Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)
Vol. XXVI (1998), pp. 357-381

Intersection Lattices and Topological Structures of Complements of Arrangements in $\mathbb{CP}^2$

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1. – Introduction

An arrangement of hyperplanes $\mathcal{A}$ is a finite collection of $\mathbb{C}$ -linear subspaces of dimension $(d-1)$ in $\mathbb{C}^d$. Associated with the arrangemant $\mathcal{A}$ is an open real $2d$-manifold the complement $M(\mathcal{A}) = \mathbb{C}^d - \cup\{H : H \in \mathcal{A}\}$. The central problem in this area is to decide to what extent the topology or differentiable structure of $M(\mathcal{A})$ is determined by the combinatorial geometry of $\mathcal{A}$ and vice versa.

The theory was first initiated in 1969 by V. I. Arnol'd [1], who calculated the Poincaré polynomial of the pure braid space M_ℓ and the cohomology ring structure of $\mathrm{H}^*(M_\ell)$, where M_ℓ is the complement of the complexified braid arrangement $\mathcal{A}_\ell$ defined by

$$\prod_{1\le i<j\le \ell} (z_i - z_j).$$

In general, for an arbitrary arrangement $\mathcal{A}$, define holomorphic differential forms $\omega_H = \frac{1}{2\pi i}\frac{d\alpha_H}{\alpha_H}$, where α_H is the linear form defining the hyperplane H for $h \in \mathcal{A}$, and let $[\omega_H]$ denote the corresponding cohomology class. Let

$$R(\mathcal{A}) = \bigoplus_{p=0}^{\ell} R_p$$

be the graded $\mathbb{C}$-algebra of holomorphic differential forms on $M(\mathcal{A})$ generated by the ω_H and 1. Arnol'd conjectured that the natural map $\eta \longrightarrow [\eta]$ of $R(\mathcal{A}) \longrightarrow \mathrm{H}^*\big(M(\mathcal{A}), \mathbb{C}\big)$ is an isomorphism of graded algebras. This was proved by Brieskorn [2] in 1971, who showed that the $\mathbf{Z}$-subalgebra of $R(\mathcal{A})$ generated by the forms ω_H and 1 is isomorphic to the singular cohomology

A.M.S. Classification: 05B35, 14B05, 14F45, 57N20
Research was partially supported by an NSF grant.
Pervenuto alla Redazione il 5 novembre 1996 e in forma definitiva il 3 settembre 1997.

$\mathrm{H}^*(M(\mathcal{A}), \mathbb{Z})$. Although Brieskorn proved the Arnol'd conjecture that $R(\mathcal{A})$ is isomorphic to $\mathrm{H}^*(M(\mathcal{A}), \mathbb{C})$ as a graded algebra for the arbitrary arrangement $\mathcal{A}$, it was not known whether the algebra R is determined by the combinatorial data of $\mathcal{A}$, since the linear forms enter the definition of $R(\mathcal{A})$. In 1980, Orlik and Solomon [16] introduced a graded algebra $A(\mathcal{A})$ to an arbitrary arrangement $\mathcal{A}$. $A(\mathcal{A})$ is a combinatorial invariant of $\mathcal{A}$. The beautiful result of Orlik and Solomon asserts that there is an isomorphism of algebra $A(\mathcal{A}) \simeq \mathcal{R}(\mathcal{A})$. This together with the Brieskorn's solution to the Arnol'd conjecture implies that the cohomological ring $\mathrm{H}^*(M(\mathcal{A}), \mathbb{C})$ is a combinatorial invariant of $\mathcal{A}$.

Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{C}^3$ and let $\mathcal{A}^*$ be the corresponding arrangement of lines in $\mathbb{CP}^2$. Then we have $M(\mathcal{A}) = M(\mathcal{A}^*) \times \mathbb{C}^*$ (cf. [18]), where $M(\mathcal{A}^*) = \mathbb{CP}^2 - \cup\mathcal{A}^*$. Topology and differentiable structure of $M(\mathcal{A}^*)$ are important in the theory of hypergeometric functions (see the work of Gel'fand [8] and his subsequent papers, the work of Deligne and Mostow [3], and subsequent papers by Mostow). Moreover, they play a role in some interesting problems in algebraic geometry (see especially the works of Hirzebruch [9] and Moishezon [13]). Although the conjecture that the homotopic type of $M(\mathcal{A}^*)$ is a combinatorial invariant of the projective arrangement of $\mathcal{A}^*$ seems disproved by G. Rybnikov [20] in 1994, we have shown [11] that for a very large class of projective arrangements in $\mathbb{CP}^2$, the diffeomorphic type of $M(\mathcal{A}^*)$ is indeed a combinatorial invariant of $\mathcal{A}^*$.

DEFINITION. Let $\mathcal{A}^*$ be a projective arrangement of lines in $\mathbb{CP}^2$, The set of all intersections of elements of $\mathcal{A}^*$ partially ordered by reverse inclusion is denoted as $L(\mathcal{A}^*)$.

It is natural to ask whether the combinatorial data $L(\mathcal{A}^*)$ of the projective arrangement are determined by the homotopic type, topological type, or diffeomorphic type of $M(\mathcal{A}^*)$. For the first question: Falk has written a series of papers [5], [6], and [7] on whether there are combinatoriably distinct arrangements that have homotopic equivalent complements. In [6], Falk constructed two projective arrangements in $\mathbb{CP}^2$, each of which has two triple points and nine double points. The homotopic equivalence of their complements was shown in [7]. In view of this example, one would like to know whether $L(\mathcal{A}^*)$ is determined by the topological type of $M(\mathcal{A}^*)$. The following theorem answers this question affirmatively.

MAIN THEOREM. *Let $\mathcal{A}_1^*$ and $\mathcal{A}_2^*$ be two projective arrangements in $\mathbb{CP}^2$. If $M(\mathcal{A}_1^*)$ is homeomorphic to $M(\mathcal{A}_2^*)$, then $L(\mathcal{A}_1^*)$ is isomorphic to $L(\mathcal{A}_2^*)$.*

In view of Falk's example mentioned above, we have the following corollary.

COROLLARY. *There exist two projective arrangements $\mathcal{A}_1^*$ and $\mathcal{A}_2^*$ in $\mathbb{CP}^2$ such that $M(\mathcal{A}_1^*)$ and $M(\mathcal{A}_2^*)$ have the same homotopic type, but they do not have the same topological type.*

In Section 2 we recall some necessary definitions and results in three-manifolds that are due to Waldhausen [22]. In Section 3 we study the boundary

of a regular neighborhood of an arrangement $\mathcal{A}^*$ in $\mathbb{CP}^2$, using Waldhausen's theory on graphed manifolds [21]. By restricting ourselves to nonexceptional projective arrangements in $\mathbb{CP}^2$, we show in Section 4 that if two such arrangements have the same topological types, then they have the same graph structures (again in the sense of Waldhausen). In Section 5 we prove the main theorem for nonexceptional arrangements. In Section 6 and Section 7 we finish the proof of the remaining part of our main theorem for the exceptional arrangements.

The second author learned this important open problem during P. Orlik's interesting lectures at CBMS conference on arrangements at Flagstaff in 1988. The main theorem of this paper was announced in [12].

Acknowledgment. We gratefully acknowledge both referees for their careful reading of this paper and especially for providing us many useful comments.

2. – Definitions and preliminaries

In this section we recall some necessary definitions and important results on three-manifolds due to Waldhausen [22].

Throughout this section, by a *manifold*, we mean an orientable compact three-dimensional manifold with or without boundary.

A *surface* is a connected two-manifold. It is compact and orientable, unless the contrary is stated explicitly. A *surface F in the manifold M* is properly embedded (i.e., $F \cap \partial M = \partial F$, where ∂ denotes boundary). A *surface in ∂M* is a submanifold of ∂M. A *system of surfaces in M or ∂M* consists of finitely many, mutually disjoint components of the above two types.

Let F be a subspace of M. $U(F)$ denotes a *regular* neighborhood of F. A regular neighborhood is always compact and sufficiently small. A typical construction is as follows. Choose a finite triangulation in which F is a subcomplex. The closed star of F in the second bary center subdivision of this triangulation is then a regular neighborhood of F.

An *isotopy deformation* of M is a level preserving map $h : M \times I \to M \times I$, $I = [0, 1]$, such that from each level $h|_{M \times t} = h_t$ is a homeomorphism from M onto itself and $h_0 =$ Identity. We often abbreviate "isotopy deformation" as "deformation."

Subspaces N_1 and N_2 in M are called *isotopic* in M if there is an isotopy deformation of $M : h_t$, $t \in I$, such that $h_1(N_1) = N_2$.

DEFINITION 2.1. Let M be a manifold. Let F be a system of surfaces in M or ∂M. F is compressible in M if either one of the following two cases hold.

(a) There is a noncontractible simple closed curve k in $\mathrm{Int}(F)$, and a disk D in M, $\mathrm{Int}(D) \subseteq \mathrm{Int}(M)$, such that $D \cap F = \partial D = k$.
(b) There is a ball E in M such that $E \cap F = \partial E$.

F is *incompressible* in M if and only if it is not compressible in M.

DEFINITION 2.2. A manifold M is called *irreducible* if every two-sphere in M is compressible.

Thus M is irreducible if and only if each two-sphere in M bounds a three-cell in M. (Remember: If M is irreducible, and $\partial M \neq \emptyset$, then either M is a ball, or else genus$(\partial M) > 0$, and hence $\mathrm{H}_1(M)$ is infinite.)

DEFINITION 2.3. A manifold M is called *boundary-irreducible* if ∂M is incompressible.

The following lemma is a well-known corollary of the sphere theorem.

LEMMA 2.1. *Suppose M is irreducible and $\pi_1(M)$ is not finite. Then M is aspherical, that is, $\pi_j(M) = 0$, for $j \geq 2$.*

Lemma 2.2 below seems to be widely known. A proof is given in [23].

LEMMA 2.2. *Let M be an irreducible manifold.*

(a) *If $\partial M \neq \emptyset$, and M is not a ball, then there exists in M an incompressible surface F such that $0 \neq [\partial F] \in \mathrm{H}_1(\partial M)$.*
(b) *If $\partial M = \emptyset$, then there exists in M an incompressible surface if and only if either $\mathrm{H}_1(M)$ is not finite or $\pi_1(M)$ is a nontrivial free product with amalgamation (or both).*

If F is a separating incompressible surface in M, $\partial M = \emptyset$, then $\pi_1(M)$ is a nontrivial free product with amalgamation, $\pi_1(M) \simeq A *_C B$, where $C \simeq \pi_1(F)$, in a natural way.

DEFINITION 2.4. Let M be an irreducible manifold that is not a ball. M is *sufficiently large* if and only if there exists an incompressible surface in M.

REMARK 2.1. There exist irreducible manifolds with infinite fundamental group, which are not sufficiently large [23].

Let $T = T_1 \cup \cdots \cup T_n (n \geq 0)$ be a system of tori in $\mathrm{Int}(M)$, and $U(T)$ be a regular neighborhood of T in M.

DEFINITION 2.5. If each component of $M - \mathrm{Int}\big(U(T)\big)$ is homeomorphic to a fiber bundle with S^1 as fiber, then T is called a *graph structure* of M. A manifold with a graph structure is called a *graph manifold.*

Let T_1 be an arbitrary fixed component of T. $U(T_1) \supseteq T_1$ is a component of $U(T)$. So $U(T_1)$ is homeomorphic to $T_1 \times I$. Let T' and T'' be the boundary surfaces of $U(T_1)$. The component of $M - \mathrm{Int}\big(U(T)\big)$ which is pasted along T'(respectively, T'') is denoted by M_1 (respectively, M_2). We can compare the homology class of curves in T' and T'' by the natural isomorphism

$$H_1(T') \leftrightarrow H_1\big(U(T_1)\big) \leftrightarrow H_1(T'').$$

Hence we can talk about intersection of homology classes of curves on T' and T''.

