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# DEFORMATIONS AND EQUITOPOLOGICAL DEFORMATIONS OF STRONGLY PSEUDOCONVEX MANIFOLDS

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## §1. Introduction

One of the main problems in complex analysis has been to determine when two open sets  $D_1$ ,  $D_2$  in  $C^n$  are biholomorphically equivalent. In [26] Poincaré studied perturbations of the unit ball  $B_2$  in  $C^2$  of a particular kind, and found necessary and sufficient conditions on a first order perturbation that the perturbed domain be biholomorphically equivalent to  $B_2$ . Recently Burns, Shnider and Wells [7] (cf. also Chern-Moser [9]) have studied the deformations of strongly pseudoconvex manifolds. They proved that there is no finite-dimensional deformation theory for M if one keeps track of the boundary.

In view of this, we have the following definition.

DEFINITION. Let M and M' be two strongly pseudoconvex manifolds with A and A' as its maximal compact analytic set. M is said to be holomorphically equivalent to M' if there exist open neighborhoods U and U' of A and A' respectively and biholomorphic map  $\varphi: U \to U'$  such that  $\varphi(A) = A'$ .

The natural question one can raise is to determine geometric conditions which imply that M and M' are holomorphically equivalent.

Let M be a strongly pseudoconvex manifold with a one-dimensional exceptional set A. Let  $\Theta$  be the holomorphic tangent sheaf to M. The general Kodaira-Spencer [18] theory shows that  $H^1(M, \Theta)$  corresponds to first order infinitesimal deformations of M and that  $H^2(M, \Theta)$  represents the obstructions to formally extending deformations to higher order.  $H^1(M, \Theta)$ is finite-dimensional since M is strongly pseudoconvex.  $H^2(M, \Theta) = 0$  since A is one-dimensional. Because of the result of Burns, Schneider and Wells we mentioned above, given a deformation of M and a compact K in M,

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we shall only worry about the deformation near K. Then in [23] Laufer proved that M has a versal deformation  $\omega: \mathfrak{M} \to Q$  with Q a manifold of dimension dim  $H^1(M, \Theta)$  in case either (i) M is of arbitrary dimension and a sufficiently small neighborhood of A [23, Theorem 2.2 and Theorem 2.5] or (ii) M is of dimension two [23, Theorem 2.8].

Let U be a compact complex submanifold of a complex manifold w. Recall that U is a stable submanifold of W in the sense of Kodaira (cf. [17]) if and only if for any complex fibre manifold  $p: \mathscr{W} \to B$  such that  $p^{-1}(o) = W$  for a point  $o \in B$ , there exist a neighborhood N of o in B and a fibre submanifold  $\mathscr{U}$  with compact fibres of the complex fibre manifold  $\mathscr{W}/N$  such that  $\mathscr{U} \cap W = U$ . This means that no small deformation of the complex structure of W makes U disappear. In [16], [17] stability of compact submanifolds of complex manifolds and other related topics are studied.

We now restrict exclusively to dimension two strongly pseudoconvex manifold M. We will assume that irreducible components  $A_i$ 's of the maximal compact analytic set A are nonsingular and have normal crossing M. Furthermore there is no  $\mathbb{CP}^1$  in M with self-intersection number -1such that blowing down this curve will not destroy the above properties of the maximal compact analytic set. In [23] (see also [15]), it was proved that there exists a semi-universal deformation of  $M = \tau^{-1}(o) \tau: \mathfrak{M}' \to Q'$ such that all irreducible components of A are stable under  $\tau$  (cf. § 2.2). Let  $\mathscr{S}$  be the sheaf of germs of vector fields which are tangential to A. Then dim  $H^1(M, \mathscr{S})$  is the number of moduli of equitopological deformation of M. (cf. § 2. 2). It is a natural question to ask for formulas for dim  $H^1(M, \Theta)$ and dim  $H^1(M, \Theta)$  and dim  $H^1(M, \mathscr{S})$ .

MAIN THEOREM. Let M be a 2-dimensional strongly pseudoconvex manifold as above (i.e., M is a minimal good resolution of normal surface singularity). Suppose the maximal compact analytic set A can be blown down to a hypersurface singularity  $o \in \{f = 0\} \subseteq C^3$ . Then

$$\dim H^{1}(M, \Theta) = -\frac{7K^{2} - 5\chi_{T}(A)}{6} + \tau - \frac{5}{6}(1 + \mu)$$
$$\dim H^{1}(M, \mathscr{S}) = -\frac{7K^{2} - 5\chi_{T}(A)}{6} + \tau - \frac{5}{6}(1 + \mu) - \sum \dim H^{1}(A_{i}, N_{A_{i}})$$

where  $K = canonical \ divisor \ of \ M$  with support on A

$$\chi_{T}(A) = topological \ Euler \ number \ of \ A$$
  
 $\mu = Milnor \ number \ of \ V \ at \ the \ origin$   
 $= \dim C\{x, y, z\} \Big/ \Big( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \Big)$   
 $\tau = number \ of \ moduli \ of \ V \ at \ o,$   
 $= \dim C\{x, y, z\} \Big/ \Big( f, \ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \Big)$   
 $N_{A_{i}} = normal \ bundle \ of \ A_{i} \ in \ M.$ 

