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MODULI AND MODULAR GROUPS OF A CLASS OF CALABI-YAU n-DIMENSIONAL MANIFOLDS, n≥3

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1. Introduction

Since the discovery of mirror symmetry in string theory by physicists, there have been tremendous activities on Calabi-Yau manifolds both by physicists and mathematicians. The reason that mirror symmetry has attracted a lot of mathematicians' attention is that it predicts successfully the number n_k of rational curves of degree k in these manifolds. This so-called Mirror Conjecture was first solved recently by Lian, Liu and Yau in their celebrated work [3]. In this paper we shall study the geometry of distinguished class of Calabi-Yau manifolds

(1.1)
$$X_s = \big\{(x_1 : \cdots : x_n) \in \boldsymbol{CP}^{n-1} : x_1^n + \cdots + x_n^n + s x_1 x_2 \ldots x_n = 0\big\}.$$

For $n = 5$, this class of Calabi-Yau 3-manifolds were studied in detail by Candelas, Ossn, Green and Parkers [1] by means of the period map. In particular, they observed that the modular group is not $SL(2, \boldsymbol{Z})$.

It is the purpose of this paper to find out the moduli and the modular group of this one-parameter family of Calabi-Yau manifolds in (1.1) for all $n \geq 5$. Our argument is uniform for all $n \geq 5$. We remark that $n = 3$ was treated by our previous paper [2] with different motivation. The crucial contribution of our paper is the introduction of some special points in Calabi-Yau manifolds.

Let ρ_i, $i = 1, 2, \ldots, n$, be n-distinct roots of $x^n = -1$. It is clear that the following $N = \frac{1}{2}n^2(n-1)$ points $Q_1, \ldots, Q_N$ of the form

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$(0,\dots,0,1,0,\dots,0,\rho_i,0,\dots,0)$, where 1, ρ_i run over all possible 2-tuple positions of $1,2,\dots,n$, are on each Calabi-Yau manifold X_s. We shall show in Proposition 2.1 that there are $(n-2)$ independent hyperplanes through Q_i in $T_{Q_i}(X_s)$, the tangent plane of X_s at Q_i, for which all the lines passing through Q_i in these $(n-2)$ independent hyperplanes have contact order n with X_s at Q_i.

Definition 1.1. A point Q in a $(n-2)$-dimensional Calabi-Yau manifold X is said to have $C-Y$ property if there are $(n-2)$ independent hyperplanes through Q in $T_Q(X)$ for which all the lines passing through Q in these $(n-2)$ independent hyperplanes have contact order at least n with X at Q. Such point Q is called a $C-Y$ point in X.

Theorem A. *For $n \geq 5$, $s \neq 0$ and $s^n \neq (-n)^n$, the $C-Y$ points on the Calabi-Yau manifolds*

$$X_s = \left\{(x_1 : \dots : x_n) \in \boldsymbol{CP}^{n-1} : x_1^n + \cdots + x_n^n + s x_1 \dots x_n = 0\right\}$$

are precisely $Q_1,\dots,Q_N$, $N = \frac{1}{2}n^2(n-1)$, of the form $(0,\dots,0,1,0,\dots,0,\rho_i,0,\dots,0)$, where $1,\rho_i,1 \leq i \leq n$, run over all possible 2-tuple positions of $1,2,\dots,n$ and $\rho_i, 1 \leq i \leq n$, are the n-distinct roots of $x^n = -1$.

Using Theorem A, we can prove the following theorem.

Theorem B. *For $n \geq 5$, $t \neq s$, s^n and $t^n \neq 0$ and $\neq (-n)^n$, the group G of biholomorphisms between*

$$X_t = \left\{(x_1 : \dots : x_n) \in \boldsymbol{CP}^{n-1} : x_1^n + \cdots + x_n^n + t x_1 \dots x_n = 0\right\}$$

and

$$X_s = \left\{(x_1 : \dots : x_n) \in \boldsymbol{CP}^{n-1} : x_1^n + \cdots + x_n^n + s x_1 \dots x_n = 0\right\}$$

consists of all projective nonsingular linear transformation $B \in PGL(n,\boldsymbol{C})$ of the following form:

$$B = \begin{pmatrix} 0 & \dots & 0 & a_{1i_1} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & a_{2i_2} & 0 & \dots & 0 & 0 & 0 & \dots \\ . & \dots & & \dots & & \dots & & \dots & & \dots & & \dots & & \dots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & a_{ni_n} & 0 & \dots \end{pmatrix}$$

where $(i_1,\dots,i_n)$ is a permutation of $(1,\dots,n)$ and $a_{1i_1},\dots,a_{ni_n}$ are n-th root of unity. Each such B induces a linear transformation on the *parameter space by sending t to $ta_{1i_1}\dots a_{ni_n}$. The group G has order $n^{n-1}(n!)$. Let N be the group of automorphisms of X_t. Then N is a normal subgroup of G of order $n^{n-2}(n!)$.*

Theorem C. *For $n \geq 5$, the modulus function of the one parameter family of Calabi-Yau manifolds*

$$X_s = \{(x_1 : \dots : x_n) \in \boldsymbol{CP}^{n-1} : x_1^n + \cdots + x_n^n + sx_1 \dots x_n = 0\}$$

is s^n, i.e. for any two parameters t, s, X_t is biholomorphically equivalent to X_s if and only if $t^n = s^n$.

2. Special points on Calabi-Yau manifolds

Let X_s be the $(n-2)$-dimension hypersurface defined by $x_1^n + \cdots + x_n^n + sx_1x_2\dots x_n = 0$ in $\boldsymbol{CP}^{n-1}$. It is easy to see that X_s is a nonsingular manifold for $s^n \neq (-n)^n$. In fact, let

$$f(x_1, \dots, x_n) = x_1^n + \cdots + x_n^n + sx_1x_2\dots x_n. \tag{2.1}$$

Then X_s is nonsingular if and only if there is no common solution to the n equations

$$\frac{\partial f}{\partial x_i} = nx_i^{n-1} + sx_1\dots x_{i-1}x_{i+1}\dots x_n = 0, \quad 1 \leq i \leq n \tag{2.2}$$

in $\boldsymbol{CP}^{n-1}$. These equations imply that

$$nx_1^n = nx_2^n = \cdots = nx_n^n = -sx_1x_2\dots x_n, \tag{2.3}$$

whence

$$(-n)^n \prod_{i=1}^n x_i^n = (s)^n \prod_{i=1}^n x_i^n. \tag{2.4}$$

If $P = (p_1 : \dots : p_n) \in \boldsymbol{CP}^{n-1}$ is a common solution of equations (2.2), then none of the p_i's may be zero by (2.3). Hence $s^n = (-n)^n$. Conversely it is easy to see that X_s is singular when $s^n = (-n)^n$.

Proposition 2.1. *Let ρ_j, $j = 1, 2, \dots, n$, be n distinct roots of $x^n = -1$. For each s with $s^n \neq (-n)^n$, let Q_i be one of the $N = n^2(n-1)/2$ points of the form $(0, \dots, 0, 1, 0, \dots, 0, \rho_j, 0, \dots, 0)$ on the Calabi-Yau manifold $X_s = \{(x_1 : \dots : x_n) \in \boldsymbol{CP}^{n-1} : x_1^n + \cdots + x_n^n + sx_1 \dots x_n = 0\}$.*

Then Q_i is a $C-Y$ point i.e., there are $(n-2)$ independent hyperplanes through Q_i in $T_{Q_i}(X_s)$ for which all the lines passing through Q_i in these $(n-2)$ independent hyperplanes have contact order at least n with X_s at Q_i.

Proof. Without loss of generality, we only check that $Q_1 = (1, \rho_1, 0, \ldots, 0)$ is a $C-Y$ point. It is clear that the tangent plane $T_{Q_1}(X_s)$ of X_s at Q_1 has equation

$$x_1 + \rho_1^{n-1} x_2 = 0. \tag{2.5}$$

Thus $T_{Q_1}(X_s) \cap X_s$ is defined by the equations

$$\begin{cases} x_1 + \rho_1^{n-1} x_2 = 0 \\ x_1^n + \cdots + x_n^n + s x_1 \ldots x_n = 0 \end{cases} \tag{2.6}$$