DEFINITION 2.6. A graph structure T of a manifold M is called *reduced* if none of the following ten situations occur. A manifold with a reduced graph structure is called a *reduced graph manifold.*

(W1) $M_1 \neq M_2$ and M_1 is a S^1-bundle over the annulus.
(W2) A fiber of M_1 is homologous in T_1 to a fiber of M_2.
(W3) $M_1 = S^1 \times D^2$ (D^2 = 2-cell) is a solid torus, and a meridian curve $\{p\} \times S^1 \subseteq \partial M_1$ has intersection number $\{\pm 1\}$ with a fiber of M_2 in T_1.
(W4) $M_1 \simeq S^1 \times D^2$ (D^2 = 2-cell) is a solid torus and a meridian curve $\{p\} \times S^1 \subseteq \partial M_1$ is homologous to a fiber of M_2 in T_1.
(W5) M_1 is a S^1-bundle over the Möbius band, and the homology class μ_1 in $\partial M_1 = T'$ of the boundary of a section of M_1 is homologous to a fiber of M_2 in T_1.
(W6) M_1 and M_2 are S^1-bundles over the Möbius band, and μ_1 is homologous to μ_2, where μ_i is defined as in (W5).
(W7) $M - \mathrm{Int}(U(T))$ has two components. One of them is a graph manifold Q, defined in Section 3 of [21], which is homeomorphic to an S^1-bundle with orientable total space over the Möbius band. The other is not a solid torus.
(W8) $M_1 = M_2 \simeq A \times S^1 (A =$ annulus$) \simeq I \times S^1 \times S^1$ and the pasting map $S^1 \times S^1 \to S^1 \times S^1$ is given by a matrix of trace ± 2.
(W9) M_1 and M_2 are solid tori.
(W10) $T = \emptyset$, and M is a S^1-bundle over S^2 or $\mathbb{RP}^2$ (real projective plane).

3. – The boundary of a regular neighborhood of an arrangement $\mathcal{A}^*$ in $\mathbb{CP}^2$

Let $\mathcal{A}^*$ be an arrangement in $\mathbb{CP}^2$ and $N(\mathcal{A}^*) = \bigcup_{\ell \in \mathcal{A}^*} \ell$. Suppose that $\mathcal{A}^*$ has $x_1, \ldots, x_k (k \geq 0)$ as multiple intersection points (i.e., multiplicity $t(x_i) \geq 3$). We blow up $\mathbb{CP}^2$ at $\{x_1, \ldots, x_k\}$. We get a set $\tilde{\mathcal{A}}^*$ of lines that includes the proper transforms of the x_i in a blown-up surface $\widetilde{\mathbb{CP}^2}$. $\tilde{\mathcal{A}}^*$ is called an associate arrangement in $\widetilde{\mathbb{CP}^2}$ induced by $\mathcal{A}^*$. Suppose that $\tilde{\mathcal{A}}^* = \{\ell_1, \ldots, \ell_n\}$. Each pair of lines of $\tilde{\mathcal{A}}^*$ intersects at most at one point. Let $N(\tilde{\mathcal{A}}^*) = \bigcup_{\ell_i \in \tilde{\mathcal{A}}^*} \ell_i$, which is connected. Let $U(\tilde{\mathcal{A}}^*)$ be a regular neighborhood of $N(\tilde{\mathcal{A}}^*)$ and $K(\tilde{\mathcal{A}}^*) = \partial(U(\tilde{\mathcal{A}}^*))$. Thus $K(\tilde{\mathcal{A}}^*)$ is a plumbed three-manifold which is homeomorphic to $K(\mathcal{A}^*)$, the boundary of a regular neighborhood of $N(\mathcal{A}^*)$ in $\mathbb{CP}^2$.

$K(\tilde{\mathcal{A}}^*)$ can be also obtained by pasting some S^1-bundles together. Consider the boundary of regular neighborhood of a line $\ell_i \in \tilde{\mathcal{A}}^*$ as a S^1-bundle $E_i \to \ell_i$. If two lines ℓ_1 and ℓ_2 of $\tilde{\mathcal{A}}^*$ intersect at a point x, let D_i be a disc in ℓ_i such that $x \in \mathrm{Int}(D_i) (i = 1, 2)$. Define $E_i' := E_i|_{\ell_i - \mathrm{Int}(D_i)}$, and glue E_1' and E_2' along $E_i|_{\partial D_i}$. In other words, we choose a trivialization of $E_i|_{\partial D_i}$ which is $S^1 \times S^1$. We then glue them together according to the map $f \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : S^1 \times S^1 \to S^1 \times S^1$, which switches the base of $E_1|_{\partial D_1}$ with the fiber of $E_2|_{\partial D_2}$. More generally, if

ℓ_i intersects n_i number of times in $\tilde{\mathcal{A}}^* - \{\ell_i\}$, then we consider the restriction of the S^1-bundle over ℓ_i to n_i-punctured sphere. The boundary of its total space is a disjoint union of n_i tori, each of which is pasted along with another S^1-bundle. Let $T(\tilde{\mathcal{A}}^*) = T_1 \coprod \ldots \coprod T_m$ be the disjoint union of all such tori in $K(\tilde{\mathcal{A}}^*)$. Thus $K(\tilde{\mathcal{A}}^*)$ is a graph manifold with graph structure $T(\tilde{\mathcal{A}}^*)$.

From this graph manifold $K(\tilde{\mathcal{A}}^*)$ and its graph structure $T(\tilde{\mathcal{A}}^*)$, we define a weighted graph $G(\tilde{\mathcal{A}}^*)$ as follows. For each ℓ_i of $\tilde{\mathcal{A}}^*$, one vertex v_i with weight $(\ell_i \cdot \ell_i)$ correspods to the self-intersection number of ℓ_i in $\widetilde{\mathbb{CP}^2}$. Each torus in the graph structure $T(\tilde{\mathcal{A}}^*)$ corresponds to an edge of $G(\tilde{\mathcal{A}}^*)$. If lines ℓ_i and ℓ_j of $\tilde{\mathcal{A}}^*$ intersect at a point, then an edge of $G(\tilde{\mathcal{A}}^*)$ is defined to have v_i and v_j as its adjacent vertices. Thus $G(\tilde{\mathcal{A}}^*)$ consists of n vertices $v_1, \ldots, v_n$ and m edges $e_1, \ldots, e_m$.

Now let us consider the case when the graph manifold $K(\tilde{\mathcal{A}}^*)$ with graph structure $T(\tilde{\mathcal{A}}^*)$ is irreducible. First we have the following lemma.

LEMMA 3.1. *If $\mathcal{A}^*$ is an arrangement such that each $\ell \in \mathcal{A}^*$ has at least three intersection points with other lines of $\mathcal{A}^*$, then $K(\tilde{\mathcal{A}}^*)$ is a reduced graph manifold with reduced graph structure $T(\tilde{\mathcal{A}}^*)$.*

PROOF. Suppose that the arrangement $\mathcal{A}^*$ in $\mathbb{CP}^2$ satisfies the condition in Lemma 2.1. Then $\tilde{\mathcal{A}}^*$ induced by $\mathcal{A}^*$ satisfies the same condition, since each added exceptional line $\mathbb{CP}^1$ from blowing up of $N(\mathcal{A}^*)$ must intersect at least three original lines in $\mathcal{A}^*$.

Let $\tilde{M} = K(\tilde{\mathcal{A}}^*) - \mathrm{Int}\big(U\big(T(\tilde{\mathcal{A}}^*)\big)\big)$, where $U\big(T(\tilde{\mathcal{A}}^*)\big)$ is a regular neighborhood of $T(\tilde{\mathcal{A}}^*)$ in $K(\tilde{\mathcal{A}}^*)$. We can see that each component M_i of $\tilde{M}$ corresponds to a line ℓ_i in $\tilde{\mathcal{A}}^*$ and M_i is an S^1-bundle over an n_i-punctured sphere B_i ($n_i \geq 3$ by our assumption). So M_i is not homeomorphic to a solid torus ($S^1 \times D^2$), an S^1-bundle over the annulus, or an S^1-bundle over the Möbius band for each $i = 1, \ldots, n$. Thus the situations from (W1) to (W9) except (W2) are excluded. (W10) is obviously not true here. With regard to the exclusion of (W2) by looking at the glue map f, one can see that fibers of M_1 and M_2 are representative of the two generators of $H_1(T_1)$, respectively, when M_1 and M_2 are glued together along $\partial\big(U(T_1)\big)$. □

On the other hand, if there is a line ℓ in $\mathcal{A}^*$ that contains at most two intersection points, then we have only the following cases.

(Case 2a) ℓ contains no intersection point. It follows that $\mathcal{A}^* = \{\ell\}$. Then $K(\mathcal{A}^*) = K(\tilde{\mathcal{A}}^*)$ is an S^1-bundle over the two-sphere, which is precisely case (W10) of Definition 2.6. So $K(\mathcal{A}^*)$ is not a reduced graph manifold.

(Case 2b) ℓ contains only one intersection point. Thus $\mathcal{A}^*$ is a pencil. The component of $\tilde{M} = K(\tilde{\mathcal{A}}^*) - \mathrm{Int}\big(U\big((A)\big)\big)$ that corresponds to ℓ is homeomorphic to $S^1 \times D^2$ and its meridian curve $\{p\} \times S^1$ in $\partial(S^1 \times D^2)$ is homologous to a fiber of an adjacent component of $\tilde{M}$ by the glue map f. So (W4) of Definition 2.6 is true, and $K(\tilde{\mathcal{A}}^*)$ is not a reduced graph manifold.

(Case 2c) ℓ contains exactly two intersection points y_1 and y_2. So their multiplicities $t(y_1)$ and $t(y_2)$ are at least two. There are two further subcases.

(Case 2c-i) $t(y_1) = t(y_2) = 2$. Then $\mathcal{A}^*$ is a triangle; that is, $\mathcal{A}^*$ has exactly three lines ℓ_1, ℓ_2, ℓ_3 and three intersection points y_1, y_2, y_3. Clearly we have $K(\tilde{\mathcal{A}}^*) = K(\mathcal{A}^*)$ with graph structure $T(\tilde{\mathcal{A}}^*) = T_1 \coprod T_2 \coprod T_3$ and $\tilde{M} = M_1 \coprod M_2 \coprod M_3 (= K(\tilde{\mathcal{A}}^*) - U(T(\tilde{\mathcal{A}}^*)))$. Their relations are as follows: $\ell_1 \leftrightarrow M_1$, $\ell_2 \leftrightarrow M_2$, $\ell_3 \leftrightarrow M_3$; $\{y_1\} = \ell_2 \cap \ell_3$, $\{y_2\} = \ell_1 \cap \ell_3$, $\{y_3\} = \ell_1 \cap \ell_2$, $T_i \leftrightarrow y_i$ $(i = 1, 2, 3)$. So M_1, M_2 and M_3 are S^1-bundle over the annulus (i.e., $T \times I$), which means that (W1), of Definition 2.6 is satisfied here. Hence $K(\tilde{\mathcal{A}}^*)$ is not a reduced graph manifold.

(Case 2c-ii) $t(y_1)$ or $t(y_2) > 2$. To fix our notation, we shall assume that $t(y_1) > 2$. Then the exceptional line ℓ_1 obtained by blowing up y_1 contains at least three intersection points. Let M (respectively, M_1) be the component of $\tilde{M}$ that corresponds to ℓ (respectively, ℓ_1). Thus M is an S^1-bundle over the annulus, and M_1 is an S^1-bundle over n_1-punctured sphere ($n_1 \geq 3$). So (W1) of Definition 2.6 is valid here.