The difficulty in obtaining formulae for dim  $H^1(M, \Theta)$  or dim  $H^1(M, \mathscr{S})$ is that they do not depend entirely on the topological data. Actually Example 3.9 below says that even though Milnor numbers (analytic invariants) for the associated singularities are the same, dim  $H^1(M, \Theta)$  can still be different. Our main observation is that the number of moduli corresponding to the associated singularities are different in this case. The crucial point in the proof of our main Theorem is to consider dim  $H^1(M, \Theta)$ and dim  $H^1(M, \mathscr{S})$  simultaneously. As a corollary to the proof of our Theorem, we have the following

COROLLARY. Let M be a two-dimensional manifold. Let  $A = \bigcup A_i$ ,  $i = 1, \dots, n$  be the maximal compact analytic set in M. Suppose the intersection matrix  $[A_i \cdot A_j]$  is negative definite,  $p(A) := 1 - \dim H^0(A, \mathcal{O}_A) + \dim H^1(A, \mathcal{O}_A) = 0$  and  $A \cdot A = -1$ . Then

$$\dim H^1(M, \Theta) = 2(n-1)$$
  
 $\dim H^1(M, \mathscr{S}) = 3n - 2 + \sum_{i=1}^n A_i \cdot A_i$ 

Deformations of resolutions of two-dimensional singularities have been studied by several authors including Artin [2], Brieskorn [6], Karras [14], [15], Laufer [20], [21], [23], Riemenschneider [27], [29], Schlessinger, and Wahl [34], [35]). A start on the higher-dimensional theory was made by Riemenschneider [29], Lieberman and Rossi [38]. Our result in this paper can be easily generalized to arbitrary dimension. However we can only get the formula for  $\sum_{i=1}^{n-1} (-1)^i \dim H^i(M, \Theta)$ . We finally remark that our result is true for complete intersection singularities and the proof is exactly the same.

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# §2. Deformations of strongly pseudoconvex manifolds and Milnor's fibration

(2.1) Let M be a strongly pseudoconvex two-dimensional manifold with maximal compact analytic set A. Let  $A = \bigcup A_i$ ,  $1 \le i \le n$  be the decomposition of A into irreducible components. We shall assume that  $A_i$ 's are nonsingular and have only normal crossings. We may further assume without loss of generality that A is connected. Let S be a reduced complex space with distinguished point o. A deformation of M is a flat holomorphic map  $\varphi \colon \mathfrak{M} \to S$  such that the fiber  $M_o = \varphi^{-1}(o)$  is isomorphic to M. We are in fact only interested in deformations near the maximal compact analytic set A of M. According to Riemenschneider [29] we may assume therefore that  $\varphi$  is 1-convex.

Let  $\Theta$  be the tangent sheaf of M. In [23], Laufer constructed a deformation  $\omega: \mathfrak{M} \to Q$  of M with smooth parameter space Q of dimension equal to dim  $H^1(M, \Theta)$ . He also proved that  $\omega: \mathfrak{M} \to Q$  is in fact a semi-universal deformation of M.

(2.2) Let  $\psi: \mathfrak{M} \to S$  be a 1-convex deformation of  $M = \psi^{-1}(o)$  with Sa reduced complex space.  $\psi$  is an *equitopological* deformation of M if all irreducible components  $A_i$ ,  $1 \leq i \leq n$  of the maximal compact analytic set A lift to  $\psi$ , that is, there are subspaces  $\mathscr{A}_i$ ,  $1 \leq i \leq n$ , of  $\mathfrak{M}$  such that  $\lambda_i$  $:= \psi/\mathscr{A}_i: \mathscr{A}_i \to S$  are flat deformations of  $A_i$  for all i. Associated to A is a weighted dual graph  $\Gamma$  (cf. [12], [19]) which completely determines the topology of A and the differentiable nature of the embedding of A into M. Observe that equitopological deformations of M do not change  $\Gamma$ .

(2.3) Let  $\mathcal{O}$  be the sheaf of germs of holomorphic functions on M. Considering the composition

$$\Theta \longrightarrow \bigoplus_i \Theta \otimes \mathscr{O}_{A_i} \longrightarrow \bigoplus_i \mathscr{O}_{A_i}(A_i) \longrightarrow 0$$

following Kodaira [17], Laufer [23] and Wahl [34], we have the following exact sequence

$$0 \longrightarrow \mathscr{S} \longrightarrow \Theta \xrightarrow{\gamma} \bigoplus_{i} \mathscr{O}_{A_{i}}(A_{i}) \longrightarrow 0$$

where  $\mathcal{S}$  is a locally free sheaf of rank two. The map

$$H^{1}(M, \Theta) \xrightarrow{\gamma_{*}} \bigoplus_{i} H^{1}(M, \mathcal{O}_{A_{i}}(A_{i}))$$

determines the obstruction to lift  $A_i$  infinitesimally, see [17]. Since  $H^{\circ}(M, \mathcal{O}_{A_i}(A_i)) = 0$  for all  $i, H^{\circ}(M, \mathcal{S}) = \text{Ker}(\gamma_*)$  and  $H^{\circ}(M, \mathcal{S})$  is the space of first-order deformations of M to which all  $A_i$  lift. Therefore, Image  $(\rho_o: T_oS \longrightarrow H^{\circ}(M, \Theta))$  is contained in  $H^{\circ}(M, \mathcal{S})$  for equitopological deformations of M.