We can think of $\big(T_{Q_1}(X_s)\big) \cap X_s$ as a hypersurface in $\boldsymbol{P}\big(T_{Q_1}(X_s)\big)$ with $(x_2 : x_3 : \ldots : x_n)$ as homogeneous coordinates. Its defining equation is

$$x_3^n + \cdots + x_n^n - s\rho_1^{n-1} x_2^2 x_3 \ldots x_n = 0 \tag{2.7}$$

Observe that x_2 coordinate of Q_1 is nonzero. Let $x_3' = \frac{x_3}{x_2}, \ldots, x_n' = \frac{x_n}{x_2}$ be the inhomogeneous coordinates. Then the inhomogeneous form of the equation of $\big(T_{Q_1}(X_s)\big) \cap X_s$ at Q_1 is

$$(x_3')^n + \cdots + (x_n')^n - s\rho_1^{n-1} x_3' \ldots x_n' = 0 \tag{2.8}$$

It is clear that all lines tangent to X_s at Q_1 are parameterized by $\boldsymbol{P}\big(T_{Q_1}(X_s)\big) = \boldsymbol{CP}^{n-3}$. Among all these lines we would like to find those lines with contact order to X_s at least n. We can write the equation of a line L as

$$\begin{cases} x_3' = \alpha_3 t \\ \vdots \\ x_n' = \alpha_n t \end{cases} \tag{2.9}$$

where $(\alpha_3 : \ldots : \alpha_n) \in \boldsymbol{P}\big(T_{Q_1}(X_s)\big) = \boldsymbol{CP}^{n-3}$. If the line L has contact order n with X_s at Q_1, the coefficients of t^k for $k \leq n-1$ have to be zero when (2.9) is substituted in (2.8). It is clear that L has contact order n with X_s at Q_1 if and only if one of the α_i has to be zero. This means that there are $(n-2)$ independent hyperplanes through Q_i in $T_{Q_i}(X_s)$ for which all the lines passing through Q_i in these $(n-2)$ independent hyperplanes have contact order at least n with X_s at Q_i. q.e.d.

We shall show that all the $C-Y$ points on X_s are exactly those $N = n^2(n-1)/2$ points listed in Proposition 2.1. For this purpose, we need to prove the following lemma.

Lemma 2.2. *Let $Q = (q_1, \ldots, q_n)$ be a $C-Y$ point in the Calabi-Yau manifold*

$$X_s = \{(x_1 : \ldots : x_n) \in \boldsymbol{CP}^{n-1} : x_1^n + \cdots + x_n^n + sx_1 \ldots x_n = 0\}.$$

Let $f = x_1^n + \cdots + x_n^n + sx_1 \ldots x_n$ and $\frac{\partial f}{\partial x_1}(Q) = b_1, \ldots, \frac{\partial f}{\partial x_n}(Q) = b_n$. Suppose $b_1 = \frac{\partial f}{\partial x_1}(Q_1) \neq 0$ and $q_2 \neq 0$. Denote $a_2 = \frac{b_2}{b_1}, \ldots, a_n = \frac{b_n}{b_1}$. Then all partial derivatives of $f(-a_2x_2 - \cdots - a_nx_n, x_2, \ldots, x_n)$ with respect to the variables $x_3, \ldots, x_n$ with order at most $n-3$ are zero at Q.

Proof. We first make a general observation. Let $g(x_2, \ldots x_n)$ be a homogeneous polynomial of degree m. Let

$$g'(x_3', \ldots, x_n') = g(1, x_3', \ldots, x_n')$$

be a homogeneous form of g where $x_3' = \frac{x_3}{x_2}, \ldots, x_n' = \frac{x_n}{x_2}$. It is easy to see that

$$\frac{\partial^p g}{\partial x_{i_1}, \ldots, \partial x_{i_p}} = (x_2)^{n-p} \frac{\partial^p g'}{\partial x_{i_1}' \ldots \partial x_{i_p}'}, \quad i_1, \ldots, i_p \in \{3, \ldots, n\}.$$

Thus in order to prove the lemma, it is enough to prove the following statement: For the inhomogeneous form $w(x_3', \ldots, x_n')$ of $f(-a_2x_2 - \cdots - a_nx_n, x_2, \ldots, x_n)$, where $x_3' = \frac{x_3}{x_2}, \ldots, x_n' = \frac{x_n}{x_2}$,

$$\left.\frac{\partial^p w(x_3', \ldots, x_n')}{\partial x_{i_1}' \ldots \partial x_{i_p}'}\right|_Q = 0 \tag{2.10}$$

for $p \leq n-3$ and $i_1, \ldots, i_p \in \{3, \ldots, n\}$.

