Appl Math Optim 34:231–266 (1996)

Applied Mathematics & Optimization

Explicit Solution of a Kolmogorov Equation*

S.-T. Yau[1] and S. S.-T. Yau[2]

[1]Department of Mathematics, Harvard University,
Cambridge, MA 02138, USA

[2]Department of Mathematics, Statistics, and Computer Science,
University of Illinois at Chicago, 851 South Morgan Street,
Chicago, IL 60607-7045, USA
U32790@UICVM.BITNET

Communicated by S. K. Mitter

Abstract. Ever since the technique of the Kalman–Bucy filter was popularized, there has been an intense interest in finding new classes of finite-dimensional recursive filters. In the late seventies, the concept of the estimation algebra of a filtering system was introduced. It has been the major tool in studying the Duncan–Mortensen–Zakai equation. Recently the second author has constructed general finite-dimensional filters which contain both Kalman–Bucy filters and Benes filter as special cases. In this paper we consider a filtering system with arbitrary nonlinear drift $f(x)$ which satisfies some regularity assumption at infinity. This is a natural assumption in view of Theorem 10 of [DTWY] in a special case. Under the assumption on the observation $h(x)$ = constant, we propose writing down the solution of the Duncan–Mortensen–Zakai equation explicitly.

Key Words. Finite-dimensional filters, Duncan–Mortensen–Zakai equation, Kolmogorov equation, Nonlinear drift.

AMS Classification. 35C06, 35J15, 60G35, 93E11.

* This research was supported by Army Grant DAAH-04-93G-0006.

1. Introduction

Until the seventies, the basic approach to nonlinear filtering theory was via "innovations methods," originally proposed by Kailath in 1967 and rigorously developed by Fujisaki *et al.* [FKK] in their seminal paper in 1972. As pointed out by Mitter, the difficulty with this approach is that the innovations process is not, in general, explicitly computable (except in the well-known Kalman–Bucy case). The idea of using estimation algebras to construct finite-dimensional nonlinear filters was first proposed in the early eighties by Brockett and Clark [BrCl], Brockett [Br], and Mitter [M]. The motivation comes from the Wei–Norman approach [WeNo] of using Lie algebraic ideas to solve time-variant linear differential equations. The extension of Wei–Norman's approach to the nonlinear filtering problem is much more complicated. Instead of an ordinary differential equation, we have to solve the Duncan–Mortensen–Zakai (DMZ) equation, which a stochastic partial differential equation. By working on the robust form of the DMZ equation we can reduce the complexity of the problem to that of solving a time-variant partial differential equation. Wong in [Wo1] constructed some new finite-dimensional estimation algebras and the Wei–Norman approach to synthesize finite-dimensional filters. However, the systems considered in [Wo1] are very specific and the question whether the Wei–Norman approach works for a general system with finite-dimensional estimation algebra remains open.

Recently, Tam *et al.* [TWY] have examined the properties of finite-dimensional estimation algebras and the Wei–Norman approach in detail. A class of filtering systems having the property that the drift term, f, of the state evolution equation is a gradient vector field was considered. In [Wo2] the concept of Ω is introduced, which is defined as the $n \times n$ matrix whose (i, j)-entry is $\partial f_j/\partial x_i - \partial f_i/\partial x_j$. In view of the Poincaré lemma, f is a gradient vector field if and only if $\Omega \doteq 0$. More recently, the second author [Y1], [Y2] considered a more general class of filtering systems having the property that $\partial f_j/\partial x_i - \partial f_i/\partial x_j$ are constant for all i, j, i.e., Ω is a skew constant matrix. These include Kalman–Bucy filtering systems and Bene's filtering systems as special cases and finite-dimensional filters were constructed explicitly. From the Lie algebraic piont of view, Chen and coworkers [ChYa], [CLY] have shown that these are most general finite-dimensional filters, at least for dimension of state space less than four. In many senses, the Lie algebraic viewponit has been remarkably successful, and the recent work has given us a deeper understanding of the DMZ equation which was essential for progress in nonlinear filtering, as well as in stochastic control. Nevertheless, the results obtained so far are quite restrictive in the sense that the drift term f has to be of the form $(l_1, \dots, l_n) + (\partial\varphi/\partial x_1, \dots, \partial\varphi/\partial x_n)$ where the l_i's are degree on polynomials and φ is a C^∞ function on R^n.

In the past decade, the Lie algebraic viewpoint played the major role in solving DMZ equations. In this paper we introduce a new method to solve DMZ equations. The advantage of this approach is that, firstly, we do not need to make any assumption on the drift term f except some regularity assumption at infinity and therefore it applies to the general class of nonlinear filtering systems. Secondly, unlike the Lie algebraic method which only reduces the DMZ equation to a finite system of ordinary differential equations, our method actually allows us to write down the formal solution on the Kolmogorov equation (i.e., DMZ equation with observation $h(x) = \text{constant}$), which is well known to be a fundamental equation in electrical engineering, explicitly in a closed form. We can give estimates of the formal solution. We can also construct a convergent solution explicitly from the truncated formal solution. Most strikingly we can actually estimate the time interval on which our solution converges. More precisely we have proven the following theorems.

Theorem A. *Let h_i, $1 \le i \le m$, be constants. Then the Kolmogorov equation*

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \left\{ \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial}{\partial x_i} - f_i(x)\right)^2 \right. \\ &\quad \left. -\frac{1}{2}\left(\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) + \sum_{i=1}^{n} f_i^2(x) + \sum_{i=1}^{m} h_i^2\right)\right\} u(t,x), \\ u(0,x) &= \sigma_0(x), \end{aligned} \tag{1.1}$$

has a formal solution on $\mathbf{R}^n$ of the following form:

$$\begin{aligned} u(t,x) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} &(2\pi t)^{n/2} \\ &\cdot \exp\left(\frac{-1}{2t}\sum_{j=1}^{n}(x_j - y_j)^2 + \int_0^1 \sum_{i=1}^{n}(x_i - y_i) f_i(y + t(x,y))\, dt\right) \\ &\cdot [1 + \tilde{a}_1(x,y)t + \tilde{a}_2(x,y)t^2 + \cdots + \tilde{a}_k(x,y)t^k + \cdots]\sigma_0(y)\, dy_1 \cdots dy_n, \end{aligned}$$

where $\tilde{a}_k(x,y) = \int_0^1 t^{k-1}\tilde{g}_k(y + t(x-y), y)\, dt$ and

$$\begin{aligned} \tilde{g}_k(x,y) &= \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 \tilde{a}_{k-1}}{\partial x_i^2}(x,y) + \sum_{i=1}^{n}\left(\frac{\partial a}{\partial x_i}(x,y) - f_i(x)\right)\frac{\partial \tilde{a}_{k-1}}{\partial x_i}(x,y) \\ &\quad + \left[\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 a}{\partial x_i^2}(x,y) + \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial a}{\partial x_i}(x,y)\right)^2\right. \\ &\qquad \left. - \sum_{i=1}^{n} f_i(x)\frac{\partial a}{\partial x_i}(x,y) - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) - \frac{1}{2}\sum_{i=1}^{m} h_i^2\right]\tilde{a}_{k-1}(x,y). \end{aligned}$$

Theorem B. *Let*

$$\begin{aligned} \tilde{\varphi}_N(t,x,y) &= (2\pi t)^{-n/2}\exp\left(a(x,y) - \frac{|x-y|^2}{2t}\right) \\ &\quad \cdot [1 + \tilde{a}_1(x,y)t + \cdots + \tilde{a}_N(x,y)t^N], \\ e_N(t,x,y) &= \frac{\partial \tilde{\varphi}_N}{\partial t}(t,x,y) - L_x \tilde{\varphi}_N(t,x,y), \end{aligned}$$

where

$$L_x = \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial}{\partial x_i} - f_i(x)\right)^2 - \frac{1}{2}\left(\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) + \sum_{i=1}^{n} f_i^2(x) + \sum_{i=1}^{m} h_i^2\right)$$

is the operator defined by the right-hand side of (1.1). *Assume that* $\operatorname{Sup}|\nabla^j f_i| \le C(j!)$, *for* $j = 1, \dots, N$, $1 \le i \le n$, *and* $\frac{1}{2}\sum_{i=1}^m h_i^2 \le C$, *where* $C > \max(2, |f(0)|)$ *and* ∇^j *denotes any* jth-*order partial differentiation in* x *variables. Then, for* $|t| \le 1$ *and* $N \ge 3nC - 1$,

(a) $\tilde{\varphi}_N(t,x,y)| \le 2(N+1)^{4N}(1+\sqrt{t}|x|)^{2N}(1+\sqrt{t}|y|)^{2N}(2\pi t)^{-n/2}\exp(a(x,y) - |x-y|^2/2t)$.

(b) $|e_N(t,x,y)| \le (N+2)^{4N+4}(1+\sqrt{t}|x|)^{2N+2}(1+\sqrt{t}|y|)^{2N+2}(2\pi t)^{-n/2} \times \exp(a(x,y) - |x-y|^2/2t)$.

Theorem C. *For* $N \ge 3nC - 1$, *let* $\varphi_N(t,x,y)$ *and* $e_N(t,x,y)$ *be defined as in Theorem B. Let*

$$\varphi(t,x,y) = \tilde{\varphi}_N(t,x,y) + \sum_{m=1}^{\infty}(-1)^{m+1}\varphi_m(t,x,y),$$

where

$$\begin{aligned}\varphi_m(t,x,y) &= \frac{1}{\sqrt{m+2}}\int_{\sum_{i=0}^{m+1}\tau_i = t}\int_{x_{m+1}}\int_{x_m}\cdots\int_{x_1}\tilde{\varphi}_N(\tau_{m+1},x,x_{m+1})e_N(\tau_m,x_{m+1},x_m)\\ &\quad\cdot e_N(\tau_{m-1},x_m,x_{m-1})\cdots e_N(\tau_0,x_1,y).\end{aligned}$$

If t *is chosen small enough so that*

$$4(N+1)t^2 + nCt < \tfrac{1}{8} \quad \text{and} \quad nC < \frac{1}{8t},$$

then the infinite series $\varphi(t,x,y)$ *converges and* $\varphi(t,x,y)$ *is the fundamental solution to the Kolmogorov equation, i.e.,*

$$\begin{cases}\dfrac{\partial\varphi}{\partial t}(t,x,y) = L_x\varphi(t,x,y),\\ \lim\limits_{t\to 0}\varphi(t,x,y) = \delta_x(y).\end{cases}$$

2. Basic Concepts

The filtering problem considered here is based on the following signal observation model:

$$\begin{cases}dx(t) = f(x(t))\,dt + g(x(t))\,dv(t), & x(0) = x_0,\\ dy(t) = h(x(t))\,dt + dw(t), & y(0) = 0,\end{cases} \tag{2.0}$$

in which x, v, y, and w, are respectively R^n-, R^p-, R^m-, and R^m-valued processes, and v and w have components which are independent, standard Brownian processes. We further assume that $n = p$; f, h are C^∞ smooth, and that g is an orthogonal matrix. We refer to $x(t)$ as the state of the system at time t and to $y(t)$ as the observation at time t. Here the Stratonovich calculus is used in (2.0).

Let $\rho(t, x)$ denote the conditional probability density of the state given the observation $y(s)$: $0 \leq s \leq t$. It is well known (see [DM], for example) that $\rho(t, x)$ is given by normalizing a function, $\sigma(t, x)$, which satisfies the following DMZ equation:

$$d\sigma(t, x) = L_0\sigma(t, x)\,dt + \sum_{i=1}^{m} L_i\sigma(t, x)\,dy_i(t), \qquad \sigma(0, x) = \sigma_0, \tag{2.1}$$

where

$$L_0 = \tfrac{1}{2}\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} - \tfrac{1}{2}\sum_{i=1}^{m} h_i^2$$

and, for $i = 1, \ldots, m$, L_i is the zero degree differential operator of multiplication by h_i. σ_0 is the probability density of the initial point, x_0.

Equation (2.1) is a stochastic partial differential equation. In real applications, we are interested in constructing robust state estimators from observed sample paths with some property of robustness. Davis [D] studied this problem and proposed some robust algorithms. In our case, his basic idea reduces to defining a new unnormalized density

$$u(t, x) = \exp\left(-\sum_{i=1}^{m} h_i(x) y_i(t)\right)\sigma(t, x).$$

It is easy to show that $u(t, x)$ satisfies the following time-varying partial differential equation:

$$\begin{aligned} \frac{\partial u}{\partial t}(t, x) &= L_0 u(t, x) + \sum_{i=1}^{m} y_i(t)[L_0, L_i]u(t, x) \\ &\quad + \tfrac{1}{2}\sum_{i,j=1}^{m} y_i(t) y_j(t)[[L_0, L_i], L_j]u(t, x), \\ u(0, x) &= \sigma_0, \end{aligned} \tag{2.2}$$

where $[\cdot, \cdot]$ is the Lie bracket defined as:

Definition. If X and Y are differential operators, the Lie bracket of X and Y, $[X, Y]$, is defined by

$$[X, Y]\xi = X(Y(\xi) - Y(X\xi)$$

for any C^∞ function ξ.

In Section 3 we write down the formal solution of (2.2) explicitly in closed form.

3. Formal Solution to the Kolmogorov Equation

The purpose of this section is to write down an asymptotic solution of the time-varying differential equation (2.2).