In [20], Laufer has developed a deformation theory for infinitesimal neighborhoods of A in M. Let  $D = \sum n_i A_i$ ,  $n_i \ge 1$ , be a cycle on A. Let  $\mathcal{O}(-D)$  be the sheaf of germs of holomorphic functions on M which vanish at least to order  $n_i$  on  $A_i$ . By  $\mathcal{O}_D$  we denote the quotient sheaf  $\mathcal{O}/\mathcal{O}(-D)$ . Then the non-reduced complex space  $A(D): = (A, \mathcal{O}_D)$  is called an infinitesimal neighborhood of A of order  $n = (n_1, \dots, n_r)$ . Let now  $\psi: \mathfrak{M} \to B$ be an equitopological deformation over a smooth parameter space B. Then, by multiplying the equations for the deformations of the  $A_i$ ,  $\psi$  gives a deformation of any infinitesimal neighborhood A(D). Let  $\mathcal{O}_D$  be the sheaf of germs of vector fields on A(D) in the sense of Grauert [10]. Observe that  $\mathcal{O}_D = \mathscr{S}/\mathcal{O}(-D)\mathscr{S}$ . The following lemma can be found in [23], [15].

LEMMA 2.4. Let M be a two-dimensional strongly pseudoconvex manifold. Then the map  $\beta: H^1(M, \mathscr{S}) \to H^1(M, \Theta_D)$  is surjective for all positive cycles D on the exceptional set of M. For all sufficiently large cycles D,  $\beta$ is an isomorphism.

Proof.  $H^{1}(M, \mathcal{O}(-D)\mathcal{S}) \xrightarrow{\sigma} H^{1}(M, \mathcal{S}) \xrightarrow{\beta} H^{1}(M, \Theta_{D}) \longrightarrow 0$  is exact since  $H^{2}(M, \mathcal{O}(-D)\mathcal{S}) = 0$ . By [10]  $\sigma$  is the zero map for all sufficiently large D.

(2.5) Let A(D) be a sufficiently large infinitesimal neighborhood of A such that the analytic type of the embedding of A in M is determined by A(D) and such that  $H^1(M, \mathscr{S}) \cong H^1(M, \Theta_D)$ . Then Laufer [23] (see also Karras [15]) construct a 1-convex equitopological deformation  $\tau \colon \mathfrak{M}' \to Q'$  of  $M = \tau^{-1}(o)$ , with Q' a manifold, such that the Kodaira-Spencer map  $\rho_o$ :  $T_oQ' \to H^1(M, \mathscr{S})$  is an isomorphism. As an easy consequence of [23, Theorem 2.5], they proved the following theorem.

THEOREM 2.6. Let M be a strongly pseudoconvex two-manifold. Let  $\tau: \mathfrak{M}' \to Q'$  be the 1-convex equitopological deformation of M as in (2.5). Then  $\tau$  is semi-universal for equitopological deformations with reduced parameter spaces.

From the above discussion, it is clear that why it is interesting to get formulae for dim  $H^{1}(M, \mathscr{S})$  and dim  $H^{1}(M, \Theta)$ .

(2.7) Now we recall Milnor's results on the topology of hypersurface singularities (cf. [24], [25]).

Let  $f: U \subseteq C^{n+1} \to C$  be an analytic function on an open neighborhood U of o in  $C^{n+1}$ . We denote

$$egin{aligned} B_{\epsilon} &= \{oldsymbol{z} \in oldsymbol{C}^{n+1} \colon \|oldsymbol{z}\| \leq arepsilon \} \ S_{\epsilon} &= \partial B_{\epsilon} = \{oldsymbol{z} \in oldsymbol{C}^{n+1} \colon \|oldsymbol{z}\| = arepsilon \} \;. \end{aligned}$$

Then:

THEOREM 2.8. For  $\varepsilon$  small enough the mapping  $\varphi_{\varepsilon}$ :  $S_{\varepsilon} - \{f = 0\} \to S^1$  defined by  $\varphi_{\varepsilon} = f(z)/|f(z)|$  is a smooth fibration.

THEOREM 2.9. For  $\varepsilon > 0$  small enough and  $\varepsilon \gg \eta > 0$  the mapping  $\psi_{\varepsilon,\eta}$ : (Int  $B_{\varepsilon}$ )  $\cap f^{-1}(\partial D_{\eta}) \rightarrow S^{1}$  defined by  $\psi_{\varepsilon,\eta}(z) = f(z)/|f(z)|$ , where  $\partial D_{\eta} = \{z \in C : |z| = \eta\}$ , is a smooth fibration isomorphic to  $\varphi_{\varepsilon}$  by an isomorphism which preserves the argument. We call the fibrations of Theorem 2.8 and 2.9 the Milnor fibrations of f at o.

COROLLARY 2.10. The fibers of  $\varphi_{\epsilon}$  have the homotopy type of an *n*-dimensional finite CW-complex.

