_j}D_k + gh\frac{\partial \omega_{kj}}{\partial x_i} - h\frac{\partial g}{\partial x_k}D_iD_j$.

LEMMA 5. (i) $[L_0, x_j + c_j] = D_j$, *where $L_0 = \frac{1}{2}(\sum_{i=1}^n D_i^2 - \eta)$.*

(ii) $[D_i, x_j + c_j] = \delta_{ij}$.

(iii) $[D_i, D_j] = \omega_{ji}$.

(iv) $Y_j := [L_0, D_j] = \sum_{i=1}^n (\omega_{ji}D_i + \frac{1}{2}\frac{\partial \omega_{ji}}{\partial x_i}) + \frac{1}{2}\frac{\partial \eta}{\partial x_j}$.

(v) $[Y_j, \omega_{kl}] = \sum_{i=1}^n \omega_{ji}\frac{\partial \omega_{kl}}{\partial x_i}$.

(vi) $[Y_j, D_k] = \sum_{i=1}^n (\omega_{ji}\omega_{ki} - \frac{\partial \omega_{ji}}{\partial x_k}D_i) - \frac{1}{2}\sum_{i=1}^n \frac{\partial^2 \omega_{ji}}{\partial x_k \partial x_i} - \frac{1}{2}\frac{\partial^2 \eta}{\partial x_k \partial x_j}$.

Consider $\tilde{x} = Rx$, where R is an orthogonal matrix. Then (2.0) becomes

$$(2.3)\qquad \begin{cases} d\tilde{x}(t) = \tilde{f}(\tilde{x}(t))dt + \tilde{g}(\tilde{x}(t))d\tilde{v}(t), & \tilde{x}(0) = \tilde{x}_0 := Rx_0, \\ d\tilde{y}(t) = \tilde{h}(\tilde{x}(t))dt + d\tilde{w}(t), & \tilde{y}(0) = 0, \end{cases}$$

where

$$\begin{aligned} \tilde{f}(\tilde{x}) &= Rf(x), & \tilde{g}(\tilde{x}) &= Rg(x), \\ \tilde{v} &= v, & \tilde{w} &= w, \\ \tilde{y} &= y, & \tilde{h}(\tilde{x}) &= h(x). \end{aligned}$$

It was observed for instance in [TWY] and [ChYa] that $\tilde{L}_0 = \frac{1}{2}\sum_{i=1}^n \frac{\partial^2}{\partial \tilde{x}_i^2} - \sum_{i=1}^n \tilde{f}_i \frac{\partial}{\partial \tilde{x}_i} - \sum_{i=1}^n \frac{\partial \tilde{f}_i}{\partial \tilde{x}_i} - \frac{1}{2}\sum_{i=1}^m \tilde{h}_m^2$ is equal to L_0. Hence the Lie algebra $\tilde{E} = \langle \tilde{L}_0, \tilde{L}_1, \dots, \tilde{L}_m \rangle_{L.A.}$ is isomorphic to $E = \langle L_0, L_1, \dots, L_m \rangle_{L.A.}$.

Let $\Omega = (\omega_{ij})$ and $\tilde{\Omega} = (\tilde{\omega}_{ij})$ where $\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$ and $\tilde{\omega}_{ij} = \frac{\partial \tilde{f}_j}{\partial \tilde{x}_i} - \frac{\partial \tilde{f}_i}{\partial \tilde{x}_j}$. It was shown in [ChYa] that the following lemma is true.

LEMMA 6. $\tilde{\Omega} = R\Omega R^{-1}$.

3. Proof of the main theorem. In this section, we shall classify all finite-dimensional estimation algebras with maximal rank for dimension of state space equal to three. By Lemma 5, we know that ω_{ij} is in E and in view of Ocone's result, ω_{ij} is a polynomial of degree at most two for all i, j. The first step is to prove ω_{ij} is a degree-one polynomial for all i, j. This step was carried out in detail in our Conference on Decision and Control paper [YaLe]. So we have

$$\begin{pmatrix} \omega_{12} \\ \omega_{13} \\ \omega_{23} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} c_{12} \\ c_{13} \\ c_{23} \end{pmatrix}.$$

Now we have to deal with the hard part of the proof. We are going to prove that ω_{ij}'s are constants. For this, we introduce an invariant $r_{\max}$ of the estimation algebra E as follows.

DEFINITION. *Let $p(x)$ be a quadratic polynomial. The rank of $p(x)$, $r(p)$ is defined as the rank of the Hessian matrix* $(\frac{\partial^2 p}{\partial x_i \partial x_j})$.

Denote

$$\begin{aligned} Q &= \text{space of homogeneous polynomials of degree 2,} \\ P_i &= \text{space of polynomials of degree at most } i, \\ U_k &= \text{space of differential operators with order at most } k. \end{aligned}$$

LEMMA 7. *Let E be a finite-dimensional estimation algebra of maximal rank. Then $P_1 \subseteq E$. If $p(x)$ is a polynomial of degree two in E, then the homogeneous degree-two part of $p(x)$ is also in E.*

Proof. This follows immediately from Lemma 5 and the definition of maximal rank. □

DEFINITION. *Let $E_Q = E \cap Q$. Define* $r_{\max} = \text{Max}\{\text{rank } p(x) : p(x) \in E_Q\}$.

Remark. Observe that $r_{\max}$ is invariant under orthogonal change of coordinates and $0 \leq r_{\max} \leq 3$ in this paper.

3.1. Case $r_{\max} = 3$. There exists homogeneous $p(x) \in E$ with rank $(p(x)) = 3$. By applying an orthogonal change of coordinates, if necessary, we may assume without loss of generality that

$$p(x) = k_1 x_1^2 + k_2 x_2^2 + k_3 x_3^2,$$

where $k_i \neq 0$ for $i = 1, 2, 3$. There are three possibilities.

Case I: all k_i's are distinct. By Lemmas 4 and 5,

$$\begin{aligned} [D_i^2, x_j^2] &= \delta_{ij}(4x_j D_j + 2), \\ [L_0, p(x)] &= \frac{1}{2}\sum_{i=1}^{3}[D_i^2, p(x)] = \sum_{j=1}^{3}(2k_j x_j D_j + 1). \end{aligned}$$

So $\sum_{j=1}^3 k_j x_j D_j \in E$ and

$$\left[\sum_{i=1}^{3} k_i x_i D_i, \frac{1}{2}p(x)\right] = \sum_{i=1}^{3} k_i^2 x_i^2 \in E.$$

Replacing $p(x)$ by $\sum_{i=1}^3 k_i^2 x_i^2$, we deduce that $\sum_{i=1}^3 k_i^3 x_i^2 \in E$. Since the matrix

$$\begin{pmatrix} k_1 & k_2 & k_3 \\ k_1^2 & k_2^2 & k_3^2 \\ k_1^3 & k_2^3 & k_3^3 \end{pmatrix}$$

is nonsingular, we conclude that $x_1^2, x_2^2, x_3^2 \in E$. Now for $i \neq j$,

$$\begin{aligned}[][L_0, x_i^2], [L_0, x_j^2]] &= [2x_iD_i + 1, 2x_jD_j + 1]] \\ &= -4x_ix_j\omega_{ij} \in E.\end{aligned}$$

Since $x_ix_j\omega_{ij}$ is a polynomial of degree at most 2 by Ocone's result, we deduce that ω_{ij} is constant.

