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Algebraic determination of isomorphism classes of the moduli algebras of $\tilde{E}_6$ singularities

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1 Introduction

A moduli algebra $A(V)$ of an isolated hypersurface singularity $(V, 0) = \{(z_0, \dots, z_n) : f(z_0, \dots, z_n) = 0\} \subseteq \boldsymbol{C}^{n+1}$ is a finite dimensional $\boldsymbol{C}$-algebra $\boldsymbol{C}\{z_0, \dots, z_n\}(f, \frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n})$. In [2], Mather and Yau proved that two germs of complex analytic hypersurfaces of the same dimension with isolated singularities are biholomorphically equivalent if a nd only if their moduli algebras are isomorphic. This means that we should be able to determine the singularity by means of studying its corresponding commutative Artinian local algebra. In his paper [3], Saito computed j-invariants for the singularities $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$. Each of these is determined by the elliptic curve which occurs as exceptinal divisor in the minimal resolution of the singularity. Seeley and Yau [4] studied certain Lie algebras as sociated with the moduli algebras of $\tilde{E}_7$ and $\tilde{E}_8$ singularities. They defined a nilpotent Lie algebra N_t for each of these singularities and showed that $N_s \cong N_t$ if and only if the singularity V_s is analytically equivalent to the V_t. They did this without reference to the resolution of the singularity, in fact even to the singularity itself. In this way, they were able to algebraically produce a modulus which was, up to a fractional linear transformation, equal to Saito's j-invariant. The case $\tilde{E}_6$ is different in that the nilpotent Lie algebra N_t is independent of the parameter t [5]. Thus we have the following challenging problem for algebraist: Find the modulus of the one parameter family of commutative Artinian local algebras arising from $\tilde{E}_6$ singularities without reference to the singularities. The

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purpose of this paper is to develop an algebraic method to understand how to see directly and algebraically how the commutative Artinian local algebra from $\tilde{E}_6$ vary. This problem may appear easy at first glance, but it is quite delicate as we discovered in our discussion with N. Jacobson, R. Howe and C. Huneke on this subject in 1984-85. This stimulated the authors to investigate this issue. The present paper is the result of this investigation.

The following is our main theorem.

Main Theorem. For $t^3 + 27 \neq 0$, let $A_t = \langle 1, x, y, z, xy, yz, zx, xyz \rangle$ as $\boldsymbol{C}$ vector space with multiplication rules: $x^2 = -\frac{t}{3}yz$, $y^2 = -\frac{t}{3}xz$, $z^2 = -\frac{t}{3}xy$ be a commutative Artinian algebras. Then, for t^3, $s^3 \neq 0$, 216 and -27, all algebra isomorphisms from A_t to A_s are given by one of the following 216 matrices in $PGL(3, \boldsymbol{C})$. Each of this isomorphism induce a linear fractional transformation on parameter space as follows.

Type I (54 matrices)

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

with $\lambda_1, \lambda_2 \in \{1, \rho, \rho^2\}$ and the induced fractional linear transformations on the parameter space given by $s = P^k(t)$ if $\lambda_1\lambda_2 = \rho^{-k}$ where $P(t) = \rho t$ and $\rho^3 = 1$.

Type II (162 matrices)

(a)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \rho^2 & \rho & 1 \\ \rho & \rho^2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \rho^2 & \rho & 1 \\ \rho & \rho^2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \rho^2 & \rho & 1 \\ \rho & \rho^2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \rho^2 & \rho & 1 \\ \rho & \rho^2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \rho^2 & \rho & 1 \\ \rho & \rho^2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \rho^2 & \rho & 1 \\ \rho & \rho^2 & 1 \end{pmatrix}$$

with $\lambda_1, \lambda_2 \in \{1, \rho, \rho^2\}$ and the induced fractional linear transformations on the parameter space given by $s = P^k R(t)$ if $\lambda_1\lambda_2 = \rho^{-k}$. Here $R(t) = \frac{3(6-t)}{3+t}$.

(b)

$$\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&\rho^2\\\rho^2&\rho&\rho^2\\\rho&\rho^2&\rho^2\end{pmatrix},\quad \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&\rho^2\\\rho^2&\rho&\rho^2\\\rho&\rho^2&\rho^2\end{pmatrix}$$

$$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&\rho^2\\\rho^2&\rho&\rho^2\\\rho&\rho^2&\rho^2\end{pmatrix},\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&\rho^2\\\rho^2&\rho&\rho^2\\\rho&\rho^2&\rho^2\end{pmatrix}$$

$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&\rho^2\\\rho^2&\rho&\rho^2\\\rho&\rho^2&\rho^2\end{pmatrix},\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&\rho^2\\\rho^2&\rho&\rho^2\\\rho&\rho^2&\rho^2\end{pmatrix}$$

with $\lambda_1, \lambda_2 \in \{1, \rho, \rho^2\}$ and the induced fractional linear transformations on the parameter space given by $s = P^k RP(t)$ if $\lambda_1\lambda_2 = \rho^{-k}$
(c)

$$\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&\rho\\\rho^2&\rho&\rho\\\rho&\rho^2&\rho\end{pmatrix},\quad \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&\rho\\\rho^2&\rho&\rho\\\rho&\rho^2&\rho\end{pmatrix}$$

$$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&\rho\\\rho^2&\rho&\rho\\\rho&\rho^2&\rho\end{pmatrix},\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&\rho\\\rho^2&\rho&\rho\\\rho&\rho^2&\rho\end{pmatrix}$$

$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&\rho\\\rho^2&\rho&\rho\\\rho&\rho^2&\rho\end{pmatrix},\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&\rho\\\rho^2&\rho&\rho\\\rho&\rho^2&\rho\end{pmatrix}$$

with $\lambda_1, \lambda_2 \in \{1, \rho, \rho^2\}$ and the induced fractional linear transformations on the parameter space given by $s = P^k RP^2(t)$ if $\lambda_1\lambda_2 = \rho^{-k}$.

Corollary 1. *The group H of fractional linear transformations generated by P and R are as follows:*

$$H=\left\{\begin{array}{llll} I(t)=t, & P(t)=\rho t, & P^2(t)=\rho^2 t, & R(t)=\frac{18-3t}{3+t},\\ PR(t)=\frac{18\rho-3\rho t}{3+t}, & P^2R(t)=\frac{18\rho^2-3\rho^2 t}{3+t}, & RP(t)=\frac{18-3\rho t}{3+\rho t} & \\ RP^2(t)=\frac{18\rho-3t}{t+3\rho}, & PRP(t)=\frac{18-3\rho t}{t+3\rho^2}, & P^2RP(t)=\frac{18\rho^2-3t}{3+\rho t} & \\ PRP^2(t)=\frac{18\rho^2-3\rho t}{t+3\rho}, & P^2RP^2(t)=\frac{18-3t\rho^2}{t+3\rho} & & \end{array}\right\}$$

H is a group of order 12 isomorphic to alternating group on four letters. H is also the group of automorphisms of the parameter space which do not change the isomorphism types of the algebras A_t. For any t and s such that $t^3, s^3 \neq 216$ and -27, if A_s is isomorphic to A_t, then s must be one of the twelve values $\{t, P(t), P^2(t), R(t), PR(t), P^2R(t), RP(t), RP^2(t), PRP(t), P^2RP(t), PRP^2(t), P^2RP^2(t)\}$.

Corollary 2. *Let U be the subgroup of $PGL(3, C)$ generated by*

$$A = \begin{pmatrix} \rho & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} \rho & \rho^2 & 1 \\ \rho^2 & \rho & 1 \\ 1 & 1 & 1 \end{pmatrix}, \ C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Then the subgroup W of U which consists of projective linear transformation inducing an identity on the parameter space is a normal subgroup of order 18 in U. The matrices in W are those matrices in Type I with $\lambda_1\lambda_2 = 1$ above. The quotient group U/W is isomorphic to the group H in Corollary 1.

Corollary 3. *Let $A_t = \langle 1, x, y, z, xy, yz, zx, xyz\rangle$ with multiplication rules: $x^2 = -\frac{t}{3}yz$, $y^2 = -\frac{t}{3}xz$ and $z^2 = -\frac{t}{3}xy$ be a one parameter family of moduli algebras for $\tilde{E}_6$ singularities. Then $k(t) = \frac{t^3(t^3-216)^3}{(t^3+27)^3}$ is a modulus of this one paramter family, i.e. $A_s \cong A_t$ if and only if $k(t) = k(s)$.*

The idea is to use the Lie algebra constructed in [5] to isolate four distinct points $0, 6, 6\rho, 6\rho^2$ where $\rho^3 = 1$ in the one parameter family of the Artinian local algebras. As a byproduct, we have enough evidence to show that Saito's computation of the j- invariant for $\tilde{E}_6$ is in error. We hope that the method developed in this paper together with those in [4] and [5] will help us to find the modulus of continuous family of isolated hypersurface singularities or commutative Artinian local algebras.

We would like to thank Professors N. Jacobson, C. Huneke and R. Howe aforementioned discussions. We also wish to thank Professor Rao Nagisetty for his helpful comments.

2 $\tilde{E}_6$ facts

The $\tilde{E}_6$ singularities are defined as germs of the zero sets of the functions

$$f_t(x, y, z) = x^3 + y^3 + z^3 + txyz.$$

The singularities f_t, for parameter values $t \in \boldsymbol{C}$ with $t^3 + 27 \neq 0$, are topologically equivalent. The moduli algebra

$$A = \boldsymbol{C}\big[[x, y, z]\big] \Big/ \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

is an analytic invariant which actually determines the analytic type of a singularity [2]. The dimension of the derivation algebra $L = \mathrm{Der}(A)$ is then also an analytic invariant.