Consider the inhomogeneous coordinate $(q_3', \ldots, q_n')$ of Q where $q_3' = \frac{q_3}{q_2}, \ldots, q_n' = \frac{q_n}{q_2}$. Let $x_3'' = x_3' - q_3', \ldots, x_n'' = x_n' - q_n'$. It is clear that (2.10) holds if and only if the following (2.11) holds

$$\left.\frac{\partial^p w(x_3'', \ldots, x_n'')}{\partial x_{i_1}'' \ldots \partial x_{i_p}''}\right|_{(0,\ldots,0)} = 0 \tag{2.11}$$

if $p \leq n-3$, $i_1, \ldots, i_p \in \{3, \ldots, n\}$.

Notice that under the new coordinates $(x_3'', \ldots, x_n'')$, the point Q is $(0, \ldots, 0)$. Consider the $(n-2)$ hyperplanes in $T_Q(X_s)$ with the special property in the Definition 1.1. Let $L_1, \ldots, L_{n-2}$ be their defining equations. Then $L_3, \ldots, L_n$ are linearly independent 1-forms in $x_3'', \ldots, x_n''$ variables. Write

$$w(x_3'', \ldots, x_n'') = w_{\geq n} + w_{\leq n-1}, \tag{2.12}$$

where $w_{\geq n}$ denotes the sum of monomials in $w(x_3'', \ldots, x_n'')$ with degrees at least n while $w_{\leq n-1}$ denotes the sum of monomials in $w(x_3'', \ldots, x_n'')$ with degree at most $n-1$. We shall prove that $w_{\leq n-1}$ can be divided by $L_3, \ldots, L_n$.

Since $L_3, \ldots, L_n$ are linearly independent, we can take $L_3, \ldots, L_n$ as new coordinates. If $w_{\leq n-1}$ is not divisible by L_3, then

$$w_{\leq n-1} = L_3 P + R,$$

where P is a polynomial in $L_3, \ldots, L_n$ and R is a polynomial in $L_4, \ldots, L_n$. Let $\alpha_4, \ldots, \alpha_n$ be such that $R(\alpha_4, \ldots, \alpha_n) \neq 0$. Consider the line L

$$\begin{cases} L_3 = 0 \\ L_4 = \alpha_4 t \\ \vdots \\ L_n = \alpha_n t. \end{cases} \tag{2.13}$$

Then $w_{\leq n-1}(0, \alpha_4 t, \ldots, \alpha_n t)$ is a polynomial of t with degree less than or equal to $n-1$. Thus the line L cannot have contact order n with $w = 0$ at Q. This is a contradiction.

From the above argument, we have proved that $w(x_3'', \ldots, x_n'')$ as polynomials of $L_3, \ldots, L_n$, contains only monomials with degree at least $n-2$. Since $L_3, \ldots, L_n$ are linear in $x_3'', \ldots, x_n''$ variables, we conclude that $w(x_3'', \ldots, x_n'')$ contains only monomials of $x_3'', \ldots, x_n''$ with degree at least $n-2$. Thus (2.11) is proved. q.e.d.

The following theorem is the key theorem of this paper.

Theorem 2.3. *For $n \geq 5$, the set $\{Q_1, \ldots, Q_N\}$ in Proposition 2.1 is precisely the set of all $C-Y$ points in the Calabi-Yau manifold $X_s = \{(x_1 : \ldots, x_n) \in \boldsymbol{CP}^{n-1} : x_1^n + \cdots + x_n^n + s x_1 \ldots x_n = 0\}$, $s \neq 0$.*

Proof. Let $Q = (q_1, \ldots, q_n)$ be a $C-Y$ point on X_s. We need to show that $Q \in \{Q_1, \ldots, Q_N\}$. We shall consider the local form of the equation of $\big(T_Q(X_s)\big) \cap X_s$ at Q. Let $f(x_1, \ldots, x_n) = x_1^n + \cdots +$