Case II: two of the k_i's are equal. In this case we may take $p(x) = k_1x_1^2 + k_2(x_2^2 + x_3^2)$. By evaluating $[[L_0, p(x)], p(x)]$, we can obtain $k_1^2x_1^2 + k_2^2(x_2^2 + x_3^2) \in E$. It follows that $x_1^2 \in E$ and $x_2^2 + x_3^2 \in E$. Since $[L_0, x_i^2] = 2x_iD_i + 1$, we have $x_1D_1, x_2D_2 + x_3D_3 \in E$. So we have

$$\begin{aligned}&- [x_1D_1, x_2D_2 + x_3D_3] = x_1x_2\omega_{12} + x_1x_3\omega_{13} \\ &= a_{11}x_1^2x_2 + a_{21}x_1^2x_3 + a_{12}x_1x_2^2 + a_{23}x_1x_3^2 + (a_{13} + a_{22})x_1x_2x_3 \qquad \text{mod } P_2 \\ &\in E.\end{aligned}$$

By Ocone's result, $[x_1D_1, x_2D_2 + x_3D_3] \in P_2$. We deduce immediately that

$$a_{11} = a_{21} = a_{12} = a_{23} = 0, \quad a_{13} + a_{22} = 0.$$

Furthermore, from the cyclic relation $\frac{\partial\omega_{12}}{\partial x_3} + \frac{\partial\omega_{23}}{\partial x_1} + \frac{\partial\omega_{31}}{\partial x_2} = 0$, we have $a_{13}+a_{31}-a_{22} = 0$ and

$$A = \begin{pmatrix} 0 & 0 & -a_{22} \\ 0 & a_{22} & 0 \\ 2a_{22} & a_{32} & a_{33} \end{pmatrix}.$$

Recall that

$$\begin{aligned}Y_1 &= [L_0, D_1] = \omega_{12}D_2 + \omega_{13}D_3 \qquad \text{mod } U_0, \\ Y_2 &= [L_0, D_2] = \omega_{21}D_1 + \omega_{23}D_3 \qquad \text{mod } U_0, \\ Y_3 &= [L_0, D_3] = \omega_{31}D_1 + \omega_{32}D_2 \qquad \text{mod } U_0.\end{aligned}$$

Then

$$\begin{aligned}&\frac{1}{2}[Y_2, x_2^2 + x_3^2] = \omega_{23}x_3 = a_{31}x_1x_3 + a_{32}x_2x_3 + a_{33}x_3^2 \qquad \text{mod } P_1 \\ &-\frac{1}{2}[Y_3, x_2^2 + x_3^2] = \omega_{23}x_2 = a_{31}x_1x_2 + a_{32}x_2^2 + a_{33}x_2x_3 \qquad \text{mod } P_1, \\ &\left[x_1D_1, \frac{1}{2}[Y_2, x_2^2 + x_3^2]\right] = a_{31}x_1x_3 \qquad \text{mod } U_0, \\ &a_{31}(x_1D_3 + x_3D_1), a_{31}x_1x_3] = a_{31}^2(x_1^2 + x_3^2) \qquad \text{mod } P_0.\end{aligned}$$

Choose k such that $k \neq \pm a_{31}^2, 0$. Then $a_{31}^2(x_1^2 + x_3^2) + k(x_2^2 + x_3^2) = a_{31}^2x_1^2 + kx_2^2 + (a_{31}^2 + k)x_3^2$ is in E. If $a_{31} \neq 0$, then we are back in Case I and we are done. So we have $a_{31} = 0 = a_{13} = a_{22}$ and

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}.$$

$a_{31} = 0$ implies $a_{32}x_2x_3 + a_{33}x_3^2$ and $a_{32}x_2^2 + a_{33}x_2x_3$ are in E.

$$\begin{aligned}
&[L_0, a_{32}x_2x_3 + a_{33}x_3^2] = a_{32}(x_2D_3 + x_3D_2) + a_{33}(2x_3D_3 + 1)\\
&\qquad \Rightarrow a_{32}x_3D_2 + (a_{32}x_2 + 2a_{33}x_3)D_3 \in E,\\
&[L_0, a_{32}x_2^2 + a_{33}x_2x_3] = a_{32}(2x_2D_2 + 1) + a_{33}(x_2D_3 + x_3D_2)\\
&\qquad \Rightarrow (2a_{32}x_2 + a_{33}x_3)D_2 + a_{33}x_2D_3 \in E,\\
&[a_{32}x_3D_2 + (a_{32}x_2 + 2a_{33}x_3)D_3, a_{32}x_2x_3 + a_{33}x_3^2]\\
&\qquad = a_{32}^2x_3^2 + a_{32}x_2(a_{32}x_2 + 2a_{33}x_3) + 2a_{33}x_3(a_{32}x_2 + 2a_{33}x_3)\\
(3.1)\quad &\qquad = a_{32}^2x_2^2 + 4a_{32}a_{33}x_2x_3 + (a_{32}^2 + 4a_{33}^2)x_3^2 \in E,\\
&[(2a_{32}x_2 + a_{33}x_3)D_2 + a_{33}x_2D_3, a_{32}x_2^2 + a_{33}x_2x_3]\\
&\qquad = (2a_{32}x_2 + a_{33}x_3)^2 + a_{33}^2x_2^2\\
(3.2)\quad &\qquad = (a_{33}^2 + 4a_{32}^2)x_2^2 + 4a_{32}a_{33}x_2x_3 + a_{33}^2x_3^2 \in E.
\end{aligned}$$

From (3.1) and (3.2), we have

$$(-a_{33}^2 - 3a_{32}^2)x_2^2 + (a_{32}^2 + 3a_{33}^2)x_3^2 \in E.$$

Recall that $x_2^2 + x_3^2 \in E$. If

$$\det\begin{pmatrix} 1 & 1 \\ -a_{33}^2 - 3a_{32}^2 & a_{32}^2 + 3a_{33}^2 \end{pmatrix} = 4(a_{32}^2 + a_{33}^2)$$

is nonzero, then x_2^2 and x_3^2 are in E. So ω_{ij} = constant for all i, j in view of the argument in Case I. On the other hand if the determinant above is zero, then $a_{32}^2 + a_{33}^2 = 0$, which implies $a_{32} = a_{33} = 0$. So $A = 0$, which means that ω_{ij}'s are constants.

Case III: all k_i's are the same. In this case, we may take $p(x) = x_1^2 + x_2^2 + x_3^2 \in E$. If there exists quadratic form $q(x)$ with $0 < \operatorname{rank}(q(x)) < 3$, we can find an orthogonal transformation R such that

$$\begin{aligned}
q(x) &\longmapsto \tilde{q}(\tilde{x}) = \tilde{x}_1^2 \quad \text{or} \quad \tilde{x}_1^2 + \tilde{x}_2^2,\\
p(x) &\longmapsto \tilde{p}(\tilde{x}) = \tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{x}_3^2,
\end{aligned}$$

so that $\tilde{E}$ contains either $\tilde{x}_1^2, \tilde{x}_2^2 + \tilde{x}_3^2$ or $\tilde{x}_1^2 + \tilde{x}_2^2, \tilde{x}_3^2$, for which the proof in Case II works. Therefore we shall assume without loss of generality that $E_Q = \langle x_1^2 + x_2^2 + x_3^2\rangle$.

Recall from Lemma 5, $Y_j = \sum_{i=1}^3 \omega_{ji}D_i \bmod U_0$ is in E.

$$\begin{aligned}
[Y_1, p(x)] &= [\omega_{12}D_2 + \omega_{13}D_3, p(x)] = 2(x_2\omega_{12} + x_3\omega_{13})\\
&= 2(a_{11}x_1x_2 + a_{12}x_2^2 + a_{13}x_2x_3 + a_{21}x_1x_3 + a_{22}x_2x_3 + a_{23}x_3^2) \mod P_1.
\end{aligned}$$

So $a_{11}x_1x_2 + a_{12}x_2^2 + (a_{13} + a_{22})x_2x_3 + a_{21}x_1x_3 + a_{23}x_3^2$ is in E_Q and hence equal to $c_1(x_1^2 + x_2^2 + x_3^2)$. Comparing coefficients of x_1^2 allows us to conclude that $c_1 = 0$. Thus $a_{11} = a_{21} = a_{12} = a_{23} = 0, a_{13} + a_{22} = 0$, and

$$A = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & -a_{13} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Similarly,

$$\begin{aligned}[Y_2, p(x)] &= [\omega_{21} D_1 + \omega_{23} D_3, p(x)] = 2(-x_1\omega_{12} + x_3\omega_{23}) \\ &= 2(-a_{11}x_1^2 - a_{12}x_1x_2 - a_{13}x_1x_3 + a_{31}x_1x_3 + a_{32}x_2x_3 + a_{33}x_3^2) \mod P_1 \\ &= 2((a_{31} - a_{13})x_1x_3 + a_{32}x_2x_3 + a_{33}x_3^2) \mod P_1.\end{aligned}$$

So $(a_{31} - a_{13})x_1x_3 + a_{32}x_2x_3 + a_{33}x_3^2$ is in E_Q and hence equal to $c_2(x_1^2 + x_2^2 + x_3^2)$. Thus $c_2 = 0$ and $a_{32} = a_{33} = 0, a_{31} = a_{13}$.

$$A = a_{13}\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Finally, the cyclic relation $\frac{\partial \omega_{12}}{\partial x_3} + \frac{\partial \omega_{23}}{\partial x_1} + \frac{\partial \omega_{31}}{\partial x_2} = 0$ allows us to conclude that $a_{13} = 0$. Therefore A is a zero matrix and we are done.