Lemma 1. *Equation* (2.2) *is equivalent to the following equation*:

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x) &= \left\{ \frac{1}{2}\sum_{i=1}^{n}\left[\frac{\partial}{\partial x_i} - \left[f_i(x) - \sum_{j=1}^{m} y_j(t)\frac{\partial h_j}{\partial x_i}(x)\right]\right]^2 \right. \\ &\quad \left. -\frac{1}{2}\left[\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) + \sum_{i=1}^{n} f_i^2(x) + \sum_{i=1}^{m} h_i^2(x)\right]\right\} u(t,x), \\ u(0,x) &= \sigma_0(x).\end{aligned} \tag{3.0}$$

Proof.

$$\begin{aligned}[L_0, h_i(x)] &= \frac{1}{2}\left[\sum_{j=1}^{n}\left(\frac{\partial}{\partial x_j} - f_j(x)\right)^2 \right. \\ &\qquad \left. - \left(\sum_{j=1}^{n}\frac{\partial f_j}{\partial x_j}(x) + \sum_{j=1}^{n} f_j^2(x) + \sum_{j=1}^{m} h_j^2(x)\right), h_i(x)\right] \\ &= \frac{1}{2}\left[\sum_{j=1}^{n}\left(\frac{\partial}{\partial x_j} - f_j(x)\right)^2, h_i(x)\right] \\ &= \frac{1}{2}\sum_{j=1}^{n}\frac{\partial^2 h_i}{\partial x_j^2}(x) + \sum_{j=1}^{n}\frac{\partial h_i}{\partial x_j}(x)\frac{\partial}{\partial x_j} - \sum_{j=1}^{n} f_j(x)\frac{\partial h_i}{\partial x_j}(x) \\ &= \sum_{j=1}^{n}\frac{\partial h_i}{\partial x_j}(x)\frac{\partial}{\partial x_j} + \frac{1}{2}\sum_{j=1}^{n}\frac{\partial^2 h_i}{\partial x_j^2}(x) - \sum_{j=1}^{n} f_j(x)\frac{\partial h_i}{\partial x_j}(x),\end{aligned}$$

$$\big[[L_0, h_i], h_j\big] = \sum_{k=1}^{n}\left[\frac{\partial h_i}{\partial x_k}(x)\right]\left[\frac{\partial h_j}{\partial x_k}(x)\right],$$

$$\begin{aligned}&L_0 + \sum_{i=1}^{m} y_i(t)[L_0, L_i] + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} y_i(t)y_j(t)[[L_0, L_i], L_j] \\ &\quad = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i(x)\frac{\partial}{\partial x_i} - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) - \frac{1}{2}\sum_{i=1}^{m} h_i^2(x) \\ &\qquad + \sum_{i=1}^{m}\sum_{j=1}^{n} y_i(t)\frac{\partial h_i}{\partial x_j}(x)\frac{\partial}{\partial x_j} \\ &\qquad + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} y_i(t)\frac{\partial^2 h_i}{\partial x_j^2}(x) - \sum_{i=1}^{m}\sum_{j=1}^{n} y_i(t) f_j(x)\frac{\partial h_i}{\partial x_j}(x) \\ &\qquad + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{n} y_i(t)y_j(t)\left[\frac{\partial h_i}{\partial x_k}(x)\right]\left[\frac{\partial h_j}{\partial x_k}(x)\right]\end{aligned}$$

$$
\begin{aligned}
&= \tfrac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{n}\left[-f_i(x) + \sum_{j=1}^{m} y_j(t)\frac{\partial h_j}{\partial x_i}(x)\right]\frac{\partial}{\partial x_i}\Bigg] \\
&\quad - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) - \tfrac{1}{2}\sum_{i=1}^{m} h_i^2(x) \\
&\quad + \tfrac{1}{2}\sum_{i=1}^{m} y_i(t)\Delta h_i(x) - \sum_{i=1}^{m}\sum_{j=1}^{n} y_i(t) f_j(x)\frac{\partial h_i}{\partial x_j}(x) \\
&\quad + \tfrac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} y_i(t)y_j(t)\sum_{k=1}^{n}\left[\frac{\partial h_i}{\partial x_k}(x)\right]\left[\frac{\partial h_j}{\partial x_k}(x)\right] \\
&= \tfrac{1}{2}\sum_{i=1}^{n}\left[\frac{\partial}{\partial x_i} - \left[f_i(x) - \sum_{j=1}^{m} y_j(t)\frac{\partial h_j}{\partial x_i}(x)\right]\right]^2 \\
&\quad + \tfrac{1}{2}\sum_{i=1}^{n}\left[\frac{\partial f_i}{\partial x_i}(x) - \sum_{j=1}^{m} y_j(t)\frac{\partial^2 h_j}{\partial x_i^2}(x)\right] \\
&\quad - \tfrac{1}{2}\sum_{i=1}^{n}\left[f_i(x) - \sum_{j=1}^{m} y_j(t)\frac{\partial h_j}{\partial x_j}(x)\right]^2 \\
&\quad - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) - \tfrac{1}{2}\sum_{i=1}^{m} h_i^2(x) + \tfrac{1}{2}\sum_{i=1}^{m} y_i(t)\Delta h_i(x) \\
&\quad - \sum_{i=1}^{m}\sum_{j=1}^{n} y_i(t) f_j(x)\frac{\partial h_i}{\partial x_j}(x) + \tfrac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} y_i(t)y_j(t)\sum_{k=1}^{n}\left[\frac{\partial h_i}{\partial x_k}(x)\right]\left[\frac{\partial h_j}{\partial x_k}(x)\right] \\
&= \tfrac{1}{2}\sum_{i=1}^{n}\left[\frac{\partial}{\partial x_i} - \left[f_i(x) - \sum_{j=1}^{m} y_j(t)\frac{\partial h_j}{\partial x_i}(x)\right]\right]^2 - \tfrac{1}{2}\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) - \tfrac{1}{2}\sum_{i=1}^{n} f_i^2(x) \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{m} f_i(x)y_j(t)\frac{\partial h_j}{\partial x_i}(x) - \tfrac{1}{2}\sum_{i=1}^{n}\left[\sum_{j=1}^{m} y_j(t)\frac{\partial h_j}{\partial x_i}(x)\right]^2 - \tfrac{1}{2}\sum_{i=1}^{m} h_i^2(x) \\
&\quad - \sum_{i=1}^{m}\sum_{j=1}^{n} y_i(t) f_j(x)\frac{\partial h_i}{\partial x_j}(x) + \tfrac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} y_i(t)y_j(t)\sum_{k=1}^{m}\left[\frac{\partial h_i}{\partial x_k}(x)\right]\left[\frac{\partial h_j}{\partial x_k}(x)\right] \\
&= \tfrac{1}{2}\sum_{i=1}^{n}\left[\frac{\partial}{\partial x_i} - \left[f_i(x) - \sum_{j=1}^{m} y_j(t)\frac{\partial h_j}{\partial x_i}(x)\right]\right]^2 \\
&\quad - \tfrac{1}{2}\left[\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) + \sum_{i=1}^{n} f_i^2(x) + \sum_{i=1}^{m} h_i^2(x)\right]. \qquad \square
\end{aligned}
$$

Theorem 2. *Equation* (3.0) *has a formal asymptotic solution on R^n if $h_i(x)$ are constants for all $1 \le i \le m$. In fact, the solution is of the following form:*

$$u(t,x) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(\sqrt{2\pi})^n} t^{-n/2} \cdot \exp\left(-\sum_{j=1}^{n} \frac{(x_j-\xi_j)^2}{2t}\right) b(t,x,\xi)\sigma_0(\xi)\, d\xi_1 \cdots d\xi_n, \tag{3.1}$$

where $b(t,x,\xi) = \sum_{k=0}^{\infty} a_k(x,\xi)t^k$.

Here $a_k(x,\xi)$ are described explicitly as follows. Let

$$a(x,\xi) = \int_0^1 \sum_{i=1}^{n} (x_i - \xi_i) f_i[\xi + t(x-\xi)]\, dt. \tag{3.2}$$

Then

$$a_0(x,\xi) = e^{a(x,\xi)}. \tag{3.3}$$

Suppose that $a_{k-1}(x,\xi)$ is given. Let

$$g_k(x,\xi) = \tfrac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 a_{k-1}}{\partial x_i^2}(x,\xi) - \sum_{i=1}^{n} f_i(x)\frac{\partial a_{k-1}}{\partial x_i}(x,\xi) - \tfrac{1}{2}\left(\sum_{i=1}^{m} h_i^2\right) a_{k-1}(x,\xi) - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(x) a_{k-1}(x,\xi). \tag{3.4}$$

Then, for $k \ge 1$,

$$a_k(x,\xi) = e^{a(x,\xi)} \int_0^1 t^{k-1} e^{-a(\xi+t(x-\xi),\xi)} g_k(\xi + t(x-\xi),\xi)\, dt$$

Proof. We prove that (3.1) is a formal solution of (3.0). Since h_i = constant for all $1 \le i \le m$, (3.0) reduces to the following equation:

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \tfrac{1}{2}\sum_{i=1}^{n}\left[\frac{\partial}{\partial x_i} - f_i(x)\right]^2 u(t,x) \\ &\quad - \tfrac{1}{2}\left[\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) + \sum_{i=1}^{n} f_i^2(x) + \sum_{i=1}^{m} h_i^2(x)\right] u(t,x), \\ u(0,x) &= \sigma_0(x). \end{aligned} \tag{3.5}$$

Putting (3.1) into (3.5) we have

$$\text{L.H.S. of (3.5)} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(\sqrt{2\pi})^n} t^{-n/2-1} \exp\left(-\sum_{j=1}^{n}\frac{(x_j-\xi_j)^2}{2t}\right) \cdot \left[\sum_{j=1}^{n}\frac{(x_j-\xi_j)^2}{2t} b(t,x,\xi) - \frac{n}{2} b(t,x,\xi) + t\frac{\partial b}{\partial t}(t,x,\xi)\right] \cdot \sigma_0(\xi)\, d\xi_1 \cdots d\xi_n.$$

On the other hand,

$$\left[\frac{\partial}{\partial x_i} - f_i(x)\right] u(t,x) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(\sqrt{2\pi})^n} t^{-n/2-1} \exp\left(-\sum_{j=1}^{n} \frac{(x_j-\xi_j)^2}{2t}\right)$$
$$\cdot \left[-(x_i-\xi_i)b(t,x,\xi) + t\frac{\partial b}{\partial x_i}(t,x,\xi) - tf_i(x)b(t,x,\xi)\right] \sigma_0(\xi$$
$$\cdot d\xi_1 \cdots d\xi_n,$$

$$\left[\frac{\partial}{\partial x_i} - f_i(x)\right]^2 u(t,x)$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(\sqrt{2\pi})^n} t^{-n/2-1} \frac{-(x_i-\xi_i)}{t} \exp\left(-\sum_{j=1}^{n} \frac{(x_j-\xi_j)^2}{2t}\right)$$
$$\cdot \left[-(x_i-\xi_i)b(t,x,\xi) + t\frac{\partial b}{\partial x_i}(t,x,\xi) - tf_i(x)b(t,x,\xi)\right] \sigma_0(\xi)\, d\xi_1 \cdots d\xi_n$$
$$+ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(\sqrt{2\pi})^n} t^{-n/2-1} \exp\left(-\sum_{j=1}^{n} \frac{(x_j-\xi_j)^2}{2t}\right)$$
$$\cdot \left[-b(t,x,\xi) - (x_i-\xi_i)\frac{\partial b}{\partial x_i}(t,x,\xi) + t\frac{\partial^2 b}{\partial x_i^2}(t,x,\xi)\right.$$
$$\left. -t\frac{\partial f_i}{\partial x_i}(x)b(t,x,\xi) - tf_i(x)\frac{\partial b}{\partial x_i}(t,x,\xi)\right] \sigma_0(\xi)\, d\xi_1 \cdots d\xi_n$$
$$+ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(\sqrt{2\pi})^n} t^{n/2-1} \exp\left(-\sum_{j=1}^{n} \frac{(x_j-\xi_j)^2}{2t}\right)$$
$$\cdot \left[(x_i-\xi_i)f_i(x)b(t,x,\xi) - tf_i(x)\frac{\partial b}{\partial x_i}(t,x,\xi) + tf_i^2(x)b(t,x,\xi)\right]$$
$$\cdot \sigma_0(\xi)\, d\xi_1 \cdots d\xi_n$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(\sqrt{2\pi})^n} t^{-n/2-1} \exp\left(-\sum_{j=1}^{n} \frac{(x_j-\xi_j)^2}{2t}\right)$$
$$\cdot \left[\frac{(x_i-\xi_i)^2}{t} b(t,x,\xi) - (x_i-\xi_i)\frac{\partial b}{\partial x_i}(t,x,\xi)\right.$$
$$+ (x_i-\xi_i)f_i(x)b(t,x,\xi) - b(t,x,\xi)$$
$$- (x_i-\xi)\frac{\partial b}{\partial x_i}(t,x,\xi) + t\frac{\partial^2 b}{\partial x_i^2}(t,x,\xi) - t\frac{\partial f_i}{\partial x_i}(x)b(t,x,\xi)$$
$$- tf_i(x)\frac{\partial b}{\partial x_i}(t,x,\xi) + (x_i-\xi_i)f_i(x)b(t,x,\xi) - tf_i(x)\frac{\partial b}{\partial x_i}(t,x,\xi)$$
$$\left. + tf_i^2(x)b(t,x,\xi)\right] \sigma_0(\xi)\, d\xi_1 \cdots d\xi_n$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(\sqrt{2\pi})^n} t^{-n/2-1} \exp\left(-\sum_{j=1}^{n} \frac{(x_j-\xi_j)^2}{2t}\right)$$

$$\cdot\left[\frac{(x_i-\xi_i)^2}{t}b(t,x,\xi)-2(x_i-\xi_i)\frac{\partial b}{\partial x_i}(t,x,\xi)+2(x_i-\xi_i)f_i(x)b(t,x,\xi)\right.$$
$$-b(t,x,\xi)+t\frac{\partial^2 b}{\partial x_i^2}(t,x,\xi)-t\frac{\partial f_i}{\partial x_i}(x)b(t,x,\xi)-2tf_i(x)\frac{\partial b}{\partial x_i}(t,x,\xi)$$
$$\left.+tf_i^2(x)b(t,x,\xi)\right]\sigma_0(\xi)\,d\xi_1\cdots d\xi_n,$$

R.H.S. of (3.5)

$$=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{1}{(\sqrt{2\pi})^n}t^{-n/2-1}\exp\left(-\sum_{j=1}^{n}\frac{(x_j-\xi_j)^2}{2t}\right)$$
$$\times\left[\frac{1}{2t}\sum_{i=1}^{n}(x_i-\xi_i)^2b(t,x,\xi)-\sum_{i=1}^{n}(x_i-\xi_i)\frac{\partial b}{\partial x_i}(t,x,\xi)\right.$$
$$+\sum_{i=1}^{n}(x_i-\xi_i)f_i(x)b(t,x,\xi)-\frac{n}{2}b(t,x,\xi)$$
$$+\frac{t}{2}\sum_{i=1}^{n}\frac{\partial^2 b}{\partial x_i^2}(t,x,\xi)-\frac{t}{2}\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x)b(t,x,\xi)-t\sum_{i=1}^{n}f_i(x)\frac{\partial b}{\partial x_i}(t,x,\xi)$$
$$\left.+\frac{t}{2}\sum_{i=1}^{n}f_i^2(x)b(t,x,\xi)-\frac{t}{2}\left(\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x)+\sum_{i=1}^{n}f_i^2(x)+\sum_{i=1}^{m}h_i^2\right)b(t,x,\xi)\right]$$
$$\cdot\sigma_0(\xi)\,d\xi_1\cdots d\xi_n.$$