$x_n^n + sx_1 \dots x_n$ and $b_1 = \frac{\partial f}{\partial x_1}(Q), \dots, b_n = \frac{\partial f}{\partial x_n}(Q)$. Without loss of generality, we shall assume $b_1 \neq 0$. Let $a_2 = \frac{b_2}{b_1}, \dots, a_n = \frac{b_n}{b_1}$. The defining equation of $\big(T_Q(X_s)\big) \cap X_s$ is

$$f(-a_2x_2 - \cdots - a_nx_n,\ x_2, \dots, x_n) = 0 \tag{2.14}$$

with homogeneous coordinates $(x_2 : \dots : x_n)$ on $\boldsymbol{P}\big(T_Q(X_s)\big)$. We assume also without loss of generality that $q_2 \neq 0$. In view of Lemma 2.2, we know that all 2^{nd} order partial derivatives of $f(-a_2x_2 - \cdots - a_nx_n, x_2, \dots, x_n)$ with respect to x_i, x_j, $i, j \in \{3, \dots, n\}$ at Q are zero because of $n \geq 5$. Hence, we have

$$\left.\frac{\partial^2 f(-a_2x_2 - \cdots - a_nx_n, x_2, \dots, x_n)}{\partial x_i \partial x_j}\right|_Q = 0,\ i, j \geq 3. \tag{2.15}$$

By chain rule, we get

$$a_ia_j\frac{\partial^2 f}{\partial x_1^2}(Q) - a_i\frac{\partial^2 f}{\partial x_1 \partial x_j}(Q) - a_j\frac{\partial^2 f}{\partial x_i \partial x_1}(Q) + \frac{\partial^2 f}{\partial x_i \partial x_j}(Q) = 0. \tag{2.16}$$

Multiplying (2.16) with $b_1^2 = \big(\frac{\partial f}{\partial x_1}(Q)\big)^2 \neq 0$, we get, for $i, j \geq 3$,

$$\begin{aligned} b_ib_j\frac{\partial^2 f}{\partial x_1^2}(Q) = {}& b_1b_i\frac{\partial^2 f}{\partial x_1 \partial x_j}(Q) + b_1b_j\frac{\partial^2 f}{\partial x_i \partial x_1}(Q) \\ & - b_1^2\frac{\partial^2 f}{\partial x_i \partial x_j}(Q), \end{aligned} \tag{2.17}$$

which can be rewritten as, for $i, j \geq 3$

$$\begin{aligned} & n(n-1)q_1^{n-2}(nq_i^{n-1} + sq_1 \dots q_{i-1}q_{i+1} \dots q_n) \\ & \qquad \cdot (nq_j^{n-1} + sq_1 \dots q_{j-1}q_{j+1} \dots q_n) \\ & = sq_2 \dots q_{i-1}q_{i+1} \dots q_{j-1}q_{j+1} \dots q_n(nq_1^{n-1} + sq_2 \dots q_n) \\ & \qquad \cdot (nq_i^n + nq_j^n - nq_1^n + sq_1 \dots q_n). \end{aligned} \tag{2.18}$$

Now we only need to prove that $q_3 = \cdots = q_n = 0$ because these will imply that $Q \in \{Q_1, \dots, Q_N\}$. There are two cases to be considered.

Case 1. $q_3, \dots, q_n$ are nonzero. If $q_1 = 0$ in this case, we have

$$nq_i^n + nq_j^n - nq_1^n + sq_1 \dots q_n = 0 \qquad \forall\, i, j \geq 3 \tag{2.19}$$

by (2.18). Thus $q_i^n + q_j^n = 0$ for any $i, j \geq 3$. In particular, if we take $i = j \geq 3$, we get $q_i^n = 0$ and hence $q_i = 0$ for $i \geq 3$. This is a contradiction.