Thus we have the following proposition.

PROPOSITION 3.2. *Suppose that $\mathcal{A}^*$ is an arrangement in $\mathbb{CP}^2$. Then $K(\tilde{\mathcal{A}}^*)$ is a reduced graph manifold with a reduced graph structure $T(\tilde{\mathcal{A}}^*)$ if and only if each line of $\mathcal{A}^*$ contains at least three intersection points.*

Recall the following theorem and lemma in Section 7 of [21].

THEOREM 3.3. *A reduced graph manifold is irreducible.*

LEMMA 3.4. *Let M be a reduced graph manifold with the graph structure $T = T_1 \cup \ldots \cup T_n$. Then T_1 is compressible if and only if one component of $M - \mathrm{Int}(U(T))$ which is pasted along $U(T_1)$ is a solid torus.*

From these results we have the following corollary.

COROLLARY 3.5. *If $\mathcal{A}^*$ is an arrangement with at least three intersection points on each of its line or $\mathcal{A}^*$ is a triangle arrangement, then $K(\tilde{\mathcal{A}}^*)$ is irreducible and $T(\tilde{\mathcal{A}}^*)$ is an incompressible surface system in $K(\tilde{\mathcal{A}}^*)$.*

4. – Two arrangements $\mathcal{A}^*$ and $\mathcal{B}^*$ in $\mathbb{CP}^2$ with the same topological type

Throughout this section, let $\mathcal{A}^*$ and $\mathcal{B}^*$ be two arrangements in $\mathbb{CP}^2$, $M(\mathcal{A}^*) = \mathbb{CP}^2 - N(\mathcal{A}^*)$, $M(\mathcal{B}^*) = \mathbb{CP}^2 - N(\mathcal{B}^*)$, and let $\varphi : M(\mathcal{A}^*) \to M(\mathcal{B}^*)$ be a homeomorphism.

PROPOSITION 4.1. *Let $U(\mathcal{A}^*)$ and $U(\mathcal{B}^*)$ be two regular neighborhoods of $\mathcal{A}^*$, and $\mathcal{B}^*$ respectively. Then $U(\mathcal{A}^*) - N(\mathcal{A}^*)$ is homotopic equivalent to $U(\mathcal{B}^*) - N(\mathcal{B}^*)$.*

PROOF. Let $U_1(\mathcal{B}^*)$ be an arbitrary regular neighborhood of $N(\mathcal{B}^*)$. Then $K := \big(\varphi^{-1}\big[(U_1(\mathcal{B}^*)^c\big]\big)^c$ is a neighborhood of $N(\mathcal{A}^*)$. There is a regular neighborhood $U(\mathcal{A}^*)$ of $N(\mathcal{A}^*)$ such that $U(\mathcal{A}^*) \subseteq K$. So $\varphi\big(\big(U(\mathcal{A}^*)\big)^c\big) \supseteq \varphi(K^c)$, i.e., $V := \big(\varphi\big[\big(U(\mathcal{A}^*)\big)^c\big]\big)^c \subseteq \big(\varphi(K^c)\big)^c = U_1(\mathcal{B}^*)$. Since V is a neighborhood of $N(\mathcal{B}^*)$, we can choose a regular neighborhood $U_2(\mathcal{B}^*)$ of $N(\mathcal{B}^*)$ such that $U_2(\mathcal{B}^*) \subseteq V$. Thus we get

$$U_2(\mathcal{B}^*) - N(\mathcal{B}^*) \subseteq V - N(\mathcal{B}^*) \subseteq U_1(\mathcal{B}^*) - N(\mathcal{B}^*).$$

Observe that $V - N(\mathcal{B}^*)$ is exactly $\varphi\big(U(\mathcal{A}^*) - N(\mathcal{A}^*)\big)$. So we have

$$U_2(\mathcal{B}^*) - N(\mathcal{B}^*) \overset{i_2}{\hookrightarrow} \varphi\big(U(\mathcal{A}^*) - N(\mathcal{A}^*)\big) \overset{i_1}{\hookrightarrow} U_1(\mathcal{B}^*) - N(\mathcal{B}^*),$$

where i_1 and i_2 are inclusion maps. Since $U_i(\mathcal{B}^*)$ $(i = 1, 2)$ are regular neighborhoods of $N(\mathcal{B}^*)$ in $\mathbb{CP}^2$ and $U_1(\mathcal{B}^*)$ can be contracted to $U_2(\mathcal{B}^*)$, the inclusion map $i = i_1 \circ i_2 : U_2(\mathcal{B}^*) - N(\mathcal{B}^*) \hookrightarrow U_1(\mathcal{B}^*) - N(\mathcal{B}^*)$ induces an isomorphism $i_* : \pi_j\big(U_2(\mathcal{B}^*) - N(\mathcal{B}^*)\big) \to \pi_j\big(U_1(\mathcal{B}^*) - N(\mathcal{B}^*)\big)$. Consider $(i_1 \circ \varphi) \circ (\varphi^{-1} \circ i_2) = i$. We have the following induced maps

$$\pi_j\big(U_2(\mathcal{B}^*) - N(\mathcal{B}^*)\big) \xrightarrow{(\varphi^{-1} \circ i_2)_*} \pi_j\big(U(\mathcal{A}^*) - N(\mathcal{A}^*)\big) \xrightarrow{(i_1 \circ \varphi)_*} \pi_j\big(U_1(\mathcal{B}^*) - N(\mathcal{B}^*)\big)$$

for $j \geq 1$. Since $i_* = (i_1 \circ \varphi)_* \circ (\varphi^{-1} \circ i_2)_*$, $(i_1 \circ \varphi)_*$ is onto and $(\varphi^{-1} \circ i_2)_*$ is one-to-one.

Similarly, we can show that $(\varphi^{-1} \circ i_2)_*$ is onto and $(i_1 \circ \varphi)_*$ is one-to-one. It follows that $\pi_j\big(U_1(\mathcal{B}^*) - N(\mathcal{B}^*)\big) \simeq \pi_j\big(U(\mathcal{A}^*) - N(\mathcal{A}^*)\big)$. In view of Whitehead theorem, $U_1(\mathcal{B}^*) - N(\mathcal{B}^*)$ is homotopic equivalent to $U(\mathcal{A}^*) - N(\mathcal{A}^*)$. Since any two regular neighborhoods of $N(\mathcal{B}^*)$ (or $N(\mathcal{A}^*)$) are homotopic equivalent, the proposition follows immediately. □

REMARK 4.1. More generally, by the same proof, the above proposition is still true for any pairs (X, K), (Y, H) of complexes, such that $X - K$ is homeomorphic to $Y - H$.

Observe that $K(\mathcal{A}^*)$, the boundary of an arbitrary regular neighborhood $U(\mathcal{A}^*)$ of $N(\mathcal{A}^*)$, is homotopic equivalent to $U(\mathcal{A}^*) - N(\mathcal{A}^*)$. So we have the following corollary.

COROLLARY 4.2. *If $K(\mathcal{A}^*)$ and $K(\mathcal{B}^*)$ are boundaries of regular neighborhoods $U(\mathcal{A}^*)$ and $U(\mathcal{B}^*)$ of $N(\mathcal{A}^*)$ and $N(\mathcal{B}^*)$ respectively, then $K(\mathcal{A}^*) \cong K(\mathcal{B}^*)$ homotopically.*

COROLLARY 4.3. *Let $\tilde{\mathcal{A}}^*$ (respectively $\tilde{\mathcal{B}}^*$) be the induced arrangement from $\mathcal{A}^*$ (respectively $\mathcal{B}^*$) by blowing up. Then $K(\tilde{\mathcal{A}}^*) \cong K(\tilde{\mathcal{B}}^*)$ homotopically.*

Before we can proceed, we need to recall a result of Waldhausen.

DEFINITION 4.1. Let M and N be compact orientable 3-manifolds. An isomorphism ψ of $\pi_1(N)$ onto $\pi_1(M)$ is said to respect the peripheral structure if for each boundary surface F of N there is a boundary surface G of M such that $\psi\big(i_*(\pi_1(F))\big) \subset R$ and R is conjugate in $\pi_1(M)$ to $i_*\big(\pi_1(G)\big)$, where i_* denotes inclusion homomorphism.

THEOREM 4.4 (6.5 of [22]). *If M and N are irreducible and boundary-irreducible compact orientable three-manifolds and ψ is an isomorphism from $\pi_1(N)$ onto $\pi_1(M)$ which respects the peripheral structure and M is sufficiently large, then there exists a homeomorphism $f : N \to M$ that induces ψ.*

LEMMA 4.5. *$K(\tilde{\mathcal{A}}^*)$ and $K(\tilde{\mathcal{B}}^*)$ are boundary irreducible, and $\varphi_* : \pi_1\big(K(\tilde{\mathcal{A}}^*)\big) \to \pi_1\big(K(\tilde{\mathcal{B}}^*)\big)$ respects the peripheral structure.*

PROOF. The lemma follows immediately from the fact that $\partial K(\tilde{\mathcal{A}}^*) = \phi = \partial K(\tilde{\mathcal{B}}^*)$. □

LEMMA 4.6. *If $\mathcal{A}^*$ is an arrangement with at least three intersection points on each of its line, then $K(\tilde{\mathcal{A}}^*)$ is reduced, irreducible, boundary irreducible and, sufficiently large.*

PROOF. By Proposition 3.2 and Corollary 3.5, we need only to show that $K(\tilde{\mathcal{A}}^*)$ is sufficiently large. In view of a result of D. Mumford [14], we know that the first Betti number of $K(\tilde{\mathcal{A}}^*)$ is at least p if $\tilde{\mathcal{A}}^*$ is p-connected (i.e., p is the minimal number such that there exist some points $P_1, \ldots, P_p \in N(\tilde{\mathcal{A}}^*)$ making $N(\tilde{\mathcal{A}}^*) - \{P_1, \ldots, P_p\}$ a tree). By Lemma 2.2 (b), $K(\tilde{\mathcal{A}}^*)$ is sufficient large. □

From Corollary 4.3 and Theorem 4.4 of Waldhausen we have the following proposition.

PROPOSITION 4.7. *If $\mathcal{A}^*$ and $\mathcal{B}^*$ are two arrangements in $\mathbb{CP}^2$ such that each of their lines contains at least three intersection points and if $M(\mathcal{A}^*)$ and $M(\mathcal{B}^*)$ are homeomorphic, then for $K(\tilde{\mathcal{A}}^*)$ and $K(\tilde{\mathcal{B}}^*)$, the boundaries of arbitrary regular neighborhoods $U(\tilde{\mathcal{A}}^*)$ and $U(\tilde{\mathcal{B}}^*)$ of $N(\tilde{\mathcal{A}}^*)$ and $N(\tilde{\mathcal{B}}^*)$, respectively, there is an isomorphism ϕ from $\pi_1\big(K(\tilde{\mathcal{A}}^*)\big)$ onto $\pi_1\big(K(\tilde{\mathcal{B}}^*)\big)$ and a homeomorphism $f : K(\tilde{\mathcal{A}}^*) \to K(\tilde{\mathcal{B}}^*)$ that induces ϕ.*

Now we need to review some results of Waldhausen [21] before we can prove our theorem.

DEFINITION 4.2. Let M be a reduced graph manifold. We say that M has the *Waldhausen property* if none of the following three cases occurs.