In our case

$$\begin{aligned} A_t &= \boldsymbol{C}\big[[x, y, z]\big]/(3x^2 + tyz, 3y^2 + txz, 3z^2 + txy) \\ &\cong \langle 1, x, y, z, xy, yz, zx, xyz \rangle \text{ as } \boldsymbol{C} \text{ vector space} \\ &\quad \text{with multiplication rules:} \\ &\quad x^2 = -\frac{t}{3} yz,\ \ y^2 = -\frac{t}{3} xz,\ \ z^2 = -\frac{t}{3} xy. \end{aligned}$$

From the multiplciation rules follow

$$x^3 = y^3 = z^3 = -\frac{t}{3} xyz$$

$$x^4 = y^4 = z^4 = x^2 y^2 = x^2 z^2 = y^2 z^2 = 0$$

$$x^2 yz = xy^2 z = xyz^2 = x^3 y = x^3 z = y^3 x = y^3 z = z^3 x = z^3 y = 0.$$

It is easy to see that in this case

$$\dim L_t = \begin{cases} 10 & \text{for} \quad t^3 \neq 0, 216, -27 \\ 12 & \text{for} \quad t^3 = 0 \text{ or } 216. \end{cases}$$

For example $x \frac{\partial}{\partial x} \in L_0$, whereas generically the only derivation of degree zero in L_t is the Euler derivation $x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}$ [5]. Thus f_6 can be analytically equivalent to (possibly) only f_0, f_{6z} or $f_{6\rho^2}$ where $\rho^3 = 1$.

3 Proof of the Theorem

We first make a few observations regarding the algebras A_t. First $A_t \cong A_{P(t)}$ where $P(t) = \rho t$ and $\rho^3 = 1$. This can be seen by making the linear change of variables

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \rho & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

One can show the isomorphism $A_{P(t)} \cong A_t$ directly by considering x', y', and z' as generators of the algebra A_t. The defining relations for $A_{P(t)}$ hold among them, so that $x \mapsto x'$, $y \mapsto y'$, $z \mapsto z'$ generates an algebra isomorphism from $A_{P(t)}$ to A_t.

Second, $A_t \cong A_{R(t)}$ where $R(t) = \frac{3(6-t)}{3+t}$. This can be seen by making the linear change of variables.

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \rho & \rho^2 & 1 \\ \rho^2 & \rho & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

One can show the isomorphism $A_{R(t)} \cong A_t$ directly by considering x', y', and z' as generators of the algebra A_t.

$$\begin{aligned}
& x'^2 + \frac{t}{3} y'z' \\
&= (\rho x + \rho^2 y + z)^2 + \frac{t}{3}(\rho^2 x + \rho y + z)(x + y + z) \\
&= \left[\rho^2 (1 + \frac{t}{3}\right) x^2 + \rho^2 \left(2 - \frac{t}{3}\right) yz\right] + \left[\rho\left(1 + \frac{t}{3}\right) y^2 + \rho\left(2 - \frac{t}{3}\right) xz\right] \\
&\quad + \left[\left(1 + \frac{t}{3}\right) z^2 + \left(2 - \frac{t}{3}\right) xy\right] \\
&= \rho^2 \left(1 + \frac{t}{3}\right)\left[x^2 + \frac{1}{3}\frac{3(6-t)}{3+t} yz\right] + \rho\left(1 + \frac{t}{3}\right)\left[y^2 + \frac{1}{3}\frac{3(6-t)}{3+t} zx\right] \\
&\quad + \left(1 + \frac{t}{3}\right)\left[z^2 + \frac{1}{3}\frac{3(6-t)}{3+t} xy\right].
\end{aligned}$$

Similarly

$$\begin{aligned}
& y'^2 + \frac{t}{3} x'z' \\
&= \rho\left(1 + \frac{t}{3}\right)\left[x^2 + \frac{1}{3}\frac{18-3t}{3+t} yz\right] + \rho^2\left(1 + \frac{t}{3}\right)\left[y^2 + \frac{1}{3}\frac{18-3t}{3+t} xz\right] \\
&\quad + \left(1 + \frac{t}{3}\right)\left[z^2 + \frac{1}{3}\frac{18-3t}{3+t} xy\right]
\end{aligned}$$

and

$$\begin{aligned}
& z'^2 + \frac{t}{3} x'y' \\
&= \left(1 + \frac{t}{3}\right)\left[x^2 + \frac{1}{3}\frac{18-3t}{3+t} yz\right] + \left(1 + \frac{t}{3}\right)\left[y^2 + \frac{1}{3}\frac{18-3t}{3+t} xz\right] \\
&\quad + \left(1 + \frac{t}{3}\right)\left[z^2 + \frac{1}{3}\frac{18-3t}{3+t} xy\right].
\end{aligned}$$

Hence $x \mapsto x'$, $y \mapsto y'$, $z \mapsto z'$ generates an algebra isomorphism from $A_{R(t)}$ to A_t. Since $R(0) = 6$, we see that $A_0 \cong A_6 \cong A_{6\rho} \cong A_{6\rho^2}$. It is essentially these few facts which will allow us to determine which algebras A_s are isomorphic to a given A_t.

Let $T(s) = \frac{as+b}{cs+d}$ be any linear fractional transformation such that A_s is isomorphic to $A_{T(s)}$. Clearly T must permute $0, 6, 6\rho, 6\rho^2$ among themselves. Represent T as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We already know of two such linear fractional transformations. These are represented as matrices

$$P = \begin{pmatrix} \rho & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} -1 & 6 \\ 1 & 3 \end{pmatrix}$$

and they generate a subgroup which acts doubly transitively on the set $\{0, 6, 6\rho, 6\rho^2\}$. Composing a given linear fractional transformation T as above with an appropriate product of P and R, we can assume, without loss of generality, $T(0) = 0$ and $T(6) = 6$. $T(0) = 0$ forces $b = 0$ and $ad \neq 0$. Let $a = 1$. $T(6) = 6$ implies $\frac{6}{6c+d} = 6$, which is equivalent to $d = 1 - 6c$. T fixes 6ρ and $6\rho^2$, or transposes them. In either case, T^2 fixes four points and must be the identity. Thus

$$\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c + cd & d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \begin{cases} c = 0, \ d = 1 & \text{or} \\ c = \frac{1}{3}, \ d = -1. \end{cases}$$

However, $c = \frac{1}{3}$, $d = -1$ implies $T(3) = \infty$, a contradiction. We conclude that T is the identity. This shows that P and R generate the group of automorphisms of the parameter space which do not change the isomorphism types of the algebras A_t. It is also clear that this group, generated by P and R, acts as the alternating subgroup of Sym_4 when restricted to the set $\{0, 6, 6\rho, 6\rho^2\}$. This group is in fact the alternating group on four letters, since only the identity linear fractional transformation fixes these 4 points.

Before proving the theorem and corollaries in the introduction, we need to establish two lemmas.

Lemma 1. *Any algebra isomorphism from* $\boldsymbol{C}\big[[x', y', z']\big]/(3x'^2 + sy'z', 3y'^2 + sx'z', 3z'^2 + sx'y')$ *to* $\boldsymbol{C}\big[[x, y, z]\big]/(3x^2 + tyz, 3y^2 + txz, 3z^2 + txy)$ *is given by the transformation*

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \in GL(3, \boldsymbol{C}) \tag{1}$$

where $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ *satisfies the following equations*

(2a) $3ta_1^2 + tsa_2a_3 = 18b_1c_1 + 3s(b_2c_3 + b_3c_2)$

(2b) $3tb_1^2 + tsb_2b_3 = 18a_1c_1 + 3s(a_2c_3 + a_3c_2)$

(2c) $3tc_1^2 + tsc_2c_3 = 18a_1b_1 + 3s(a_2b_3 + a_3b_2)$

(2d) $3ta_2^2 + tsa_1a_3 = 18b_2c_2 + 3s(b_1c_3 + b_3c_1)$

(2e) $3tb_2^2 + tsb_1b_3 = 18a_2c_2 + 3s(a_1c_3 + a_3c_1)$

(2f) $3tc_2^2 + tsc_1c_3 = 18a_2b_2 + 3s(a_1b_3 + a_3b_1)$

(2g) $3ta_3^2 + tsa_1a_2 = 18b_3c_3 + 3s(b_1c_2 + b_2c_1)$

(2h) $3tb_2^2 + tsb_1b_2 = 18a_3c_3 + 3s(a_1c_2 + a_2c_1)$

(2i) $3tc_3^2 + tsc_1c_2 = 18a_3b_3 + 3s(a_1b_2 + a_2b_1)$

Proof. By the homogeneity of the ideals $(3x^2 + tyz, 3y^2 + tzx, 3z^2 + txy)$ and $(3x'^2 + sy'z', 3y'^2 + sz'x', 3z'^2 + tx'y')$, it is easy to show that there is a 3×3 matrix (b_{ij}) such that the isomorphism from A_s to A_t is given by (1) and the following equations hold

(3a) $3(a_1x + b_1y + c_1z)^2 + s(a_2x + b_2y + c_2z)(a_3x + b_3y + c_3z)$
$= b_{11}(3x^2 + tyz) + b_{12}(3y^2 + tzx) + b_{13}(3z^2 + txy)$

(3b) $3(a_2x + b_2y + c_2z)^2 + s(a_3x + b_3y + c_3z)(a_1x + b_1y + c_1z)$
$= b_{21}(3x^2 + tyz) + b_{22}(3y^2 + tzx) + b_{23}(3z^2 + txy)$

(3c) $3(a_2x + b_3y + c_3z)^2 + s(a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z)$
$= b_{31}(3x^2 + tyz) + b_{22}(3y^2 + tzx) + b_{33}(3z^2 + txy).$