3.2. Case $r_{\max} = 2$. There exists homogeneous polynomial $p(x) \in E$ with $\operatorname{rank}(p(x)) = 2$. Without loss of generality, we shall assume that

$$p(x) = k_1x_1^2 + k_2x_2^2,$$

where $k_1k_2 \neq 0$. We remark that E cannot contain x_3^2 since $r_{\max} = 2$.

Case I: $k_1 \neq k_2$. By evaluating $[[L_0, p(x)], p(x)]$, we can obtain $k_1^2x_1^2 + k_2^2x_2^2$ in E. It follows that x_1^2, x_2^2 are in E.

$$\begin{aligned}&[L_0, x_1^2] = 2x_1D_1 + 1 \qquad \text{and} \qquad [L_0, x_2^2] = 2x_2D_2 + 1 \\ &\Rightarrow x_1D_1 \in E \qquad \text{and} \qquad x_2D_2 \in E \\ &\Rightarrow x_1x_2\omega_{12} = -[x_1D_1, x_2D_2] \in E \\ &\Rightarrow \omega_{12} = c_{12} = \text{constant} \qquad \text{by Ocone's result.}\end{aligned}$$

LEMMA 8. *Suppose that x_1D_1, x_2D_2 are in E. If $q(x) = q_{11}x_1^2 + q_{12}x_1x_2 + q_{13}x_1x_3 + q_{22}x_2^2 + q_{23}x_2x_3 + q_{33}x_3^2$ is in E, then each individual $q_{ij}x_ix_j$ is in E.*

Proof.

$$\begin{aligned}[x_1D_1, q(x)] &= x_1\frac{\partial q}{\partial x_1} = 2q_{11}x_1^2 + q_{12}x_1x_2 + q_{13}x_1x_3 \\ [x_1D_1, [x_1D_1, q(x)]] &= 4q_{11}x_1^2 + q_{12}x_1x_2 + q_{13}x_1x_3.\end{aligned}$$

These imply $q_{11}x_1^2 \in E$ and $q_{12}x_1x_2 + q_{13}x_1x_3 \in E$.

$$[x_2D_2, q_{12}x_1x_2 + q_{13}x_1x_3] = q_{12}x_1x_2 \in E.$$

This implies $q_{13}x_1x_3 \in E$.

$$\begin{aligned}&[x_2D_2, q_{22}x_2^2 + q_{23}x_2x_3 + q_{33}x_3^2 = 2q_{22}x_2^2 + q_{23}x_2x_3 \in E \\ &[x_2D_2, 2q_{22}x_2^2 + q_{23}x_2x_3] = 4q_{22}x_2^2 + q_{23}x_2x_3 \in E.\end{aligned}$$

These imply $q_{22}x_2^2 \in E, q_{23}x_2x_3 \in E$ and $q_{23}x_3^2 \in E$. □

We now claim that $x_1x_3 \notin E$. If $x_1x_3 \in E$, then

$$\begin{aligned}
&[L_0, x_1x_3] = \frac{1}{2}[D_1^2 + D_2^2 + D_3^2, x_1x_3] = x_3D_1 + x_1D_3 \in E \\
&\Rightarrow\ [x_1D_3 + x_3D_1, x_1x_3] = x_1^2 + x_3^2 \in E \\
&\Rightarrow\ x_1^2 + x_2^2 + x_3^2 \in E \qquad \text{and} \qquad \operatorname{rank}(x_1^2 + x_2^2 + x_3^2) = 3.
\end{aligned}$$

This gives a contradiction. So we conclude that $x_1x_3 \notin E$. Similarly we conclude that $x_2x_3 \notin E$. Clearly $x_3^2 \notin E$. In view of Lemma 8, we have

$$\langle x_1^2, x_2^2\rangle \subseteq E_Q \subseteq \langle x_1^2, x_2^2, x_1x_2\rangle.$$

By Lemma 5,

$$-[x_1D_1, D_3] = x_1\omega_{13} = a_{21}x_1^2 + a_{22}x_1x_2 + a_{23}x_1x_3 \in E.$$

In view of Lemma 8, we have $a_{23}x_1x_3 \in E$, which implies $a_{23} = 0$. Similarly,

$$[x_2D_2, D_3] = x_2\omega_{23} = a_{31}x_1x_2 + a_{32}x_2^2 + a_{33}x_2x_3 \in E$$

implies $a_{33} = 0$. Then

$$A = \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix}.$$

Let $Z_1 = x_1D_1$,

$$\begin{aligned}
Z_2 := [L_0, Z_1] &= \frac{1}{2}\sum_{i=1}^{3}\left(2\frac{\partial x_1}{\partial x_i}D_iD_1 - 2x_1\omega_{i1}D_i\right) \quad \mathrm{mod}\ U_0 \\
&= D_1^2 + c_{12}x_1D_2 + x_1\omega_{13}D_3 \quad \mathrm{mod}\ U_0, \\
Z_3 := [L_0, Z_2] &= \frac{1}{2}\sum_{i=1}^{3}[D_i^2, D_1^2 + c_{12}x_1D_2 + x_1\omega_{13}D_3] \quad \mathrm{mod}\ U_1 \\
&= \sum_{i=1}^{3}\left(2\omega_{1i}D_iD_1 + \frac{\partial(c_{12}x_1)}{\partial x_i}D_iD_2 + \frac{\partial(x_1\omega_{13})}{\partial x_i}D_iD_3\right) \quad \mathrm{mod}\ U_1 \\
&= 2\omega_{12}D_2D_1 + 2\omega_{13}D_3D_1 + c_{12}D_1D_2 + \left(\omega_{13} + x_1\frac{\partial\omega_{13}}{\partial x_1}\right)D_1D_3 \\
&\quad + x_1\frac{\partial\omega_{23}}{\partial x_2}D_2D_3 \quad \mathrm{mod}\ U_1 \\
&= 3c_{12}D_1D_2 + (4a_{21}x_1 + 3a_{22}x_2 + 3c_{13})D_1D_3 + a_{32}x_1D_2D_3 \quad \mathrm{mod}\ U_1, \\
[Z_3, Z_1] &= [3c_{12}D_1D_2 + (4a_{21}x_1 + 3a_{22}x_2 + 3c_{13})D_1D_3 + a_{32}x_1D_2D_3, x_1D_1] \\
&\qquad \mathrm{mod}\ U_1 \\
&= 3c_{12}D_1D_2 + (4a_{21}x_1 + 3a_{22}x_2 + 3c_{13})D_1D_3 - 4a_{21}x_1D_1D_3 - a_{32}x_1D_2D_3 \\
&\qquad \mathrm{mod}\ U_1 \\
&= 3c_{12}D_1D_2 + 3a_{22}x_2 + 3c_{13})D_1D_3 - a_{32}x_1D_2D_3 \quad \mathrm{mod}\ U_1 \\
Z_4 &= \frac{1}{2}[Z_3, Z_1] + \frac{1}{2}Z_3 = 3c_{12}D_1D_2 + (2a_{21}x_1 + 3a_{22}x_2 + 3c_{13})D_1D_3 \quad \mathrm{mod}\ U_1, \\
[Z_4, Z_1] &= [3c_{12}D_1D_2 + (2a_{21}x_1 + 3a_{22}x_2 + 3c_{13})D_1D_3, x_1D_1] \quad \mathrm{mod}\ U_1
\end{aligned}$$