(E1) $M - \mathrm{Int}\big(U(T)\big)$ consists of the bundle over the two-sphere with three-punctures and three solid tori,

(E2) $M - \mathrm{Int}\big(U(T)\big)$ consists of the bundle over the Möbius band and one solid torus.

(E3) $T \neq \phi$ and $M - \mathrm{Int}\big(U(T)\big)$ is torus $\times$ interval.

LEMMA 4.8. *If $\mathcal{A}^*$ is an arrangement with at least three intersection points on each of its lines, then $K(\tilde{\mathcal{A}}^*)$ is a reduced graph manifold with the Waldhausen property.*

PROOF. This follows from Lemma 4.6 and Definition 4.2. □

In Section 9 of [21], for a reduced graph manifold M that satisfies the Waldhausen property, a weighted graph $G(M)$ was introduced. It can be described axiomatically as follow.

(G1) $G(M)$ has only finitely many weighted vertices $\mu_1, \mu_2, \ldots$ and finitely many directed edges $\tau_1, \tau_2, \ldots$. For each edge, each of its end point is incident with one vertex. $G(M)$ is connected.

(G2) Each vertex μ_j is assigned a triple of integers (g_j, r_j, s_j). Here r_j is nonnegative. When $r_j = 0$, s_j is arbitrary. When if $r_j > 0$, s_j is replaced by a dash (or is omitted). g_j is arbitrary. (As for the graph manifolds discussed in our paper, each vertex μ_j corresponds to a component M_j of $M - \mathrm{Int}(U(T))$ with weight given by (g_j, r_j, s_j). Here g_j is the genus of the base of M_j, r_j is the number of boundary surfaces that are not connected to any component of T, and s_j is the cross-section obstruction when r_j is zero).

(G3) If a vertex of degree one is assigned the triple $(0, 0, s_j)$, it is replaced by a dash.

(G4) (a) If $G(M)$ has only two vertices, both two vertices are not weighted by a dash.

(b) A vertex of degree zero is not assigned the triple $(0, 0, s_j)$ or $(-1, 0, s_j)$.

(c) A vertex of degree two is not assigned the triple $(0, 0, s_j)$.

(d) A vertex of degree one is not assigned the triple $(0, 1, -)$.

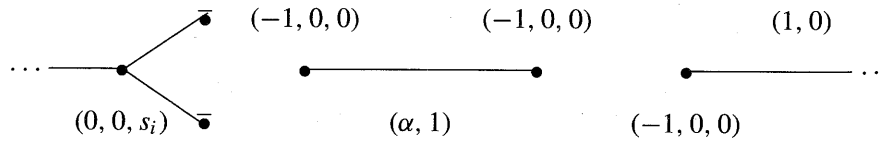
(e) If $G(M)$ has three vertices that are weighted by dashes, and only three edges, then the fourth vertex is not assigned $(0, 0, s)$.

(f) If $G(M)$ has one vertex that is weighted by a dash, and only one edge, then the second vertex cannot be weighted $(-1, 0, s_j)$.

(G5) If edge τ_i is incident with a vertex weighted by a dash, then τ_i is directed to this vertex and is weighted by a pair of integers (α_i, β_i), where α_i, β_i are co-prime and $1 \leq \beta_i < \alpha_i$.

(G6) If the vertices that are incident with τ_i are not weighted by dashes, then the edge τ_i is weighted with a pair of integers (α_i, β_i), where α_i and β_i are co-prime and $0 \leq \beta_i < \alpha_i$.

(G7) The following graphs or subgraphs are not considered.



(G8) Let G^* (possibly disconnected) be the subgraph of $G(M)$ where all the vertices with $g_j < 0$ and adjacent edges are removed. The homology group $H_1(G^*)$ is weighted by a homomorphism to $\mathbb{Z}_2$.

Suppose that the arrangement $\mathcal{A}^*$ in $\mathbb{CP}^2$ is chosen such that each line of $\mathcal{A}^*$ contains at least three intersection points. Recall that for each ℓ_i of $\tilde{\mathcal{A}}^*$ there is a corresponding vertex μ_i with weight $(0, 0, s_i)$, where $s_i = \ell_i \cdot \ell_i$, the self-intersection number of ℓ_i in $\widetilde{\mathbb{CP}^2}$. In $G(\tilde{\mathcal{A}}^*)$ each vertex μ_i has degree at least three. There is no dash for any vertex in $G(\tilde{\mathcal{A}}^*)$. So the definition of $G(\tilde{\mathcal{A}}^*)$ has no conflict with (G3), (G4), and (G7). For (G5) and (G6), first we describe the direction of weight of τ_i.

Let $U(T_i)$ be a component of $U(T)$ containing T_i. The boundary surfaces are T_i^- and T_i^+ with their orientation induced by inclusion $T_i^- \hookrightarrow M - \mathrm{Int}(U(T))$ and $T_i^+ \hookrightarrow M - \mathrm{Int}(U(T))$. The direction and weight of edge τ_i are decided by the gluing homeomorphism from T_i^- to T_i^+. Let the direction of τ_i be from T_i^- to T_i^+. From the orientation of T_i^- and T_i^+ we choose bases $\{a_1, b_1\}$ and $\{a_2, b_2\}$ (b_j is represented by a fiber of M_j, $j = 1, 2$, $T_i^- \in M_1$, $T_i^+ \in M_2$) for $H_1(T_i^-)$ and $H_1(T_i^+)$. The gluing homeomorphism induces an isomorphism from $H_1(T_i^-)$ to $H_1(T_i^+)$, which is expressed by

$$\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \epsilon \begin{pmatrix} \gamma_i & \delta_i \\ \alpha_i & \beta_i \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix},$$

where $\epsilon_i = 1$ or -1, $\det \begin{pmatrix} \gamma_i & \delta_i \\ \alpha_i & \beta_i \end{pmatrix} = -1$, $0 \le \beta_i < \alpha_i$, $(\alpha_i, \beta_i) = 1$.
Then τ_i is weighted by (α_i, β_i).
In our situation, we have simply $\alpha_i = \delta_i = 1$, $\beta_i = \gamma_i = 0$.

For (G8) the definition of assigning $H_1(G^*)$ a weight that is a homomorphism to $\mathbb{Z}_2$ is as follows. Let M^* be a (possibly disconnected) submanifold of M obtained by taking away all those S^1-bundle over nonorientable surfaces together with those $U(T_i)$ which pasted along on them from $M = U(T) \cup M_1 \cup \ldots \cup M_m$. $H_1(M^*) \to H_1(G^*)$ is surjective. The kernel is generated by $H_1(M_j)$ for $M_j \subseteq M^*$. For a closed path ℓ in M^*, we define $\rho'(\ell) = \sum_i \sigma(\ell, T_i)\epsilon_i'$ where $\sigma(\ell, T_i)$ is the intersection number of ℓ and T_i module 2, $\epsilon_i' \in \mathbb{Z}_2$,

$$\epsilon_i' = \begin{cases} 1 & \text{if } \epsilon_i = -1 \\ 0 & \text{if } \epsilon_i = \ \ 1. \end{cases}$$

Then $\rho : H_1(G^*) \to \mathbb{Z}_2$ is the desired homomorphism such that the following diagram commutes.

$$\begin{array}{ccc} H_1(M^*) & \longrightarrow & H_1(G^*) \\ {\scriptstyle \rho'}\searrow & & \swarrow{\scriptstyle \rho} \\ & \mathbb{Z}_2 & \end{array}$$

DEFINITION 4.3. Let G_1 and G_2 be two graphs with properties (G1) to (G8). G_1 and G_2 are said to be equivalent if there is $1-1$ incidence preserving map $\phi : G_1 \to G_2$ with the following properties.

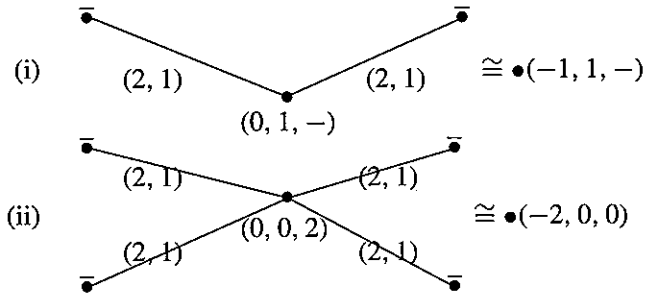
(a) ϕ carries one vertex to a vertex with same weight

(b) ϕ carries one weighted edge with (α, β) to a edge with weight (α, β^{ω}), where $\omega = +$ or -1 according to the edge orientation being preserved or not. β^{-1} is the standard representative of the coset (mod α) of the inverse coset of β. We use the convention that $0^{-1} = 0$.

(c) The induced homeomorphism $\phi : G_1^* \to G_2^*$ by ϕ induces a commutative diagram.

$$\begin{array}{ccc} H_1(G_1^*) & \longrightarrow & H_1(G_2^*) \\ & \searrow \quad \swarrow & \\ & Z_2 & \end{array}$$

(d) Each of the following pair of graphs is equivalent.



REMARK 4.2. Condition (d) above does not occur in our graphs $G(\tilde{\mathcal{A}}^*)$ or $G(\tilde{\mathcal{B}}^*)$, since they are not weighted by dashes.

We are ready to state the main theorem of [21] (cf. (9.4) of [21]), which is essential to the proof of our theorem.

THEOREM 4.9. *An oriented reduced graph manifold that satisfies the Waldhausen property (Definition* 4.2*) determines and is determined by its weighted graph with the properties* (G1)-(G8).

Thus, two oriented reduced graph manifolds are homeomorphic if and only if the corresponding graphs are equivalent.

COROLLARY 4.10. *Let $\mathcal{A}^*$ and $\mathcal{B}^*$ be two arrangements in $\mathbb{CP}^2$ such that each of their lines contains at least three intersection points. If $M(\mathcal{A}^*)$ is homeomorphic to $M(\mathcal{B}^*)$, then $G(\tilde{\mathcal{A}}^*)$ is equivalent to $G(\tilde{\mathcal{B}}^*)$.*

PROOF. This follows from Proposition 4.7, Lemma 4.8, and Theorem 4.9. □

5. – Proof of the main theorem for non-exceptional arrangements in $\mathbb{CP}^2$

In this section we shall prove a weak form (Theorem 5.4) of our main theorem.

THEOREM 5.1. *Let $\mathcal{A}^*$ and $\mathcal{B}^*$ be two arrangements in $\mathbb{CP}^2$. By blowing up their multiple points (multiplicity ≥ 3), we obtain two associated arrangements $\mathcal{A}^*$*

and $\mathcal{B}^*$ in some blownup surfaces $\widetilde{\mathbb{CP}^2}$. Let $L(\mathcal{A}^*)$ and $L(\tilde{\mathcal{A}}^*)$ be the set of all intersections of elements of $\mathcal{A}^*$ and $\tilde{\mathcal{A}}^*$, respectively, which are partially ordered by $X \leq Y \Leftrightarrow Y \subseteq X$. Then $L(\mathcal{A}^*) \cong L(\mathcal{B}^*)$ if and only if $L(\tilde{\mathcal{A}}^*) \cong L(\tilde{\mathcal{B}}^*)$, which preserves weights (self-intersection numbers) of lines in $\tilde{\mathcal{A}}^*$ and $\tilde{\mathcal{B}}^*$.

PROOF. Obviously $L(\mathcal{A}^*) \cong L(\mathcal{B}^*) \Rightarrow L(\tilde{\mathcal{A}}^*) \cong L(\tilde{\mathcal{B}}^*)$.