THEOREM 2.11. Let  $V_o$ : f = 0. For  $\varepsilon > 0$  small enough,  $S_{\epsilon}$  cuts the smooth part of the algebraic set  $V_o$  transversally. If o is an isolated critical point of f, then the pairs  $(S_{\epsilon}, S_{\epsilon} \cap V_o)$  for any  $\varepsilon$  small enough are diffeomorphic, and  $(B_{\epsilon}, B_{\epsilon} \cap V_o)$  is homeomorphic to  $(B_{\epsilon}, C(S_{\epsilon} \cap V_o))$ , where  $C(S_{\epsilon} \cap V_o)$  is the cone which is the union of real line segments joining o and points of  $S_{\epsilon} \cap V_o$ .

### §3. Formulae for dim $H^{1}(M, \Theta)$ and dim $H^{1}(M, \mathscr{S})$

Given a 2-dimensional strongly pseudoconvex manifold M it is easy to see that dim  $H^1(M, \Theta)$  depends not only on the  $C^{\infty}$  structure of M but also on the complex structure of M. In this section we shall prove that dim  $H^1(M, \Theta)$  depends only on the  $C^{\infty}$  structure of M and numerical invariants of the isolated singularity associated to M.

THEOREM 3.1. Let f(x, y, z) be holomorphic in N, a Stein neighborhood of (0, 0, 0) with f(0, 0, 0) = 0. Let  $V = N \cap f^{-1}(0)$  have the origin as its only singular point. Let  $\pi: M \to V$  be the minimal resolution of V such that the

irreducible components of  $A = \pi^{-1}(0, 0, 0) = \bigcup_{i=1}^{n} A_i$  are nonsingular with normal crossings. Let  $\Theta$  be the tangent sheaf of M. Let  $\mathscr{S}$  be the sheaf of germs of vector fields that are tangential along A. Then

(3.1) 
$$\dim H^{1}(M,\Theta) = -\frac{7K^{2} - 5\chi_{T}(A)}{6} + \tau - \frac{5}{6}(1+\mu)$$

(3.2) 
$$\dim H^{1}(M, \mathscr{S}) = -\frac{7K^{2} - 5\chi_{T}(A)}{6} + \tau - \frac{5}{6}(1 + \mu) - \sum_{i=1}^{n} \dim H^{1}(A_{i}, N_{A_{i}})$$

where K = canonical divisor of M with support on A  $\chi_r(A) = \text{topological Euler characteristic of } A$   $\mu = Milnor number of V at the origin$   $= \dim C\{x, y, z\} / \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$   $\tau = number of moduli of <math>V = \dim C\{x, y, z\} / \left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$  $N_{A_i} = normal bundle of A_i in M.$ 

*Proof.* Any holomorphic function which agrees with f to sufficiently high order defines a holomorphically equivalent singularity at (0, 0, 0), [3]. So we may take f to be a polynomial. Compactify  $C^3$  to  $P^3$ . Let  $\overline{V}_t$  be the closure in  $P^3$  of

$$V_t = \{(x, y, z) \in C^3 : f(x, y, z) = t\}$$
.

By adding a suitably general high order homogeneous term of degree e to the polynomial f, we may additionally assume that  $\overline{V}_o$  has  $(0, 0, 0) \in \mathbb{C}^3$  as its only singularity and that  $\overline{V}_t$  is non-singular for small  $t \neq 0$ . We may also assume that the highest order terms of f define, in homogeneous coordinates, a nonsingular hypersurface of order e in  $\mathbb{P}^2 = \mathbb{P}^3 - \mathbb{C}^3$ .  $\overline{V}_t$  is then necessarily irreducible for all small t. Without loss of generality, we take  $N = \mathbb{C}^3$ . Then  $V = V_o$ .

Let  $\mathscr{I}_t$  be the ideal sheaf of  $\overline{V}_t$  in  $P^3$ . We denote  $\Theta_t$  and  $\Theta_{P^3}$  the tangent sheaves of  $\overline{V}_t$  and  $P^3$  respectively. From the second fundamental exact sequence

$$(3.3) \qquad \qquad \mathscr{I}_{o}/\mathscr{I}_{o}^{2} \longrightarrow \mathscr{Q}_{P^{3}}^{1}/\overline{V}_{o} \longrightarrow \mathscr{Q}_{V_{o}}^{1} \longrightarrow 0 ,$$

we get

$$(3.4) 0 \longrightarrow \Theta_o \longrightarrow \Theta_{P3}/\overline{V}_o \longrightarrow N_{\overline{V}_o} \longrightarrow T^1_{V_o} \longrightarrow 0$$

where  $N_{\mathcal{V}_o}$  is the normal sheaf of  $\overline{V}_o$  in  $P^3$  and  $T^1_{\mathcal{V}_o}$  is the set of isomorphism classes of first order infinitesimal deformations of  $V_o$  at the origin (cf. [30]). It is well known that

$$T^1_{{}^{V}{}_o}\cong {}^{\mathcal{O}}{}_{{\mathcal{C}}^3}\Big/\Big(f, rac{\partial f}{\partial x}, rac{\partial f}{\partial y}, rac{\partial f}{\partial z}\Big)\,.$$

It follows that

(3.5) 
$$\chi(\overline{V}_o, \Theta_o) = \chi(\overline{V}_o, \Theta_{P^3}/\overline{V}_o) - \chi(\overline{V}_o, N_{\overline{V}_o}) + \tau$$

where  $\chi(\chi, \mathscr{F}) = \sum_{i} (-1)^{i} \dim H^{i}(X, \mathscr{F})$  for any coherent sheaf  $\mathscr{F}$  over the compact complex space X.