$$
\begin{aligned}
&= 3c_{12}D_1D_2 + (2a_{21}x_1 + 3a_{22}x_2 + 3c_{13})D_1D_3 - 2a_{21}x_1D_1D_3 \mod U_1 \\
&= 3c_{12}D_1D_2 + (3a_{22}x_2 + 3c_{13})D_1D_3 \mod U_1, \\
Z_5 &= \frac{1}{2}(Z_4 - [Z_4, Z_1]) = a_{21}x_1D_1D_3 \mod U_1, \\
[L_0, Z_5] &= \frac{1}{2}[D_1^2 + D_2^2 + D_3^2, a_{21}x_1D_1D_3] \mod U_2 \\
&= a_{21}D_1^2D_3 \mod U_2, \\
[[L_0, Z_5], Z_5] &= [a_{21}D_1^2D_3, a_{21}x_1D_1D_3] \mod U_3 \\
&= a_{21}^2D_1^2D_3^2 \mod U_3.
\end{aligned}
$$

By induction, we get infinite elements in E of the form

$$(-1)^n Ad_{Z_5}^n(L_0) = a_{21}^n D_1^2 D_3^n \mod U_{n+1}.$$

Since E is finite dimensional, we conclude that

$$a_{21} = 0. \tag{3.3}$$

$$
\begin{aligned}
W_1 &:= x_2D_2 \in E \\
W_2 &:= [L_0, W_1] = \frac{1}{2}\sum_{i=1}^{3}[D_i^2, x_2D_2] \mod U_0 \\
&= \frac{1}{2}\sum_{i=1}^{3}\left(2\frac{\partial x_2}{\partial x_i}D_iD_2 - 2x_1\omega_{i2}D_i\right) \mod U_0 \\
&= D_2^2 - x_1c_{12}D_1 + x_1\omega_{23}D_3 \mod U_0, \\
W_3 &:= [L_0, W_2] \\
&= \frac{1}{2}\sum_{i=1}^{3}[D_i^2, D_2^2 - c_{12}x_1D_1 + x_1\omega_{23}D_3] \mod U_1 \\
&= \sum_{i=1}^{3}\left(2\omega_{2i}D_iD_2 - \frac{\partial(c_{12}x_1)}{\partial x_i}D_iD_1 + \frac{\partial(x_1\omega_{23})}{\partial x_i}D_iD_3\right) \mod U_1 \\
&= 2\omega_{21}D_1D_2 + 2\omega_{23}D_3D_2 - c_{12}D_1^2 + \left(\omega_{23} + x_1\frac{\partial\omega_{23}}{\partial x_1}\right)D_1D_3 \\
&\quad + x_1\frac{\partial\omega_{23}}{\partial x_2}D_2D_3 \mod U_1 \\
&= (-2a_{22}x_2)D_1D_2 + (2a_{31}x_1 + 2a_{32}x_2)D_2D_3 - c_{12}D_1^2 \\
&\quad + (2a_{31}x_1 + a_{32}x_2)D_1D_3 + a_{32}x_1D_2D_3 \mod U_1 \\
&= (-2a_{22}x_2)D_1D_2 + [(2a_{31} + a_{32})x_1 + 2a_{32}x_2]D_2D_3 - c_{12}D_1^2 \qquad (3.4) \\
&\quad + (2a_{31}x_1 + a_{32}x_2)D_1D_3 \mod U_1, \\
[W_3, W_1] &= [(-2a_{22}x_2)D_1D_2 + ((2a_{31} + a_{32})x_1 + 2a_{32}x_2)D_2D_3 - c_{12}D_1^2 \\
&\quad + (2a_{31}x_1 + a_{32}x_2)D_1D_3, x_2D_2] \mod U_1 \\
&= -2a_{22}x_2D_1D_2 + ((2a_{31} + a_{32})x_1 + 2a_{32}x_2)D_2D_3 + 2a_{22}x_2D_1D_2 \\
&\quad - 2a_{32}x_2D_2D_3 - a_{32}x_2D_1D_3 \mod U_1 \\
&= (2a_{31} + a_{32})x_1D_2D_3 - a_{32}x_2D_1D_3 \mod U_1,
\end{aligned}
$$

$$[[W_3,W_1],Z_1] = [(2a_{31}+a_{32}x_1D_2D_3 - a_{32}x_2D_1D_3, x_1D_1] \mod U_1$$
$$= -a_{32}x_2D_1D_3 - (2a_{31}+a_{32})x_1D_2D_3 \mod U_1$$

(3.5) $$-\frac{1}{2}([W_3,W_1]+[[W_3,W_1],Z_1]) = a_{32}x_2D_1D_3 \mod U_1.$$

(3.6) $$\frac{1}{2}([W_3,W_1]-[[W_3,W_1],Z_1]) = (2a_{31}+a_{32})x_1D_2D_3 \mod U_1.$$

It follows from (3.4), (3.5), and (3.6) that

$$W_4 := -2a_{22}x_2D_1D_2 + 2a_{32}x_2D_2D_3 - c_{12}D_1^2 + 2a_{31}x_1D_1D_3 \mod U_1,$$
$$[W_4,Z_1] = [-2a_{22}x_2D_1D_2 + 2a_{32}x_2D_2D_3 - c_{12}D_1^2 + 2a_{31}x_1D_1D_3, x_1D_1] \mod U_1$$
$$= -2a_{22}x_2D_1D_2 - 2c_{12}D_1^2 \mod U_1,$$
$$W_5 := -\frac{1}{2}[W_4,Z_1] \mod U_1$$
$$= a_{22}x_2D_1D_2 + c_{12}D_1^2 \mod U_1,$$
$$[L_0,W_5] = \frac{1}{2}[D_1^2+D_2^2+D_3^2, a_{22}x_2D_1D_2 + c_{12}D_1^2] \mod U_2$$
$$= a_{22}D_1D_2^2 \mod U_2,$$
$$[[L_0,W_5],W_5] = [a_{22}D_1D_2^2, a_{22}x_2D_1D_2 + c_{12}D_1^2] \mod U_3$$
$$= 2a_{22}^2D_1^2D_2^2 \mod U_3.$$

By induction, we have

$$(-1)^n Ad_{W_5}^n(L_0) = 2^{n-1}a_{22}^nD_1^nD_2^2 \mod U_{n+1}.$$

Since E is finite dimensional, we conclude that

(3.7) $$a_{22}=0.$$

By the cyclic relation $\frac{\partial\omega_{12}}{\partial x_3}+\frac{\partial\omega_{23}}{\partial x_1}+\frac{\partial\omega_{31}}{\partial x_2}=0$, we get

$$a_{13}+a_{31}-a_{22}=0.$$

From (3.3) and (3.7), we get $a_{31}=0$. It follows that

$$W_4 = 2a_{32}x_2D_2D_3 - c_{12}D_1^2 \mod U_1,$$
$$\left[L_0,\frac{1}{2}W_4\right] = \frac{1}{2}\left[D_1^2+D_2^2+D_3^2, a_{32}x_2D_2D_3 - \frac{1}{2}c_{12}D_1^2\right] \mod U_2$$
$$= a_{32}D_2^2D_3 \mod U_2,$$
$$\left[\left[L_0,\frac{1}{2}W_4\right],\frac{1}{2}W_4\right] = \left[a_{32}D_2^2D_3, a_{32}x_2D_2D_3 - \frac{1}{2}c_{12}D_1^2\right] \mod U_3$$
$$= 2a_{32}^2D_2^2D_3^2 \mod U_3,$$
$$(-1)^n Ad_{\frac{1}{2}W_4}^n(L_0) = 2^{n-1}a_{32}^nD_2^2D_3^n \mod U_{n+1}.$$

Since E is finite dimensional, we have $a_{32}=0$. Therefore, the ω_{ij}'s are constants for all i,j.