It follows that (3.1) is a solution of (3.5) if

$$t\frac{\partial b}{\partial t}(t,x,\xi)=-\sum_{i=1}^{n}(x_i-\xi_i)\frac{\partial b}{\partial x_i}(t,x,\xi)+\sum_{i=1}^{n}(x_i-\xi_i)f_i(x)b(t,x,\xi)$$
$$+\frac{t}{2}\sum_{i=1}^{n}\frac{\partial^2 b}{\partial x_i^2}(t,x,\xi)$$
$$-\frac{t}{2}\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x)b(t,x,\xi)-t\sum_{i=1}^{n}f_i(x)\frac{\partial b}{\partial x_i}(t,x,\xi)$$
$$+\frac{t}{2}\sum_{i=1}^{n}f_i^2(x)b(t,x,\xi)$$
$$-\frac{t}{2}\left(\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x)+\sum_{i=1}^{n}f_i^2(x)+\sum_{i=1}^{m}h_i^2\right)b(t,x,\xi),$$

i.e., if

$$t\frac{\partial b}{\partial t}(t,x,\xi)=-\sum_{i=1}^{n}(x_i-\xi_i)\frac{\partial b}{\partial x_i}(t,x,\xi)+\sum_{i=1}^{n}(x_i-\xi_i)f_i(x)b(t,x,\xi)$$
$$+\frac{t}{2}\sum_{i=1}^{n}\frac{\partial^2 b}{\partial x_i^2}(t,x,\xi)-t\sum_{i=1}^{n}f_i(x)\frac{\partial b}{\partial x_i}(t,x,\xi)$$

$$+\frac{t}{2}\left[-2\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x)-\sum_{i=1}^{m}h_i^2\right]b(t,x,\xi). \tag{3.6}$$

Put $b(t,x,\xi)=\sum_{k=0}^{\infty}a_k(x,\xi)t^k$ in (3.6). We have

R.H.S. of (3.6)

$$\begin{aligned}
&=-\sum_{i=1}^{n}\sum_{k=0}^{\infty}(x_i-\xi_i)\frac{\partial a_k}{\partial x_i}(x,\xi)t^k+\sum_{i=1}^{n}\sum_{k=0}^{\infty}(x_i-\xi_i)f_i(x)a_k(x,\xi)t^k\\
&\quad+\sum_{i=1}^{n}\sum_{k=0}^{\infty}\tfrac{1}{2}\frac{\partial^2 a_k}{\partial x_i^2}(x,\xi)t^{k+1}-\sum_{i=1}^{n}\sum_{k=0}^{\infty}f_i(x)\frac{\partial a_k}{\partial x_i}(x,\xi)t^{k+1}\\
&\quad-\sum_{k=0}^{\infty}\left(\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x)+\tfrac{1}{2}\sum_{i=1}^{m}h_i^2\right)a_k(x,\xi)t^{k+1}\\
&=-\sum_{k=0}^{\infty}\sum_{i=1}^{n}(x_i-\xi_i)\frac{\partial a_k}{\partial x_i}(x,\xi)t^k+\sum_{k=0}^{\infty}\sum_{i=1}^{n}(x_i-\xi_i)f_i(x)a_k(x,\xi)t^k\\
&\quad+\sum_{k=1}^{\infty}\sum_{i=1}^{n}\frac{1}{2}\frac{\partial^2 a_{k-1}}{\partial x_i^2}(x,\xi)t^k-\sum_{k=1}^{\infty}\sum_{i=1}^{n}f_i(x)\frac{\partial a_{k-1}}{\partial x_i}(x,\xi)t^k\\
&\quad-\sum_{k=1}^{\infty}\left(\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x)+\tfrac{1}{2}\sum_{i=1}^{n}h_i^2\right)a_{k-1}(x,\xi)t^k\\
&=-\sum_{i=1}^{n}(x_i-\xi_i)\frac{\partial a_0}{\partial x_i}(x,\xi)+\sum_{i=1}^{n}(x_i-\xi_i)f_i(x)a_0(x,\xi)\\
&\quad+\sum_{k=1}^{\infty}\Bigg[-\sum_{i=1}^{n}(x_i-\xi_i)\frac{\partial a_k}{\partial x_i}(x,\xi)+\sum_{i=1}^{n}(x_i-\xi_i)f_i(x)a_k(x,\xi)\\
&\qquad+\tfrac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 a_{k-1}}{\partial x_i^2}(x,\xi)-\sum_{i=1}^{n}f_i(x)\frac{\partial a_{k-1}}{\partial x_i}(x,\xi)\\
&\qquad-\left(\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}+\tfrac{1}{2}\sum_{i=1}^{m}h_i^2\right)a_{k-1}(x,\xi)\Bigg]t^k,
\end{aligned}$$

$$\text{L.H.S. of (3.6)}=\sum_{k=1}^{\infty}ka_k(x,\xi)t^k.$$

Therefore (3.1) is a solution of (3.5) if (3.7) and (3.8) are satisfied:

$$\sum_{i=1}^{n}(x_i-\xi_i)\frac{\partial a_0}{\partial x_i}(x,\xi)=\sum_{i=1}^{n}(x_i-\xi_i)f_i(x)a_0(x,\xi). \tag{3.7}$$

For $k\geq 1$,

$$\left(k-\sum_{i=1}^{n}(x_i-\xi_i)f_i(x)\right)a_k(x,\xi)+\sum_{i=1}^{n}(x_i-\xi_i)\frac{\partial a_k}{\partial x_i}(x,\xi)$$

$$= \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 a_{k-1}}{\partial x_i^2}(x,\xi) - \sum_{i=1}^{n} f_i(x)\frac{\partial a_{k-1}}{\partial x_i}(x,\xi)$$
$$- \left(\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) + \frac{1}{2}\sum_{i=1}^{m} h_i^2\right) a_{k-1}(x,\xi). \tag{3.8}$$

Differentiate (3.2) with respect to x_i,

$$\frac{\partial a}{\partial x_i}(x,\xi) = \int_0^1 f_i\,(\xi + t(x-\xi))\,dt + \int_0^1 \sum_{j=1}^{n}(x_j-\xi_j)\frac{\partial f_j[\xi + t(x-\xi)]}{\partial x_i}\,dt$$
$$= \int_0^1 f_i(\xi + t(x-\xi))\,dt + \int_0^1 \sum_{j=1}^{n}(x_j-\xi_j)\sum_{k=1}^{n}\frac{\partial f_j}{\partial x_k}(\xi + t(x-\xi))$$
$$\cdot\frac{\partial(\xi_k + t(x_k-\xi_k))}{\partial x_i}\,dt$$
$$= \int_0^1 f_i(\xi + t(x-\xi))\,dt$$
$$+ \int_0^1 \sum_{j=1}^{n}(x_j-\xi_j)\sum_{k=1}^{n} t\delta_{ik}\frac{\partial f_j}{\partial x_k}(\xi + t(x-\xi))\,dt$$
$$= \int_0^1 f_i(\xi + t(x-\xi))\,dt$$
$$+ \int_0^1 t\sum_{j=1}^{n}(x_j-\xi_j)\frac{\partial f_j}{\partial x_i}(\xi + t(x-\xi))\,dt,$$

$$\sum_{i=1}^{n}(x_i-\xi_i)\frac{\partial a}{\partial x_i}(x,\xi)$$
$$= \sum_{i=1}^{n}(x_i-\xi_i)\int_0^1 f_i(\xi + t(x-\xi))\,dt$$
$$+ \int_0^1 t\sum_{j=1}^{n}(x_j-\xi_j)\sum_{i=1}^{n}(x_i-\xi_i)\frac{\partial f_j}{\partial x_i}(\xi + t(x-\xi))\,dt$$
$$= \sum_{i=1}^{n}(x_i-\xi_i)\int_0^1 f_i(\xi + t(x-\xi))\,dt$$
$$+ \int_0^1 t\sum_{j=1}^{n}(x_j-\xi_j)\,df_j(\xi + t(x-\xi))$$
$$= \sum_{j=1}^{n}(x_j-\xi_j)f_j(x). \tag{3.9}$$

Let $a_0(x,\xi) = e^{a(x,\xi)}$ as in (3.3). Then

$$\frac{\partial a_0}{\partial x_i}(x,\xi) = \frac{\partial a}{\partial x_i}(x,\xi)e^{a(x,\xi)},$$

$$\begin{aligned}\sum_{i=1}^{n}(x_i-\xi_i)\frac{\partial a_0}{\partial x_i}(x,\xi) &= \sum_{i=1}^{n}(x_i-\xi_i)\frac{\partial a}{\partial x_i}(x,\xi)e^{a(x,\xi)}\\ &= \sum_{j=1}^{n}(x_j-\xi_j)f_j(x)a_0(x,\xi) \qquad \text{in view of (3.9).}\end{aligned}$$

So (3.7) is satisfied. Let

$$\begin{aligned}g_k(x,\xi) &= \tfrac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 a_{k-1}}{\partial x_i^2}(x,\xi) - \sum_{i=1}^{n} f_i(x)\frac{\partial a_{k-1}}{\partial x_i}(x,\xi) - \tfrac{1}{2}\left(\sum_{i=1}^{m}h_i^2\right)a_{k-1}(x,\xi)\\ &\quad -\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x)a_{k-1}(x,\xi)\end{aligned}$$

as in (3.4) and, for $k \geq 1$,

$$a_k(x,\xi) = e^{a(x,\xi)}\int_0^1 t^{k-1}e^{-a(\xi+t(x-\xi),\xi)}g_k(\xi+t(x-\xi),\xi)\,dt.$$

Then

$$\begin{aligned}\frac{\partial a_k}{\partial x_i}(x,\xi) &= \frac{\partial a}{\partial x_i}(x,\xi)e^{a(x,\xi)}\int_0^1 t^{k-1}e^{-a(\xi+t(x-\xi),\xi)}g_k(\xi+t(x-\xi),\xi)\,dt\\ &\quad + e^{a(x,\xi)}\int_0^1 t^{k-1}\frac{\partial}{\partial x_i}F(\xi+t(x-\xi),\xi)\,dt\\ &\qquad \text{(where } F(\xi+t(x-\xi),\xi) = e^{-a(\xi+t(x-\xi),\xi)}g_k(\xi+t(x-\xi),\xi))\\ &= \frac{\partial a}{\partial x_i}(x,\xi)e^{a(x,\xi)}\int_0^1 t^{k-1}e^{-a(\xi+t(x-\xi),\xi)}g_k(\xi+t(x-\xi),\xi)\,dt\\ &\quad + e^{a(x,\xi)}\int_0^1 t^{k-1}\frac{\partial F}{\partial x_i}(\xi+t(x-\xi),\xi)\frac{\partial}{\partial x_i}(\xi_j+t(x_j-\xi_j))\,dt\\ &= \frac{\partial a}{\partial x_i}(x,\xi)e^{a(x,\xi)}\int_0^1 t^{k-1}e^{-a(\xi+t(x-\xi),\xi)}g_k(\xi+t(-\xi),\xi)\,dt\\ &\quad + e^{a(x,\xi)}\int_0^1 t^{k}\frac{\partial F}{\partial x_i}(\xi+t(x-\xi),\xi)\,dt,\end{aligned}$$

$$\begin{aligned}&\sum_{i=1}^{n}(x_i-\xi_i)\frac{\partial a_k}{\partial x_i}(x,\xi)\\ &= \sum_{i=1}^{n}(x_i-\xi_i)\frac{\partial a}{\partial x_i}(x,\xi)e^{a(x,\xi)}\int_0^1 t^{k-1}F(\xi+t(x-\xi),\xi)\,dt\\ &\quad + e^{a(x,\xi)}\int_0^1 t^k\sum_{i=1}^{n}(x_i-\xi_i)\frac{\partial F}{\partial x_i}(\xi+t(x-\xi),\xi)\,dt\\ &= \sum_{i=1}^{n}(x_i-\xi_i)f_i(x)a_k(x,\xi) + e^{a(x,\xi)}\int_0^1 t^k\,dF(\xi+t(x-\xi),\xi)\end{aligned}$$

$$= \sum_{i=1}^{n}(x_i - \xi_i) f_i(x) a_k(x,\xi) + e^{a(x,\xi)} F(x,\xi)$$
$$- k e^{a(x,\xi)} \int_0^1 t^{k-1} F(\xi + t(x-\xi), \xi)\, dt$$
$$= \sum_{i=1}^{n}(x_i - \xi_i) f_i(x) a_k(x,\xi) + e^{a(x,\xi)} e^{-a(x,\xi)} g_k(x,\xi) - k a_k(x,\xi)$$
$$\Rightarrow \quad \sum_{i=1}^{n}(x_i - \xi_i) \frac{\partial a_k}{\partial x_i}(x,\xi)$$
$$= \sum_{i=1}^{n}(x_i - \xi_i) f_i(x) a_k(x,\xi) + g_k(x,\xi) - k a_k(x,\xi)$$
$$\Rightarrow \quad \left(k - \sum_{i=1}^{n}(x_i - \xi_i) f_i(x)\right) a_k(x,\xi)$$
$$+ \sum_{i=1}^{n}(x_i - \xi_i) \frac{\partial a_k}{\partial x_i}(x,\xi) = g_k(x,\xi).$$

So (3.8) is also satisfied. □

Lemma 3. *Let $\tilde{a}_0(x,\xi) = 1$ and $\tilde{a}_{k-1}(x,\xi) = e^{-a(x,\xi)} a_{k-1}(x,\xi)$. Let $\tilde{g}_k(x,\xi) = e^{-a(x,\xi)} g_k(x,\xi)$. Then*

$$\tilde{a}_k(x,\xi) = \int_0^1 t^{k-1} \tilde{g}_k(\xi + t(x-\xi), \xi)\, dt,$$

where

$$\tilde{g}_k(x,\xi) = \tfrac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 \tilde{a}_{k-1}}{\partial x_i^2}(x,\xi) + \sum_{i=1}^{n} \left(\frac{\partial a}{\partial x_i}(x,\xi) - f_i(x) \right) \frac{\partial \tilde{a}_{k-1}}{\partial x_i}(x,\xi)$$
$$+ \left[\tfrac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 a}{\partial x_i^2}(x,\xi) + \tfrac{1}{2} \sum_{i=1}^{n} \left(\frac{\partial a}{\partial x_i}(x,\xi) \right)^2 - \sum_{i=1}^{n} f_i(x) \frac{\partial a}{\partial x_i}(x,\xi) \right.$$
$$\left. - \tfrac{1}{2} \left(\sum_{i=1}^{m} h_i^2 \right) - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(x) \right] \tilde{a}_{k-1}(x,\xi).$$

Proof.