On the other hand if $q_1 \neq 0$, we shall consider (2.18) for $i, j \geq 3$ and $k, j \geq 3$. By dividing these two equalities, we have

$$\frac{nq_i^n + sq_1 \dots q_n}{nq_k^n + sq_1 \dots q_n} = \frac{nq_i^n + sq_1 \dots q_n + nq_j^n - nq_1^n}{nq_k^n + sq_1 \dots q_n + nq_j^n - nq_1^n}, \tag{2.20}$$

which implies

$$\frac{nq_i^n + sq_1 \dots q_n}{nq_k^n + sq_1 \dots q_n} = \frac{n(q_j^n - q_1^n)}{n(q_j^n - q_1^n)} = 1. \tag{2.21}$$

Hence we have $x_i^n = x_k^n$ for $i, k \geq 3$. Similarly by exchanging the roles of the indices 2 and 3, (recall that $q_3 \neq 0$ is assumed), we have $q_2^n = q_3^n = \cdots = q_n^n$. If $b_2 = \frac{\partial f}{\partial x_2}(Q) = nq_2^{n-1} + sq_1 q_3 \dots q_n \neq 0$, then by exchanging the roles of the indices of 1 and 2, (note $q_1 \neq 0$), we have $q_1^n = q_2^n = \cdots = q_n^n$. Since $(q_1, \dots, q_n)$ satisfies the following equation

$$q_1^n + \cdots + q_n^n + sq_1 \dots q_n = 0 \tag{2.22}$$

we have $q_2(nq_2^{n-1} + sq_1 q_2 \dots q_n) = 0$. This contradicts our assumption that $b_2 = nq_2^{n-1} + sq_1 q_3 \dots q_n \neq 0$. If

$$b_2 = \frac{\partial f}{\partial x_2}(Q) = nq_2^{n-1} + sq_1 q_3 \dots q_n = 0,$$

then $nq_2^n + sq_1 q_2 \dots q_n = 0$. (2.22) implies $q_1^n + (n-1)q_2^n + sq_1 \dots q_n = 0$. By adding q_2^n in both sides of this equation, we get $q_1^n = q_2^n$. Thus $q_1^n = \cdots = q_n^n$. (2.22) implies $nq_i^n + sq_1 \dots q_n = 0$ for $1 \leq i \leq n$. This implies that Q is a singular point of X_s, a contradiction.

Case 2. At least one of $q_3, \dots, q_n$ is equal to zero. Without loss of generality, we shall assume $q_3 = 0$. Since

$$b_1 = \frac{\partial f}{\partial x_1}(Q) = nq_1^n - sq_2 \dots q_n \neq 0$$

is assumed, we have $q_1 \neq 0$. Consider (2.18) for $i = j \geq 4$. The right-hand side of (2.18) becomes zero because of our assumption that at least one of $q_3, \dots, q_n$ is equal to zero. It follows that $nq_i^{n-1} + sq_1 \dots q_{i-1} q_{i+1} \dots q_n = 0$, $4 \leq i \leq n$. These $n-3$ equations together with the assumption that at least one of $q_3, \dots, q_n$ is zero imply $q_4 = q_5 = \cdots = q_n = 0$. If q_3 is nonzero, then at least one of $q_2, q_4, \dots, q_n$ is zero. By considering (2.18) with $i = j = 3$, we have

$$\begin{aligned} n(n-1)q_1^{n-2}&(nq_3^{n-1} + sq_1q_2q_4 \dots q_n)^2 \\ &= sq_2q_4 \dots q_n(nq_1^{n-1} + sq_2 \dots q_n)(2nq_3^n - nq_1^n + sq_1 \dots q_n) = 0, \end{aligned}$$

which implies $q_3 = 0$. Thus we have shown $q_3 = q_4 = \cdots = q_n = 0$ and Q has to be in $\{Q_1, Q_2, \dots, Q_n\}$. q.e.d.

3. Moduli and modular group of Calabi-Yau manifolds

We shall use Theorem 2.3 to study the moduli and modular group of Calabi-Yau manifolds.