Case II: $k_1 = k_2$. Without loss of generality, we may take $p(x) = x_1^2 + x_2^2$. In view of Case I, we shall assume that E does not contain x_1^2, x_2^2.

LEMMA 9. *Under the Case* II *assumption,* $\langle x_1^2 + x_2^2 \rangle \subseteq E_Q \subseteq \langle x_1^2, x_2^2, x_1x_2 \rangle$.

Proof. Let $q(x) \in E_Q$. Then

$$q(x) = q_{11}x_1^2 + q_{22}x_2^2 + q_{33}x_3^2 + q_{12}x_1x_2 + q_{13}x_1x_3 + q_{23}x_2x_3.$$

Recall that $x_1D_1 + x_2D_2$ is in E. By applying $x_1D_1 + x_2D_2$ repeatedly to $q(x)$, we see immediately that $q_{11}x_1^2 + q_{22}x_2^2 + q_{12}x_1x_2$, $q_{13}x_1x_3 + q_{23}x_2x_3, q_{33}x_3^2 \in E$. These imply $q_{33} = 0$ (since $r_{\max} = 2$) and $\frac{1}{2}(x_1^2 + x_2^2) + (q_{13}x_1x_3 + q_{23}x_2x_3) \in E$.

$$Hess\left[\frac{1}{2}(x_1^2 + x_2^2) + (q_{13}x_1x_3 + q_{23}x_2x_3)\right] = \begin{pmatrix} 1 & 0 & q_{13} \\ 0 & 1 & q_{23} \\ q_{13} & q_{23} & 0 \end{pmatrix}.$$

The determinant of the above matrix is $-(q_{13}^2 + q_{23}^2)$. Since $r_{\max} = 2 < 3$, we have $q_{13}^2 + q_{23}^2 = 0$ which implies $q_{13} = 0 = q_{23}$. □

We deduce from Lemma 9 that $1 \leq \dim E_Q \leq 3$.

If $\dim E_Q = 3$, then $E_Q = \langle x_1^2, x_2^2, x_1x_2 \rangle$ and we are in Case I.

If $\dim E_Q = 2$, then we may take $E_Q = \langle x_1^2 + x_2^2, q_{11}x_1^2 + q_{12}x_1x_2 \rangle$. If $q_{12} = 0$, then E_Q contains both x_1^2 and x_2^2 and we are back in Case I. Therefore we can assume that $q_{12} \neq 0$. Furthermore if $q_{11} = 0$, then E_Q is actually $\langle x_1^2 + x_2^2, x_1x_2 \rangle$. We consider the following particular orthogonal transformation:

$$\tilde{x} = Rx \qquad R = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

such that it gives rise to

$$\begin{aligned} x &= R^T\tilde{x} \\ x_1^2 + x_2^2 &\longmapsto \tilde{x}_1^2 + \tilde{x}_2^2 \\ x_1x_2 &\longmapsto \frac{\tilde{x}_1^2 + \tilde{x}_2^2}{\sqrt{2}} \frac{-\tilde{x}_1^2 + \tilde{x}_2^2}{\sqrt{2}} = \frac{-\tilde{x}_1^2 + \tilde{x}_2^2}{2} \\ E_Q &\longmapsto \tilde{E}_Q = \left\langle \tilde{x}_1^2 + \tilde{x}_2^2, \frac{-\tilde{x}_1^2 + \tilde{x}_2^2}{2} \right\rangle = \langle \tilde{x}_1^2, \tilde{x}_2^2 \rangle. \end{aligned}$$

Thus, $\tilde{E}$ contains $\tilde{x}_1^2$ and $\tilde{x}_2^2$. By Case I, the $\tilde{\omega}_{ij}$'s are constants and so are the ω_{ij}'s as $\Omega = R^T\tilde{\Omega}R$. Hence we may also assume that $q_{11} \neq 0$. So $E_Q = \langle x_1^2 + x_2^2, x_1^2 + 2kx_1x_2 \rangle$ for $k \neq 0$. Observe that if we can find a quadratic form $p_0 \in E_Q$ with $r(p_0) = 1$, then there exists an orthogonal transformation such that E_Q is mapped into $\tilde{E}_Q$, which contains both $\tilde{x}_1^2$ and $\tilde{x}_2^2$, and we are done. So we try to find such a p_0 below. Consider

$$p_0 = \lambda(x_1^2 + x_2^2) + \sigma(x_1^2 + 2kx_1x_2).$$

Its underlying symmetric matrix is

$$A_{p_0} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sigma \begin{pmatrix} 1 & k \\ k & 0 \end{pmatrix} = \begin{pmatrix} \lambda + \sigma & \sigma k \\ \sigma k & \lambda \end{pmatrix}$$

and $\det A_{p_0} = \lambda^2 + \sigma\lambda - \sigma^2 k^2$. Fix $\sigma \neq 0$ (say $\sigma = k$) and choose

$$\lambda = \sigma \frac{-1 + \sqrt{1 + 4k^2}}{2}.$$

Then $r(p_0) = 1$. We are done for $\dim E_Q = 2$.

If $\dim E_Q = 1$, then $E_Q = \langle x_1^2 + x_2^2 \rangle$. Recall from Lemma 5 that Y_j's are in E where $Y_1 = \omega_{12} D_2 + \omega_{13} D_3 \mod U_0$, $Y_2 = \omega_{21} D_1 + \omega_{23} D_3 \mod U_0$, and $Y_3 = \omega_{31} D_1 + \omega_{32} D_2 \mod U_0$.

$$\begin{aligned}
\frac{1}{2}[Y_1, x_1^2 + x_2^2] &= x_2\omega_{12} = a_{11}x_1x_2 + a_{12}x_2^2 + a_{13}x_1x_3 \mod P_1 \\
\Rightarrow \quad & a_{11}x_1x_2 + a_{12}x_2^2 + a_{13}x_1x_3 \in E_Q = \langle x_1^2 + x_2^2 \rangle \\
\Rightarrow \quad & a_{11} = a_{12} = a_{13} = 0, \\
-\frac{1}{2}[Y_3, x_1^2 + x_2^2] &= x_1\omega_{13} + x_2\omega_{23} \\
&= a_{21}x_1^2 + (a_{22} + a_{31})x_1x_2 + a_{32}x_2^2 + a_{23}x_1x_3 + a_{33}x_2x_3 \mod P_1 \\
\Rightarrow \quad & a_{21}x_1^2 + (a_{22} + a_{31})x_1x_2 + a_{32}x_2^2 + a_{23}x_1x_3 + a_{33}x_2x_3 \in \langle x_1^2 + x_2^2 \rangle \\
\Rightarrow \quad & a_{21} = a_{32}, a_{22} + a_{31} = 0, a_{23} = a_{33} = 0.
\end{aligned}$$

By the cyclic relation $\frac{\partial \omega_{12}}{\partial x_3} + \frac{\partial \omega_{23}}{\partial x_1} + \frac{\partial \omega_{31}}{\partial x_2} = 0$, we have $a_{13} + a_{31} - a_{22} = 0$. It follows that $a_{22} = a_{31} = 0$ and

$$A = a_{21} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

In order to prove that $a_{21} = 0$, we consider the following sequence of elements in E.