$$a_{k-1}(x,\xi) = e^{a(x,\xi)} \tilde{a}_{k-1}(x,\xi),$$

$$\frac{\partial a_{k-1}}{\partial x_i}(x,\xi) = e^{a(x,\xi)} \left[\frac{\partial a}{\partial x_i}(x,\xi) \tilde{a}_{k-1}(x,\xi) + \frac{\partial \tilde{a}_{k-1}}{\partial x_i}(x,\xi) \right],$$

$$
\begin{aligned}
\frac{\partial^2 a_{k-1}}{\partial x_i^2}(x,\xi) &= \frac{\partial a}{\partial x_i}(x,\xi)e^{a(x,\xi)}\left[\frac{\partial a}{\partial x_i}(x,\xi)\tilde{a}_{k-1}(x,\xi)+\frac{\partial \tilde{a}_{k-1}}{\partial x_i}(x,\xi)\right]\\
&\quad + e^{a(x,\xi)}\left[\frac{\partial^2 a}{\partial x_i^2}(x,\xi)\tilde{a}_{k-1}(x,\xi)+\frac{\partial a}{\partial x_i}(x,\xi)\frac{\partial \tilde{a}_{k-1}}{\partial x_i}(x,\xi)\right.\\
&\quad \left. + \frac{\partial^2 \tilde{a}_{k-1}}{\partial x_i^2}(x,\xi)\right]\\
&= e^{a(x,\xi)}\left[\left(\frac{\partial^2 a}{\partial x_i^2}+\left(\frac{\partial a}{\partial x_i}(x,\xi)\right)^2\right)\tilde{a}_{k-1}(x,\xi)\right.\\
&\quad \left. + 2\frac{\partial a}{\partial x_i}(x,\xi)\frac{\partial \tilde{a}_{k-1}}{\partial x_i}(x,\xi)+\frac{\partial^2 \tilde{a}_{k-1}}{\partial x_i^2}(x,\xi)\right],
\end{aligned}
$$

$$
\begin{aligned}
\tilde{g}_k(x,\xi) &= e^{-a(x,\xi)}g_k(x,\xi)\\
&= e^{-a(x,\xi)}\left[\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 a_{k-1}}{\partial x_i^2}(x,\xi)-\sum_{i=1}^{n} f_i(x)\frac{\partial a_{k-1}}{\partial x_i}(x,\xi)\right.\\
&\quad \left. -\frac{1}{2}\left(\sum_{i=1}^{m} h_i^2\right)a_{k-1}(x,\xi)-\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x)a_{k-1}(x,\xi)\right]\\
&= \frac{1}{2}\sum_{i=1}^{n}\left[\left(\frac{\partial^2 a}{\partial x_i^2}(x,\xi)+\left(\frac{\partial a}{\partial x_i}(x,\xi)\right)^2\right)\tilde{a}_{k-1}(x,\xi)\right.\\
&\quad \left. + 2\frac{\partial a}{\partial x_i}(x,\xi)\frac{\partial \tilde{a}_{k-1}}{\partial x_i}(x,\xi)+\frac{\partial^2 \tilde{a}_{k-1}}{\partial x_i^2}(x,\xi)\right]\\
&\quad -\sum_{i=1}^{n} f_i(x)\left[\frac{\partial a}{\partial x_i}(x,\xi)\tilde{a}_{k-1}(x,\xi)+\frac{\partial \tilde{a}_{k-1}}{\partial x_i}(x,\xi)\right]\\
&\quad -\frac{1}{2}\left(\sum_{i=1}^{m} h_i^2\right)\tilde{a}_{k-1}(x,\xi)-\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x)\tilde{a}_{k-1}(x,\xi)\\
&= \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 \tilde{a}_{k-1}}{\partial x_i^2}(x,\xi)+\sum_{i=1}^{n}\left(\frac{\partial a}{\partial x_i}(x,\xi)-f_i(x)\right)\frac{\partial \tilde{a}_{k-1}}{\partial x_i}(x,\xi)\\
&\quad +\left[\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 a}{\partial x_i^2}(x,\xi)+\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial a}{\partial x_i}(x,\xi)\right)^2\right.\\
&\quad \left. -\sum_{i=1}^{n} f_i(x)\frac{\partial a}{\partial x_i}(x,\xi)-\frac{1}{2}\left(\sum_{i=1}^{m} h_i^2\right)-\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x)\right]\tilde{a}_{k-1}(x,\xi).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& a_k(x,\xi) = e^{a(x,\xi)}\int_0^1 t^{k-1}e^{-a(\xi+t(x-\xi),\xi)}g_k(\xi+t(x-\xi),\xi)\,dt\\
\Rightarrow\quad & \tilde{a}_k(x,\xi) = \int_0^1 t^{k-1}\tilde{g}_k(\xi+t(x-\xi),\xi)\,dt.
\end{aligned}
$$

□

Theorem 4. *Equation* (3.0) *has a formal solution on* R^n *if* $h_i(x)$ *are constants for all* $1 \le i \le m$. *In fact the solution is of the following form:*

$$\begin{aligned} u(t,x) = &\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(\sqrt{2\pi})^n} t^{-n/2} \\ &\cdot \exp\left(-\frac{1}{2t} \sum_{j=1}^{n} (x_j - y_j)^2 + \int_0^1 \sum_{i=1}^{n} (x_i - y_i) f_i(y + t(x-y))\, dt \right) \\ &\cdot [1 + \tilde{a}_1(x,y)t + \tilde{a}_2(x,y)t^2 + \cdots + \tilde{a}_k(x,y)t^k + \cdots] \\ &\cdot \sigma_0(y)\, dy_1 \cdots dy_n, \end{aligned} \tag{3.10}$$

where $\tilde{a}_k(x,y) = \int_0^1 t^{k-1} \tilde{g}_k(y + t(x-y), y)\, dt$ *and*

$$\begin{aligned} \tilde{g}_k(x,y) = &\tfrac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 \tilde{a}_{k-1}}{\partial x_i^2}(x,y) + \sum_{i=1}^{n} \left(\frac{\partial a}{\partial x_i}(x,y) - f_i(x) \right) \frac{\partial \tilde{a}_{k-1}}{\partial x_i}(x,y) \\ &+ \left[\tfrac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 a}{\partial x_i^2}(x,y) + \tfrac{1}{2} \sum_{i=1}^{n} \left(\frac{\partial a}{\partial x_i}(x,y) \right)^2 - \sum_{i=1}^{n} f_i(x) \frac{\partial a}{\partial x_i}(x,y) \right. \\ &\left. - \tfrac{1}{2} \left(\sum_{i=1}^{m} h_i^2 \right) - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(x) \right] \tilde{a}_{k-1}(x,y). \end{aligned}$$

Lemma 5. *Let* $\alpha = (\alpha_1, \ldots, \alpha_n)$ *and* $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$,

$$\begin{aligned} &\frac{\partial^{\alpha_1 + \cdots + \alpha_n} a}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}(x,y) \\ &\quad = \alpha^1 \int_0^1 t^{|\alpha|-1} \frac{\partial^{|\alpha|-1} f_1}{\partial x_1^{\alpha_1 - 1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}(y + t(x-y))\, dt \\ &\quad + \cdots + \alpha_i \int_0^1 t^{|\alpha|-1} \frac{\partial^{|\alpha|-1} f_i}{\partial x_1^{\alpha_1} \cdots \partial x_{i-1}^{\alpha_{i-1}} \partial x_i^{\alpha_i - 1} \partial x_{i+1}^{\alpha_{i+1}} \cdots \partial x_n^{\alpha_n}}(y + t(x-y))\, dt \\ &\quad + \cdots + \alpha_n \int_0^1 t^{|\alpha|-1} \frac{\partial^{|\alpha|-1} f_n}{\partial x_1^{\alpha_1} \cdots \partial x_{n-1}^{\alpha_{n-1}} \partial x_n^{\alpha_n - 1}}(y + t(x-y))\, dt \\ &\quad + \int_0^1 t^{|\alpha|} \sum_{j=1}^{n} (x_j - y_j) \frac{\partial^{|\alpha|} f_j}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}(y + t(x-y))\, dt. \end{aligned}$$

Proof.

$$\frac{\partial a}{\partial x_i} = \int_0^1 f_i(y + t(x-y))\, dt + \int_0^1 t \sum_{j=1}^{n} (x_j - \xi_j) \frac{\partial f_j}{\partial x_i}(y + t(x-y)).$$

The proof follows from induction. □

4. Estimates of the Formal Asymptotic Solution

Let

$$\tilde{\varphi}_N(t,x,y) = (2\pi t)^{-n/2}\exp\left(a(x,y) - \frac{|x-y|^2}{2t}\right) \cdot [1 + \tilde{a}_1(x,y)t + \cdots + \tilde{a}_N(x,y)t^N]. \tag{4.1}$$

We estimate $|\tilde{\varphi}_N(t,x,y)|$. Our basic assumption is that

$$\begin{aligned} &\operatorname{Sup}|\nabla^j f_i| \le C(j!), \qquad j = 1,\ldots,N,\\ &\tfrac{1}{2}\sum_{i=1}^{m} h_i^2 \le C, \end{aligned} \tag{4.2}$$

where $C > \max(2, |f(0)|)$. Here ∇^j denotes the partial differentiation of order j with respect to x variables. We begin with the following lemma.

Lemma 6.

(a) $|f_i(x) - f_i(y)| \le \sqrt{n}C|x-y|$.
(b) $|(\partial a/\partial x_i)(x,y)| \le |f_i(y)| + \sqrt{n}C|x-y|$.
(c) *For* $j \ge 2$, $|\nabla^j a(x,y)| \le (j-1)!\,(C + \sqrt{n}C|x-y|)$.
(d) $|(\partial a/\partial x_i)(x,y) - f_i(x)| \le \sqrt{n}|x-y|C$.
(e) *For* $j \ge 1$, $\nabla^j((\partial a/\partial x_i)(x,y) - f_i(x))| \le j!\,(\sqrt{n}C|x-y| + 2C)$.

Proof. (a) Let $g_i(t) = f_i(y + t(x-y))$. Then

$$\begin{aligned} f_i(x) - f_i(y) &= g_i(1) - g_i(0) = g'(\tau_i) \qquad \text{for some} \quad 0 < \tau_i < 1\\ &= \sum_{j=1}^{n}(x_j - y_j)\frac{\partial f_i}{\partial x_j}(y + \tau_i(x-y)). \end{aligned} \tag{4.3}$$

Hence

$$|f_i(x) - f_i(y)| \le \sqrt{n}C|x-y| \qquad \text{by the Schwartz inequality.}$$

(b) Recall

$$a(x,y) = \int_0^1 \sum_{j=1}^{n}(x_j - y_j) f_j(y + t(x-y))\,dt,$$

$$\begin{aligned} \left|\frac{\partial a}{\partial x_i}(x,y)\right| &\le \left|\int_0^1 f_i(y + t(x-y))\,dt\right|\\ &\quad + \left|\int_0^1 t\sum_{j=1}^{n}(x_j - y_j)\frac{\partial f_j}{\partial x_i}(y + t(x-y))\,dt\right|\\ &\le \int_0^1 (|f_i(y)| + \sqrt{n}tC|x-y|)\,dt + \int_0^1 t|x-y|\sqrt{n}C\,dt\\ &= |f_i(y)| + \sqrt{n}C|x-y|. \end{aligned}$$

(c) Note that, by Lemma 5, if $|\alpha| = \alpha_1 + \cdots + \alpha_n \geq 2$, then

$$\begin{aligned}\left|\frac{\partial^{|\alpha|}a}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}(x,y)\right| &\leq \frac{\alpha_1 C[(|\alpha|-1)!]}{|\alpha|} + \cdots + \frac{\alpha_n C[(|\alpha|-1)!]}{|\alpha|} \\ &\quad + \frac{\sqrt{n}C|x-y|(|\alpha|!)}{|\alpha|+1} \\ &= C[(|\alpha|-1)!] + \frac{\sqrt{n}C|x-y|(|\alpha|!)}{|\alpha|+1} \\ &\leq (|\alpha|-1)!\,[C+\sqrt{n}C|x-y|].\end{aligned}$$

(d)

$$\begin{aligned}&\left|\frac{\partial a}{\partial x_i}(x,y) - f_i(x)\right| \\ &\quad = \left|\int_0^1 f_i(y+t(x-y))\,dt \right. \\ &\qquad \left. + \int_0^1 t\sum_{j=1}^n (x_j-y_j)\frac{\partial f_j}{\partial x_i}(y+t(x-y))\,dt - \int_0^1 f_i(x)\,dt\right| \\ &\quad \leq \int_0^1 |f_i(y+t(x-y)) - f_i(x)|\,dt + \int_0^1 t|x-y|\sqrt{n}C\,dt \\ &\quad \leq \int_0^1 \sqrt{n}C(1-t)|x-y|\,dt + \frac{\sqrt{n}C}{2}|x-y| \\ &\quad = \sqrt{n}|x-y|C.\end{aligned}$$

(e)

$$\begin{aligned}\left|\nabla^j\left(\frac{\partial a}{\partial x_i}(x,y) - f_i(x)\right)\right| &\leq j!\text{ as }(C+\sqrt{n}C|x-y|) + j!\,C \\ &= j!\,(\sqrt{n}C|x-y| + 2C).\end{aligned}$$

□

Proposition 7. *Let*

$$\begin{aligned}A_i(x,y) &= \frac{\partial a}{\partial x_i}(x,y) - f_i(x), \\ B(x,y) &= \tfrac{1}{2}\sum_{i=1}^n \frac{\partial^2 a}{\partial x_i^2}(x,y) + \tfrac{1}{2}\sum_{i=1}^n\left(\frac{\partial a}{\partial x_i}(x,y)\right)^2 \\ &\quad - \sum_{i=1}^n f_i(x)\frac{\partial a}{\partial x_i}(x,y) - \tfrac{1}{2}\sum_{i=1}^m h_i^2 - \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}(x).\end{aligned}$$

Then:

(a) $|\nabla^j A_i(x,y)| \leq j!\,(\sqrt{n}C|x-y| + 2C)$.
(b) $|\nabla^j B(x,y)| \leq (j+1)!\,n(|f(y)| + 2\sqrt{n}C|x-y| + 2C)^2$.
(c) $|\nabla^j \tilde{a}_k(x,y)| \leq (j+2k)!\,n^k(|f(y)| + 2\sqrt{n}C|x-y| + 2C)^{2k}$.

Proof. Part (a) follows from (d) and (e) of Lemma 6.