Comparing the coefficients of x^2, xy, xz, y^2, yz, and z^2, we see that

(4a) $3a_1^2 + sa_2a_3 = 3b_{11}$

(4b) $6a_1b_1 + s(a_2b_3 + a_3b_2) = tb_{13}$

(4c) $6a_1c_1 + s(a_2c_3 + a_3c_2) = tb_{12}$

(4d) $3b_1^2 + sb_2b_3 = 3b_{12}$

(4e) $6b_1c_1 + s(b_2c_3 + b_3c_2) = tb_{11}$

(4f) $3c_1^2 + sc_2c_3 = 3b_{13}$

(5a) $3a_2^2 + sa_1a_3 = 3b_{21}$

(5b) $6a_2b_2 + s(a_3b_1 + a_1b_3) = tb_{23}$

(5c) $6a_2c_2 + s(a_3c_1 + a_1c_3) = tb_{22}$

(5d) $3b_2^2 + sb_1b_3 = 3b_{22}$

(5e) $6b_2c_2 + s(b_3c_1 + b_1c_3) = tb_{21}$

(5f) $3c_2^2 + sc_1c_3 = 3b_{23}$

(6a) $3a_3^2 + sa_1a_2 = 3b_{31}$

(6b) $6a_3b_3 + s(a_1b_2 + a_2b_1) = tb_{23}$

(6c) $6a_3c_3 + s(a_1c_2 + a_2c_1) = tb_{32}$

(6d) $3b_3^2 + sb_1b_2 = 3b_{32}$

(6e) $6b_3c_3 + s(b_1c_2 + b_2c_1) = tb_{31}$

(6f) $3c_3^2 + sc_1c_2 = 3b_{33}$

By eliminating b_{ij}' in (4a)-(4f), (5a)-(5f) and (6a)-(6f), we get equation (2a)-(2i). $Q.E.D.$

Lemma 2. *With the notation in lemma 1 and* $t^3 \neq 0$, *216 and* -27, *the set* $\{P_1 = (a_1 - b_1 : a_2 - b_2 : a_3 - b_3), P_2 = (b_1 - c_1 : b_2 - c_2 : b_3 - c_3), P_3 = (c_1 - a_1 : c_2 - a_2 : c_3 - a_3), T_1 = (a_1 - \rho b_1 : a_2 - \rho b_2 : a_3 - \rho b_3), T_2 = (b_1 - \rho c_1 : b_2 - \rho c_2 : b_3 - \rho c_3), T_3 = (c_1 - \rho a_1 : c_2 - \rho a_2 : c_3 - \rho a_3), W_1 = (a_1 - \rho^2 b_1 : a_2 - \rho^2 b_2 : a_3 - \rho^2 b_3), W_2 = (b_1 - \rho^2 c_1 : b_2 - \rho^2 c_2 : b_3 - \rho^2 c_3), W_3 = (c_1 - \rho^2 a_1 : c_2 - \rho^2 a_2 : c_3 - \rho^2 a_3)\}$ *of 9 points in* $\boldsymbol{CP}^2$ *coincides with the following set of 9 points*

$$\{(\rho^i : -1 : 0),\ (0 : \rho^i : -1),\ (-1 : 0 : \rho^i)\ \text{ for }\ i = 1, 2, 3\}.$$

Proof. By considering $[a_1 \cdot (2a) + a_2 \cdot (2d) + a_3(2g)]$ and $[b_1 \cdot (2b) + b_2 \cdot (2e) + b_3 \cdot (2h)]$, we have

(7) $3t(a_1^3 + a_2^3 + a_3^3 + sa_1a_2a_3) = 18(a_1b_1c_1 + a_2b_2c_2 + a_3b_3c_3) + 3s(a_1b_2c_3 + a_1b_3c_2 + a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 + a_3b_2c_1)$

and

(8) $3t(b_1^3 + b_2^3 + sb_1b_2b_3) = 18(a_1b_1c_1 + a_2b_2c_2 + a_3b_3c_3)$
$+3s(a_2b_1c_3 + a_3b_1c_2 + a_1b_2c_3 + a_3b_2c_1 + a_1b_3c_2 + a_2b_3c_1)$.

It follows from (7) and (8) that

(9) $a_1^3 + a_2^3 + a_3^3 + sa_1a_2a_3 = b_1^3 + b_2^3 + b_3^3 + sb_1b_2b_3.$

Similar, we can derive

(10) $b_1^3 + b_2^3 + b_3^3 + sb_1b_2b_3 = c_1^3 + c_2^3 + c_3^3 + sc_1c_2c_3.$

By considering $b_1 \cdot (2a) + b_2 \cdot (2d) + b_3 \cdot (2g)$, $c_1 \cdot (2a) + c_2 \cdot (2d) + c_3 \cdot (2g)$, $a_1 \cdot (2b) + a_2(2e) + a_3 \cdot (2h)$, $c_1 \cdot (2b) + c_2(2e) + c_3 \cdot (2h)$, $a_1(2c) + a_2 \cdot (2f) + a_3 \cdot (2i)$ and $b_1 \cdot (2c) + b_2 \cdot (2f) + b_3 \cdot (2i)$, we have, respectively, the following

(11a) $t\left[3(a_1^2b_1 + a_2^2b_2 + a_3^2b_3) + s\,(a_2b_1 + a_1b_2)a_3 + a_1a_2b_3)\right]$
$-6\left[3(b_1^2c_1 + b_2^2c_2 + b_3^2c_3) + s\big((b_1c_2 + b_2c_1)b_3 + b_1b_2c_3\big)\right] = 0$

(11b) $t\left[3(a_1^2c_1 + a_2^2c_2 + a_3^2c_3) + s\big((a_1c_2 + a_2c_1)a_3 + a_1a_2c_3\big)\right]$

$$-6\big[3(c_1^2b_1+c_2^2b_2+c_3^2b_3)+s\big((b_1c_2+c_1b_2)c_3+c_1c_2b_3\big)\big]=0$$

(11c) $t\big[3(b_1^2a_1+b_2^2a_2+b_3^2a_3)+s\big((a_1b_2+b_1a_2)b_3+b_1b_2a_3\big)\big]$
$-6\big[3(a_1^2c_1+a_2^2c_2+a_3^2c_3)+s\big((a_1c_2+a_2c_1)a_3+a_1a_2c_3\big)\big]=0$

(11d) $t\big[3(b_1^2c_1+b_2^2c_2+b_3^2c_3)+s\big((b_1c_2+b_2c_1)b_3+b_1b_2c_3\big)\big]=0$
$-6\big[3(c_1^2a_1+c_2^2a_2+c_3^2a_3)+s\big((a_1c_2+a_2c_1)c_3+c_1c_2a_3\big)\big]=0$

(11e) $t\big[3(c_1^2d_1+c_2^2a_2+c_3^2a_3)+s\big((a_1c_2+a_2c_1)c_3+c_1c_2a_3\big)\big]$
$-6\big[3(a_1^2b_1+a_2^2b_2+a_3^2b_3)+s\big((a_2b_1+a_1b_2)a_3+a_1a_2b_3\big)\big]=0$

(11f) $t\big[3(c_1^2b_1+c_2^2b_2+c_3^2b_3)+s\big((b_1c_2+c_1b_2)c_3+c_1c_2b_3\big)\big]$
$-6\big[3(b_1^2a_1+b_2^2a_2+b_3^2a_3)+s\big((a_1b_2+a_2b_1)b_3+b_1b_2a_3\big)\big]=0.$

Since

$$\det\begin{pmatrix} t&0&0&6&0&0\\ 0&t&0&0&0&6\\ 0&6&t&0&0&0\\ 0&0&0&t&6&0\\ 6&0&0&0&t&0\\ 0&0&6&0&0&t \end{pmatrix}=(t^3-216)^2\neq 0,$$

(11a)-(11f) imply

(12a) $3a_1^2b_1+3a_1^2b_2+3a_3^2b_3+s\big((a_1b_2+a_2b_1)a_3+a_1a_2b_3\big)=0$

(12b) $3a_1^2c_1+3a_2^2c_2+3a_3^2c_3+s\big((a_1c_2+a_2c_1)a_3+a_1a_2c_3\big)=0$

(12c) $3b_1^2a_1+3b_2^2a_2+3b_3^2a_3+s\big((a_1b_2+a_2b_1)b_3+b_1b_2a_3\big)=0$

(12d) $3b_1^2c_1+3b_2^2c_2+3b_3^2c_3+s\big((b_1c_2+b_2c_1)b_3+b_1b_2c_3\big)=0$

(12e) $3c_1^2a_1+3c_2^2a_2+3c_3^2a_3+s\big((a_1c_2+a_2c_1)c_3+c_1c_2a_3\big)=0$

(12f) $3c_1^2b_1+3c_2^2b_2+3c_3^2b_3+s\big((b_1c_2+b_2c_1)c_3+c_1c_2b_3\big)=0.$

By considering (9)-(12a)-(12c), (10)-(12d)-(12f) and (7)-(8) respectively, we have the following

(13) $(a_1-b_1)^3+(a_2-b_2)^3+(a_3-b_3)^3+s(a_1-b_1)(a_2-b_2)(a_3-b_3)=0$

(14) $(b_1-c_1)^3+(b_2-c_2)^3+(b_3-c_3)^3+s(b_1-c_1)(b_2-c_2)(b_3-c_3)=0$

(15) $(c_1-a_1)^3+(c_2-a_2)^3+(c_3-a_3)^3+s(c_1-a_1)(c_2-a_2)(c_3-a_3)=0.$

Thus the points P_1, P_2 and P_3 are on the elliptic curve

$(*)$ $u^3+v^3+w^3+suvw=0.$

Observe that the matrices

$$\begin{pmatrix} a_1&\rho b_1&c_1\\ a_2&\rho b_2&c_2\\ a_3&\rho b_3&c_3 \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} a_1&b_1&\rho^2c_1\\ a_2&b_2&\rho^2c_2\\ a_3&b_3&\rho^2c_3 \end{pmatrix}$$

satisfy equations (2a)-(2i) with t replaced by $\rho^2 t$ and ρt respectively and hence induced algebra isomorphisms from A_s to $A_{\rho^2 t}$ and from A_s to $A_{\rho t}$ respectively. Thus we have

(16) $(a_1-\rho b_1)^3+(a_2-\rho b_2)^3+(a_3-\rho b_3)^3+s(a_1-\rho b_1)(a_2-\rho b_2)(a_3-\rho b_3) = 0$

(17) $(b_1-\rho^2 c_1)^3+(b_2-\rho^2 c_2)^3+(b_3-\rho^2 c_3)^3+s(b_1-\rho^2 c_1)(b_2-\rho^2 c_2)(b_3-\rho^2 c_3) = 0$

and the points T_1, T_2, T_3, W_1, W_2, and W_3 are on the elliptic curve $(*)$.