$$\begin{aligned}
K_1 :&= x_1 D_1 + x_2 D_2, \\
K_2 :&= [L_0, K_1] = \frac{1}{2}\sum_{i=1}^{3}[D_i^2, x_1 D_1] + \frac{1}{2}\sum_{i=1}^{3}[D_i^2, x_2 D_2] \mod U_0 \\
&= D_1^2 + x_1\omega_{12}D_2 + x_1\omega_{13}D_3 + D_2^2 + x_2\omega_{21}D_1 + x_2\omega_{23}D_3 \mod U_0 \\
&= D_1^2 + D_2^2 + x_2\omega_{21}D_1 + x_1\omega_{12}D_2 + (x_1\omega_{13} + x_2\omega_{23})D_3 \mod U_0, \\
K_3 :&= [L_0, K_2] \\
&= \frac{1}{2}\sum_{i=1}^{3}[D_i^2, D_1^2 + D_2^2 + x_2\omega_{21}D_1 + x_1\omega_{12}D_2 + (x_1\omega_{13} + x_2\omega_{23})D_3] \\
&\quad \mod U_1 \\
&= 2(\omega_{12}D_1D_2 + \omega_{13}D_1D_3 + \omega_{21}D_2D_1 + \omega_{23}D_2D_3) \\
&\quad - c_{12}D_2D_1 + c_{12}D_1D_2 \\
&\quad + (\omega_{13}D_1D_3 + a_{21}x_1D_1D_3 + \omega_{23}D_2D_3 + a_{32}x_2D_2D_3) \mod U_1 \\
&= (3\omega_{13} + a_{21}x_1)D_1D_3 + (3\omega_{23} + a_{21}x_2)D_2D_3 \mod U_1 \\
&= 4a_{21}(x_1D_1 + x_2D_2)D_3 \mod U_1, \\
(-1)Ad_{K_3}(K_2) &= [K_2, K_3] \\
&= [D_1^2 + D_2^2, 4a_{21}(x_1D_1 + x_2D_2)D_3] \mod U_2 \\
&= 4a_{21}([D_1^2, x_1D_1] + [D_2^2, x_2D_2])D_3 \mod U_2 \\
&= 8a_{21}(D_1^2 + D_2^2)D_3 \mod U_2.
\end{aligned}$$

Inductively, we have

$$(-1)Ad^n_{K_3}(K_2) = (8a_{21})^n(D_1^2 + D_2^2)D_3^n \mod U_{n+1}.$$

Since $\dim E < \infty$, $a_{21} = 0$ and we have $A = 0$. So ω_{ij}'s are constant for all i, j.

3.3. Case $r_{\max} = 1$. In this case, we may assume that $p(x) = x_1^2 \in E$ and $E_Q = \langle x_1^2 \rangle$.

$$[Y_2, p(x)] = [\omega_{21}D_1 + \omega_{23}D_3, x_1^2] = 2\omega_{21}x_1 \in E_Q$$
$$[Y_3, p(x)] = [\omega_{31}D_1 + \omega_{32}D_2, x_1^2] = 2\omega_{31}x_1 \in E_Q.$$

Thus, ω_{12} and ω_{13} depend only on x_1 because $E_Q = \langle x_1^2 \rangle$. So

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

The cyclic relation $\frac{\partial \omega_{12}}{\partial x_3} + \frac{\partial \omega_{23}}{\partial x_1} + \frac{\partial \omega_{31}}{\partial x_2} = 0$ implies $a_{31} = 0$ and implies $a_{31} = 0$ and

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now $x_1^2 \in E$ implies $x_1 D_1 \in E$. Let

$$\begin{aligned}
X_1 &:= x_1 D_1, \\
X_2 &:= [L_0, X_1] = \frac{1}{2}\sum_{i=1}^{3}[D_i^2, x_1 D_1] \mod U_0 \\
&= \frac{1}{2}\sum_{i=1}^{3}\left(2\frac{\partial x_1}{\partial x_i}D_i D_1 - 2x_1\omega_{i1}D_i + \frac{\partial^2 x_1}{\partial x_i^2}D_1\right) \mod U_0 \\
&= D_1^2 + x_1\omega_{12}D_2 + x_1\omega_{13}D_3 \mod U_0 \\
&= D_1^2 + (a_{11}x_1^2 + c_{12}x_1)D_2 + (a_{21}x_1^2 + c_{13}x_1)D_3 \mod U_0 \\
&= D_1^2 \mod U_1, \\
X_3 &:= [L_0, X_2] \\
&= \frac{1}{2}\sum_{i=1}^{3}[D_i^2, D_1^2 + (a_{11}x_1^2 + c_{12}x_1)D_2 + (a_{21}x_1^2 + c_{13}x_1)D_3] \mod U_1 \\
&= \sum_{i=1}^{3}\left(2\omega_{1i}D_1 D_i + \frac{\partial(a_{11}x_1^2 + c_{12}x_1)}{\partial x_i}D_i D_2 + \frac{\partial(a_{21}x_1^2 + c_{13}x_1)}{\partial x_i}D_i D_3\right) \mod U_1 \\
&= 2(a_{11}x_1 + c_{12})D_1 D_2 + 2(a_{21}x_1 + c_{13})D_1 D_3 + (2a_{11}x_1 + c_{12})D_1 D_2 \\
&\quad + (2a_{21}x_1 + c_{13})D_1 D_3 \mod U_1 \\
&= (4a_{11}x_1 + 3c_{12})D_1 D_2 + (4a_{21}x_1 + 3c_{13})D_1 D_3 \mod U_1.
\end{aligned}$$

We are going to show that $a_{11} = 0$. Suppose $a_{11} \neq 0$. Denote $a = \frac{a_{21}}{a_{11}}$ and define

$$\alpha := \frac{1}{4a_{11}}X_3 = \left(x_1 + \frac{3c_{12}}{a_{11}}\right)D_1 D_2 + \left(ax_1 + \frac{3c_{13}}{a_{11}}\right)D_1 D_3 \mod U_1.$$

LEMMA 10. *If* $a_{11} \neq 0$, *let* $\alpha := \frac{1}{4a_{11}} X_3, a = \frac{a_{21}}{a_{11}}$. *Then for* $j \geq 1$

$$\frac{(-1)^j Ad_\alpha^j(X_2)}{2^j} = D_1^2(D_2 + aD_3)^j \mod U_{j+1}.$$

Proof. We shall prove this by induction.

$$\begin{aligned}
\frac{(-1)Ad_\alpha(X_2)}{2} &= \frac{1}{2}[X_2, \alpha] \\
&= \frac{1}{2}\left[D_1^2, \left(x_1 + \frac{3c_{12}}{a_{11}}\right)D_1D_2 + \left(ax_1 + \frac{3c_{13}}{a_{11}}\right)D_1D_3\right] \mod U_2 \\
&= \frac{\partial}{\partial x_1}\left(x_1 + \frac{3c_{12}}{a_{11}}\right)D_1^2D_2 + \frac{\partial}{\partial x_1}\left(ax_1 + \frac{3c_{13}}{a_{11}}\right)D_1^2D_3 \mod U_2 \\
&= D_1^2(D_2 + aD_3) \mod U_2, \\
\frac{(-1)^{j+1}Ad_\alpha^{j+1}(X_2)}{2^{j+1}} &= \left(-\frac{1}{2}\right)Ad_\alpha \frac{(-1)^j Ad_\alpha^j(X_2)}{2^j} \\
&= \frac{1}{2}\left[\frac{(-1)^j Ad_\alpha^j(X_2)}{2^j}, \alpha\right] \\
&= \frac{1}{2}\left[D_1^2(D_2 + aD_3)^j, \left(x_1 + \frac{3c_{13}}{a_{11}}\right)D_1D_2 + \left(ax_1 + \frac{3c_{13}}{a_{11}}\right)D_1D_3\right] \\
&\quad \mod U_{j+2} \\
&= D_1^2(D_2 + aD_3)^j D_2 + aD_1^2(D_2 + aD_3)^j D_3 \mod U_{j+2} \\
&= D_1^2(D_2 + aD_3)^j(D_2 + aD_3) \mod U_{j+2} \\
&= D_1^2(D_2 + aD_3)^{j+1} \mod U_{j+2}. \quad \square
\end{aligned}$$