To prove the converse, for the sake of convenience but without loss of generality, we simply assume that $\tilde{\mathcal{A}}^* = \tilde{\mathcal{B}}^*$. Let $\mathcal{A}_1 = \mathcal{A}^*$, $\mathcal{A}_2 = \mathcal{B}^*$ and $\phi_i : \tilde{\mathcal{A}}^* \longrightarrow \mathcal{A}_i$ be a blowing up map $(i = 1, 2)$. An element of $\tilde{\mathcal{A}}^*$ that corresponds to an element of $\mathcal{A}_i$ by ϕ_i is called *regular with respect to* ϕ_i. The set of all regular elements of $\tilde{\mathcal{A}}^*$ with respect to ϕ_i is denoted by R_i. The remaining elements of $\tilde{\mathcal{A}}^*$ that are blown down to points by ϕ_i are called *exceptional with respect to* ϕ_i. The set that consists of all such elements is denoted by E_i.

For fixed i $(i = 1$ or $2)$, we list the following three basic properties for elements in $\tilde{\mathcal{A}}^*$.

(P1) For each $e \in E_i$, it has the self intersection number $e^2 = -1$; for $r \in R_i$, $r^2 = 1 - |\{e \in E_i : e \cap r \neq \phi\}|$. Here $|S|$ denotes the cardinal number of the set S.

(P2) Either two elements in R_i intersect with exactly one exceptional element in E_i without intersecting to each other themselves, or they intersect exactly at one double point.

(P3) For each $e \in E_i$, $\big(\text{star}(e) - \{e\}\big) \subseteq R_i$ and $|\text{star}(e) - \{e\}| \geq 3$, where $\text{star}(e)$ is the set consists of all elements in $\tilde{\mathcal{A}}^*$ that intersect with e.

LEMMA 5.2. *For an $\ell \in \tilde{\mathcal{A}}^*$, if $\ell^2 \neq -1$, then $\ell \in R_1 \cap R_2$. If $\ell^2 = -1$ and at most one element in* $\text{star}(\ell) - \{\ell\}$ *has the same self intersection, then $\ell \in E_1 \cap E_2$ and* $\text{Star}(\ell) - \{\ell\} \subseteq R_1 \cap R_2$.

PROOF. By (P1), $\ell \notin E_1 \cup E_2$ if $\ell^2 \neq -1$. Hence $\ell \in \tilde{\mathcal{A}}^* - (E_1 \cup E_2) = R_1 \cap R_2$. If $\ell^2 = -1$, then $\ell \in E_1 \cap E_2$ by (P1) and $\big(\text{star}(\ell) - \{\ell\}\big) \subseteq R_1 \cap R_2$ by (P3). □

Clearly we have $\tilde{\mathcal{A}}^* = (R_1 \nabla R_2) \cup (R_1 \cap R_2) \cup (E_1 \cap E_2)$ as the union of three disjoint sets. (Recall $R_1 \nabla R_2 = (R_1 - R_2) \cup (R_2 - R_1)$). We also have $R_1 - R_2 = E_2 - E_1$, $R_2 - R_1 = E_1 - E_2$, $R_1 \nabla R_2 = E_1 \nabla E_2$, etc.

If $R_1 \nabla R_2 = \phi$ (i.e., $E_1 \nabla E_2 = \phi$), then $E_1 = E_2$, $\phi_1 = \phi_2$ and $\mathcal{A}_1 = A_2$. So in this case, we have nothing to prove. If $R_1 \nabla R_2 \neq \phi$, then we wish to find out what it looks like.

Define a set $K(\ell) = \big((\text{Star}(\ell) - \{\ell\}\big) \cap (R_1 \nabla R_2)$ for each $\ell \in \tilde{\mathcal{A}}^*$

LEMMA 5.3. (i) *If $\ell \in E_1 \cap E_2$, $K(\ell) = \phi$.*

(ii) *If $\ell \in R_1 \cap R_2$, then $|K(\ell) \cap E_1| = |K(\ell) \cap E_2|$.*

(iii) *If $\ell \in R_1 \nabla R_2$, then $|K(\ell)| = 2$. $K(\ell) \subseteq R_1 - R_2$ if $\ell \in R_2 - R_1$ and $K(\ell) \subseteq R_2 - R_1$ if $\ell \in R_1 - R_2$,*

(iv) *$K(\ell)$ is discrete for each $\ell \in \tilde{\mathcal{A}}^*$ in the sense that any two elements of $K(\ell)$ do not intersect to each other in $\tilde{\mathcal{A}}^*$.*

PROOF. (i) It is clear because $E_1 \cap E_2$ is disjoint from $R_1 \nabla R_2$ (P3).
(ii) In view of (P1), we have

$$\begin{aligned} 1-\ell^2 &= \big|\{e \in E_1 : e \cap \ell \neq \phi\}\big| \\ &= \big|\big(\mathrm{star}(\ell) - \{\ell\}\big) \cap E_1\big| \\ &= \big|\big[\big(\mathrm{star}(\ell) - \{\ell\}\big) \cap E_1 \cap E_2\big] \cup \big[\big(\mathrm{star}(\ell) - \{\ell\}\big) \cap (E_1 - E_2)\big]\big| \\ &= \big|\big[\big(\mathrm{star}(\ell) - \{\ell\}\big) \cap E_1 \cap E_2\big] \cup \big(K(\ell) \cap E_1\big)\big| \\ &= \big|\big(\mathrm{star}(\ell) - \{\ell\}\big) \cap E_1 \cap E_2\big| + \big|K(\ell) \cap E_1\big|. \end{aligned}$$

Similarly

$$1-\ell^2 = \big|\big(\mathrm{star}(\ell) - \{\ell\}\big) \cap E_1 \cap E_2\big| + \big|K(\ell) \cap E_2\big|.$$

Thus we have $\big|K(\ell) \cap E_1\big| = \big|K(\ell) \cap E_2\big|$
(iii) When $\ell \in R_1 \nabla R_2$, we have $\ell^2 = -1$ by Lemma 5.2. If $\ell \in E_1 - E_2 = R_2 - R_1$, then we have $-1 = \ell^2 = 1 - \big|\{e \in E_2 : e \cap \ell \neq \phi\}\big|$ by (P1); that is, $\big|\{e \in E_2 : e \cap \ell \neq \phi\}\big| = 2$. On the other hand, by (P3) we know that ℓ does not intersect with any element in E_1. So

$$\big|K(\ell)\big| = \big|\big(\mathrm{Star}(\ell) - \{\ell\}\big) \cap (E_1 - E_2)\big| + \big|\big(\mathrm{Star}(\ell) - \{\ell\}\big) \cap (E_2 - E_1)\big| = 0 + 2 = 2$$

and $K(\ell) \subseteq E_2 - E_1 = R_1 - R_2$. Similarly, one can show that if $\ell \in E_2 - E_1$, then $\big|K(\ell)\big| = 2$ and $K(\ell) \subseteq R_2 - R_1$
(iv) Since E_i is discrete by (P3), $K(\ell) = \big(K(\ell) \cap E_1\big) \cup \big(K(\ell) \cap E_2\big)$ and $K(\ell) \cap E_i = \phi$ for $\ell \in E_i$ by (P3), we need only to show that if $\ell \in R_1 \cap R_2$ and $\ell_i \in K(\ell) \cap E_i$ (for $i = 1, 2$), then $\ell_1 \cap \ell_2 = \phi$. However, since $\ell_1, \ell \in R_2$ intersect and ℓ intersects with $\ell_2 \in E_2$, we have $\ell_1 \cap \ell_2 = \phi$ in view of (P2). □

Now we shall continue the proof of Theorem 5.1. In view of (iii) of Lemma 5.3, we have $|R_1 \nabla R_2| = 2|R_2 - R_1| = 2|R_1 - R_2| \geq 4$, and $R_1 \nabla R_2$ must consist of some cycles. In fact, the elements of $R_1 - R_2$ and $R_2 - R_1$ form the edges of each cycle alternately. But by (P2), any two elements in $R_1 - R_2$ must either intersect at exactly one double point or intersect with exactly one element in $R_2 - R_1$ without intersecting each other themselves. Thus $R_1 \nabla R_2$ must consist of only one hexagon (fig. 1). Let us label the edges of this hexagon by $a_1 \leftrightarrow e_2 \leftrightarrow a_3 \leftrightarrow e_1 \leftrightarrow a_2 \leftrightarrow e_3 (\leftrightarrow a_1)$, where $a_j \in R_1 - R_2$, $e_j \in E_1 - E_2 (j = 1, 2, 3)$.

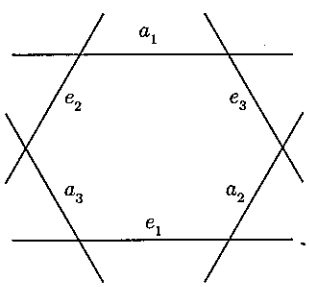


Fig. 1. Hexagon

For $\ell \in R_1 \cap R_2$, consider ℓ as a line in $\mathcal{A}_1$. If ℓ passes through the intersection points of $a_1 \cap a_2$, $(a_2 \cap a_3$ or $a_1 \cap a_3)$ in $\mathcal{A}_1 \subseteq \mathbb{CP}^2$, then clearly ℓ intersects with e_3, $(e_1$ or $e_2)$ in $\tilde{\mathcal{A}}^* \subseteq \widetilde{\mathbb{CP}^2}$. On the other hand, if ℓ does not pass through the intersection points of $a_1 \cap a_2$, $a_2 \cap a_3$ or $a_1 \cap a_3$ in $\mathcal{A}_1 \subseteq \mathbb{CP}^2$, then clearly ℓ must intersect with a_1, a_2, and a_3 in $\mathcal{A}_1 \subseteq \mathbb{CP}^2$. It follows that, if we consider ℓ as a line in $\mathcal{A}_2$, ℓ passes through $e_1 \cap e_2$, $e_2 \cap e_3$, and $e_3 \cap e_1$ in $\mathcal{A}_2 \subseteq \mathbb{CP}^2$, which is impossible because these three points are not colinear. So only the first situation is allowed. By (ii) and (iv) of Lemma 5.3, we conclude that $|K(\ell)| = 2$. In fact, ℓ must intersect with $R_1 \nabla R_2$ on exactly two opposite edges namely, $|\ell \cap a_1| = 1 = |\ell \cap e_1|$ or $|\ell \cap a_2| = 1 = |\ell \cap e_2|$ or $|\ell \cap a_3| = 1 = |\ell \cap e_3|$. Thus we can define

$$\phi : (E_1 \cap E_2) \cup (R_1 \cap R_2) \cup \{a_1, a_2, a_3\} \longmapsto (E_1 \cap E_2) \cup (R_1 \cap R_2) \cup \{e_1, e_2, e_3\}$$

by $\phi|_{E_1 \cap E_2} =$ Identity, $\phi|_{R_1 \cap R_2} =$ Identity, $\phi(a_j) = e_j$ for $j = 1, 2, 3$. It is clear that ϕ induces an isomorphism from $L(\mathcal{A}_1)$ to $L(\mathcal{A}_2)$. Thus the theorem is proved. A configuration for illustration of $\phi_2 \circ \phi_1^{-1} : L(\mathcal{A}_1) \to L(\mathcal{A}_2)$ can be shown as in figure 2. $R_1 \cap R_2 = \{\ell_1, \ell_2, \ldots, \ell_9\}$, $R_2 - R_1 = \{e_1, e_2, e_3\}$, $R_1 - R_2 = \{a_1, a_2, a_3\}$, $E_1 \cap E_2 =$ exceptional sets obtained by blowing up three points $\ell_1 \cap \ell_6 \cap \ell_9$, $\ell_2 \cap \ell_5 \cap \ell_9$, $\ell_4 \cap \ell_6 \cap \ell_8$. □