For part (b), we use Lemma 6(b)–(d) to get the following estimates:

$$|B(x, y|) \le \tfrac{1}{2}n(C + \sqrt{n}C|x - y|) + n(\sqrt{n}C|x - y|)(|f(y)| + \sqrt{n}C|x - y|) \\ +\tfrac{1}{2}n(|f(y)| + \sqrt{n}C|x - y|)^2 + (n + 1)C. \tag{4.4}$$

On the other hand,

$$\begin{aligned} &n(|f(y)| + 2\sqrt{n}C|x - y| + 2C)^2 \\ &\quad= n(|f(y)| + 2\sqrt{n}C|x - y|)^2 + 4nC(|f(y)| + 2\sqrt{n}C|x - y|) + 4nC^2 \\ &\quad= n(|f(y)| + \sqrt{n}C|x - y|)^2 + 2n(\sqrt{n}C|x - y|)(|f(y)| + \sqrt{n}C|x - y|) \\ &\qquad+ n(\sqrt{n}C|x - y|)^2 + 4nC(|f(y)| + 2\sqrt{n}C|x - y|) + 4nC^2 \\ &\quad\ge \tfrac{1}{2}n(|f(y)| + \sqrt{n}C|x - y|)^2 + n(\sqrt{n}C|x - y|)(|f(y)| + \sqrt{n}C|x - y|) \\ &\qquad+ \tfrac{1}{2}n\sqrt{n}C|x - y| + \tfrac{1}{2}nC + (n + 1)C \\ &\quad= \tfrac{1}{2}n(C + \sqrt{n}C|x - y|) + n(\sqrt{n}C|x - y|)(|f(y)| + \sqrt{n}C|x - y|) \\ &\qquad+ \tfrac{1}{2}n(|f(y)| + \sqrt{n}C|x - y|)^2 + (n + 1)C. \end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5), we have

$$|B(x, y)| \le n(|f(y)| + 2\sqrt{n}C|x - y| + 2C)^2.$$

For $j \ge 1$,

$$\begin{aligned} &|\nabla^j B(x, y)| \\ &\quad= \left| \tfrac{1}{2}\sum_{i=1}^{n} \nabla^j \frac{\partial^2 a}{\partial x_i^2}(x, y) + \sum_{i=1}^{n} \nabla^j \left[\left(\frac{\partial a}{\partial x_i}(x, y) - f_i(x) \right) \frac{\partial a}{\partial x_i}(x, y) \right] \right. \\ &\qquad \left. -\tfrac{1}{2}\sum_{i=1}^{n} \nabla^j \left[\left(\frac{\partial a}{\partial x_i}(x, y) \right)^2 \right] - \sum_{i=1}^{n} \nabla^{j+1} f_i(x) \right| \\ &\quad\le \frac{n}{2} \cdot (j + 1)!\, (C + \sqrt{n}C|x - y|) \\ &\qquad+ \sum_{i=1}^{n} \sum_{p+q=j} \frac{j!}{p!\, q!} \left| \nabla^p \left(\frac{\partial a}{\partial x_i}(x, y) - f_i(x) \right) \right| \cdot |\nabla^{q+1} a(x, y)| \\ &\qquad+ \frac{n}{2} \sum_{p+q=j} \frac{j!}{p!\, q!} |\nabla^{p+1} a(x, y)| \cdot |\nabla^{q+1} a(x, y)| + n \cdot (j + 1)!\, C \\ &\quad\le \frac{n}{2} \cdot (j + 1)!\, (C + \sqrt{n}C|x - y|) \\ &\qquad+ n \sum_{p+q=j} \frac{j!}{p!\, q!} p!\, (\sqrt{n}C|x - y| + 2C) \cdot (q!)(|f(y)| + C + \sqrt{n}C|x - y|) \\ &\qquad+ \frac{n}{2} \sum_{p+q=j} \frac{j!}{p!\, q!} p!\, (|f(y)| + \sqrt{n}|x - y| + C) \\ &\qquad\cdot (q!)(|f(y)| + C + \sqrt{n}C|x - y|) + n \cdot (j + 1)!\, C \end{aligned}$$

$$
\begin{aligned}
&= (j+1)!\,\frac{n}{2}(\sqrt{n}C|x-y|+C) + (j+1)!\,n(\sqrt{n}C|x-y|+2C)\\
&\quad \cdot (|f(y)| + \sqrt{n}C|x-y|+C)\\
&\quad + (j+1)!\,\frac{n}{2}(|f(y)|+\sqrt{n}C|x-y|+C)^2 + (j+1)!\,nC\\
&\le (j+1)!\,n(|f(y)|) + \sqrt{n}C|x-y|+C)^2 + (j+1)!\,2n(\sqrt{n}C|x-y|+C)\\
&\quad \cdot (|f(y)| + \sqrt{n}C|x-y|+C)\\
&\quad + (j+1)!\,n(\sqrt{n}C|x-y|+C)^2\\
&= (j+1)!\,n(|f(y)|+2\sqrt{n}C|x-y|+2C)^2.
\end{aligned}
$$

This proves part (b).

Part (c) we prove by induction on k. For $k = 1$,

$$
\begin{aligned}
\tilde{a}_1(x,y) &= \int_0^1 \tilde{g}_1(y+t(x-y),y)\,dt\\
&= \int_0^1 B(y+t(x-y),y)\,dt.
\end{aligned}
$$

It follows from part (b) that

$$
|\nabla^j \tilde{a}_1(x,y)| \le (j+1)!\,n(|f(y)|+2\sqrt{n}C|x-y|+2C)^2.
$$

For $k \ge 1$,

$$
\begin{aligned}
\tilde{a}_{k+1}(x,y) &= \int_0^1 t^k \tilde{g}_{k+1}(y+t(x-y),y)\,dt\\
&= \int_0^1 t^k \Bigg\{ \tfrac{1}{2}\sum_{i=1}^n \frac{\partial^2 \tilde{a}_k}{\partial x_i^2}(y+t(x-y),y)\\
&\qquad\qquad + \sum_{i=1}^n A_i(y+t(x-y),y)\frac{\partial \tilde{a}_k}{\partial x_i}(t+t(x-y),y)\\
&\qquad\qquad + B(y+t(x-y),y)\tilde{a}_k(y+t(x-y),y) \Bigg\}\,dt,
\end{aligned}
$$

$$
\begin{aligned}
|\nabla^j \tilde{a}_{k+1}(x,y)| \le{}& \int_0^1 t^{j+k}\frac{n}{2}|\nabla^{j+2}\tilde{a}_k(y+t(x-y),y)\,dt|\\
&+ \int_0^1 \sum_{i=1}^n t^{j+k} \sum_{p+q=j} \frac{j!}{p!\,q!}|\nabla^q A_i(y+t(x-y),y)|\\
&\cdot |\nabla^{p+1}\tilde{a}_k(y+t(x-y),y)|\,dt\\
&+ \int_0^1 t^{j+k} \sum_{p+q=j} \frac{j!}{p!\,q!}|\nabla^q B(y+t(x-y),y)|\\
&\cdot |\nabla^p \tilde{a}_k(y+t(x-y),y)|\,dt
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{n}{2(j+k+1)} n^k \cdot (j+2+2k)!\,(|f(y)| \\
&\quad +2\sqrt{n}C|x-y|+2C)^{2k} + \frac{n}{j+k+1} \\
&\quad \cdot \sum_{p+q=j} \frac{j!}{p!\,q!} q!\,(\sqrt{n}C|x-y|+2C) n^k \\
&\quad \cdot (p+1+2k)!\,(|f(y)|+2\sqrt{n}|x-y|+2C)^{2k} \\
&\quad + \frac{1}{j+k+1} \sum_{p+q=j} \frac{j!}{p!\,q!} n^k \\
&\quad \cdot (p+2k)!\,(|f(y)|+2\sqrt{n}C|x-y|+2C)^{2k} \cdot (q+1)! \\
&\quad \cdot n(|f(y)|+2\sqrt{n}C|x-y|+2C)^2 \\
&= \frac{n^{k+1}}{2(j+k+1)} (j+2k+2)!\,(|f(y)|+2\sqrt{n}C|x-y|+2C)^{2k} \\
&\quad + \frac{n^{k+1}}{j+k+1} \sum_{p+q=j} \frac{j!}{p!} (p+1+2k)!\,(\sqrt{n}C|x-y|+2C) \\
&\quad \cdot (|f(y)|+2\sqrt{n}C|x-y|+2C)^{2k} \\
&\quad + \frac{n^{k+1}}{j+k+1} \sum_{p+q=j} \frac{j!\,(q+1)}{p!} \\
&\quad \cdot (p+2k)!\,(|f(y)|+2\sqrt{n}C|x-y|+2C)^{2k+2} \\
&\leq \frac{n^{k+1}}{2(j+k+1)} (j+2(k+1))!\,(|f(y)|+2\sqrt{n}C|x-y|+2C)^{2k} \\
&\quad + \frac{n^{k+1}}{j+k+1} (j+1) \\
&\quad \cdot (j+1+2k)!\,(|f(y)|+2\sqrt{n}C|x-y|+2C)^{2k+1} \\
&\quad + \frac{n^{k+1}}{j+k+1} (j+1)^2 \\
&\quad \cdot (j+2k)!\,(|f(y)|+2\sqrt{n}C|x-y|+2C)^{2k+2} \\
&\leq n^{k+1} \cdot (j+2(k+1))!\,(|f(y)|+2\sqrt{n}C|x-y|+2C)^{2k+2} \\
&\quad \cdot \left(\frac{1}{2(j+k+1)(2C)^2} + \frac{j+1}{(j+k+1)(j+2k+2)2C} \right. \\
&\qquad \left. + \frac{(j+1)^2}{(j+k+1)(j+2k+2)(j+2k+1)} \right) \\
&\leq n^{k+1} \cdot (j+2(k+1))!\,(|f(y)|+2\sqrt{n}C|x-y|+2C)^{2k+2}. \qquad \square
\end{aligned}
$$

We can now give an estimate of $\tilde{\varphi}_N(t,x,y)$.

Proposition 8.

(a) $|f(y)| + 2\sqrt{n}C|x-y| + 2C \le 3\sqrt{n}C(1+|x|+|y|)$.

(b) *For* $|t| \le 1$ *and* $N \ge 3nC - 1$,

$$\left|\sum_{i=1}^{N} \tilde{a}_i(x,y)t^i\right| \le 2(N+1)^{4N}(1+\sqrt{t}|x|)^{2N}(1+\sqrt{t}|y|)^{2N}.$$

(c) *For* $N \ge 3nC - 1$ *and* $|t| \le 1$,

$$|\tilde{\varphi}_N(t,x,y)| \le 2(N+1)^{4N}(1+\sqrt{t}|x|)^{2N}(1+\sqrt{t}|y|)^{2N}(2\pi t)^{-n/2} \cdot \exp\left(a(x,y) - \frac{|x-y|^2}{2t}\right).$$

Proof.

$$\begin{aligned} |f(y)| + 2\sqrt{n}C|x-y| + 2C &\le |f(0)| + \sqrt{n}C|y| + 2\sqrt{n}C|x| + 2\sqrt{n}C|y| + 2C \\ &\le 3C + 3\sqrt{n}C|y| + 2\sqrt{n}C|x| \\ &\le 3\sqrt{n}C(1+|x|+|y|). \end{aligned}$$

This proves part (a). In view of part (c) of Proposition 7, we have

$$\begin{aligned} \left|\sum_{i=0}^{N} \tilde{a}_i(t,x,y)t^i\right| &\le \sum_{i=0}^{N} n^i[(2i)!](3\sqrt{n}C)^{2i}(1+|x|+|y|)^{2i}t^i \\ &= \sum_{i=0}^{N} n^i[(2i)!]3^{2i}n^iC^{2i}(1+\sqrt{t}|x|)^{2i}(1+\sqrt{t}|y|)^{2i} \\ &\le \sum_{i=0}^{N} n^{2i}(i+1)^{2i}3^{2i}C^{2i}(1+\sqrt{t}|x|)^{2i}(1+\sqrt{t}|y|)^{2i} \\ &\le (N+1)^{2N}\sum_{i=0}^{N}[3nC(1+\sqrt{t}|x|)(1+\sqrt{t}|y|)]^{2i}. \end{aligned}$$

Observe that if $r \ge 3$, then $\sum_{i=0}^{N} r^{2i} \le 2r^{2N}$. So we have

$$\begin{aligned} \left|\sum_{i=1}^{N} \tilde{a}_i(t,x,y)t^i\right| &\le (N+1)^{2N}2[3nC(1+\sqrt{t}|x|)(1+\sqrt{t}|y|)]^{2N} \\ &= 2(N+1)^{2N}3^{2N}n^{2N}C^{2n}(1+\sqrt{t}|x|)^{2N}(1+\sqrt{t}|y|)^{2N}. \end{aligned}$$

If $N \ge 3nC - 1$, then we have

$$\left|\sum_{i=0}^{N} \tilde{a}_i(t,x,y)t^i\right| \le 2(N+1)^{4N}(1+\sqrt{t}|x|)^{2N}(1+\sqrt{t}|y|)^{2N}.$$

So part (b) is proven. Since

$$\tilde{\varphi}_N(t, x, y) = (2\pi t)^{-n/2} \exp\left(a(x, y) - \frac{|x-y|^2}{2t}\right) \sum_{i=0}^{N} \tilde{a}_i(t, x, y) t^i.$$

Part (c) follows immediately from part (b). □

We now estimate

$$e_N := \frac{\partial \tilde{\varphi}_N}{\partial t} - L_x \tilde{\varphi}_N,$$

where

$$L_x = \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial}{\partial x_i} - f_i(x)\right)^2 - \frac{1}{2}\left(\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) + \sum_{i=1}^{n} f_i^2(x) + \sum_{i=1}^{m} h_i^2\right)$$

is the operator defined by the right-hand side of (3.5).