The above lemma implies that E is infinite dimensional, contradicting the finite-dimensionality of E. Hence $a_{11} = 0$. Then

$$A = \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix},$$

$$X_2 = D_1^2 + c_{12}x_1D_2 + (a_{21}x_1^2 + c_{13}x_1)D_3 \mod U_0,$$

$$X_3 = 3c_{12}D_1D_2 + (4a_{21}x_1 + 3c_{13})D_1D_3 \mod U_1.$$

Next we shall see that $a_{21} = 0$. Suppose $a_{21} \neq 0$. Consider

$$\beta := \frac{1}{4a_{21}}X_3 = \frac{3c_{12}}{4a_{21}}D_1D_2 + \left(x_1 + \frac{3c_{13}}{4a_{21}}\right)D_1D_3 \mod U_1,$$

$$\begin{aligned}
(-1)Ad_\beta(X_2) = [X_2, \beta] &= \left[D_1^2, \frac{3c_{12}}{4a_{21}}D_1D_2 + \left(x_1 + \frac{3c_{13}}{4a_{21}}\right)D_1D_3\right] \mod U_2 \\
&= 2D_1^2D_3 \mod U_2.
\end{aligned}$$

We claim that $(-1)^j Ad_\beta^j(X_2) = 2^j D_1^2 D_3^j \mod U_{j+1}$. This can be seen by induction.

$$\begin{aligned}
(-1)^{j+1}Ad_\beta^{j+1}(X_2) &= (-1)Ad_\beta((-1)Ad_\beta^j(X_2)) \\
&= [(-1)^j Ad_\beta^j(X_2), \beta] \\
&= \left[2^j D_1^2 D_3^j, \frac{3c_{12}}{4a_{21}}D_1D_2 + \left(x_1 + \frac{3c_{13}}{4a_{21}}\right)D_1D_3\right] \mod U_{j+2} \\
&= 2^{j+1}D_1^2D_3^{j+1} \mod U_{j+2}.
\end{aligned}$$

Then E contains an infinite-dimensional subspace, which is impossible. Hence $a_{21} = 0$ and

$$A = \begin{pmatrix} o & o & o \\ 0 & 0 & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}.$$

Consider the expression of $[Y_j, D_k]$ in Lemma 5(vi). Noting that ω_{ij}'s are linear, the following elements belong to E:

$$K_{jk} = \sum_{i=1}^{3} \omega_{ji}\omega_{ki} - \frac{1}{2}\frac{\partial^2 \eta}{\partial x_k \partial x_j}.$$

K_{jk} is symmetric about j, k (Table 1) and is a polynomial of degree at most two, which in turn forces η to be a polynomial of degree at most four.

TABLE 1.

(j,k)	K_{jk}
(1,1)	$\omega_{12}^2 + \omega_{13}^2 - \frac{1}{2}\frac{\partial^2\eta}{\partial x_1^2}$
(1,2)	$\omega_{13}\omega_{23} - \frac{1}{2}\frac{\partial^2\eta}{\partial x_1 \partial x_2}$
(1,3)	$\omega_{12}\omega_{32} - \frac{1}{2}\frac{\partial^2\eta}{\partial x_1 \partial x_3}$
(2,2)	$\omega_{21}^2 + \omega_{23}^2 - \frac{1}{2}\frac{\partial^2\eta}{\partial x_2^2}$
(2,3)	$\omega_{21}\omega_{31} - \frac{1}{2}\frac{\partial^2\eta}{\partial x_2 \partial x_3}$
(3,3)	$\omega_{31}^2 + \omega_{32}^2 - \frac{1}{2}\frac{\partial^2\eta}{\partial x_3^2}$

Recall our notation:

$$\begin{pmatrix} \omega_{12} \\ \omega_{13} \\ \omega_{23} \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} c_{12} \\ c_{13} \\ c_{23} \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}.$$

Since $K_{jk} \in P_2$ and $E_Q = \langle x_1^2 \rangle$ in this case $r_{\max} = 1$, we have

$$K_{jk} = kx_1^2 \mod P_1.$$

So we can form the following relationships:

$$\frac{1}{2}\frac{\partial^2\eta}{\partial x_2^2} = a_{32}^2 x_2^2 + 2a_{32}a_{33}x_2x_3 + a_{33}x_3^2 + ax_1^2 \mod P_1,$$
$$\frac{1}{2}\frac{\partial^2\eta}{\partial x_3^2} = a_{32}^2 x_2^2 + 2a_{32}a_{33}x_2x_3 + a_{33}x_3^2 + bx_1^2 \mod P_1,$$
$$\frac{1}{2}\frac{\partial^2\eta}{\partial x_2 \partial x_3} = cx_1^2 \mod P_1.$$

Observe that the term $a_{33}x_3^2$ in $\frac{1}{2}\frac{\partial^2\eta}{\partial x_2^2}$ must come from the $x_2^2x_3^2$ term in η. Let η contain the term $\alpha x_2^2 x_3^2$. Since

$$\frac{1}{2}\frac{\partial^2 \alpha x_2^2x_3^2}{\partial x_2^2} = \alpha x_3^2, \quad \frac{1}{2}\frac{\partial^2 \alpha x_2^2x_3^2}{\partial x_3^2} = \alpha x_2^2, \quad \frac{1}{2}\frac{\partial^2 \alpha x_2^2x_3^2}{\partial x_2 \partial x_3} = 2\alpha x_2x_3,$$

by comparing coefficients we obtain

$$\alpha = a_{33}^2, \quad \alpha = a_{32}^2, \quad 2\alpha = 0.$$

So, $a_{32} = a_{33} = 0$ and accordingly

$$\Omega = O_{3\times 3} \mod P_0.$$

Hence Case $r_{\max} = 1$ is done.

3.4. Case $r_{\max} = 0$. In this case $E_Q = \phi$. All functions in E are automatically linear.

Recall that

$$m_{jk} = -\sum_{i=1}^{3} \omega_{ji}\omega_{ki} + \frac{1}{2}\frac{\partial^2 \eta}{\partial x_k \partial x_j} \in E_Q.$$

This expression is written in element form. It's more insightful to view it in matrix form.

Let $M = (m_{jk})_{3\times 3}$ and note that the Ω matrix is antisymmetric. Then we have

$$\begin{aligned} M &= -\Omega\Omega^T + \frac{1}{2}Hess(\eta) \\ &= \Omega^2 + \frac{1}{2}Hess(\eta), \end{aligned}$$

where $Hess(\eta) = (\frac{\partial^2 \eta}{\partial x_k \partial x_j})_{3\times 3}$ is the Hessian matrix of η.

Let $\Omega = Dx_1 + Bx_2 + Cx_3$ (mod P_0), where $D = (\alpha_{ij})_{3\times 3}, B = (\beta_{ij})_{3\times 3}, C = (\gamma_{ij})_{3\times 3}$ are skew-symmetric matrices. We make use of $\Omega^2 + \frac{1}{2}Hess(\eta) = 0 \mod P_1$ to infer that $D = B = C = (0)_{3\times 3}$ as follows. Writing

$$\begin{aligned} H &= \Omega^2 \\ &= H_{11}x_1^2 + H_{22}x_2^2 + H_{33}x_3^2 + H_{12}x_1x_2 + H_{13}x_1x_3 + H_{23}x_2x_3 \\ &= D^2x_1^2 + B^2x_2^2 + C^2x_3^2 + (DB+BD)x_1x_2 + (DC+CD)x_1x_3 \\ &\quad + (BC+CB)x_2x_3, \end{aligned}$$

we have

$$H_{11} = D^2 = -DD^T, \qquad \text{where} \quad D = \begin{pmatrix} 0 & \alpha_{12} & \alpha_{13} \\ -\alpha_{12} & 0 & \alpha_{23} \\ -\alpha_{13} & -\alpha_{23} & 0 \end{pmatrix}.$$

So

$$H_{11} = -\begin{pmatrix} \alpha_{12}^2 + \alpha_{13}^2 & \alpha_{13}\alpha_{23} & -\alpha_{12}\alpha_{23} \\ \alpha_{13}\alpha_{23} & \alpha_{12}^2 + \alpha_{23}^2 & \alpha_{12}\alpha_{13} \\ -a_{12}a_{23} & a_{12}a_{13} & a_{13}^2 + a_{23}^2 \end{pmatrix}.$$

The other H_{ij} matrices can be obtained similarly and they are listed explicitly at the end of this section.