Theorem 3.1. *For $n \geq 5$ and any nonzero $t \neq s$, the biholomorphism between $X_t = \{(x_1 : \dots : x_n) \in \boldsymbol{CP}^{n-1} : x_1^n + \cdots + x_n^n + tx_1 \dots x_n = 0, t^n \neq (-n)^n\}$ and $X_s = \{(x_1 : \dots : x_n) \in \boldsymbol{CP}^{n-1} : x_1^n + \cdots + x_n^n + sx_1 \dots x_n = 0, s^n \neq (-n)^n\}$ is induced by a projective nonsingular linear transformation $B \in PGL(n, \boldsymbol{C})$ on coordinates with only one nonzero entry in each row and each column. Moreover, these entries is B are n-th roots of unity. Conversely any matrix B of the above form will send X_t to X_s where $s = tc_1c_2 \dots c_n$, being $c_1, \dots c_n$ the nonzero entries of B.*

Proof. It is well known that any biholomorphism between X_t and X_s is induced by a projective nonsingular linear transformation $B = (b_{ij})$, $1 \leq i, j \leq n$, in $PGL(n, \boldsymbol{C})$. For any $C-Y$ point Q in X_t, it is clear that $B(Q)$, the image of Q under B, is also a $C-Y$ point on X_s. In view of Theorem 2.3, we have $\{B(Q_1), \dots, B(Q_N)\} = \{Q_1, \dots, Q_N\}$ where $N = \frac{1}{2}n^2(n-1)$.

We now consider the set of first coordinates of the points $B(Q_1), \dots, B(Q_N)$. This set consists of $N = \frac{1}{2}n^2(n-1)$ elements of the form $a_{1i} + \rho_m a_{1j}$, with $1 \leq i < j \leq n$, $1 \leq m \leq n$. We know that there are $\frac{1}{2}n(n-1)(n-2)$ of N first coordinates of those points

$$\{B(Q_1), \dots, B(Q_N)\} = \{Q_1, \dots, Q_N\}$$

equal to zero. Hence there are $\frac{1}{2}n(n-1)(n-2)$ of $a_{1i} + \rho_m a_{1j}$, with $1 \leq i < j \leq n$, $1 \leq m \leq n$, equal to zero. Suppose that k of n numbers $a_{11}, \dots, a_{1n}$ are zero. Notice that for nonzero complex numbers c and

d, there is at most one zero among n complex numbers $c+\rho_m d$. We also note that if precisely only one of c, d is zero, then $c+\rho_m d$ can never be zero for $1 \le m \le n$. Thus among N complex numbers $a_{1i}+\rho_m a_{ij}$, $1 \le i < j \le n$, $1 \le m \le n$, there are at most $\frac{1}{2}nk(k-1)+\frac{1}{2}(n-k)(n-k-1)$ of them are zero. It follows that we have the following inequality

$$\frac{nk(k-1)}{2} + \frac{(n-k)(n-k-1)}{2} \ge \frac{n(n-1)(n-2)}{2}. \tag{3.1}$$

(3.1) implies $k > 0$. It follows that $nk \ge n-k$ because k is a positive integer. Thus, in view of (3.1) we have

$$\begin{aligned} \frac{nk(n-2)}{2} &= \frac{nk(k-1)}{2} + \frac{nk(n-k-1)}{2} \\ &\ge \frac{nk(k-1)}{2} + \frac{(n-k)(n-k-1)}{2} \\ &\ge \frac{n(n-1)(n-2)}{2}. \end{aligned} \tag{3.2}$$

(3.3) implies $k \ge n-1$. Since B is a nonsingular matrix, we have $k = n-1$. Therefore we have proved that there is only one nonzero entry in the first row. Similarly, we can prove that there is only one nonzero entry in each row. Since B is nonsingular, there is only one nonzero entry in each column.

Let $a_{1i_1}, a_{2i_2}, \ldots, a_{ni_n}$ be the nonzero entries of the 1^{st} row, 2^{nd} row, $\ldots$, and n^{th} row of the matrix B respectively. Consider the action of B on the point $P = (0, \ldots, 0, \rho_m, 0, \ldots, 0, 1, 0, \ldots, 0)$ where $1 \le m \le n$, ρ_m is the i_1-coordinate of P while 1 is the i_2-coordinate of P. Clearly $B(P) = (a_{1i_1}\rho_m, a_{2i_2}, 0, \ldots, 0)$ is a $C-Y$ point. In view of Theorem 2.3, we have $\rho_m a_{1i_1}/a_{2i_2} \in \{\rho_1, \ldots, \rho_n\}$. This implies a_{1i_1}/a_{2i_2} is a n^{th} root of unity. Similarly we can show that all ratios between $a_{1i_1}, a_{2i_2}, \ldots, a_{ni_n}$ are n^{th} root of unity. The first part of Theorem 3.1 follows immediately.