From the exact sequence

$$0 \longrightarrow \mathcal{O}_{P^{\mathfrak{g}}}(-e) \longrightarrow \mathcal{O}_{P^{\mathfrak{g}}} \longrightarrow \mathcal{O}_{P^{\mathfrak{g}}}/\overline{V}_{o} \longrightarrow 0$$

we have

(3.6) 
$$\chi(\overline{V}_o, \Theta_{P3}/\overline{V}_o) = \chi(P^3, \Theta_{P3}) - \chi(P, \Theta_{P3}(-e)) \\ = \chi(\overline{V}_t, \Theta_{P3}/\overline{V}_t), \quad t \neq 0$$

Let  $\overline{V}$  be the hypersurface defined by

$$w^{e}f\left(\frac{x}{w},\frac{y}{w},\frac{z}{w}\right)-tw^{e}=0$$

in  $P^3 \times D_{\epsilon}$  where  $D_{\epsilon}$  is a disk of radius  $\varepsilon$  in C. Since the normal sheaf of  $\overline{V}$  in  $P^3 \times D_{\epsilon}$  is locally free,  $N_{\overline{v}}$  is torsion-free and in particular a  $\pi$ -flat coherent analytic sheaf where  $\pi: \overline{V} \to D_{\epsilon}$  is the natural projection. Therefore

(3.7) 
$$\chi(\overline{V}_o, N_{\overline{V}_o}) = \chi(\overline{V}_t, N_{\overline{V}_t}) \quad \text{for } t \neq 0.$$

On the other hand  $\overline{V}_t$  is smooth for  $t \neq 0$ , so the exact sequence

 $0 \longrightarrow \mathcal{O}_t \longrightarrow \mathcal{O}_{P^3/\overline{v}_t} \longrightarrow N_{\overline{v}_t} \longrightarrow 0$ 

gives

(3.8) 
$$\chi(\overline{V}_{t}, \Theta_{t}) = \chi(\overline{V}_{t}, \Theta_{P^{3}}/\overline{V}_{t}) - \chi(\overline{V}_{t}, N_{\overline{V}_{t}})$$

Putting (3.11)-(3.14) together, we have

(3.9) 
$$\chi(\overline{V}_o, \Theta_o) = \chi(\overline{V}_t, \Theta_t) + \tau, \quad t \neq 0.$$

Let  $\overline{M}$  be the resolution of  $\overline{V}_{\circ}$  which has M as an open subset. Let  $\mathscr{S}$  be the sheaf of germs of vector fields on  $\overline{M}$  which are tangential along each  $A_i \subseteq A$ . As in (2.3), we have

$$(3.10) 0 \longrightarrow \mathscr{S} \longrightarrow \varTheta \longrightarrow \bigoplus_{i=1}^n N_{A_i} \longrightarrow 0$$

where  $N_{A_i}$  is the normal bundle of  $A_i$  in M. If A is given by xy = 0, where (x, y) is a local coordinate system of M, then  $\mathscr{S}$  is generated by  $x(\partial/\partial x)$  any  $y(\partial/\partial y)$ ; if A is given by x = 0, then  $\mathscr{S}$  is generated by  $x(\partial/\partial x)$  and  $\partial/\partial y$ . From (3.10), since the normal bundle of  $A_i$  are negative,

$$(3.11) \Gamma(M,\mathscr{S}) \xrightarrow{} \Gamma(M,\Theta)$$

(3.12) 
$$\dim H^{1}(M, \Theta) - \dim H^{1}(M, \mathscr{S}) = \bigoplus_{i=1}^{n} \dim H^{1}(A_{i}, N_{A_{i}}).$$

In [11], Hironaka has proved that every normal singularity  $V_o$  admits an equivariant  $\pi: M \to V_o$ , i.e., one for which  $\pi_* \Theta_M \cong \Theta_{V_o}$ . For a proof in dimension two, see [8], 1.2; [34], 4.2; and also [31]. Moreover in this case a minimal good resolution in equivariant (cf. [34], 4.2). By Leray spectral sequence, we have

$$(3.13) \quad \dim H^{1}(M, \Theta) = \chi(\overline{V}, \Theta_{\overline{V}}) - \chi(\overline{M}, \Theta_{\overline{M}}) = \tau + \chi(\overline{V}_{\iota}, \Theta_{\overline{V}_{\iota}}) - \chi(\overline{M}, \Theta) \quad t \neq 0.$$

It is easy to see that

(3.14) 
$$\chi_T(\overline{M}) = \chi_T(\overline{V}_o) + \chi_T(A) - 1$$

where  $\chi_T(X)$  denote the topological Euler characteristic of X.