We consider terms in η and relationships derived from $\Omega^2 + \frac{1}{2}Hess(\eta) = 0 \mod P_1$ in terms of entries in H_{ij} matrices. The coefficient of $x_1^2x_2^2$ in $-\eta = H_{11}[2,2] =$

$H_{22}[1,1] = \frac{1}{2}H_{12}[1,2]$. Similarly, $H_{11}[3,3] = H_{33}[1,1] = \frac{1}{2}H_{13}[1,3]$ and $H_{22}[3,3] = H_{33}[2,2] = H_{23}[2,3]$ ($H_{ij}[p,q]$ means the (p,q)-entry of matrix H_{ij}). We have

$$\alpha_{12}^2 + \alpha_{23}^2 = \beta_{12}^2 + \beta_{13}^2 = \frac{1}{2}(\alpha_{13}\beta_{23} + \alpha_{23}\beta_{13}), \tag{3.8}$$

$$\alpha_{13}^2 + \alpha_{23}^2 = \gamma_{12}^2 + \gamma_{13}^2 = -\frac{1}{2}(\alpha_{12}\gamma_{23} + \alpha_{23}\gamma_{12}), \tag{3.9}$$

$$\beta_{13}^2 + \beta_{23}^2 = \gamma_{12}^2 + \gamma_{23}^2 = \frac{1}{2}(\beta_{12}\gamma_{13} + \beta_{13}\gamma_{12}). \tag{3.10}$$

Together with the simple majorization relationship between any two real numbers, $2ab \le a^2 + b^2$, we can rewrite (3.8), (3.9) and (3.10) to obtain

$$\begin{aligned}
2(\alpha_{12}^2 + \alpha_{23}^2 + \beta_{12}^2 + \beta_{13}^2) &= 2(\alpha_{13}\beta_{23} + \alpha_{23}\beta_{13}) \le \alpha_{13}^2 + \beta_{23}^2 + \alpha_{23}^2 + \beta_{13}^2,\\
2(\alpha_{13}^2 + \alpha_{23}^2 + \gamma_{12}^2 + \gamma_{13}^2) &= -2(\alpha_{12}\gamma_{23} + \alpha_{23}\gamma_{12}) \le \alpha_{12}^2 + \gamma_{23}^2 + \alpha_{23}^2 + \gamma_{12}^2,\\
2(\beta_{13}^2 + \beta_{23}^2 + \gamma_{12}^2 + \gamma_{23}^2) &= 2(\beta_{12}\gamma_{13} + \beta_{13}\gamma_{12}) \le \beta_{12}^2 + \gamma_{13}^2 + \beta_{13}^2 + \gamma_{12}^2.
\end{aligned}$$

Summing these three inequalities and simplifying, we have

$$\alpha_{12}^2 + \alpha_{13}^2 + 2\alpha_{23}^2 + \beta_{12}^2 + 2\beta_{13}^2 + \beta_{23}^2 + 2\gamma_{12}^2 + \gamma_{13}^2 + \gamma_{23}^2 \le 0,$$

which implies that

$$\alpha_{12} = \alpha_{13} = \alpha_{23} = \beta_{12} = \beta_{13} = \beta_{23} = \gamma_{12} = \gamma_{13} = \gamma_{23} = 0,$$

i.e.,

$$D = B = C = O_{3\times 3}.$$

Hence

$$\Omega = O_{3\times 3} \quad \bmod P_0.$$

Case $r_{\max} = 0$ is done.

For reference we list the H_{ij} matrices below:

$$H_{11} = -\begin{pmatrix} \alpha_{12}^2 + \alpha_{13}^2 & \alpha_{13}\alpha_{23} & -\alpha_{12}\alpha_{23} \\ \alpha_{13}\alpha_{23} & \alpha_{12}^2 + \alpha_{23}^2 & \alpha_{12}\alpha_{13} \\ -\alpha_{12}\alpha_{23} & \alpha_{12}\alpha_{13} & \alpha_{13}^2 + \alpha_{23}^2 \end{pmatrix};$$

$$H_{22} = -\begin{pmatrix} \beta_{12}^2 + \beta_{13}^2 & \beta_{13}\beta_{23} & -\beta_{12}\beta_{23} \\ \beta_{13}\beta_{23} & \beta_{12}^2 + \beta_{23}^2 & \beta_{12}\beta_{13} \\ -\beta_{12}\beta_{23} & \beta_{12}\beta_{13} & \beta_{13}^2 + \beta_{23}^2 \end{pmatrix};$$

$$H_{33} = -\begin{pmatrix} \gamma_{12}^2 + \gamma_{13}^2 & \gamma_{13}\gamma_{23} & -\gamma_{12}\gamma_{23} \\ \gamma_{13}\gamma_{23} & \gamma_{12}^2 + \gamma_{23}^2 & \gamma_{12}\gamma_{13} \\ -\gamma_{12}\gamma_{23} & \gamma_{12}\gamma_{13} & \gamma_{13}^2 + \gamma_{23}^2 \end{pmatrix};$$

$$H_{12} = -\begin{pmatrix} 2\alpha_{12}\beta_{12} + 2\alpha_{13}\beta_{13} & \alpha_{13}\beta_{23} + a_{23}b_{13} & -\alpha_{12}\beta_{23} - \beta_{23}\beta_{12} \\ \alpha_{13}\beta_{23} + \alpha_{23}\beta_{13} & 2\alpha_{12}\beta_{12} + 2\alpha_{23}\beta_{23} & \alpha_{12}\beta_{13} + \alpha_{13}\beta_{12} \\ -\alpha_{12}\beta_{23} - \alpha_{23}\beta_{12} & \alpha_{12}\beta_{13} + \alpha_{13}\beta_{12} & 2\alpha_{13}\beta_{13} + 2\alpha_{23}\beta_{23} \end{pmatrix};$$

$$H_{13} = -\begin{pmatrix} 2\alpha_{12}\gamma_{12} + 2\alpha_{13}\gamma_{13} & \alpha_{13}\gamma_{23} + \alpha_{23}\gamma_{13} & -\alpha_{12}\gamma_{23} - \alpha_{23}\gamma_{12} \\ \alpha_{13}\gamma_{23} + \alpha_{23}\gamma_{13} & 2\alpha_{12}\gamma_{12} + 2\alpha_{23}\gamma_{23} & \alpha_{12}\gamma_{13} + \alpha_{13}\gamma_{12} \\ -\alpha_{12}\gamma_{23} - \alpha_{23}\gamma_{12} & \alpha_{12}\gamma_{13} + \alpha_{13}\gamma_{12} & 2\alpha_{13}\gamma_{13} + 2\alpha_{23}\gamma_{23} \end{pmatrix};$$

$$H_{23} = -\begin{pmatrix} 2\beta_{12}\gamma_{12} + 2\beta_{13}\gamma_{13} & \beta_{13}\gamma_{23} + \beta_{23}\gamma_{13} & -\beta_{12}\gamma_{23} - \beta_{23}\gamma_{12} \\ \beta_{13}\gamma_{23} + \beta_{23}\gamma_{13} & 2\beta_{12}\gamma_{12} + 2\beta_{23}\gamma_{23} & \beta_{12}\gamma_{13} + \beta_{13}\gamma_{12} \\ -\beta_{12}\gamma_{23} - \beta_{23}\gamma_{12} & \beta_{12}\gamma_{13} + \beta_{13}\gamma_{12} & 2\beta_{13}\gamma_{13} + 2\beta_{23}\gamma_{23} \end{pmatrix}.$$

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