Proposition 9. *For $N \geq 3nC - 2$ and $|t| \leq 1$, let $e_N(t, x, y) = (\partial\tilde{\varphi}_N/\partial t)(t, x, y) - L_x\tilde{\varphi}_N(t, x, y)$. Then*

$$|e_N(t, x, y)| \leq (2\pi t)^{-n/2}\left[\exp\left(a(x, y) - \frac{|x-y|^2}{2t}\right)\right] \cdot (N+2)^{4N+4}(1+\sqrt{t}|x|)^{2N+2}(1+\sqrt{t}|y|)^{2N+2}.$$

Proof. In view of the computation in the proof of Theorem 2, we have

$$\begin{aligned}
e_N(t, x, y) &= -(2\pi)^{-n/2} t^{-n/2}\left[\exp\left(-\frac{|x-y|^2}{2t}\right)\right] \\
&\quad \cdot\left\{\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 a_N}{\partial x_i^2}(x, y) - \sum_{i=1}^{n} f_i(x)\frac{\partial a_N}{\partial x_i}(x, y)\right. \\
&\quad \left. - \left(\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) + \frac{1}{2}\sum_{i=1}^{m} h_i^2\right) a_N(x, y)\right\} t^{N+1} \\
&= -(2\pi)^{-n/2} t^{-n/2}\left[\exp\left(a(x, y) - \frac{|x-y|^2}{2t}\right)\right] \\
&\quad \cdot\left\{\sum_{i=1}^{n}\frac{1}{2}\frac{\partial^2 \tilde{a}_N}{\partial x_i^2}(x, y) + \sum_{i=1}^{n}\left(\frac{\partial a}{\partial x_i}(x, y) - f_i(x)\right)\frac{\partial \tilde{a}_N}{\partial x_i}\right. \\
&\quad + \left[\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 a}{\partial x_i^2}(x, y) + \sum_{i=1}^{n}\frac{1}{2}\left(\frac{\partial a}{\partial x_i}(x, y)\right)^2 - \sum_{i=1}^{n} f_i(x)\frac{\partial a}{\partial x_i}(x, y)\right. \\
&\quad \left.\left. - \left(\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) + \frac{1}{2}\sum_{i=1}^{m} h_i^2\right)\right]\tilde{a}_N(x, y)\right\} t^{N+1}.
\end{aligned}$$

By applying the estimates in Lemma 6 and Proposition 7, we get

$$
\begin{aligned}
|e_N(t,x,y)| \le\ & (2\pi)^{-n/2}t^{-n/2}\left[\exp\left(a(x,y)-\frac{|x-y|^2}{2t}\right)\right] \\
& \cdot\Big\{\frac{n}{2}n^N[(2N+2)!](|f(y)|+2\sqrt{n}C|x-y|+2C)^{2N} \\
& \quad + n(\sqrt{n}C|x-y|)n^N[(2N+1)!] \\
& \quad \cdot(|f(y)|+2\sqrt{n}C|x-y|+2C)^{2N} \\
& \quad + n(|f(y)|+2\sqrt{n}C|x-y|+2C)^2 n^N[(2N)!] \\
& \quad \cdot(|f(y)|+2\sqrt{n}C|x-y|+2C)^{2N}\Big\}t^{N+1} \\
\le\ & (2\pi t)^{-n/2}\left[\exp\left(a(x,y)-\frac{|x-y|^2}{2t}\right)\right] \\
& \cdot\big\{\tfrac{1}{2}n^{N+1}[(2N+2)!](|f(y)|+2\sqrt{n}C|x-y|+2C)^{2N} \\
& \quad + n^{N+1}[(2N+1)!](|f(y)|+2\sqrt{n}C|x-y|+2C)^{2N+1} \\
& \quad + n^{N+1}[(2N)!](|f(y)|+2\sqrt{n}C|x-y|+2C)^{2N+2}\big\}\,t^{N+1} \\
\le\ & (2\pi t)^{-n/2}\left[\exp\left(a(x,y)-\frac{|x-y|^2}{2t}\right)\right]n^{N+1}[(2N+2)!] \\
& \cdot(|f(y)|+2\sqrt{n}C|x-y|+2C)^{2N+2} \\
& \cdot\left\{\frac{1}{2}\cdot\frac{1}{4C^2}+\frac{1}{(2N+2)2C}+\frac{1}{(2N+1)(2N+2)}\right\}t^{N+1} \\
\le\ & (2\pi t)^{-n/2}\left[\exp\left(a(x,y)-\frac{|x-y|^2}{2t}\right)\right] \\
& \cdot n^{N+1}(N+2)^{2N+2}(3\sqrt{n}C)^{2N+2}(1+|x|+|y|)^{2N+2} \\
& \cdot\left\{\tfrac{1}{32}+\tfrac{1}{8}+\tfrac{1}{2}\right\}t^{N+1} \\
\le\ & (2\pi t)^{-n/2}\left[\exp\left(a(x,y)-\frac{|x-y|^2}{2t}\right)\right] \\
& \cdot(N+2)^{2N+2}(3nC)^{2N+2}(1+\sqrt{t}|x|)^{2N+2}(1+\sqrt{t}|y|)^{2N+2}.
\end{aligned}
$$

Therefore if $N \ge 3nC-2$, then

$$
\begin{aligned}
|e_N(t,x,y)| \le\ & (2\pi t)^{-n/2}\left[\exp\left(a(x,y)-\frac{|x-y|^2}{2t}\right)\right] \\
& \cdot(N+2)^{4N+4}(1+\sqrt{t}|x|)^{2N+2}(1+\sqrt{t}|y|)^{2N+2}.
\end{aligned}
$$

□

5. From Formal Solution to Convergent Solution

From the truncated formal solution

$$
\begin{aligned}
\tilde{\varphi}_N(t,x,y) =\ & (2\pi t)^{-n/2}\left[\exp\left(a(x,y)-\frac{|x-y|^2}{2t}\right)\right] \\
& \cdot\left[1+\tilde{a}_1(x,y)t+\cdots+\tilde{a}_N(x,y)t^N\right],
\end{aligned}
$$

which satisfies the following properties,

$$|\tilde{\varphi}_N(t,x,y)| \le 2(N+1)^{4N}(2\pi t)^{-n/2}(1+\sqrt{t}|x|)^{2N}(1+\sqrt{t}|y|)^{2N} \cdot \exp\left(a(x,y) - \frac{|x-y|^2}{2t}\right)$$

and

$$\begin{aligned} &\left|\frac{\partial \tilde{\varphi}_N}{\partial t}(t,x,y) - L_x\tilde{\varphi}_N(t,x,y)\right| \\ &\quad = |e_N(t,x,y)| \\ &\quad \le (\sqrt{2\pi})^{-n} t^{-n/2}(N+1)^{4N} \\ &\qquad \cdot \left[\exp\left(a(x,y) - \frac{|x-y|^2}{2t}\right)\right](1+\sqrt{t}|x|)^{2N}(1+\sqrt{t}|y|)^{2N}, \end{aligned}$$

where $t \le 1$ and $N \ge 3nC - 1$, we construct a convergent solution. In fact we claim that, for $t \le \varepsilon/N$ so that (5.5) and (5.6) in Theorem 13 are satisfied, the following series converges:

$$\begin{aligned} &\tilde{\varphi}_N(t,x,y) + \sum_{m=0}^{\infty}(-1)^{m+1}\frac{1}{\sqrt{m+2}}\int_{\sum_{i=0}^{m+1}\tau_i = t}\int_{x_{m+1}}\int_{x_m}\cdots\int_{x_1}\tilde{\varphi}_N(\tau_{m+1},x,x_{m+1}) \\ &\cdot e_N(\tau_m,x_{m+1},x_m)e_N(\tau_{m-1},x_m,x_{m-1})\cdots e_N(\tau_0,x_1,y) \end{aligned} \qquad (5.1)$$

and it represents a kernel $\varphi(t,x,y)$ which satisfies

$$\frac{\partial \varphi}{\partial t}(t,x,y) = L_x\varphi(t,x,y),$$

$$\lim_{t\to 0}\varphi(t,x,y) = \delta_x(y).$$

In this way, for $t \le \varepsilon/N$, we have found an explicit kernel for the equation. When time is equal to T which may be large, we can find the kernel up to time T by the formula

$$\varphi(T,x,y) = \int_{x_1}\int_{x_2}\cdots\int_{x_K}\varphi\left(\frac{T}{K},x,x_1\right)\varphi\left(\frac{T}{K},x_1,x_2\right)\cdots\varphi\left(\frac{T}{K},x_K,y\right).$$

Here K is the smallest integer greater than TN/ε.

Lemma 10.

(a) $|a(x,y) - \sum_{i=1}^{n}(x_i - y_i)f_i(x)| \le nC|x-y|^2$.
(b) $|a(x,x_{m+1}) + \sum_{i=1}^{m}a(x_{i+1},x_i) + a(x_1,y) - (x-y)f(x)| \le nC[(t/\tau_0 + 1) \cdot |y - x_1|^2 + \sum_{i=1}^{m}(t/\tau_i + 1)|x_i - x_{i+1}|^2 + (t/\tau_{m+1} + 1)|x_{m+1} - x|^2]$, *where* $f(x) = (f_1(x),\ldots,f_n(x))$ *and* $t = \sum_{i=1}^{m+1}\tau_i$.

Proof.

$$\begin{aligned}
&\left|a(x,y) - \sum_{i=1}^{n}(x_i - y_i) f_i(x)\right| \\
&\quad = \left|\sum_{i=1}^{n}(x_i - y_i)\int_0^1 (f_i(y + t(x-y)) - f_i(x))\,dt\right| \\
&\quad \le \int_0^1 (1-t)\sqrt{n}C|x-y|^2\,dt \\
&\quad = \frac{\sqrt{n}}{2}C|x-y|^2.
\end{aligned}$$

By summation

$$\begin{aligned}
&a(x, x_{m+1}) + a(x_{m+1}, x_m) + \cdots + a(x_1, y) \\
&\quad = (x - x_{m+1})f(x) + (x_{m+1} - x_m)f(x_{m+1}) + \cdots + (x_1 - y)f(x_1) \\
&\qquad + O(|x - x_{m+1}|^2 + |x_{m+1} - x_m|^2 + \cdots + |x_1 - y|^2),
\end{aligned}$$

where

$$\begin{aligned}
&O(||x - x_{m+1}|^2 + |x_{m+1} - x_m|^2 + \cdots + |x_1 - y|^2) \\
&\quad \le \frac{\sqrt{n}}{2}C(|x - x_{m+1}|^2 + |x_{m+1} - x_m|^2 + \cdots + |x_1 - y|^2) \\
&\quad = (x - x_{m+1})\cdot f(x) + \sum_{i=1}^{m}(x_{i+1} - x_i)\cdot f(x_{i+1}) \\
&\qquad + (x_1 - y)\cdot f(x_1) - (x - x_{m+1})\cdot f(x) \\
&\qquad - \sum_{i=1}^{m}(x_{i+1} - x_i)\cdot f(x) - (x_1 - y)\cdot f(x) + (x - y)\cdot f(x) \\
&\qquad + O\left(|x - x_{m+1}|^2 + \sum_{i=1}^{m}|x_{i+1} - x_i|^2 + |x_1 - y|^2\right) \\
&\quad = \sum_{i=1}^{m}(x_{i+1} - x_i)\cdot(f(x_{i+1}) - f(x)) \\
&\qquad + (x_1 - y)\cdot(f(x_1) - f(x)) + (x - y)\cdot f(x) \\
&\qquad + O\left(|x - x_{m+1}|^2 + \sum_{i=1}^{m}|x_{i+1} - x_i|^2 + |x_1 - y|^2\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left|a(x, x_{m+1}) + \sum_{i=1}^{m} a(x_{i+1}, x_i) + a(x_1, y) - (x - y)\cdot f(x)\right| \\
&\quad \le nC\sum_{i=1}^{m}|x_{i+1} - x_i||x_{i+1} - x| + nC|x_1 - y||x_1 - x|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\sqrt{n}}{2}C\left(|x-x_{m+1}|^2+\sum_{i=1}^{m}|x_{i+1}-x_i|^2+|x_1-y|^2\right)\\
&\le nC\sum_{i=1}^{m}|x_{i+1}-x_i|(|x_{i+1}-x_{i+2}|+|x_{i+2}-x_{i+3}|+\cdots+|x_{m+1}-x|)\\
&\quad+nC|x_1-y|(|x_1-x_2|+\cdots+|x_{m+1}-x|)\\
&\quad+\frac{\sqrt{n}}{2}C\left(|x-x_{m+1}|^2+\sum_{i=1}^{m}|x_{i+1}-x_i|^2+|x_1-y|^2\right)\\
&\le nC(|y-x_1|+|x_1-x_2|+\cdots+|x_m-x_{m+1}|+|x_{m+1}-x|)^2\\
&\quad+\frac{\sqrt{n}}{2}C\left(|x-x_{m+1}|^2+\sum_{i=1}^{m}|x_{i+1}-x_i|^2+|x_1-y|^2\right)\\
&= nC\left(\sqrt{\tau_0}\frac{|y-x_1|}{\sqrt{\tau_0}}+\sqrt{\tau_1}\frac{|x_1-x_2|}{\sqrt{\tau_1}}+\cdots\right.\\
&\qquad\left.+\sqrt{\tau_m}\,\frac{|x_m-x_{m+1}|}{\sqrt{\tau_m}}+\sqrt{\tau_{m+1}}\frac{|x_{m+1}-x|^2}{\sqrt{\tau_{m+1}}}\right)\\
&\quad+\frac{\sqrt{n}}{2}C\left(|x-x_{m+1}|^2+\sum_{i=1}^{m}|x_{i+1}-x_i|^2+|x_1-y|^2\right)\\
&\le nC\left(\sum_{i=1}^{m+1}\tau_i\right)\left(\frac{|y-x_1|^2}{\tau_0}+\sum_{i=1}^{m}\frac{|x_i-x_{i+1}|^2}{\tau_i}+\frac{|x_{m+1}-x|^2}{\tau_{m+1}}\right)\\
&\quad+\frac{\sqrt{n}}{2}C\left(|x-x_{m+1}|^2+\sum_{i=1}^{m}|x_{i+1}-x_i|^2+|x_1-y|^2\right)\\
&\le nC\left[\left(\frac{t}{\tau_0}+1\right)|y-x_1|^2+\sum_{i=1}^{m}\left(\frac{t}{\tau_i}+1\right)|x_i-x_{i+1}|^2\right.\\
&\qquad\left.+\left(\frac{t}{\tau_{m+1}}+1\right)|x_{m+1}-x|^2\right].
\end{aligned}
$$

□

Lemma 11.

$$
\begin{aligned}
&(1+\sqrt{\tau_{m+1}}|x|)^{2N}\prod_{i=1}^{m+1}(1+\sqrt{\tau_i}|x_i|)^{2N}\prod_{i=0}^{m}(1+\sqrt{\tau_i}|x_{i+1}|)^{2N}(1+\sqrt{\tau_0}|y|)^{2N}\\
&\le(1+\sqrt{t}|x|)^{2N}(1+\sqrt{t}|y|)^{2N}2^{4N(m+1)}\\
&\quad\cdot\exp\left[4Nt^2\left(\sum_{j=1}^{m+1}\frac{|x_{j+1}-x_j|^2}{\tau_j}+\frac{|x_1-y|^2}{\tau_0}\right)+4Nt|y|^2\right].
\end{aligned}
$$

Proof. Note that

$$
1+\sqrt{\tau_i}|x_i|\le 2\exp\left(\frac{\tau_i|x_i|^2}{2}\right).
$$

Hence

$$\prod_{i=1}^{m+1}(1+\sqrt{\tau_i}|x_i|)^{2N} \leq 2^{2N(m+1)} \exp\left(N \sum_{i=1}^{m+1} \tau_i |x_i|^2\right). \tag{5.2}$$

Since

$$\begin{aligned} |x_i - y|^2 &\leq (|x_i - x_{i-1}| + |x_{i-1} - x_{i-2}| + \cdots + |x_1 - y|)^2 \\ &= \left(\sqrt{\tau_{i-1}}\frac{|x_i - x_{i-1}|}{\sqrt{\tau_{i-1}}} + \sqrt{\tau_{i-2}}\frac{|x_{i-1} - x_{i-2}|}{\sqrt{\tau_{i-2}}} + \cdots + \sqrt{\tau_0}\frac{|x_1 - y|}{\tau_0}\right)^2 \\ &\leq \left(\sum_{j=0}^{i-1}\tau_j\right)\left(\sum_{j=1}^{i}\frac{|x_j - x_{j-1}|^2}{\tau_j} + \frac{|x_1 - y|^2}{\tau_0}\right) \\ &\leq t\left(\sum_{j=1}^{m+1}\frac{|x_{j+1} - x_j|^2}{\tau_j} + \frac{|x_1 - y|^2}{\tau_0}\right), \end{aligned}$$

we have

$$\begin{aligned} \sum_{i=1}^{m+1}\tau_i|x_i|^2 &\leq 2\sum_{i=1}^{m+1}\tau_i|y|^2 + 2\sum_{i=1}^{m+1}\tau_i|x_i - y|^2 \\ &\leq 2\sum_{i=1}^{m+1}\tau_i|y|^2 + 2\sum_{i=1}^{m+1}\tau_i t\left(\sum_{j=1}^{m+1}\frac{|x_{j+1} - x_j|^2}{\tau_j} + \frac{|x_1 - y|^2}{\tau_0}\right) \\ &= 2t|y|^2 + 2t^2\left(\sum_{j=1}^{m+1}\frac{|x_{j+1} - x_j|^2}{\tau_j} + \frac{|x_1 - y|^2}{\tau_0}\right). \end{aligned} \tag{5.3}$$

Similarly, we have

$$\sum_{i=0}^{m}\tau_i|x_{i+1}|^2 \leq 2t|y|^2 + 2t^2\left(\sum_{j=1}^{m+1}\frac{|x_{j+1} - x_j|^2}{\tau_j} + \frac{|x_1 - y|^2}{\tau_0}\right). \tag{5.4}$$

Lemma 11 follows easily from (5.2)–(5.4). □

We are now ready to estimate the infinite series (5.1).