Recall that elliptic curve $(*)$ has abelian group structure as follows. For any three points Q_1, Q_2, Q_3 on the elliptic curve $(*)$

$$Q_1+Q_2+Q_3=0 \Longleftrightarrow Q_1, Q_2 \text{ and } Q_3 \text{ are collinear.}$$

One checks easily the following relations

(18a) $P_1+P_2+P_3=0$, (18g) $P_1+T_2+W_3=0$
(18b) $T_1+T_2+T_3=0$, (18h) $P_1+T_3+W_2=0$
(18c) $W_1+W_2+W_3=0$, (18i) $P_2+T_1+W_3=0$
(18d) $P_1+T_1+W_1=0$, (18j) $P_2+T_3+W_2=0$
(18e) $P_2+T_2+W_2=0$, (18k) $P_3+T_2+W_1=0$
(18f) $P_3+T_3+W_3=0$, (18l) $P_3+T_1+W_2=0$

Observe that

$$W_3-W_2=(P_1+T_2+W_3)-(P_1+T_3+W_2)+(T_3-T_2)=T_3-T_2$$

and

$$W_2-W_1=(P_3+T_1+W_2)-(P_3+T_2+W_1)+(T_2-T_1)=T_2-T_1.$$

Since

$$3W_1+(W_2-W_1)+(W_3-W_2)+(W_2-W_1)=W_1+W_2+W_3=0$$
$$=T_1+T_2+T_3=3T_1+(T_2-T_1)+(T_3-T_2)+(T_2-T_1),$$

we have

$$3W_1=3T_1.$$

By similar computations, we have

(17a) $3P_1=3W_1=3T_1$
(17b) $3P_2=3W_2=3T_2$
(17c) $3P_3=3W_3=3T_3$.

By considering (18a)+(18d)+(18h)-(18e)-(18f), we get

$$3P_1 = 0 = 3W_1 = 3T_1.$$

Recall that an inflection point on $(*)$ is an order 3 point on $(*)$ or equivalently a point where there exists a line tangent to $(*)$ at order 3 (see [1] p. 322, example 4.8.3). Hence P_1, W_1 and T_1 are inflection points on $(*)$. Similarly, we can show that P_2, W_2, T_2, P_3, W_3, and T_3 are inflection points on $(*)$.

On the other hand, we can check that the line

$$3\rho^i u + 3\rho^{2i} v = sw$$

intersects

$$u^3 + v^3 + w^3 + suvw = 0$$

at $(\rho^i, -1, 0)$ with order 3. Thus $(\rho^i, -1, 0)$, $1 \leq i \leq 3$, are inflection points of $(*)$. Similarly, we can show that $(0, -1, \rho^i)$, $(-1, 0, \rho^i)$, $1 \leq i \leq 3$, are inflection points on $(*)$.

Observe that the nine points $P_1, P_2, P_3, T_1, T_2, T_3, W_1, W_2$ and W_3 are distinct since $\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_3 \\ a_3 & b_3 & c_3 \end{pmatrix} \neq 0$.

Let

$$A = \begin{pmatrix} \rho & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} \rho & \rho^2 & 1 \\ \rho^2 & \rho & 1 \\ 1 & 1 & 1 \end{pmatrix}, \ C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

One can check that A, B, C, D and E are solutions of (2a)-(2i) and they induce moduli algebra isomorphisms

$A: \quad A_t \cong A_{\rho t}$
$B: \quad A_t \cong A_{R(t)}$
$C: \quad A_t \cong A_t$
$D: \quad A_t \cong A_t$
$E: \quad A_t \cong A_t.$

Let $U \subset GL(3, \boldsymbol{C})$ be the subgroup generated by A, B, C, D and E.

Theorem 3. *For $t^3, s^3 \neq 0$, 216 and -27, all algebra isomorphisms from A_s to A_t in Lemma 1 are in U up to a scalar multiplication.*

Proof. From Lemma 2, we know that the set of nine points $\{P_1, P_2, P_3, T_1, T_2, T_3, W_1, W_2$ and $W_3\}$ coinsides with the set of nine points $\{(\rho^i, -1, 0), (0, \rho^i, -1), (-1, 0, \rho^i) : i = 1, 2, 3\}$. Thus among $a_1 - b_1$, $b_1 - c_1$, $c_1 - a_1$, $a_1 - \rho b_1$, $b_1 - \rho c_1$, $c_1 - \rho a_1$, $a_1 - \rho^2 b_1$, $b_1 - \rho^2 c_1$, and $c_1 - \rho^2 a_1$, three of them have to be zero.

Observe that except for the case $a_1 = b_1 = 0$, there is at most one of $a_1 - b_1$, $a_1 - \rho b_1$, $a_1 - \rho^2 b_1$ which can be zero. Similarly, there is at most one of $b_1 - c_1$, $b_1 - \rho c_1$, $b_1 - \rho^2 c_1$ (respectively $c_1 - a_1$, $c_1 - \rho a_1$, $c_1 - \rho^2 a_1$) which can be zero unless $b_1 = c_1 = 0$ (respectively $c_1 = a_1 = 0$). Hence if $(a_1, b_1) \neq (0, 0)$, $(b_1, c_1) \neq (0, 0)$ and $(c_1, a_1) \neq (0, 0)$, then there is exactly one of $a_1 - b_1$, $a_1 - \rho b_1$, $a_1 - \rho^2 b_1$ (respectively one of $b_1 - c_1$, $b_1 - \rho c_1$, $b_1 - \rho^2 c_1$; one of $c_1 - a_1$, $c_1 - \rho a_1$, $c_1 - \rho^2 a_1$) is zero.

We claim that if $a_1 = b_1 = 0$, then either $b_2 = c_2 = 0$ or $c_2 = a_2 = 0$. For if $a_1 = b_1 = 0$, then the three points $(0, a_2 - b_2, a_3 - b_3)$, $(0, a_2 - \rho b_2, a_3 - \rho b_3)$, $(0, a_2 - \rho^2 b_2, a_3 - \rho^2 b_3)$ are exactly $(0, \rho^i, -1)$, $i = 1, 2, 3$. These imply $a_2 - b_2$, $a_2 - \rho b_2$ and $a_2 - \rho^2 b_2$ are nonzero. On the other hand, there is at most one zero for $b_2 - c_2$, $b_2 - \rho c_2$, and $b_2 - \rho^2 c_2$ (respectively, one zero for $c_2 - a_2$, $c_2 - \rho a_2$, $c_2 - \rho^2 a_2$) if b_2 and c_2 (respect ively c_2 and a_2) are non-zero. By comparing the second components of the two sets of nine inflection points, we conclude that our claim is true.

Similarly we have $b_3 = c_3 = 0$ or $c_3 = a_3 = 0$ if $a_1 = b_1 = 0$. This is proved by comparing the third components of the two sets of nine inflection points.

From the above arguments, we have the following cases

Case 1. $a_1 = b_1 = 0$. Then the matrix $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ in Lemma 1 is of the following form

$$\text{Case (1a)} \quad \begin{pmatrix} 0 & 0 & c_1 \\ a_2 & 0 & 0 \\ 0 & b_3 & 0 \end{pmatrix} \quad \text{or} \quad \text{Case (1b)} \quad \begin{pmatrix} 0 & 0 & c_1 \\ 0 & b_2 & 0 \\ a_3 & 0 & 0 \end{pmatrix}.$$

In Case (1a), in view of Lemma 2, we know that the set of nine points $\{(0 : a_2 : -b_3), (-c_1 : 0 : b_3), (c_1 : -a_2 : 0), (0 : a_2 : -\rho b_3), (-\rho c_1 : 0 : b_3), (c_1 : -\rho a_2 : 0), (0 : a_2 : -\rho^2 b_3), (-\rho^2 c_1 : 0 : b_3), (c_1 : -\rho^2 a_2 : 0)\}$ in $\boldsymbol{CP}^2$ coincides with the set of none points $\{(0 : \rho^i : -1), (-1 : 0 : \rho^i), (\rho^i : -1 : 0), i = 0, 1, 2\}$. Clearly a necessary and sufficient condition for the above two sets to be equal are $c_1/b_3 \in \{1, \rho, \rho^2\}$ and $a_2/b_3 \in \{1, \rho, \rho^2\}$. Hence

$$(21) \quad \begin{pmatrix} 0 & 0 & c_1 \\ a_2 & 0 & 0 \\ 0 & b_3 & 0 \end{pmatrix} = \begin{pmatrix} b_3 & 0 & 0 \\ 0 & b_3 & 0 \\ 0 & 0 & b_3 \end{pmatrix} \begin{pmatrix} c_1/b_3 & 0 & 0 \\ 0 & a_2/b_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

where $c_1/b_3, a_2/b_3 \in \{1, \rho, \rho^2\}$. Observe that

$$(22)\quad CDAE = \begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix} \begin{pmatrix} \rho&0&0\\ 0&1&0\\ 0&0&1 \end{pmatrix} \begin{pmatrix} 0&0&1\\ 0&1&0\\ 1&0&0 \end{pmatrix}$$

$$= \begin{pmatrix} 0&0&1\\ 1&0&0\\ 0&1&0 \end{pmatrix} \begin{pmatrix} 0&0&\rho\\ 0&1&0\\ 1&0&0 \end{pmatrix} = \begin{pmatrix} 1&0&0\\ 0&0&\rho\\ 0&1&0 \end{pmatrix}$$

$$(23)\quad CDAED = \begin{pmatrix} 1&0&0\\ 0&0&\rho\\ 0&1&0 \end{pmatrix} \begin{pmatrix} 1&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix} = \begin{pmatrix} 1&0&0\\ 0&\rho&0\\ 0&0&1 \end{pmatrix}.$$