Recall from Theorem 2.9 that the intersection of  $V_t$  with the open  $\varepsilon$ -ball is diffeomorphic with the fiber  $F_o$ . So the manifold with boundary  $V_t \cap B_{\varepsilon}$ is connected, with 2<sup>nd</sup> Betti number equal to  $\mu$ , and with Euler number

$$\chi_T(V_t \cap B_{\epsilon}) = 1 + \mu$$
.

Since the two manifolds  $V_t \cap B_t$  and  $\overline{V}_t$  – Int  $B_t$  have union  $\overline{V}_t$  and intersection  $K_t$ , we have the Euler number of  $\overline{V}_t$ 

$$\chi_T(\overline{V}_t) = \chi_T(V_t \cap B_\epsilon) + \chi_T(\overline{V}_t - \operatorname{Int} B_\epsilon) - \chi(K_t)$$
  
= 1 +  $\mu$  +  $\chi_T(\overline{V}_o - \operatorname{Int} B_\epsilon) - \chi_T(V_o \cap S_\epsilon)$ 

by the differentiable triviality of the family  $\{V_i\}$  away from  $(0, 0, 0) \in \mathbb{C}^3$ . Hence STEPHEN S.-T. YAU

$$egin{aligned} \chi_{T}(\overline{V}_{t}) &= 1 + \mu + \chi_{T}(\pi^{-1}(\overline{V}_{0} - \operatorname{Int}B_{\epsilon})) + \chi_{T}(\pi^{-1}(V_{0} \cap B_{\epsilon})) \ &- \chi_{T}(V_{0} \cap S_{\epsilon}) - \chi_{T}(\pi^{-1}(V_{0} \cap B_{\epsilon})) \ &= 1 - \mu + \chi_{T}(\overline{M}) - \chi_{T}(A) \end{aligned}$$

since  $\pi^{-1}(V_0 \cap B_{\epsilon})$  contracts to A. Thus we have

(3.15) 
$$\chi_T(\overline{V}_t) - \chi_T(\overline{M}) = 1 + \mu - \chi_T(A), \quad t \neq 0.$$

Let

$$\omega = rac{dx \wedge dy}{\partial f / \partial z} = rac{dy \wedge dz}{\partial f / \partial x} = rac{dz \wedge dy}{\partial f / \partial y}$$

is a non-zero holomorphic 2-form on  $V_o - \{o\}$ .  $\pi^*(\omega)$  extends to a meromorphic 2-form on M with a pole set contained in A. K, the divisor of  $\omega$  and also called the canonical divisor, may be characterized topologically by the adjunction formula [32]

$$A_i \cdot K = -A_i \cdot A_i + 2g_i - 2 + 2\delta_i$$

where  $A_i$  is an irreducible component of A,  $g_i$  is the genus of  $A_i$ , and  $\delta_i$  is the "number" of nodes and cusps on  $A_i$ .

 $\omega$ , defined above, is a non-zero holomorphic 2-form on  $V_t$ ,  $t \neq 0$ , and on  $V_o - \{o\}$ . Let  $K_{\omega,t}$  be the part of the divisor of  $\omega$  on  $\overline{V}_t$  which is supported on  $\overline{V}_t - V_t$ , for t small.  $K_{\omega,t} \cdot K_{\omega,t}$  is independent of t since the family  $\{\overline{V}_t\}$  is differentiably trivial away from  $(0, 0, 0) \in C^3$ . Let  $K_{\omega} \cdot K_{\omega}$ denote this constant value for  $K_{\omega,t} \cdot K_{\omega,t}$ . Riemann-Roch Theorem [13] says

$$\chi(\overline{V}_{t},\Theta_{t}) - \chi(\overline{M},\Theta) = \frac{1}{6}(7K_{\infty}\cdot K_{\infty} - 5\chi_{T}(\overline{V}_{t})) - \frac{1}{6}(7K_{\infty}\cdot K_{\infty} + 7K\cdot K - 5\chi_{T}(\overline{M}))$$

$$(3.16) = -\frac{7K^{2}}{6} - \frac{5}{6}(\chi_{T}(\overline{V}_{t}) - \chi_{T}(\overline{M}))$$

$$= -\frac{7K^{2}}{6} - \frac{5}{6}(1 + \mu - \chi_{T}(A))$$

$$= -\frac{7K^{2} - 5\chi_{T}(A)}{6} - \frac{5}{6}(1 + \mu).$$

Put this into (3.13), and we have

dim 
$$H^{1}(M, \Theta) = - \frac{7K^{2} - 5\chi_{T}(A)}{6} + \tau - \frac{5}{6}(1 + \mu)$$
.

Hence from (3.12)

$$\dim H^{1}(M, \mathscr{S}) = -\frac{7K^{2} - 5\chi_{T}(A)}{6} + \tau - \frac{5}{6}(1 + \mu) - \sum_{i=1}^{n} \dim H^{1}(A_{i}, N_{A_{i}})$$
Q.E.D.

We recall a proposition by Laufer [23, Proposition 4.14].

LEMMA 3.2 (Laufer). Let M be a strongly pseudoconvex two-dimensional manifold with exceptional set A. Let  $\pi: M' \to M$  be a quadratic transformation of M at a singular point p of A. Then there is a canonical map  $\pi_1: H^1(M', \Theta') \to H^1(M, \Theta)$ .  $\pi_1$  is onto with kernel of dimension 2. The canonical map  $\pi_2: \Gamma(M', \mathscr{S}') \to \Gamma(M, \mathscr{S})$  is an isomorphism.