Theorem 12. *If t is chosen small enough so that*

$$4(N+1)t^2 + nCt < \tfrac{1}{8} \quad \textit{and} \quad nC < \frac{1}{8t},$$

then the general term in (5.1) *has the following estimate:*

$$\begin{aligned} \int_{\sum_{i=0}^{m+1}\tau_i = t}\int_{x_{m+1}}\int_{x_m}\cdots\int_{x_1} &|\tilde{\varphi}_N(\tau_{m+1}, x, x_{m+1}) e_N(\tau_m, x_{m+1}, x_m) \\ &\cdot e_N(\tau_{m-1}, x_m, x_{m-1}) \cdots e_N(\tau_0, x_1, y)| \end{aligned}$$

$$\leq 2(N+2)^{4(m+2)N+4(m+1)}(2\pi)^{-(m+2)(n/2)}2^{4(N+1)(m+1)}$$
$$\cdot (1+\sqrt{t}|x|)^{2N+2}(1+\sqrt{t}|y|)^{2N+2}$$
$$\cdot (4\pi)^{(n/2)(m+1)} \cdot \sqrt{m+2}\frac{t^{-n/2+m+1}}{(m+1)!}$$
$$\cdot \exp\left[(x-y)f(x)+4(N+1)t|y|^2-\frac{|x-y|^2}{4t}\right].$$

Proof.

$$\int_{\sum_{i=0}^{m+1}\tau_i=t}\int_{x_{m+1}}\int_{x_m}\cdots\int_{x_1}|\tilde{\varphi}_N(\tau_{m+1},x,x_{m+1})e_N(\tau_m,x_{m+1},x_m)$$
$$\cdot e_N(\tau_{m-1},x_m,x_{m-1})\cdots e_N(\tau_0,x_1,y)|$$
$$\leq \int_{\sum_{i=0}^{m+1}\tau_i=t}\int_{x_{m+1}}\int_{x_m}\cdots\int_{x_1} 2(N+1)^{4N}(2\pi)^{-n/2}(\tau_{m+1})^{-n/2}(1+\sqrt{\tau_{m+1}}|x|)^{2N}$$
$$\cdot (1+\sqrt{\tau_{m+1}}|x_{m+1}|)^{2N}\exp\left(a(x,x_{m+1})-\frac{|x-x_{m+1}|^2}{2\tau_{m+1}}\right)$$
$$\cdot (2\pi)^{-n/2}(\tau_m)^{-n/2}(N+2)^{4N+4}(1+\sqrt{\tau_m}|x_{m+1}|)^{2N+2}(1+\sqrt{\tau_m}|x_m|)^{2N+2}$$
$$\cdot \exp\left(a(x_{m+1},x_m)-\frac{|x_{m+1}-x_m|^2}{2\tau_m}\right)$$
$$\cdot (2\pi)^{-n/2}(\tau_{m-1})^{-n/2}(N+2)^{4N+4}(1+\sqrt{\tau_{m-1}}|x_m|)^{2N+2}$$
$$\cdot (1+\sqrt{\tau_{m-1}}|x_{m-1}|)^{2N+2}\exp\left(a(x_m,x_{m-1})-\frac{|x_m-x_{m-1}|^2}{2\tau_{m-1}}\right)$$
$$\vdots$$
$$\cdot (2\pi)^{-n/2}(\tau_0)^{-n/2}(N+2)^{4N+4}(1+\sqrt{\tau_0}|x_1|)^{2N+2}$$
$$\cdot (1+\sqrt{\tau_0}|y|)^{2N+2}\exp\left(a(x_1,y)-\frac{|x_1-y|^2}{2\tau_0}\right)$$
$$\leq 2(N+2)^{4(m+2)N+4(m+1)}(2\pi)^{-(m+2)(n/2)}$$
$$\cdot \int_{\sum_{i=0}^{m+1}\tau_i=t}\int_{x_{m+1}}\int_{x_m}\cdots\int_{x_1}(\tau_0\cdots\tau_{m+1})^{-n/2}$$
$$\cdot (1+\sqrt{\tau_{m+1}}|x|)^{2N+2}\prod_{i=1}^{m+1}(1+\sqrt{\tau_i}|x_i|)^{2N+2}\prod_{i=0}^{m}(1+\sqrt{\tau_i}|x_{i+1}|)^{2N+2}$$
$$\cdot (1+\sqrt{\tau_0}|y|)^{2N+2}\exp\left(a(x,x_{m+1})+\sum_{i=1}^{m}a(x_{i+1},x_i)+a(x_1,y)\right)$$
$$\cdot \exp\left(-\frac{|x-x_{m+1}|^2}{2\tau_{m+1}}-\sum_{i=1}^{m}\frac{|x_{i+1}-x_i|^2}{2\tau_i}-\frac{|x_1-y|^2}{2\tau_0}\right)$$
$$\leq 2(N+2)^{4(m+2)N+4(m+1)}(2\pi)^{-(m+2)(n/2)}$$
$$\cdot \int_{\sum_{i=0}^{m+1}\tau_i=t}\int_{x_{m+1}}\int_{x_m}\cdots\int_{x_1}(\tau_0\cdots\tau_{m+1})^{-n/2}(1+\sqrt{t}|x|)^{2N+2}$$

$$\begin{aligned}
&\cdot (1+\sqrt{t}|y|)^{2N+2} 2^{4(N+1)(m+1)} \\
&\cdot \exp\left[4(N+1)t^2\left(\sum_{j=1}^{m+1}\frac{|x_{j+1}-x_j|^2}{\tau_j}+\frac{|x_1-y|^2}{\tau_0}\right)+4(N+1)t|y|^2\right] \\
&\cdot \exp\left\{(x-y)f(x)+nC\left[\left(\frac{t}{\tau_0}+1\right)|y-x_1|^2\right.\right. \\
&\qquad +\sum_{i=1}^{m}\left(\frac{t}{\tau_i}+1\right)|x_i-x_{i+1}|^2 \\
&\qquad \left.\left.+\left(\frac{t}{\tau_{m+1}}+1\right)|x_{m+1}-x|^2\right]\right\} \\
&\cdot \exp\left(-\frac{|x-x_{m+1}|^2}{2\tau_{m+1}}-\sum_{i=1}^{m}\frac{|x_{i+1}-x_i|^2}{2\tau_i}-\frac{|x_1-y|^2}{2\tau_0}\right) \\
&= 2(N+2)^{4(m+2)N+4(m+1)}(2\pi)^{-(m+2)(n/2)}2^{4(N+1)(m+1)} \\
&\cdot (1+\sqrt{t}|x|)^{2N+2}(1+\sqrt{t}|y|)^{2N+2} \\
&\cdot \exp[(x-y)f(x)]\int_{\sum_{i=0}^{m+1}\tau_i=t}\int_{x_{m+1}}\int_{x_m}\cdots\int_{x_1}(\tau_0\cdots\tau_{m+1})^{-n/2} \\
&\cdot \exp\left\{\left[4(N+1)\frac{t^2}{\tau_0}+nC\left(\frac{t}{\tau_0}+1\right)-\frac{1}{2\tau_0}\right]|x_1-y|^2\right. \\
&\qquad +\sum_{i=1}^{m}\left[4(N+1)\frac{t^2}{\tau_i}+nC\left(\frac{t}{\tau_i}+1\right)-\frac{1}{2\tau_i}\right]|x_{i+1}-x_i|^2 \\
&\qquad \left.+\left[nC\left(\frac{t}{\tau_{m+1}}+1\right)-\frac{1}{2\tau_{m+1}}\right]|x-x_{m+1}|^2+4(N+1)t|y|^2\right\}.
\end{aligned}$$

Since t is chosen small enough so that

$$4(N+1)t^2+nCt<\tfrac{1}{8} \quad \text{and} \quad nC<\frac{1}{8t},$$

it follows that

$$\frac{4(N+1)t^2}{\tau_i}+nC\left(\frac{t}{\tau_i}+1\right)-\frac{1}{2\tau_i}<\frac{-1}{4\tau_i}$$

for all i. So the general term in our series is estimated by

$$\begin{aligned}
&2(N+2)^{4(m+2)N+4(m+1)}(2\pi)^{-(m+2)(n/2)}2^{4(N+1)(m+1)} \\
&\qquad \cdot (1+\sqrt{t}|x|)^{2N+2}(1+\sqrt{t}|y|)^{2N+2}\exp[(x-y)f(x)+4(N+1)t|y|^2] \\
&\qquad \cdot \int_{\sum_{i=0}^{m+1}\tau_i=t}\int_{x_{m+1}}\int_{x_m}\cdots\int_{x_1}(\tau_0\cdots\tau_{m+1})^{-n/2} \\
&\qquad \cdot \exp\left(-\frac{|x-x_{m+1}|^2}{4\tau_{m+1}}-\frac{|x_{m+1}-x_m|^2}{4\tau_m}-\cdots-\frac{|x_1-y|^2}{4\tau_0}\right) \\
&= 2(N+2)^{4(m+2)N+4(m+1)}(2\pi)^{-(m+2)(n/2)}2^{4(N+1)(m+1)}
\end{aligned}$$

$$\cdot\,(1+\sqrt{t}|x|)^{2N+2}(1+\sqrt{t}|y|)^{2N+2}$$
$$\cdot\,\exp[(x-y)f(x)+4(N+1)t|y|^2]$$
$$\cdot\int_{\sum_{i=0}^{m+1}\tau_i=t}\int_{x_{m+1}}\int_{x_m}\cdots\int_{x_1}(4\pi)^{(n/2)(m+2)}H(\tau_{m+1},x-x_{m+1})$$
$$\star\, H(\tau_m,x_{m+1}-x_m)\star\cdots\star H(\tau_0,x-y),$$

where, for $1\le i\le m$, $H(\tau_i,x)$ is the kernel $(4\pi\tau_i)^{-n/2}\exp(-|x|^2/4\tau_i)$, $H(\tau_0,x)$ is the kernel $(4\pi\tau_0)^{-n/2}\exp(-|x|^2/4\tau_0)$, and $H(\tau_{m+1},x)$ is the kernel $(4\pi\tau_{m+1})^{-n/2}\times\exp(-|x|^2/4\tau_{m+1})$. Each $H(\tau_i,x)$ defines an integral operator acting on $L^2(\mathbf{R}^n)$. In view of the semigroup property of H (see Theorem 3 on p. 33 of [Wi]), the convolution $H(\tau_{m+1},x)\star\cdots\star H(\tau_0,x)$ is given by a kernel of the form

$$H\left(\sum_{i=0}^{m+1}\tau_i,x-y\right)=(4\pi)^{-n/2}\left(\sum_{i=0}^{m+1}\tau_i\right)^{-n/2}\exp\left(\frac{-|x-y|^2}{4\sum_{i=0}^{m+1}\tau_i}\right).$$

Hence we have an estimate of the general term in series (5.1) of the form

$$2(N+2)^{4(m+2)N+4(m+1)}(2\pi)^{-(m+2)(n/2)}2^{4(N+1)(m+1)}(1+\sqrt{t}|x|)^{2N+2}$$
$$\cdot(1+\sqrt{t}|y|)^{2N+2}\exp[(x-y)f(x)+4(N+1)t|y|^2]$$
$$\cdot(4\pi)^{(n/2)(m+1)}t^{-n/2}\exp\left(\frac{-|x-y|^2}{4t}\right)\int_{\sum_{i=0}^{m+1}\tau_i=t}1.$$

Notice that

$$\int_{\sum_{i=0}^{m+1}\tau_i=t}1=\mathrm{Vol}\left(\sum_{i=0}^{m+1}\tau_i=t,\,\tau_i\ge 0\right)$$
$$=\int_{\sum_{i=0}^{m}\tau_i\le t,\tau_i\ge 0}\sqrt{1+|\operatorname{grad}\tau_{m+1}|^2}\,d\tau_m\cdots\tau_0,$$

where $\tau_{m+1}=-(\tau_0+\tau_1+\cdots+\tau_m)$ is viewed as a function of $\tau_0,\tau_1,\ldots,\tau_m$. Therefore

$$\mathrm{Vol}\left(\sum_{i=0}^{m+1}\tau_i=t\right)=\sqrt{m+2}\cdot\mathrm{Vol}\left(\sum_{i=0}^{m}\tau_i\le t,\,\tau_i\ge 0\right)$$
$$=\sqrt{m+2}\cdot\frac{t^{m+1}}{(m+1)!}.$$

Therefore the general term in series (5.1) is estimated by

$$2(N+2)^{4(m+2)N+4(m+1)}(2\pi)^{-(m+2)(n/2)}2^{4(N+1)(m+1)}(1+\sqrt{t}|x|)^{2N+2}$$
$$\cdot(1+\sqrt{t}|y|)^{2N+2}\exp[(x-y)f(x)+4(N+1)t|y|^2](4\pi)^{(n/2)(m+1)}\sqrt{m+2}$$
$$\cdot\exp\left(-\frac{|x-y|^2}{4t}\right)\frac{t^{-n/2+m+1}}{(m+1)!}.$$

□

Theorem 13. *If t is chosen small enough so that*

$$4(N+1)t^2+nCt<\tfrac{1}{8}\tag{5.5}$$

and

$$nC < \frac{1}{8C}, \tag{5.6}$$

then the infinite series (5.1) *converges.*

Proof. This follows easily from the root test for convergent power series, the estimate $(m+1)! \geq (m+1)^{(m+1)/2}$ and Theorem 12. □

Theorem 14.