In view of (23), all solutions (21) in Case (1a) are in the group U up to a scalar multiplication. These nine projective linear transformations induce algebra isomorphisms from A_t to A_s where

$$s = \rho^k t \qquad \text{i.e. } s = P^k(t) \text{ if } \frac{c_1}{b_3} \cdot \frac{a_2}{b_2} = \rho^{2-k}.$$

In Case (lb), by using the same argument as above, we have

$$(24)\quad \begin{pmatrix} 0&0&c_1\\ 0&b_2&0\\ a_3&0&0 \end{pmatrix} = \begin{pmatrix} a_3&0&0\\ 0&a_3&0\\ 0&0&a_3 \end{pmatrix} \begin{pmatrix} c_1/a_3&0&0\\ 0&b_2/a_3&0\\ 0&0&1 \end{pmatrix} \begin{pmatrix} 0&0&1\\ 0&1&0\\ 1&0&0 \end{pmatrix}$$

where c_1/a_3 and b_2/a_3 are in $\{1, \rho, \rho^2\}$. All solutions (24) in Case (1b) are in the group U up to a scalar multiplication. These nine projective linear transformations induce algebra isomorphisms from A_t to A_s where

$$s = \rho^k t, \qquad \text{i.e. } s = P^k(t), \text{ if } \frac{c_1}{a_3} \cdot \frac{b_2}{a_3} = \rho^{2-k}.$$

Case 2. $b_1 = c_1 = 0$. Then the matrix $\begin{pmatrix} a_1&b_1&c_1\\ a_2&b_2&c_2\\ a_3&b_3&c_3 \end{pmatrix}$ in Lemma 1 is of the following from

$$\text{Case (2a)} \quad \begin{pmatrix} a_1&0&0\\ 0&b_2&0\\ 0&0&c_3 \end{pmatrix} \quad \text{or} \quad \text{Case (2b)} \quad \begin{pmatrix} a_1&0&0\\ 0&0&c_2\\ 0&b_3&0 \end{pmatrix}.$$

In Case (2a), by using the same argument as above, we have

$$(25)\quad \begin{pmatrix} a_1&0&0\\ 0&b_2&0\\ 0&0&c_3 \end{pmatrix} = \begin{pmatrix} c_3&0&0\\ 0&c_3&0\\ 0&0&c_3 \end{pmatrix} \begin{pmatrix} a_1/c_3&0&0\\ 0&b_2/c_3&0\\ 0&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0\\ 0&1&0\\ 0&0&1 \end{pmatrix}$$

where $a_1/c_3, b_2/c_3 \in \{1, \rho, \rho^2\}$. All solutions (25) in Case (2a) are in the group U up to scalar multiplication. These nine projective linear transformations induce algebra isomorphisms from A_t to A_s where

$$s = \rho^k t \qquad \text{i.e. } s = P^k(t), \qquad \text{if } \frac{a_1}{c_3} \cdot \frac{b_2}{c_3} = \rho^{2-k}.$$

In Case (2b), by using the same argument as above, we have

$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & 0 & c_2 \\ 0 & b_3 & 0 \end{pmatrix} = \begin{pmatrix} b_3 & 0 & 0 \\ 0 & b_3 & 0 \\ 0 & 0 & b_3 \end{pmatrix} \begin{pmatrix} a_1/b_3 & 0 & 0 \\ 0 & c_2/b_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{26}$$

where $a_1/b_3, c_2/b_3 \in \{1, \rho, \rho^2\}$. All solutions (26) in Case (2b) are in the group U up to scalar multiplication. These nine projective linear transformations induce algebra isomorphisms from A_t to A_s where

$$s = \rho^k t, \qquad \text{i.e. } s = P^k(t), \qquad \text{if } \frac{a_1}{b_3} \cdot \frac{c_2}{b_3} = \rho^{2-k}.$$

Case 3. $c_1 = a_1 = 0$. Then the matrix $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ in Lemma 1 is of the following from

$$\text{Case (3a)} \quad \begin{pmatrix} 0 & b_1 & 0 \\ a_2 & 0 & 0 \\ 0 & 0 & c_3 \end{pmatrix} \quad \text{or} \quad \text{Case (3b)} \quad \begin{pmatrix} 0 & b_1 & 0 \\ 0 & 0 & c_2 \\ a_3 & 0 & 0 \end{pmatrix}.$$

In Case (3a), by using the same argument as before, we have

$$\begin{pmatrix} 0 & b_1 & 0 \\ a_2 & 0 & 0 \\ 0 & 0 & c_3 \end{pmatrix} = \begin{pmatrix} c_3 & 0 & 0 \\ 0 & c_3 & 0 \\ 0 & 0 & c_3 \end{pmatrix} \begin{pmatrix} b_1/c_3 & 0 & 0 \\ 0 & a_2/c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{27}$$

where $b_1/c_3, a_2/c_3 \in \{1, \rho, \rho^2\}$. All solutions (27) in Case (3a) are in the group U up to scalar multiplication. These nine projective linear transformations induce algebra isomorphisms from A_t to A_s where

$$s = \rho^k t, \qquad \text{i.e. } s = P^k(t), \qquad \text{if } \frac{a_1}{b_3} \cdot \frac{c_2}{b_3} = \rho^{2-k}.$$

In Case (3b), by using the same argument as before, we have

$$\begin{pmatrix} 0 & b_1 & 0 \\ 0 & 0 & c_2 \\ a_3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_3 & 0 & 0 \\ 0 & a_3 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \begin{pmatrix} b_1/a_3 & 0 & 0 \\ 0 & c_2/a_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{28}$$

where $b_1/a_3, c_2/a_3 \in \{1, \rho, \rho^2\}$. All solutions (28) in Case (3b) are in the group U up to scalar multiplication. These nine projective linear transformations induce algebra isomorphisms from A_t to A_s where

$$s = \rho^k t, \qquad \text{i.e. } s = P^k(t), \qquad \text{if } \frac{b_1}{a_3} \cdot \frac{c_2}{a_3} = \rho^{2-k}.$$

Case 4. $(a_1, b_1) \neq (0, 0)$, $(b_1, c_1) \neq (0, 0)$ and $(c_1, a_1) \neq (0, 0)$. In this case, there is exactly one of $a_1 = b_1$, $a_1 - \rho b_1$, $a_1 - \rho^2 b_1$ (respectively one of $b_1 - c_1$, $b_1 - \rho c_1$, $b_1 - \rho^2 c_1$ and one of $c_1 - a_1$, $c_1 - \rho a_1$, $c_1 - \rho^2 a_1$) is zero. Hence we conclude that none of a_1, b_1 or c_1 can be zero. By repeating the same arguments as in Case 1 to Case 3, we conclude that none of a_2, b_2, c_2, a_3, b_3 or c_3 can be zero.

Case (4a). $a_1 - b_1 = 0$. Then $a_1 - \rho b_1 \neq 0$ and $a_1 - \rho^2 b_1 \neq 0$. $a_1 - \rho b_1 \neq 0$ implies $a_2 - \rho b_2 = 0$ or $a_3 - \rho b_3 = 0$ since $(a_1 - \rho b_1, a_2 - \rho b_2, a_3 - \rho b_3)$ is one of the nine inflection points. Similarly $a_1 - \rho^2 b_1 \neq 0$ implies $a_2 - \rho^2 b_2 = 0$ or $a_3 - \rho^2 b_3 = 0$. Therefore we have only two cases: either $a_1 = b_1$, $a_2 = \rho b_2$, $a_3 = \rho^2 b_3$ or $a_1 = b_1$, $a_2 = \rho^2 b_2$, $a_3 = \rho b_3$.

Case (4b). $a_1 - \rho b_1 = 0$. Then $a_1 - b_1 \neq 0$ and $a_1 = \rho^2 b_1 \neq 0$. $a_1 - b_1 \neq 0$ implies $a_2 - b_2 = 0$ or $a_3 - b_3 = 0$. $a_1 - \rho^2 b_1 \neq 0$ implies $a_2 - \rho^2 b_2 = 0$ or $a_3 - \rho^2 b_3 = 0$. Therefore we have only two cases: $a_1 = \rho b_1$, $a_2 = b_2$, $a_3 = \rho^2 b_3$ or $a_1 = \rho b_1$, $a_2 = \rho^2 b_2$, $a_3 = b_3$.

Case (4c). $a_1 - \rho^2 b_1 = 0$. We can prove similarly that there are only two cases: $a_1 = \rho^2 b_1$, $a_2 = b_2$, $a_3 = \rho b_3$ or $a_1 = \rho^2 b_1$, $a_2 = \rho^2 b_2$, $a_3 = b_3$.