From Theorem 3.1 and Lemma 3.2, we have the following Theorem 3.3.

THEOREM 3.3. Let f(x, y, z) be holomorphic in N, a Stein neighborhood of (0, 0, 0) with f(0, 0, 0) = 0. Let  $V = N \cap f^{-1}(o)$  have the origin as its only point. Let  $\pi \colon M \to V$  be any resolution of V which is obtained as follows: let  $\pi_0 \colon M_0 \to V$  be the minimal resolutions of V, let  $\pi_1 \colon M_1 \to M_0$  be a quadratic transformation of M at a singular point  $p_1$  of the exceptional set in  $M_0, \cdots$ , let  $\pi_r \colon M_r \to M_{r-1}$  be a quadratic transformation of  $M_{r-1}$  at a singular point  $p_r$  of the exceptional set in  $M_{r-1}$ . Then  $M = M_r$  and  $\pi = \pi_0 \circ \pi_1 \circ \cdots \pi_r$ . Let  $\Theta$  be the tangent sheaf of M. Then

$$\dim H^{1}(M,\Theta) = -\frac{7K^{2}-5\chi_{T}(A)}{6} + \tau - \frac{5}{6}(1+\mu) .$$

If the irreducible components of  $A = \pi^{-1}(0, 0, 0) = \bigcup_{i=1}^{n} A_i$  are nonsingular with normal crossings, then

$$\dim H^{1}(M, \mathscr{S}) = -\frac{7K^{2} - 5\chi_{T}(A)}{6} + \tau - \frac{5}{6}(1 + \mu)$$
$$-\sum_{i=1}^{n} \dim H^{1}(A_{i}, N_{A_{i}})$$
$$= number of moduli of equitopological deforma-tion of M.$$

THEOREM 3.4. Let M be a two-dimensional manifold. Let  $A = \bigcup A_i$ ,  $i = 1, \dots, n$  be the maximal compact analytic set in M. Suppose the intersection matrix  $[A_i \cdot A_j]$  is negative definite,

$$p(A)$$
: = 1 - dim  $H^0(A, \mathcal{O}_A)$  + dim  $H^1(A, \mathcal{O}_A) = 0$ , and  $A \cdot A = -1$ .  
Then

$$\dim H^{\scriptscriptstyle 1}(M,\Theta)=2(n-1)$$

$$\dim H^{i}(M, \mathscr{S}) = 2(n-1) + \sum_{i=1}^{n} A_{i} \cdot A_{i} + n$$
$$= 3n - 2 + \sum_{i=1}^{n} A_{i} \cdot A_{i} .$$

**Proof.** By Artin's result, we know that A is an exceptional curve of the first kind. So  $\pi: M \to C^2$  is just a point modification of  $C^2$  at origin. It is easy to see that  $K^2 = -n$  and  $\chi_T(A) = n + 1$ . By the proof of Theorem 3.1, we get

(3.17) 
$$\dim H^{1}(M, \Theta) = -\frac{7K^{2} - 5\chi_{T}(A)}{6} - \frac{5}{6} - \dim \operatorname{coker} \rho$$

and

(3.18) 
$$\dim H^{1}(M, \mathscr{S}) = -\frac{7K^{2} - 5\chi_{T}(A)}{6} - \frac{5}{6} - \dim \operatorname{coker} \rho$$
$$-\sum_{i=1}^{n} \dim H^{1}(A_{i}, N_{A_{i}})$$

where  $\rho: \pi_* \Theta_M \to \Theta_{C^2}$  is the inclusion (cf. (1.2) of [8]) and  $N_{A_i}$  is the normal bundle of  $A_i$ . Easy computation shows that the image of  $\rho$  is those sheaf of germs of vector fields on  $C^2$  which vanish at origin. Hence coker  $\rho$  has dimension 2. From (3.17), we have

$$\dim H^{i}(M, \Theta) = -\frac{7(-n) - 5(n+1)}{6} - \frac{5}{6} - 2 = 2n - 2$$
$$\dim H^{i}(M, \mathscr{S}) = 2n - 2 - \sum_{i=1}^{n} H^{i}(A_{i}, N_{A_{i}})$$
$$= 2n - 2 - \left[ -\sum_{i=1}^{n} (A_{i} \cdot A_{i} + 1) \right]$$
$$= 3n - 2 + \sum_{i=1}^{n} A_{i} \cdot A_{i} .$$

The following corollary is first proved by Laufer [23, Corollary 2.6].

COROLLARY 3.5. Let M be a two-dimensional manifold. Let A be a submanifold of M which is a compact Riemann surface of genus 0 with  $A \cdot A = -1$ . Let  $\lambda: \mathfrak{M} \to S$  be a deformation of  $M = \lambda^{-1}(o)$ . Then in a neighborhood of A in  $\mathfrak{M}$ ,  $\lambda$  is the trivial deformation.

Let M be a strongly pseudoconvex manifold. In [37], we introduce a bunch of numerical invariants associated to M. In the following we introduce the  $\gamma$  invariant for M.