(i) $\lim_{t\to 0} \tilde{\varphi}_N(t,x,y) = \delta_x(y)$.
(ii) $\lim_{t\to 0} \varphi(t,x,y) = \delta_x(y)$ *where* $\varphi(t,x,y)$ *denotes the infinite series* (5.1).

Proof. (i) For any differentiable function $\sigma(x)$ on R^n, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \tilde{\varphi}_N(t,x,y)\sigma(y)\,dy_1\cdots dy_n \\
&\quad = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \tilde{\varphi}_N(t,x,x-y)\sigma(x-y)\,dy_1\cdots dy_n \\
&\quad = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} (2\pi t)^{-n/2}\left[\exp\left(a(x,x-y) - \frac{|y|^2}{2t}\right)\right] \\
&\qquad \cdot [1 + \tilde{a}_1(x,x-y)t + \cdots + \tilde{a}_N(x,x-y)t^N]\sigma(x-y)\,dy_1\cdots dy_n.
\end{aligned}$$

Let $y = \sqrt{2t}r$ where $r = (r_1,\ldots,r_n)$. Then

$$\begin{aligned}
a(x,x-y) &= \int_0^1 \sum_{i=1}^n y_i f_i((x-y)+ty)\,dt \\
&= \int_0^1 y\cdot f(x+(t-1)y)\,dt.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \tilde{\varphi}_N(t,x,y)\sigma(y)\,dy_1\cdots dy_n \\
&\quad = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \pi^{-n/2}\left[\exp\left(\int_0^1 \sqrt{2t}r\cdot f(x+(t-1)\sqrt{2t}r) - |r|^2\right)\right] \\
&\qquad \cdot [1 + \tilde{a}_1(x,x-\sqrt{2t}r)t + \cdots + \tilde{a}_N(x,x-\sqrt{2t}r)t^N]\sigma(x-\sqrt{2t}r) \\
&\qquad \cdot dr_1\cdots dr_n.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\lim_{t\to 0}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \tilde{\varphi}_N(t,x,y)\sigma(y)\,dy_1\cdots dy_n \\
&\quad = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \pi^{-n/2}\exp\left(-\sum_{i=1}^n r_i^2\right)\sigma(x)\,dr_1\cdots dr_n = \sigma(x).
\end{aligned}$$

So (i) is proven.

(ii) follows immediately from (i) and (5.8) below. □

Theorem 15. *Let $\varphi(t, x, y)$ denote the infinite series* (5.1). *Then $\varphi(t, x, y)$ is the fundamental solution to the Kolmogorov equation, i.e.,*

$$\begin{aligned} &\frac{\partial \varphi}{\partial t}(t, x, y) = L_x \varphi(t, x, y), \\ &\lim_{t \to 0} \varphi(t, x, y) = \delta_x(y), \end{aligned} \tag{5.7}$$

where L_x is defined by the right-hand side of (3.5).

Proof. In view of Theorem 13, there is no problem for convergence of the infinite series (5.1) and its derivatives. We can differentiate the series (5.1) term by term. We rewrite the integral

$$\begin{aligned} &\int_{\sum_{i=0}^{m+1} \tau_i = t} \int_{x_{m+1}} \int_{x_m} \cdots \int_{x_1} \frac{1}{\sqrt{m+2}} \tilde{\varphi}_N(\tau_{m+1}, x, x_{m+1}) e_N(\tau_m, x_{m+1}, x_m) \\ &\quad \cdot e_N(\tau_{m-1}, x_m, x_{m-1}) \cdots e_N(\tau_0, x_1, y) \end{aligned}$$

as

$$\begin{aligned} \varphi_m(t, x, y) = &\int_0^t \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_m} \int_{x_{m+1}} \int_{x_m} \cdots \int_{x_1} \tilde{\varphi}_N(t - t_1, x, x_{m+1}) \\ &\cdot e_N(t_1 - t_2, x_{m+1}, x_m) \\ &\cdot e_N(t_2 - t_3, x_m, x_{m-1}) \cdots e_N(t_m - t_{m+1}, x_2, x_1) \\ &\cdot e_N(t_{m+1}, x_1, y)\, dt_{m+1}, dt_m \cdots dt_1. \end{aligned} \tag{5.8}$$

It follows easily that

$$\begin{aligned} &\frac{\partial \varphi_m}{\partial t}(t, x, y) \\ &\quad = \int_0^t \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_m} \int_{x_{m+1}} \int_{x_m} \cdots \int_{x_1} \frac{\partial \tilde{\varphi}_N}{\partial t}(t - t_1, x, x_{m+1}) \\ &\qquad \cdot e_N(t_1 - t_2, x_{m+1}, x_m) e_N(t_2 - t_3, x_m, x_{m-1}) \cdots e_N(t_m - t_{m+1}, x_2, x_1) \\ &\qquad \cdot e_N(t_{m+1}, x_1, y), dt_{m+1} dt_m \cdots dt_1 \\ &\qquad + \lim_{t_1 \to t} \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_m} \int_{x_{m+1}} \int_{x_m} \cdots \int_{x_1} \tilde{\varphi}_N(t - t_1, x, x_{m+1}) \\ &\qquad \cdot e_N(t_1 - t_2, x_{m+1}, x_m) e_N(t_2 - t_3, x_m, x_{m-1}) \cdots e_N(t_m - t_{m+1}, x_2, x_1) \\ &\qquad \cdot e_N(t_{m+1}, x_1, y)\, dt_{m+1}\, dt_m \cdots dt_2 \\ &\quad = \int_0^t \int_0^{t_1} \cdots \int_0^{t_m} \int_{x_{m+1}} \int_{x_m} \cdots \int_{x_1} \frac{\partial \tilde{\varphi}_N}{\partial t}(t - t_1, x, x_{m+1}) e_N(t_1 - t_2, x_{m+1}, x_m) \\ &\qquad \cdot e_N(t_2 - t_3, x_m, x_{m-1}) \cdots e_N(t_m - t_{m+1}, x_2, x_1) \\ &\qquad \cdot e_N(t_{m+1}, x_1, y)\, dt_{m+1}\, dt_m \cdots dt_1 \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t\int_0^{t_2}\cdots\int_0^{t_m}\int_{x_m}\int_{x_{m-1}}\cdots\int_{x_1} e_N(t-t_2,x,x_m)\\
&\cdot e_N(t_2-t_3,x_m,x_{m-1})\cdots e_N(t_m-t_{m+1},x_2,x_1)\\
&\cdot e_N(t_{m+1},x_1,y)\,dt_{m+1}\,dt_m\cdots dt_2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-L_x\right)\varphi_m(t,x,y)\\
&\quad=\int_0^t\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_m}\int_{x_{m+1}}\int_{x_m}\cdots\int_{x_1}\left(\frac{\partial}{\partial t}-L_x\right)\tilde{\varphi}_N(t-t_1,x,x_{m+1})\\
&\qquad\cdot e_N(t_1-t_2,x_{m+1},x_m)\\
&\qquad\cdot e_N(t_2-t_3,x_m,x_{m-1})\cdots e_N(t_m-t_{m+1},x_2,x_1)\\
&\qquad\cdot e_N(t_{m+1},x_1,y)\,dt_{m+1}\,dt_m\cdots dt_1\\
&\qquad+\int_0^t\int_0^{t_2}\cdots\int_0^{t_m} e_N(t-t_2,x,x_m)e_N(t_2-t_3,x_m,x_{m-1})\\
&\qquad\cdots e_N(t_m-t_{m+1},x_2,x_1)e_N(t_{m+1},x_1,y)\,dt_{m+1}\,dt_m\cdots dt_2\\
&\quad=\int_0^t\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_m}\int_{x_{m+1}}\int_{x_m}\cdots\int_{x_1} e_N(t-t_1,x,x_{m+1})\\
&\qquad\cdot e_N(t_1-t_2,x_{m+1},x_m)e_N(t_2-t_3,x_m,x_{m+1})\cdots e_N(t_m-t_{m+1},x_2,x_1)\\
&\qquad\cdot e_N(t_{m+1},x_1,y)\,dt_{m+1}\cdots dt_2\,dt_1+\int_0^t\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_m} e_N(t-t_2,x,x_m)\\
&\qquad\cdot e_N(t_2-t_3,x_m,x_{m-1})\cdots e_N(t_m-t_{m+1},x_2,x_1)\\
&\qquad\cdot e_N(t_{m+1},x_1,y)\,dt_{m+1}\,dt_m\cdots dt_2.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-L_x\right)\left(\tilde{\varphi}_N(t,x,y)+\sum_{m=0}^{K}(-1)^{m+1}\varphi_m\right)\\
&\quad=(-1)^{K+1}\int_0^t\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_K}\int_{x_{K+1}}\int_{x_K}\cdots\int_{x_1} e_N(t-t_1,x,x_{K+1})\\
&\qquad\cdot e_N(t_1-t_2,x_{K+1},x_K)\\
&\qquad\cdot e_N(t_2-t_3,x_K,x_{K+1})\cdots e_N(t_{K+1},x_1,y)\,dt_{K+1}\cdots dt_2\,dt_1. \qquad \square
\end{aligned}
$$

However, a similar estimate as in Theorem 12 shows that the above expression tends to zero uniformly as $K\to\infty$. Hence

Corollary 16. *The fundamental solution $\varphi(t,x,y)$ in Theorem 15 is approximated by*

$$
\tilde{\varphi}_N(t,x,y)+\sum_{m=0}^{K}(-1)^{m+1}\varphi_m(t,x,y)
$$

which is readily computable. Here $\varphi_m(t,x,y)$ is given by (5.8). The error for such an approximation is given by

$$\sum_{m=K+1}^{\infty}(-1)^{m+1}\varphi_m(t,x,y)$$

which can be estimated by

$$\begin{aligned}&(1+\sqrt{t}|x|)^{2N+2}(1+\sqrt{t}|y|)^{2N+2}\exp\left[(x-y)\cdot f(x)+4(N+1)t|y|^2\right]\\&\quad\cdot\exp\left(-\frac{|x-y|^2}{4t}\right)\sum_{m=K+1}^{\infty}2(N+2)^{4(m+2)N+4(m+1)}\sqrt{m+2}\\&\quad\cdot(\sqrt{2\pi})^{-(m+2)(n/2)}2^{4(N+1)(m+1)}(\sqrt{4\pi})^{(m+1)(n/2)}\frac{t^{-n/2+m+1}}{(m+1)!},\end{aligned}$$

which clearly tends to zero rapidly if t is small and K is large.

Proof. This follows immediately from Theorem 12. □

References

[Be] V. Benes, Exact finite-dimensional filters for certain diffusions with nonlinear drift. Stochastics, 5 (1981) 65–92.

[Br] R. W. Brockett, Nonlinear systems and nonlinear estimation theory, in The Mathematics of Filtering and Identification and Applications, M. Hazewinkel and J. S. Willems, eds., Reidel, Dordrecht, 1981.

[BrCl] R. W. Brockett and J. M. C. Clark, The geometry of the conditional density functions, in Analysis and Optimization of Stochastic Systems, O. L. R. Jacobs et al., eds., Academic Press, New York, 1980, pp. 299–309.

[ChMi] M. Chaleyat-Maurel and D. Michel, Des resultats de non-existence de filtre de dimension finie, Stochastics, 13 (1984), 83–102.

[ChYa] W. L. Chiou and S. S.-T. Yau, Finite-dimensional filters with nonlinear drift, II: Brockett's problem on classification of finite-dimensional estimation algebras, SIAM J. Control Optim., 32(1) (1994), 297–310.

[CLY] J. Chen, C. W. Leung, and S. S.-T. Yau, Finite-dimensional filters with nonlinear drift, IV: Classification of finite-dimensional estimation algebras of maximal rank with state space dimension 3, SIAM J. Control Optim., 34(1) (1996), 179–198.

[C] P. C. Collingwood, Some remarks on estimation algebras, Systems Control Lett., 7 (1986), 217–224.

[D] M. H. A. Davis, On a multiplicative functional transformation arising in nonlinear filtering theory, Z. Wahrsch Verw. Gebiete, 54 (1980), 125–139.

[DM] M. H. A. Davis and S. I. Marcus, An introduction to nonlinear filtering, in The Mathematics of Filtering and Identification and Applications, M. Hazewinkel and J. S. Willems, eds., Reidel, Dordrecht, 1981.

[DTWY] R. T. Dong, L. F. Tam, W. S. Wong, and S. S-T. Yau, Structure and classification theorems of finite-dimensional exact estimation algebras, SIAM J. Control Optim., 29(4) (1991), 866–877.

[FKK] M. Fujisaki, G. Kallianpur, and H. Kunita, Stochastic differential equations for the nonlinear filtering problem, Osaka J. Math., 1 (1972), 19–40.

[M] S. K. Mitter, On the analogy between mathematical problems of nonlinear filtering and quantum physics, Ricerche Automat. 10(2) (1979), 163–216.

[O] D. L. Ocone, Finite-dimensional estimation algebras in nonlinear filtering, in The Mathematics of Filtering and Identification and Applications, M. Hazewinkel and J. S. Willems, eds., Reidel, Dordrecht, 1981.

[S] S. Steinberg, Applications of the Lie algebraic formulas of Baker, Campbell, Hausdorff, and Zassenhaus to the calculation of explicit solutions of partial differential equations, J. Differential Equations, 26 (1979), 404–434.

[TWY] L. F. Tam, W. S. Wong, and S. S.-T. Yau, On a necessary and sufficient condition for finite dimensionality of esitmation algebras, SIAM J. Control Optim., 28(1) (1990), 173–185.

[WeNo] J. Wei and E. Norman, On global representations of the solutions of linear differential equations as a product of exponentials, Proc. Amer. Math. Soc., 15 (1964), 327–334.

[Wi] D. V. Widder, The Heat Equation, Mathematics 67, Academy Press, 1975.

[Wo1] W. S. Wong, New classes of finite-dimensional nonlinear filters, Systems Control Lett., 3 (1983), 155–164.

[Wo2] W. S. Wong, On a new class of finite-dimensional estimation algebras, Systems Control Lett., 9 (1987), 79–83.

[Wo3] W. S. Wong, Theorems on the structure of finite-dimensional estimation algebras, Systems Control Lett., 9 (1987), 117–124.

[Y1] S. S.-T. Yau, Recent results on nonlinear filtering, New class of finite-dimensional filters, Proceedings of the 29th IEEE Conference on Decision and Control, Honolulu, HI, Dec. 5–7, 1990, pp. 231–233.

[Y2] S. S.-T. Yau, Finite-dimensional filters with nonlinear drift, I: A class of filters including both Kalman–Bucy filters and Benes filters, J. Math. Systems Estim. Control, 4 (1994), 181–203.

[YC] S. S.-T. Yau and W. L. Chiou, Recent results on classification of finite-dimensional estimation algebras: dimension of state space ≤ 2, Proceedings of the 30th IEEE Conference on Decision and Control, Brighton, Dec. 11–13, 1991, pp. 2758–2760.

Accepted 26 *April* 1995