In all these cases, we have shown that $\left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}\right\} = \{1, \rho, \rho^2\}$ as sets. Similarly we have $\left\{\frac{b_1}{c_1}, \frac{b_2}{c_2}, \frac{b_3}{c_3}\right\} = \{1, \rho, \rho^2\}$ and $\left\{\frac{c_1}{a_1}, \frac{c_2}{a_2}, \frac{c_3}{a_3}\right\} = \{1, \rho, \rho^2\}$ as sets. We shall discuss each of these possibilities.

Case 1. $\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}\right) = (1, \rho, \rho^2)$.

Case 1 (i). $\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}\right) = (1, \rho, \rho^2)$, $\left(\frac{b_1}{c_1}, \frac{b_2}{c_2}, \frac{b_3}{c_3}\right) = (1, \rho, \rho^2)$. These imply $\left(\frac{c_1}{a_1}, \frac{c_2}{a_2}, \frac{c_3}{a_3}\right) = (1, \rho, \rho^2)$. In this case, the set of three points $\big\{(0 : a_2 - b_2 : a_3 - b_3), (0 : b_2 - c_2 : b_3 - c_3), (0 : c_2 - a_2 : c_3 - a_3)\big\}$ in $\boldsymbol{P}^2$ is the same as the set of three points $\big\{(0 : \rho^i : -1) : i = 0, 1, 2\big\}$ in $\boldsymbol{P}^2$.

Recall that $a_2 = \rho b_2 = \rho^2 c_2, a_3 = \rho^2 b_3 = \rho c_3$. It follows that

$$\begin{aligned}&\big\{(0 : a_2 - b_2 : a_3 - b_3), (0 : b_2 - c_2 : b_3 - c_3), (0 : c_2 - a_2 : c_3 - a_3)\big\}\\ &= \big\{\big(0 : \rho c_2(\rho - 1) : \rho c_3(1 - \rho)\big), \big(0 : c_2(\rho - 1) : c_3(\rho - 1)(\rho + 1)\big),\\ &\quad \big(0 : c_2(1 - \rho)(1 + \rho) : c_3(1 - \rho)\big)\big\}\\ &= \big\{(0 : c_2 : -c_3), (0 : c_2 : -c_3\rho^2), (0 : c_2 : -c_3\rho)\big\}\end{aligned}$$

In order for this set of three points to coincide with the set $\big\{(0 : \rho^i : -1) : i = 0, 1, 2\big\}$, a necessary and sufficient condition is that $\frac{c_3}{c_2} = \gamma \in \{1, \rho, \rho^2\}$. Hence

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & \gamma c_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \rho^2 & \rho & 1 \\ \rho & \rho^2 & 1 \end{pmatrix} = \begin{pmatrix} c_1 & c_1 & c_1 \\ c_2\rho^2 & c_2\rho & c_2 \\ \gamma c_2\rho & \gamma c_2\rho^2 & \gamma c_2 \end{pmatrix}.$$

Equations (2a)–(2c) imply

$$t = \frac{3(6c_1^2 - s\gamma c_2^2)}{3c_1^2 + s\gamma c_2^2}. \tag{19}$$

Equations (2d)–(2i) imply

$$t = \frac{3(6\gamma^2 c_2 - sc_1)}{3\gamma^2 c_2 + sc_1}. \tag{20}$$

By comparing (19) and (20) and using the fact that $\gamma^3 = 1$, we have $c_1^3 = c_2^3$ i.e., $c_2/c_1 \in \{1, \rho, \rho^2\}$. Thus we have shown that c_2/c_1 and c_3/c_1 are in $\{1, \rho, \rho^2\}$.

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_1 & 0 \\ 0 & 0 & c_1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2/c_1 & 0 \\ 0 & 0 & c_3/c_1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \rho^2 & \rho & 1 \\ \rho & \rho^2 & 1 \end{pmatrix} \tag{21}$$

where $c_1 \in \boldsymbol{C}^*$ and $c_2/c_1, c_3/c_1 \in \{1, \rho, \rho^2\}$. Hence we have nine solutions in $PGL(3, \boldsymbol{C})$. We can list them as follows.

$$\begin{pmatrix} 1 & 1 & 1 \\ \rho^2 & \rho & 1 \\ \rho & \rho^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \rho^2 & \rho \\ \rho & \rho^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \rho^2 & \rho & 1 \\ \rho^2 & 1 & \rho \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \rho^2 & \rho \\ \rho^2 & 1 & \rho \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & \rho^2 & \rho \\ 1 & \rho & \rho^2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \rho & 1 & \rho^2 \\ \rho^2 & 1 & \rho \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \rho & 1 & \rho^2 \\ 1 & \rho & \rho^2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \rho & 1 & \rho^2 \\ \rho & \rho^2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ \rho^2 & \rho & 1 \\ 1 & \rho & \rho^2 \end{pmatrix}.$$

Observe that

$$EB = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \rho & \rho^2 & 1 \\ \rho^2 & \rho & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \rho^2 & \rho & 1 \\ \rho & \rho^2 & 1 \end{pmatrix} \tag{22}$$

$$\begin{aligned} CDAE &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \rho & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \rho \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \rho \\ 0 & 1 & 0 \end{pmatrix} \end{aligned} \tag{23}$$

$$CDAED = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \rho \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{24}$$

$$DCDAE = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \rho \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho \end{pmatrix}. \tag{25}$$

In view of (22), (23), (24) and (25), all solutions (21) is Case I (i) are in the group U up to a scalar multiplication. These nine projective linear transformations induce algebra isomorphisms from A_t to A_s where the corresponding transformations on the parameter space are respectively

$$s = \frac{18 - 3t}{3 + t} \qquad \big(s = R(t)\big)$$

$$s = \frac{18\rho^2 - 3\rho^2 t}{3 + t} \qquad \big(s = P^2R(t)\big)$$

$$s = \frac{18\rho^2 - 3\rho^2 t}{3 + t} \qquad \big(s = P^2R(t)\big)$$

$$s = \frac{18\rho - 3\rho t}{3 + t} \qquad \big(s = PR(t)\big)$$

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$$s = \frac{18\rho - 3\rho t}{3 + t} \qquad \big(s = PR(t)\big)$$

$$s = \frac{18\rho - 3\rho t}{3+t} \qquad \big(s = PR(t)\big).$$

Case 1 (ii). $(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{c_3}) = (1, \rho, \rho^2)$, $(\frac{b_1}{c_1}, \frac{b_2}{c_2}, \frac{b_3}{c_3}) = (1, \rho^2, \rho)$. These imply $(\frac{c_1}{a_1}, \frac{c_2}{a_2}, \frac{c_3}{a_3}) = (1, 1, 1)$, which contradicts the fact that $\{\frac{c_1}{a_1}, \frac{c_2}{a_2}, \frac{c_3}{a_3}\} = \{1, \rho, \rho^2\}$ as sets.

Case 1 (iii). $(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}) = (1, \rho, \rho^2)$, $(\frac{b_1}{c_1}, \frac{b_2}{c_2}, \frac{b_3}{c_3}) = (\rho, 1, \rho^2)$. These imply $(\frac{c_1}{a_1}, \frac{c_2}{a_2}, \frac{c_3}{a_3}) = (\rho^2, \rho^2, \rho^2)$, which contradicts the fact that $\{\frac{c_1}{a_1}, \frac{c_2}{a_2}, \frac{c_3}{a_3}\} = \{1, \rho, \rho^2\}$.

Case 1 (iv). $(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}) = (1, \rho, \rho^2)$, $(\frac{b_1}{c_1}, \frac{b_2}{c_2}, \frac{b_3}{c_3}) = (\rho, \rho^2, 1)$. These imply $(\frac{c_1}{a_1}, \frac{c_2}{a_2}, \frac{c_3}{a_3}) = (\rho^2, 1, \rho)$. Let $(c_1', c_2', c_3') = (\rho c_1, \rho c_2, \rho c_3)$. Then we have $(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}) = (1, \rho, \rho^2)$, $(\frac{b_1}{c_1'}, \frac{b_2}{c_2'}, \frac{b_3}{c_3'}) = (1, \rho, \rho^2)$, and $(\frac{c_1'}{a_1}, \frac{c_2'}{a_2}, \frac{c_3'}{a_3}) = (1, \rho, \rho^2)$.

So we are in Case 1 (i) again. Thus the solutions in this case are obtained by multiplying $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho^2 \end{pmatrix}$ on the right of solutions of Case 1 (i). They are of the form

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} c_1' & 0 & 0 \\ 0 & c_1' & 0 \\ 0 & 0 & c_1' \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2'/c_1' & 0 \\ 0 & 0 & c_3'/c_1' \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \rho^2 & \rho & 1 \\ \rho & \rho^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho^2 \end{pmatrix}$$

$$= \begin{pmatrix} c_1' & 0 & 0 \\ 0 & c_1' & 0 \\ 0 & 0 & c_1' \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2'/c_1' & 0 \\ 0 & 0 & c_3'/c_1' \end{pmatrix} \begin{pmatrix} 1 & 1 & \rho^2 \\ \rho^2 & \rho & \rho^2 \\ \rho & \rho^2 & \rho^2 \end{pmatrix}$$

where $c_1' \in \boldsymbol{C}^*$ and $c_2'/c_1', c_3'/c_1' \in \{1, \rho, \rho^2\}$. The nine solutions in $PGL(3, \boldsymbol{C})$ are

$$\begin{pmatrix} 1 & 1 & \rho \\ \rho^2 & \rho & \rho^2 \\ \rho & \rho^2 & \rho^2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & \rho^2 \\ 1 & \rho^2 & 1 \\ \rho & \rho^2 & \rho^2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & \rho^2 \\ \rho^2 & \rho & \rho^2 \\ \rho^2 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & \rho^2 \\ 1 & \rho^2 & 1 \\ \rho^2 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & \rho^2 \\ \rho & 1 & \rho \\ \rho & \rho^2 & \rho^2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & \rho^2 \\ \rho^2 & \rho & \rho^2 \\ 1 & \rho & \rho \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & \rho^2 \\ 1 & \rho^2 & 1 \\ 1 & \rho & \rho \end{pmatrix}, \begin{pmatrix} 1 & 1 & \rho^2 \\ \rho & 1 & \rho \\ 1 & \rho & \rho \end{pmatrix}.$$