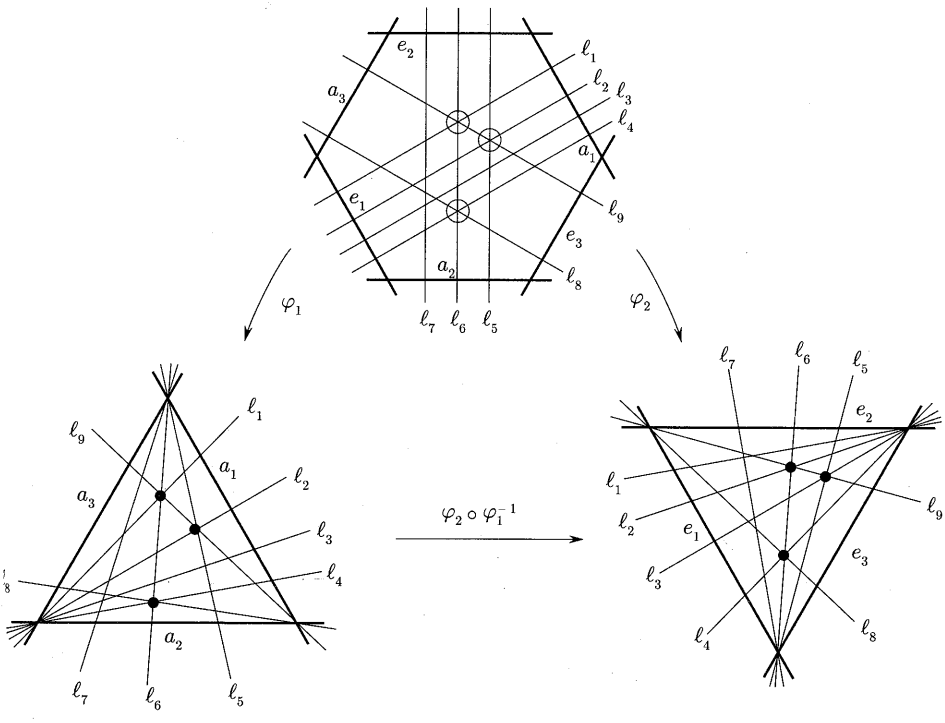


Fig. 2. Rational map

THEOREM 5.4. *Let $\mathcal{A}^*$ and $\mathcal{B}^*$ be two arrangements in $\mathbb{CP}^2$ such that each of their lines contains at least three intersection points. If $M(\mathcal{A}^*)$ is homeomorphic to $M(\mathcal{B}^*)$, then $L(\mathcal{A}^*)$ is isomorphic to $L(\mathcal{B}^*)$.*

PROOF. In view of Corollary 4.10 we know that $G(\tilde{\mathcal{A}}^*)$ is equivalent to $G(\tilde{\mathcal{B}}^*)$. It means that each ℓ of $\tilde{\mathcal{A}}^*$ with weight $(\ell \cdot \ell)$ (self-intersection number) is one-to-one correspondent to a line of $\tilde{\mathcal{B}}^*$ with the same weight and $L(\tilde{\mathcal{A}}^*)$ is equivalent to $L(\tilde{\mathcal{B}}^*)$. Thus Theorem 5.4 follows immediately from Theorem 5.1. □

6. – Exceptional arrangements in $\mathbb{CP}^2$

An arrangement in $\mathbb{CP}^2$ is called *exceptional* if one of its lines has at most two intersection points. We have shown in Section 4 that the topological type of the complement of a nonexceptional arrangement in $\mathbb{CP}^2$ determines the lattice of this arrangement. In this section we shall study the cohomology rings as well as the fundamental groups of complements of exceptional arrangements.

We shall first study the cohomology algebra of arrangement in $\mathbb{C}^3$. In general, let $\mathcal{A}$ be an arrangement in $\mathbb{C}^\ell$. Recall that the cohomology of its complement $M(\mathcal{A})$ as an algebra is isomorphic to Orlik-Solomon algebra over $\mathbb{C}$ which is defined as follows.

Let $\mathbb{E} = \mathbb{E}(\mathcal{A}) = \wedge(E_1)$ be the exterior algebra of $E_1 := \bigoplus_{H\in\mathcal{A}} \mathbb{C}e_H$. Write $uv = u \wedge v$ for $u, v \in E$. If $|\mathcal{A}| = n$, then $\mathbb{E}$, as a graded algebra, can be written as $\mathbb{E} = \bigoplus_{p=0}^{n} E_p$, where $E_0 = \mathbb{C}$, E_p is spanned by all $e_{H_1} \dots e_{H_p}$ with $H_k \in \mathcal{A}$. Define a map $\partial : \mathbb{E} \to \mathbb{E}$ by $\partial(1) = 0$, $\partial(e_H) = 1$ and $\partial(e_{H_1} \dots e_{H_p}) = \sum_{k=1}^{p} (-1)^{k-1} e_{H_1} \dots \hat{e}_{H_k} \dots e_{H_p}$ $(p \geq 2)$ for all $H_1, \dots, H_p \in \mathcal{A}$.

Consider a p-tuple of hyperplanes $S = (H_1, \dots, H_p)$. Write $|S| = p$, $e_S = e_{H_1} \dots e_{H_p} \in E$ and $\cap S = H_1 \cap \dots \cap H_p \in L(\mathcal{A})$ (Lattice of $\mathcal{A}$).

DEFINITION 5.1. Let $S = (H_1, \dots, H_p)$, $r(\cap S) =$ codimension of $\cap S$. S is said to be *independent* if $r(\cap S) = p$ and dependent if $r(\cap S) < p$. Let $\mathbb{S}_p$ be the set of all p-tuple and $\mathbb{S} = \bigcup_{p\geq 0} \mathbb{S}_p$. Let $I = I(\mathcal{A})$ be the ideal of $\mathbb{E}$ generated by ∂e_s for all dependent $s \in \mathbb{S}$. *Orlik-Solomon algebra* of $\mathcal{A}$ is defined as $\mathbb{A} = \mathbb{A}(\mathcal{A}) = \mathbb{E}/I$.

I is also a graded ideal. If we let $I_p = I \cap E_p$, then $I = \bigoplus_{p<0}^{n} I_p$ and $\partial I_p \subset I_{p-1}$ $(p \geq 1)$. Let $\phi : \mathbb{E} \to \mathbb{A}$ be the natural homomorphism. Let $A_p = \phi(E_p)$, $a_H = \phi(e_H)$ for $H \in \mathcal{A}$ and $a_s = \phi(e_s)$ for $s \in \mathbb{S}$.

Now let $n = 3$, $\mathcal{A}$ be an arrangement in $\mathbb{C}^3$ corresponding to an exceptional arrangement $\mathcal{A}^*$ in $\mathbb{CP}^2$. Write

$$\mathcal{A} = \{H_0, H_1, \dots, H_m, \ H_{m+1}, \dots, H_{m+n}\},$$
$$\mathcal{A}^* = \{\ell_0, \ell_1, \dots, \ell_m, \ell_{m+1}, \dots, \ell_{m+n}\}$$

where $\bigcap_{i=0}^{m} \ell_i \neq \phi$ and $\bigcap_{j=m+1}^{m+n} \ell_j \cap \ell_0 \neq \phi$. Thus any three hyperplanes

in $\{H_0, \ldots, H_m\}$ (or $\{H_0, H_{m+1}, \ldots, H_{m+n}\}$) are dependent. Such $\mathcal{A}$ is an exceptional arrangement in $\mathbb{C}^3$.

LEMMA 6.1. *Let $\mathcal{A}$ be an exceptional arrangement in $\mathbb{C}^3$ as above. Then the cohomology algebra of the complement of $\mathcal{A}$ is isomorphic to $\mathbb{A}(\mathcal{A}) = \bigoplus_{p=0}^{3} A_p$ where $A_0 = \mathbb{C}$, $A_1 = \bigoplus_{i=0}^{m+n} \mathbb{C}a_i$ (where $a_i = a_{H_2}$), $A_2 = \left(\bigoplus_{k=1}^{m+n} \mathbb{C}a_0a_k\right) \oplus \left(\bigoplus_{1\le i\le m<j\le m+n} \mathbb{C}a_ia_j\right)$ and $A_3 = \bigoplus_{1\le i\le m<j\le m+n} \mathbb{C}a_0a_ia_j$.*

PROOF. $A_0 = \mathbb{C}$ follows from $I_0 = 0$ and $E_0 = \mathbb{C}$.

Since any two distinct hyperplanes are independent, (H_i, H_i) are the only dependent elements in $\mathbb{S}_2$. But $\partial(e_ie_i) = 0$. So $I_1 = 0$ and $A_1 \cong E_1$, that is, $A_1 = \bigoplus_{i=0}^{m+n} \mathbb{C}a_i$.

For A_2, we know that it is spanned by a_ia_j for $0 \le i < j \le m+n$. Let $e_i = e_{H_i}$, $i = 0, \ldots, m+n$. For $1 \le i < j \le m$ or $m < i < j \le m+n$, we have $e_ie_j + e_je_0 + e_0e_i = \partial(e_ie_je_0) \in I$. So

$$a_ia_j = a_ia_0 + a_0a_j \quad \text{in} \quad \mathbb{A} \quad \text{for} \quad 1 \le i < j \le m \quad \text{or} \quad m < i < j \le m+n. \tag{6.1}$$

Thus A_2 is spanned by a_0a_ℓ, $0 < \ell \le m+n$ and a_ia_j, $1 \le i \le m < j \le m+n$. Next we need to show that if there are c_ℓ, $1 \le \ell \le m+n$ and c_{ij} $1 \le i \le m < j \le m+n$ in $\mathbb{C}$ such that

$$\sum_{\ell=1}^{m+n} c_\ell a_0a_\ell + \sum_{\substack{1\le i\le m\\ m<j\le m+n}} c_{ij}a_ia_j = 0, \tag{6.2}$$

then all c_ℓ, c_{ij} are zero. From equation 6.2 we have $\sum_{\ell=1}^{m+n} c_\ell e_0e_\ell + \sum_{\substack{1\le i\le m\\ m<j\le m+n}} c_{ij}e_ie_j \in I_2$. Remember that I_2 is spanned by $\{\partial(e_ie_je_k) : 0 \le i < j < k \le m \text{ or } m < i < j < k \le m+n+1\}$, where we set $e_{m+n+1} = e_0$. There are $c_{rst} \in \mathbb{C}$ such that

$$\sum_{\substack{0\le r<s<t\le m\\ \text{or } m\le r<s<t\le m+n+1}} c_{rst}\partial(e_re_se_t) = \sum_{\ell=1}^{m+n} c_\ell e_0e_\ell + \sum_{\substack{1\le i\le m\\ m<j\le m+n}} c_{ij}e_ie_j,$$

which is equivalent to

$$\sum_{\substack{0\le r<s<t\le m\\ \text{or } m\le r<s<t\le m+n+1}} c_{rst}(e_se_t - e_re_t + e_re_s) - \sum_{\ell=1}^{m+n} c_\ell e_0e_\ell = \sum_{\substack{1\le i\le m\\ m<j\le m+n}} c_{ij}e_ie_j.$$

Since $\{e_ie_j,\ 0 \le i < j \le m+n\}$ is a base of E_2, the above equation implies immediately $c_{ij} = 0$ for $1 \le i \le m < j \le m+n$. So

$$\sum_{\ell=1}^{m+n} c_\ell e_0e_\ell = \partial\left(\sum_{\substack{0\le r<s<t\le m\\ \text{or } m\le r<s<t\le m+n+1}} c_{rst}e_re_se_t\right).$$

As $\partial^2 = 0$, we have

$$0 = \partial\left(\sum_{\ell=1}^{m+n} c_\ell(e_0 e_\ell)\right) = \sum_{\ell=1}^{m+n} c_\ell(e_\ell - e_0).$$

It follows that $c_\ell = 0$.