Conversely, suppose that B is a nonsingular matrix given by

$$B : (x_1, x_2, \ldots, x_n) \longmapsto (a_{1i_1}x_{i_1}, a_{2i_2}x_{i_2}, \ldots, a_{ni_n}x_{i_n}),$$

where $a_{1i_1}, a_{2i_2}, \ldots, a_{ni_n}$ are n^{th} roots of unity and $(i_1, i_2, \ldots, i_n)$ is a permutation of $(1, 2, \ldots, n)$. Then clearly $X_t : x_1^n + \cdots + x_n^n + tx_1 \ldots x_n = 0$ is sent to $X_s : x_1^n + \cdots + x_n^n + sx_1 \ldots x_n = 0$ where $s = ta_{1i_1} \ldots a_{ni_n}$.

q.e.d.

Corollary 3.2. *For $n \ge 5$, $t \ne s$, s^n and $t^n \ne 0$ and $(-n)^n$, the group G of biholomorphisms between $X_t = \big\{(x_1 : \cdots : x_n) \in \boldsymbol{CP}^{n-1} :$*

$x_1^n + \cdots + x_n^n + tx_1 \ldots x_n = 0\big\}$ *and* $X_s = \big\{(x_1 : \cdots : x_n) \in \boldsymbol{CP}^{n-1} : x_1^n + \cdots + x_n^n + sx_1 \ldots x_n = 0\big\}$ *consists of all projective nonsingular linear transformation* $B \in PGL(n, G)$ *of the following form:*

$$B = \begin{pmatrix} 0 & \ldots & 0 & a_{1i_1} & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & a_{2i_2} & 0 & \ldots & 0 & 0 & 0 & \ldots \\ . & \ldots & & \ldots & & \ldots & & \ldots & & \ldots & & \ldots & & \ldots \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & a_{ni_n} & 0 & \ldots \end{pmatrix},$$

where $(i_1, \ldots, i_n)$ *is a permutation of* $(1, \ldots, n)$ *and* $a_{1i_1}, \ldots, a_{ni_n}$ *are* n^{th} *root of unity. Each such* B *induces a linear transformation on the parameter space by sending* t *to* $ta_{1i_1} \ldots a_{ni_n}$*. The group* G *has order* $n^{n-1}(n!)$*. Let* N *be the group of automorphisms of* X_t*. Then* N *is a normal subgroup of* G *of order* $n^{n-2}(n!)$*.*

Proof. To compute the order of G, consider the first row of B. We can pick any number from 1 to n as i_1 and we can assign a_{1i_1} to be any number in the set of n^{th} roots of unity. So we have n^2 choices. In the second row, we can pick i_2 to be any number from 1 to n except i_1 and assign a_{2i_2} to be any number in the set of n^{th} roots of unity. So we have $n(n-1)$ choices. By continuing this argument, we see that there are $(n!)n^n$ elements. By dividing the scalar multiplications, we conclude that the order of the group G is $(n!)n^{n-1}$.

To compute the order of the automorphism group N of X_t, we observe that $B \in N$ if and only if $a_{1i_1}a_{2i_2} \ldots a_{ni_n} = 1$. Thus the order of N is $(n!)n^{n-2}$. q.e.d.

Theorem 3.3. *For* $n \geq 5$*, the modulus function of the one parameter family of Calabi-Yau manifolds* $X_s = \big\{(x_1 : \cdots : x_n) \in \boldsymbol{CP}^{n-1} : x_1^n + \cdots + x_n^n + sx_1 \ldots x_n = 0\big\}$ *is* s^n*, i.e., for any two parameters* t*,* s*,* X_t *is biholomorphically equivalent to* X_s *if and only if* $t^n = s^n$*.*

Proof. It is easy to see that X_t is biholomorphically equivalent to X_{tr} for any n^{th} root of unity r. Conversely, we know that if X_t is biholomorphically equivalent to X_s, then $s = tr$ for some n^{th} root of unity r in view of Corollary 3.2. Hence the modulus function of the one-parameter family of Calabi-Yau manifold X_s is s^n. q.e.d.

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