In view of (22), (24) and (25), we see easily that all solutions are in the group U up to scalar multiplication. These nine projective transformations induce moduli algebra isomorphisms from A_t to A_s where the corresponding transformations on the parameter space are respectively

$$s = \frac{18 - 3\rho t}{3 + \rho t} \qquad (s = R\,P(t))$$

$$s = \frac{18\rho^2 - 3t}{3 + \rho t} \qquad (s = P^2 R\,P(t))$$

$$s = \frac{18\rho^2 - 3t}{3 + \rho t} \qquad (s = P^2 R\,P(t))$$

$$s = \frac{18\rho - 3\rho^2 t}{3 + \rho t} \qquad (s = PR\,P(t))$$

$$s = \frac{18\rho - 3\rho^2 t}{3 + \rho t} \qquad (s = PR\,P(t))$$

$$s = \frac{18\rho - 3\rho^2 t}{3 + \rho t} \qquad (s = PR\,P(t))$$

$$s = \frac{18 - 3\rho t}{3 + \rho t} \qquad (s = R\,P(t))$$

$$s = \frac{18 - 3\rho t}{3 + \rho t} \qquad (s = R\,P(t))$$

$$s = \frac{18\rho^2 - 3t}{3 + \rho t} \qquad (s = P^2 R\,P(t)).$$

Case 1 (v). $(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}) = (1, \rho, \rho^2)$, $(\frac{b_1}{c_1}, \frac{b_2}{c_2}, \frac{b_3}{c_3}) = (\rho^2, 1, \rho)$. These imply $(\frac{c_1}{a_1}, \frac{c_2}{a_2}, \frac{c_3}{a_3}) = (\rho, \rho^2, 1)$. Let $(c_1', c_2', c_3') = (\rho^2 c_1, \rho^2 c_2, \rho^2 c_3)$. Then we have $(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}) = (1, \rho, \rho^2)$, $(\frac{b_1}{c_1'}, \frac{b_2}{c_2'}, \frac{b_3}{c_3'}) = (1, \rho, \rho^2)$, and $(\frac{c_1'}{a_1}, \frac{c_2'}{a_2}, \frac{c_3'}{a_3}) = (1, \rho, \rho^2)$. We are in Case 1 (i) again. Thus the solutions in this case are obtained by multiplying $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho \end{pmatrix}$ on the right of solutions of Case 1 (i). They are of the form

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} c_1' & 0 & 0 \\ 0 & c_1' & 0 \\ 0 & 0 & c_1' \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2'/c_1' & 0 \\ 0 & 0 & c_3'/c_1' \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \rho^2 & \rho & 1 \\ \rho & \rho^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho \end{pmatrix}$$

$$= \begin{pmatrix} c_1' & 0 & 0 \\ 0 & c_1' & 0 \\ 0 & 0 & c_1' \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2'/c_1' & 0 \\ 0 & 0 & c_3'/c_1' \end{pmatrix} \begin{pmatrix} 1 & 1 & \rho \\ \rho^2 & \rho & \rho \\ \rho & \rho^2 & \rho \end{pmatrix}$$

where $c_1' \in \boldsymbol{C}^*$ and $c_2'/c_1', c_3'/c_1' \in \{1, \rho, \rho^2\}$. The nine solutions in $PGL(3, \boldsymbol{C})$ are

$$\begin{pmatrix} 1 & 1 & \rho \\ \rho^2 & \rho & \rho \\ \rho & \rho^2 & \rho \end{pmatrix}, \begin{pmatrix} 1 & 1 & \rho \\ 1 & \rho^2 & \rho^2 \\ \rho & \rho^2 & \rho \end{pmatrix}, \begin{pmatrix} 1 & 1 & \rho \\ \rho^2 & \rho & \rho \\ \rho^2 & 1 & \rho^2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & \rho \\ \rho & 1 & 1 \\ \rho & \rho^2 & \rho \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & \rho \\ \rho^2 & \rho & \rho \\ 1 & \rho & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & \rho \\ 1 & \rho^2 & \rho^2 \\ \rho^2 & 1 & \rho^2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & \rho \\ \rho & 1 & 1 \\ \rho^2 & 1 & \rho^2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & \rho \\ 1 & \rho^2 & \rho^2 \\ 1 & \rho & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & \rho \\ \rho & 1 & 1 \\ 1 & \rho & 1 \end{pmatrix}.$$

In view of (22), (24) and (25), we see easily that all solutions are in the group U up to scalar multiplication. These nine projective transformations induce moduli algebra isomorphisms from A_t to A_s where the corresponding transformations on the parameter space are respectively

$$s = \frac{18 - 3\rho^2 t}{3 + \rho^2 t} \qquad (s = R\,P^2(t))$$

$$s = \frac{18\rho^2 - 3\rho t}{3 + \rho^2 t} \qquad (s = P^2 R\,P^2(t))$$

$$s = \frac{18\rho^2 - 3\rho t}{3 + \rho^2 t} \qquad (s = P^2 R\,P^2(t))$$

$$s = \frac{18\rho - 3t}{3 + \rho^2 t} \qquad (s = PR\,P^2(t))$$

$$s = \frac{18\rho - 3t}{3 + \rho^2 t} \qquad (s = PR\,P^2(t))$$

$$s = \frac{18\rho - 3t}{3 + \rho^2 t} \qquad (s = PR\,P^2(t))$$

$$s = \frac{18 - 3\rho^2 t}{3 + \rho^2 t} \qquad (s = R\,P^2(t))$$

$$s = \frac{18 - 3\rho^2 t}{3 + \rho^2 t} \qquad (s = R\,P^2(t))$$

$$s = \frac{18\rho^2 - 3\rho t}{3 + \rho^2 t} \qquad (s = P^2 R\,P^2(t)).$$

Case 1 (vi). $\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}\right) = (1, \rho, \rho^2)$, $\left(\frac{b_1}{c_1}, \frac{b_2}{c_2}, \frac{b_3}{c_3}\right) = (\rho^2, \rho, 1)$. These imply $\left(\frac{c_1}{a_1}, \frac{c_2}{a_2}, \frac{c_3}{a_3}\right) = (\rho, \rho, \rho)$, which contradicts the fact that $\left\{\frac{c_1}{a_1}, \frac{c_2}{a_2}, \frac{c_3}{a_3}\right\} = \{1, \rho, \rho^2\}$ as sets.

Case 2. $\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}\right) = (1, \rho^2, \rho)$.

In this case if we multiply $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ on the left by the matrix $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, then we are in Case 1 again. Thus the solutions in Case 2 are obtained by multiplying $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ on the left to the solutions of Case 1. There are 27 solutions in $PGL(3, \boldsymbol{C})$. They are of th e following forms

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \rho^2 & \rho & 1 \\ \rho & \rho^2 & 1 \end{pmatrix}$$

where $\lambda_1, \lambda_2 \in \{1, \rho, \rho^2\}$ and the corresponding transformations on the parameter space are

$$s = P^k R(t) \qquad \text{with} \qquad \begin{cases} k = 0 \text{ if } \lambda_1\lambda_2 = 1 \\ k = 2 \text{ if } \lambda_1\lambda_2 = \rho \\ k = 1 \text{ if } \lambda_1\lambda_2 = \rho^2 \end{cases}$$

and

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & \rho^2 \\ \rho^2 & \rho & \rho^2 \\ \rho & \rho^2 & \rho^2 \end{pmatrix}$$

where $\lambda_1, \lambda_2 \in \{1, \rho, \rho^2\}$ and the corresponding transformations on the parameter space are

$$s = P^k R\, P(t) \qquad \text{with} \qquad \begin{cases} k = 0 \text{ if } \lambda_1\lambda_2 = 1 \\ k = 2 \text{ if } \lambda_1\lambda_2 = \rho \\ k = 1 \text{ if } \lambda_1\lambda_2 = \rho^2 \end{cases}$$

and

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & \rho \\ \rho^2 & \rho & \rho \\ \rho & \rho^2 & \rho \end{pmatrix}$$

where $\lambda_1, \lambda_2 \in \{1, \rho, \rho^2\}$ and the corresponding transformations on the parameter space are

$$s = P^k R\, P^2(t) \qquad \text{with} \qquad \begin{cases} k = 0 \text{ if } \lambda_1\lambda_2 = 1 \\ k = 2 \text{ if } \lambda_1\lambda_2 = \rho \\ k = 1 \text{ if } \lambda_1\lambda_2 = \rho^2 \end{cases}$$

Since all solutions of Case 1 are in U and the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = D$ is in U, all solutions in Case 2 are in U.

Case 3. $\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}\right) = (\rho, 1, \rho^2)$.