We next prove that $A_3 = \bigoplus_{\substack{1\le i\le m\\ m+1\le j\le m+n}} \mathbb{C}a_0a_ia_j$. Since any four hyperplanes in $\mathbb{C}^3$ are linearly dependent, we have

$$e_je_ke_0 - e_ie_ke_0 + e_ie_je_0 - e_ie_je_k = \partial(e_ie_je_ke_0) \in I.$$

So in A, we have

$$a_ia_ja_k = a_oa_ja_k + a_ia_oa_k + a_ia_ja_o. \tag{6.3}$$

On the other hand,

$$e_i\partial(e_ie_je_k) = e_i(e_je_k - e_ie_k + e_ie_j) = e_ie_je_k.$$

If (H_i, H_j, H_k) is dependent, then $e_ie_je_k \in I_3$, which means $a_ia_ja_k = 0$. In view of this statement and equation 6.3, we see that A_3 is spanned by $\{a_oa_ia_j : 1 \le i \le m < j \le m+n\}$.

We have left to check that if there are $c_{ij} \in \mathbb{C}$ such that $\sum_{\substack{1\le i\le m\\ m<j\le m+n}} c_{ij}a_oa_ia_j = 0$, then $c_{ij} = 0$.

Let us denote $e_\alpha e_\beta e_\gamma e_\delta$ and $e_ie_je_k$ by $e_{\alpha\beta\gamma\delta}$ and e_{ijk}, respectively. By the definition of I, we have

$$I_3 = \langle \partial(e_{\alpha\beta\gamma\delta}),\ e_\ell\partial(e_{ijk}) : 0 \le \alpha < \beta < \gamma < \delta \le m+n,$$
$$0 \le i < j < k \le m \quad \text{or} \quad m < i < j < k \le m+n+1\rangle,$$

where we still use the convention $e_{m+n+1} = e_o$. The relation $\sum_{\substack{1\le i\le m\\ m<j\le m+n}} c_{ij}a_oa_ia_j = 0$ means $\sum_{\substack{1\le i\le m\\ m<j\le m+n}} c_{ij}e_{oij} \in I$, which implies there are $c_{\alpha\beta\gamma\delta}$ and $c'_{ijk\ell} \in \mathbb{C}$ such that

$$\sum_{\substack{1\le i\le m\\ m<j\le m+n}} c_{ij}e_{oij} = \sum_{0\le\gamma<\beta<\gamma<\delta\le m+n} c_{\alpha\beta\gamma\delta}\partial(e_{\alpha\beta\gamma\delta}) + \sum_{\substack{0\le\ell\le m+n\\ 0\le i<j<k\le m\\ \text{or } m<i<j<k\le m+n+1}} c'_{ijk\ell}e_\ell\partial(e_{ijk}).$$

Applying ∂ on both sides of the above equation, we have

$$\sum_{\substack{1\le i\le m\\ m<j\le m+n}} c_{ij}(e_{ij} + e_{jo} + e_{oi}) = \sum_{\substack{0\le i<j<k\le m\\ \text{or } m<i<j<k\le m+n+1}} c''_{ijk}\partial(e_{ijk}),$$

where $c''_{ijk} = \sum_{\ell=0}^{m+n} c'_{ijk\ell}$. The above equation can be written as

$$\sum_{\substack{1\le i\le m\\ m<j\le m+n}} c_{ij}e_{ij} = \sum_{\substack{1\le i\le m\\ m<j\le m+n}} c_{ij}(e_{io} + e_{oj}) + \sum_{\substack{0\le i<j<k\le m\\ \text{or } m<i<j<k\le m+n+1}} c''_{ijk}(e_{jk} + e_{ki} + e_{ij}).$$

Since the terms e_{ij} for $1 \le i \le m < j \le m+n$ do not appear in the right hand side of this equation, we conclude that $c_{ij} = 0$ for $1 \le i \le m < j \le m+n$. Thus $A_3 = \bigoplus_{\substack{1\le i\le m\\ m<j\le m+n}} \mathbb{C}a_oa_ia_j$. □

In fact, from the proof of Lemma 6.1, we see that for any given arrangement $\mathcal{A} = \{H_1, \ldots, H_k\}$ in $\mathbb{C}^3$, the ideal of exterior algebra $E(\mathcal{A})$ is $I = I_2 \bigoplus I_3$ where I_2 is spanned by the set of ∂e_{ijk} with dependent $(H_i, H_j, H_k) \in \mathbb{S}$ and I_3 is spanned by the set of all $\partial(e_{ijk\ell})$ and the set of all $e_\delta \partial(e_{\alpha\beta\gamma})$ with dependent $(H_\alpha, H_\beta, H_\gamma) \in \mathbb{S}$. Thus we have the following proposition.

PROPOSITION 6.2. *Let $\mathcal{A} = \{H_1, \ldots, H_k\}$ be a central arrangement in $\mathbb{C}^3$. Then the cohomology algebra of the complement of $\mathcal{A}$ in $\mathbb{C}^3$ is isomorphic to $\mathbb{A}(\mathcal{A}) = A_0 \bigoplus A_1 \bigoplus A_2 \bigoplus A_3$ where $A_0 = \mathbb{C}$, $A_1 = \bigoplus_{i=1}^k \mathbb{C}a_i$, A_2 is spanned by the set of all $a_i a_j (i < j)$ with relations $a_\alpha a_\beta + a_\beta a_\gamma + a_\gamma a_\alpha = 0$ for dependent $(H_\alpha, H_\beta, H_\gamma) \in \mathbb{S}$ and A_3 is spanned by $a_\alpha a_\beta a_\gamma (\alpha < \beta < \gamma)$ subject to the relations $a_\alpha a_\beta a_\gamma = a_\delta a_\beta a_\gamma + a_\alpha a_\delta a_\gamma + a_\alpha a_\beta a_\delta$ and the relations $a_x(a_u a_v + a_v a_w + a_w a_u) = 0$ with dependent $(H_u, H_v, H_w) \in \mathbb{S}$.*

LEMMA 6.3. *Let $\mathcal{A}$ be a central arrangement in $\mathbb{C}^3$. Let $\mathcal{A}^* = \{\ell_1, \ell_2, \ldots, \ell_n\}$ be the corresponding projective arrangement in $\mathbb{CP}^2$. Suppose that $n = |\mathcal{A}^*| \geq 3$. If $\mathcal{A}^*$ is not a pencil, then $b_3\big(M(\mathcal{A})\big)$, the third Betti number of $M(\mathcal{A})$, is nonzero.*

PROOF. We need only to show that $A_3(\mathcal{A})$ is nonzero, where $A_3(\mathcal{A})$ is the third graded piece of the Orlik-Solomon algebra. Choose three lines ℓ_1, ℓ_2, ℓ_3 in general position from $\mathcal{A}^*$. So ℓ_1, ℓ_2, ℓ_3 do not form a pencil. Then we claim that $a_{123} \neq 0$ in $A_3(\mathcal{A})$. If $a_{123} = 0$, that is, $e_{123} \in I_3$ where $I = \bigoplus_{i=0}^3 I_i$ is the ideal generated by ∂e_s for dependent s, then we would have some $c_{ijk\ell}$ and $c_{\alpha\beta\gamma\delta} \in \mathbb{C}$ such that

$$e_{123} = \sum_{1 \leq i < j < k < \ell \leq n} c_{ijk\ell} \partial(e_{ijk\ell}) + \sum_{\delta,\ \text{dependent } (\alpha,\beta,\gamma)} c_{\alpha\beta\gamma\delta} e_\delta \partial(e_{\alpha\beta\gamma}).$$

By acting ∂ on each side of the above equation, we get

$$e_{12} + e_{23} + e_{31} = \sum_{(\alpha,\beta,\gamma)\ \text{dependent}} c_{\alpha\beta\gamma} \partial(e_{\alpha\beta\gamma}),$$

where $c_{\alpha\beta\gamma} = \sum_\delta c_{\alpha\beta\gamma\delta}$. Observe that each dependent (α, β, γ) corresponds to a vertex $P_{(\alpha,\beta,\gamma)} = \ell_\alpha \cap \ell_\beta \cap \ell_\gamma$ in $\mathcal{A}^*$ and that if $P_{(\alpha,\beta,\gamma)} \neq P_{(\alpha',\beta',\gamma')}$, then $\{\pm e_{\alpha\beta}, \pm e_{\alpha\gamma}, \pm e_{\beta\gamma}\}$ is disjoint from $\{\pm e_{\alpha'\beta'}, \pm e_{\alpha'\gamma'}, \pm e_{\beta'\gamma'}\}$. Therefore we have

$$e_{12} = \sum_{\substack{\text{dependent } (\alpha,\beta,\gamma) \\ \ell_\alpha \cap \ell_\beta \cap \ell_\gamma = \ell_2 \cap \ell_1}} c_{\alpha\beta\gamma} \partial(e_{\alpha\beta\gamma}).$$

By taking ∂ on both sides of the above equation, we get $e_2 - e_1 = 0$, which is absurd. □

7. – End of the proof of the main theorem

Let $\mathcal{A}_1^*$ and $\mathcal{A}_2^*$ be two arrangements in $\mathbb{CP}^2$ and let at least one of them be exceptional. In this section we shall finish the proof of the main theorem. In view of Theorem 5.4, it remains to prove that under the above hypothesis, if $L(\mathcal{A}_1)$ is not isomorphic to $L(\mathcal{A})$, then $M(\mathcal{A}_1^*)$ is not homeomorphic to $M(\mathcal{A}_2^*)$.

CASE 7.1. Both $\mathcal{A}_1^*$ and $\mathcal{A}_2^*$ are exceptional. Write

$$\mathcal{A}_1^* = \{H_0, H_1, \ldots, H_p, H_{p+1}, \ldots, H_{p+q}\},$$
$$\mathcal{A}_2^* = \{G_0, G_1, \ldots, g_s, G_{s+1}, \ldots, G_{s+t}\},$$

where H_0 (respectively G_0) intersects with $H_1, \ldots, H_p$ (respectively, $G_1, \ldots, G_s$) at one point and intersects with $H_{p+1}, \ldots, H_{p+q}$ (respectively, $G_{s+1}, \ldots, G_{s+t}$) at another point.

If $M(\mathcal{A}_1^*)$ is homeomorphic to $M(\mathcal{A}_2^*)$, then $M(\mathcal{A}_1)$ is homeomorphic to $M(\mathcal{A}_2)$. By Lemma 6.1, we have $p+q = s+t$ and $pq = st$ which imply either $(p,q) = (s,t)$ or $(p,q) = (t,s)$. Thus $L(\mathcal{A}_1)$ is isomorphic to $L(\mathcal{A}_2)$.

CASE 7.2. $\mathcal{A}_1^*$ is exceptional but $\mathcal{A}_2^*$ is not. In this case we shall show that $M(\mathcal{A}_1^*)$ is not homeomorphic to $M(\mathcal{A}_2^*)$ by considering the following subcases.

Case 7.2.a $\mathcal{A}_1^*$ consists of at most three lines.

Case 7.2.b $\mathcal{A}_1^*$ is a pencil, and $|\mathcal{A}_1| \geq 4$ (fig. 3).

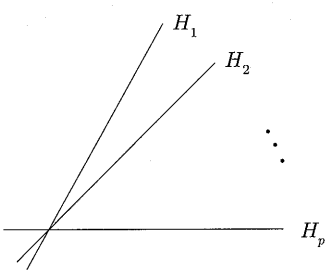


Fig. 3. Pencil

Case 7.2.c $\mathcal{A}_1^*$ consists of a pencil and a line in general position, and $|\mathcal{A}_1^*| \geq 4$ (fig. 4).