DEFINITION 3.6. Let M be a strongly pseudoconvex manifold of dimension 2. Suppose the irreducible components of the maximal compact analytic set A in M are nonsingular and have normal crossings. Let  $\pi$ :  $M \to V$  be the bimeromorphic morphism such that  $\pi$  blows down A to an isolated singularity. Let  $\Theta_M, \Theta_V$  be tangent sheaves of M and V respectively. Then  $\gamma$  is defined to be dim coker  $\rho$  where  $\rho$ :  $\pi_* \Theta_M \to \Theta_V$  is the natural inclusion. (cf. (1.2) of [8]).

The proof of the following theorem is the same as Theorem 3.1.

THEOREM 3.7. Let M be a strongly pseudoconvex manifold of dimension 2. Suppose the irreducible components of the maximal compact analytic set A in M are nonsingular and have normal crossings. Let  $\pi: M \to V$  be the bimeromorphic morphism, such that  $\pi$  blows down A to an isolated hypersurface singularity. Then

$$\dim H^{1}(M, \Theta) = -\frac{7K^{2} - 5\chi_{T}(A)}{6} + \tau - \frac{5}{6}(1 + \mu) - \gamma$$
$$\dim H^{1}(M, \mathscr{S}) = -\frac{7K^{2} - 5\chi_{T}(A)}{6} + \tau - \frac{5}{6}(1 + \mu) - \gamma$$
$$-\sum \dim H^{1}(A_{i}, N_{A_{i}}) .$$

The following example was suggested to us by Laufer.

EXAMPLE 3.9. Consider the singularities given at (0, 0, 0) by

$$V = \{(x, y, z): z^2 + x^3 + y^7 = 0\}$$
  
 $V' = \{(x, y, z): z^2 + x^3 + y^7 + kxy^5 = 0, k \neq 0\}$ 

Both V and V' resolve to have exceptional sets A and A' which have the same weighted dual graph.



The genera of the compact Riemann surfaces are zero.

$$egin{aligned} K^2 &= -4 & K'^2 &= -4 \ \chi_T(A) &= 5 & \chi_T(A') &= 5 \ \mu &= 12 & \mu' &= 12 \ au &= 12 & au' &= 11 \ \dim H^1(M,\, \Theta) &= 10 & \dim H^1(M',\, \Theta) &= 9 \ \dim H^1(M,\, \mathscr{S}) &= 1 & \dim H^1(M',\, \mathscr{S}) &= 0 \end{aligned}$$

This example says that dim  $H^{1}(M, \Theta)$  and dim  $H^{1}(M, \mathscr{S})$  depend not only on the topological structure of the strongly pseudoconvex manifold but also on the complex structure of the manifold.

PROPOSITION 3.9. Let M be a strongly pseudoconvex two-dimensional manifold with exceptional set A. Let  $\rho: M' \to M$  be a quadratic transformation of M at any point p of A. Then

$$\dim H^{\scriptscriptstyle 1}(M', \Theta) - \dim H^{\scriptscriptstyle 1}(M, \Theta) = 2 - \dim \operatorname{coker} \, i$$

where i:  $\Gamma(M, \Theta_o) \rightarrow \Gamma(M, \Theta)$  and  $\Theta_o =$  sheaf of germs of tangent vector fields which vanish at p.

**Proof.**<sup>(\*)</sup> Let  $A_1 = \rho^{-1}(p)$  and N be the normal bundle of  $A_1$  in M'. Then we have  $N \cong \mathcal{O}_{A_1}(-1)$ ,  $A_1 \cong CP^1$  and an exact sequence

$$0 \longrightarrow \Theta_{M'} \longrightarrow \rho^* \Theta_M \longrightarrow N^* \longrightarrow 0$$

$$\begin{array}{c} & & \\ & \\ & &$$

The associated cohomology sequence

then gives the proposition immediately.

COROLLARY 3.10. Let M be a strongly pseudoconvex two-manifold with exceptional set A. Let  $\rho: M' \to M$  be a quadratic transformation of M at any smooth point p of A. Suppose p is in  $A_1$ , an irreducible component of A. If one of the following conditions holds,

(1)  $A_1$  is a compact Riemann surface of genus zero, and  $A_1$  intersects with the other irreducible components of A in at least three points.

(2)  $A_1$  is a compact Riemann surface of genus one and  $A_1$  intersects with the other irreducible components of A in at least one point.

(3)  $A_1$  is a compact Riemann surface of genus  $\geq 2$ .

<sup>&</sup>lt;sup>(\*)</sup> The above simplified proof was suggested to us by the referee.

Then  $\dim H^1(M', \Theta) = \dim H^1(M, \Theta) + 2$ .

Proof. Since the global vector fields of M must be tangential to the exceptional set, the automorphism given by integrating along a vector field must map any singular point q of A into itself. Since automorphism of compact Riemann surface of genus one fixing three points must be an identity, statement (1) above follows from Proposition 3.9. As the tangent bundle of compact Riemann surface of genus 1 is trivial, statement (2) follows also from Proposition 3.9. The statement (3) above is trivial because there are no global vector fields on compact Riemann surface of genus  $\geq 2$ .

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