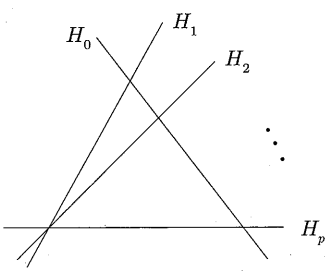


Fig. 4. Pencil and a line

Case 7.2.d $\mathcal{A}_1^* = \{H_0, H_1, \ldots, H_p, H_{p+1}, \ldots, H_{p+q}\}$ where $\bigcap_{i=0}^{p} H_i$ and $H_0 \cap \left(\bigcap_{i=p+1}^{p+q} H_i\right)$ are two different nonempty intersections, $p > 1$, $q > 1$ (fig. 5).

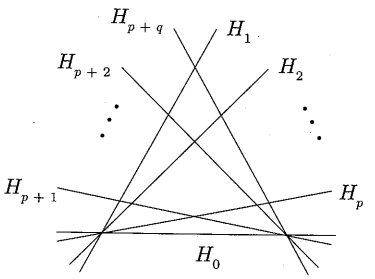


Fig. 5. Double pencils

From Proposition 6.2 we know that the first Betti number of $M(\mathcal{A})$ in $\mathbb{C}^3$ is precisely $|\mathcal{A}|$, the number of elements in $\mathcal{A}$. In case of (7.2.a), we have $|\mathcal{A}_1^*| \leq 3 < |\mathcal{A}_2^*|$. Thus $M(\mathcal{A}_1^*)$ is not homeomorphic to $M(\mathcal{A}_2^*)$.

In case (7.2.b), we have the third Betti number of $M(\mathcal{A}_1) = 0$ by Lemma 6.1 while the third Betti number of $M(\mathcal{A}_2)$ is nonzero by Lemma 6.3. Thus $M(\mathcal{A}_1^*)$ is not homeomorphic to $M(\mathcal{A}_2^*)$.

For Case (7.2.c), let $\mathcal{A}_1^* = \{H_0, H_1, \ldots, H_p\}$ $p \geq 3$, where $H_1, \ldots, H_p$ form a pencil and H_0 is in general position. Let $G(\mathcal{A}_1^*)$ be the weighted graph of $\mathcal{A}_1^*$, and $K(\mathcal{A}_1^*)$ be the manifold that is the boundary of a tubular neighborhood of $\mathcal{A}_1^*$ in $\mathbb{CP}^2$. $G(\mathcal{A}_1^*)$ is obtained by blowing up $\bigcap_{i=1}^{p} H_i$ as an exceptional line E. We denote the vertices by $G(\mathcal{A}_1^*)$ $v, v_0, v_1, \ldots, v_p$ corresponding to E, $H_0, H_1, \ldots, H_p$, respectively. The pictures of $\tilde{\mathcal{A}}_1^*$ and $G(\mathcal{A}_1^*)$ are shown as in figure 6 and the left-hand side of figure 7, where (q_i, r_i, s_i) is the weight of the vertex v_i which is defined in Section 4. By the Neuman's calculus of plumbing in Section 2 of [15], we can reduce $G(\mathcal{A}_1^*)$ to G' and then to a single vertex graph G'' with certain weight $(g, 0, 0)$ (fig. 8). Notice that if we apply R4 or R5 in the ith step, the weight of $V + V_0$ will change from $(g_i, 0, 0)$ to

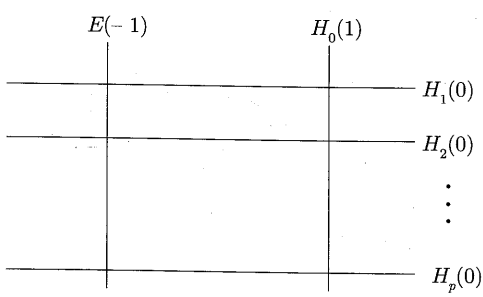


Fig. 6. Blown-up pencil

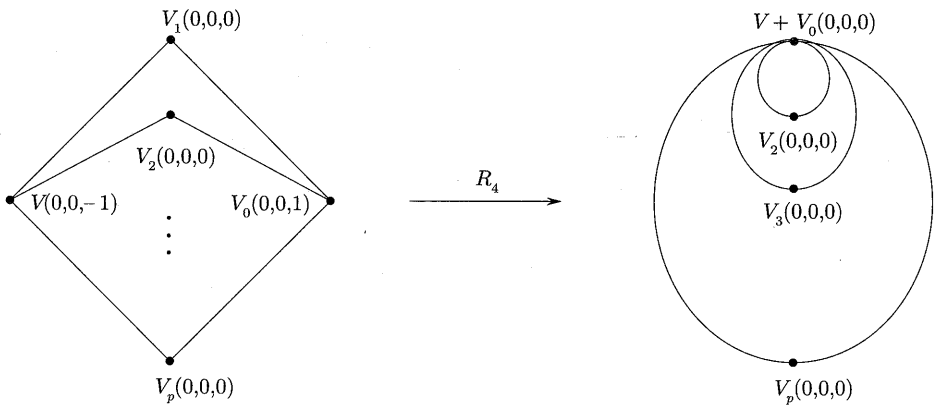


Fig. 7. $G(\mathcal{A}^*)$ to G'

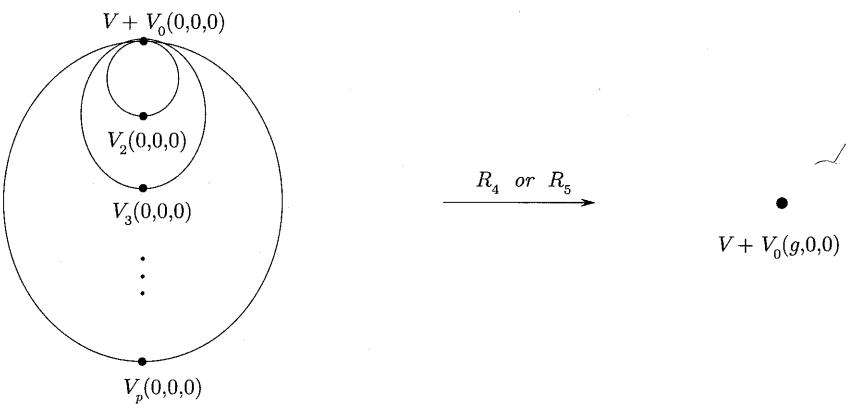


Fig. 8. G' to G''

$(g_i\#(-2), 0, 0)$ (for R4) or $(g_i\#1, 0, 0)$ (for R5), where

$$g_i\#(-2) = \begin{cases} -2 & \text{if } g_i = 0 \\ -2g_i - 2 & \text{if } g_i > 0 \\ g_i - 2 & \text{if } g_i < 0 \end{cases} \qquad g_i\#1 = \begin{cases} 1 & \text{if } g_i = 0 \\ g_i + 1 & \text{if } g_i > 0 \\ g_i - 2 & \text{if } g_i < 0. \end{cases}$$

(R4) (R5)

So if we use only step R5, then we have $g \geq 2$, since $p \geq 3$. On the other hand, R4 will bring any g_i down to a number ≤ -2. Once g_i is negative, both step R4 and R5 will bring g_i to a smaller number less than -2. Thus we conclude that $|g| \geq 2$. The plumbed three-manifold $M(V + V_0)$ corresponding to $V + V_0$ is diffeomorphic to $K(\mathcal{A}_1^*)$ by Proposition 2.1 of [15]. Observe that $M(V + V_0)$

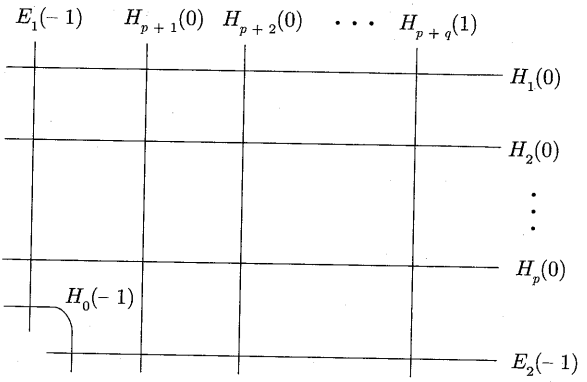


Fig. 9. Blown-up pencils

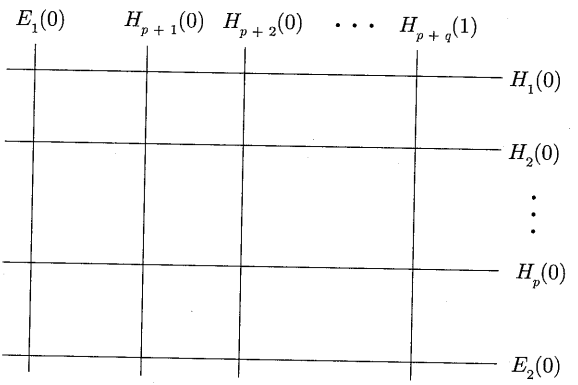


Fig. 10. Blow-down H_0

is a reduced graph manifold with reduced graph structure $T = \phi$ (one needs to check only the last condition of the definition of reduced graph manifold). and it has the Waldhausen property (cf. Definition 4.2). Now by Theorem 3.12 of Waldhausen (cf. (9.4) of [21]), $K(\mathcal{A}_1^*)$ homeomorphic to $K(\mathcal{A}_2^*)$ will imply that $G(\mathcal{A}_2^*)$ is equivalent to G''. Since $G(\mathcal{A}_2^*)$ is not equivalent to G'', we conclude that $M(\mathcal{A}_1^*)$ is not homeomorphic to $M(\mathcal{A}_2^*)$.

For the last case (7.2.d), we blow up the points $\bigcap_{i=1}^{p} H_i$ and $\bigcap_{j=p+1}^{p+q} H_j$ and get exceptional line E_1 and E_2, respectively (fig. 9). Number in the parentheses is the self-intersection number. We blow down H_0 to a point and get figure 10. The graph manifold is then reduced with respect to the graph G (fig. 11), since $p \geq 2$ and $q \geq 2$ in view of Lemma 4.6. Notice that G has weights $(0, 0, 0)$ for all its vertices. If $M(\mathcal{A}_1^*)$ were homeomorphic to $M(\mathcal{A}_2^*)$, this would imply $K(\mathcal{A}_1^*)$ homeomorphic to $K(\mathcal{A}_2^*)$ in view of Proposition 4.7, which in turn would

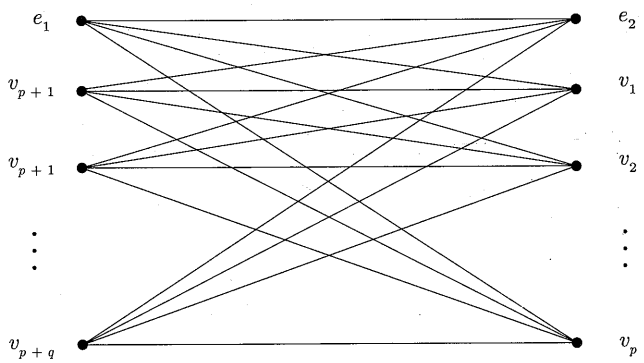


Fig. 11. Graph **G**

imply G equivalent to $G(\mathcal{A}_2^*)$. The last assertion is not possible because $G(\mathcal{A}_2^*)$ has nonzero weights. Therefore we conclude that $M(\mathcal{A}_1^*)$ is not homeomorphic to $M(\mathcal{A}_2^*)$. This finishes the proof of our main theorem.

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