In this case, if we multiply $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ on the left by the matrix $C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then we are in Case 1 again. Thus the solutions in Case 3 are obtained by multiplying $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ on the left to the solutions of Case 1. There are 27 solutions in $PGL(3, C)$. They are of the forms

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \rho^2 & \rho & 1 \\ \rho & \rho^2 & 1 \end{pmatrix} \qquad \text{with} \qquad \lambda_1, \lambda_2 \in \{1, \rho, \rho^2\}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & \rho^2 \\ \rho^2 & \rho & \rho^2 \\ \rho & \rho^2 & \rho^2 \end{pmatrix} \qquad \text{with} \qquad \lambda_1, \lambda_2 \in \{1, \rho, \rho^2\}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & \rho \\ \rho^2 & \rho & \rho \\ \rho & \rho^2 & \rho \end{pmatrix} \qquad \text{with} \qquad \lambda_1, \lambda_2 \in \{1, \rho, \rho^2\}$$

Since $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = C$ is in U, all solutions in Case 3 are in U. These solutions respectively induce linear fractional transformations on the parameter space as

follows

$$s = P^k R(t) \qquad \text{with} \qquad \begin{cases} k = 0 \text{ if } \lambda_1\lambda_2 = 1 \\ k = 2 \text{ if } \lambda_1\lambda_2 = \rho \\ k = 1 \text{ if } \lambda_1\lambda_2 = \rho^2 \end{cases}$$

$$s = P^k R\, P(t) \qquad \text{with} \qquad \begin{cases} k = 0 \text{ if } \lambda_1\lambda_2 = 1 \\ k = 1 \text{ if } \lambda_1\lambda_2 = \rho \\ k = 2 \text{ if } \lambda_1\lambda_2 = \rho^2 \end{cases}$$

$$s = P^k R\, P^2(t) \qquad \text{with} \qquad \begin{cases} k = 0 \text{ if } \lambda_1\lambda_2 = 1 \\ k = 2 \text{ if } \lambda_1\lambda_2 = \rho \\ k = 1 \text{ if } \lambda_1\lambda_2 = \rho^2. \end{cases}$$

Case 4. $\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}\right) = (\rho, \rho^2, 1)$.

In this case, if we multiply $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ on the left by the matrix $CD = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, then we are in Case 1 again. Thus the solutions in Case 4 are obtained by multiplying $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = (CD)^{-1} = D^{-1}C^{-1} = DC$ on the left to the solutions of Case 1. There are 27 solutions in $PGL(3, \boldsymbol{C})$. They are of the forms

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \rho^2 & \rho & 1 \\ \rho & \rho^2 & 1 \end{pmatrix} \qquad \text{with} \qquad \lambda_1, \lambda_2 \in \{1, \rho, \rho^2\}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & \rho^2 \\ \rho^2 & \rho & \rho^2 \\ \rho & \rho^2 & \rho^2 \end{pmatrix} \qquad \text{with} \qquad \lambda_1, \lambda_2 \in \{1, \rho, \rho^2\}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & \rho \\ \rho^2 & \rho & \rho \\ \rho & \rho^2 & \rho \end{pmatrix} \qquad \text{with} \qquad \lambda_1, \lambda_2 \in \{1, \rho, \rho^2\}$$

Since $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = DC$ is in U, all solutions in Case 4 are in U. These solutions respectively induce linear fractional transformations on the parameter space as

follows

$$s = P^k R(t) \qquad \text{with} \qquad \begin{cases} k = 0 \text{ if } \lambda_1\lambda_2 = 1 \\ k = 2 \text{ if } \lambda_1\lambda_2 = \rho \\ k = 1 \text{ if } \lambda_1\lambda_2 = \rho^2 \end{cases}$$

$$s = P^k R\, P(t) \qquad \text{with} \qquad \begin{cases} k = 0 \text{ if } \lambda_1\lambda_2 = 1 \\ k = 2 \text{ if } \lambda_1\lambda_2 = \rho \\ k = 1 \text{ if } \lambda_1\lambda_2 = \rho^2 \end{cases}$$

$$s = P^k R\, P(t) \qquad \text{with} \qquad \begin{cases} k = 0 \text{ if } \lambda_1\lambda_2 = 1 \\ k = 2 \text{ if } \lambda_1\lambda_2 = \rho \\ k = 1 \text{ if } \lambda_1\lambda_2 = \rho^2. \end{cases}$$

Case 5. $\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}\right) = (\rho^2, 1, \rho)$.

In this case, if we multiply $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ on the left by the matrix $DC = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, then we are in Case 1 again. Thus the solutions in Case 5 are obtained by multiplying $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = (DC)^{-1} = CD$ on the left to the solutions of Case 1. There are 27 solutions in $PGL(3, \boldsymbol{C})$. They are of the forms

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \rho^2 & \rho & 1 \\ \rho & \rho^2 & 1 \end{pmatrix} \qquad \text{with} \qquad \lambda_1, \lambda_2 \in \{1, \rho, \rho^2\}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & \rho^2 \\ \rho^2 & \rho & \rho^2 \\ \rho & \rho^2 & \rho^2 \end{pmatrix} \qquad \text{with} \qquad \lambda_1, \lambda_2 \in \{1, \rho, \rho^2\}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & \rho \\ \rho^2 & \rho & \rho \\ \rho & \rho^2 & \rho \end{pmatrix} \qquad \text{with} \qquad \lambda_1, \lambda_2 \in \{1, \rho, \rho^2\}$$

Since $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = CD$ is in U, all solutions in Case 5 are in U. These solutions respectively induce linear fractional transformations on the parameter space as

follows

$$s = P^k R(t) \qquad \text{with} \qquad \begin{cases} k = 0 \text{ if } \lambda_1\lambda_2 = 1 \\ k = 2 \text{ if } \lambda_1\lambda_2 = \rho \\ k = 1 \text{ if } \lambda_1\lambda_2 = \rho^2 \end{cases}$$

$$s = P^k R\, P(t) \qquad \text{with} \qquad \begin{cases} k = 0 \text{ if } \lambda_1\lambda_2 = 1 \\ k = 2 \text{ if } \lambda_1\lambda_2 = \rho \\ k = 1 \text{ if } \lambda_1\lambda_2 = \rho^2 \end{cases}$$

$$s = P^k R\, P^2(t) \qquad \text{with} \qquad \begin{cases} k = 0 \text{ if } \lambda_1\lambda_2 = 1 \\ k = 2 \text{ if } \lambda_1\lambda_2 = \rho \\ k = 1 \text{ if } \lambda_1\lambda_2 = \rho^2. \end{cases}$$

Case 6. $\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}\right) = (\rho^2, \rho, 1)$.

In this case, if we multiply $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ on the left by the matrix $E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, then we are in Case 1 again. Thus the solutions in Case 6 are obtained by multiplying $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = E$ on the left to the solutions of Case 1. There are 27 solutions in $PGL(3, \boldsymbol{C})$. They are of the forms

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \rho^2 & \rho & 1 \\ \rho & \rho^2 & 1 \end{pmatrix} \qquad \text{with} \qquad \lambda_1, \lambda_2 \in \{1, \rho, \rho^2\}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & \rho^2 \\ \rho^2 & \rho & \rho^2 \\ \rho & \rho^2 & \rho^2 \end{pmatrix} \qquad \text{with} \qquad \lambda_1, \lambda_2 \in \{1, \rho, \rho^2\}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & \rho \\ \rho^2 & \rho & \rho \\ \rho & \rho^2 & \rho \end{pmatrix} \qquad \text{with} \qquad \lambda_1, \lambda_2 \in \{1, \rho, \rho^2\}$$

Since $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = E$ is in U, all solutions in Case 6 are in U. These solutions respectively induce linear fractional transformations on the parameter space as

follows

$$s = P^k R(t) \qquad \text{with} \qquad \begin{cases} k = 0 \text{ if } \lambda_1\lambda_2 = 1 \\ k = 2 \text{ if } \lambda_1\lambda_2 = \rho \\ k = 1 \text{ if } \lambda_1\lambda_2 = \rho^2 \end{cases}$$

$$s = P^k R\, P(t) \qquad \text{with} \qquad \begin{cases} k = 0 \text{ if } \lambda_1\lambda_2 = 1 \\ k = 1 \text{ if } \lambda_1\lambda_2 = \rho \\ k = 2 \text{ if } \lambda_1\lambda_2 = \rho^2 \end{cases}$$

$$s = P^k R\, P^2(t) \qquad \text{with} \qquad \begin{cases} k = 0 \text{ if } \lambda_1\lambda_2 = 1 \\ k = 2 \text{ if } \lambda_1\lambda_2 = \rho \\ k = 1 \text{ if } \lambda_1\lambda_2 = \rho^2. \end{cases}$$

From all the above arguments, we have proven the conclusion of Theorem 3. *Q.E.D.*

In the course of the proof of Theorem, we have proved the following theorem.

Theorem 4. *For $t^3, s^3 \neq 0$, 216 and -27, all algebra isomorphisms from A_t to A_s are given by one of the following 216 matrices (up to scalar multiplication). Each of the isomorphism induce a linear fractional transformation on parameter space as follows.*

Type I (54 matrices)

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

with $\lambda_1, \lambda_2 \in \{1, \rho, \rho^2\}$ and the induced fractional linear transformations on the parameter space given by $s = P^k(t)$ if $\lambda_1\lambda_2 = \rho^{-k}$.

Type II (162 matrices)
(a)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \rho^2 & \rho & 1 \\ \rho & \rho^2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \rho^2 & \rho & 1 \\ \rho & \rho^2 & 1 \end{pmatrix}$$

$$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&1\\\rho^2&\rho&1\\\rho&\rho^2&1\end{pmatrix},\ \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&1\\\rho^2&\rho&1\\\rho&\rho^2&1\end{pmatrix}$$

$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&1\\\rho^2&\rho&1\\\rho&\rho^2&1\end{pmatrix},\ \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&1\\\rho^2&\rho&1\\\rho&\rho^2&1\end{pmatrix}$$

with $\lambda_1, \lambda_2 \in \{1, \rho, \rho^2\}$ and the induced fractional linear transformations on the parameter space given by $s = P^k R(t)$ if $\lambda_1\lambda_2 = \rho^{-k}$.
(b)

$$\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&\rho^2\\\rho^2&\rho&\rho^2\\\rho&\rho^2&\rho^2\end{pmatrix},\qquad \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&\rho^2\\\rho^2&\rho&\rho^2\\\rho&\rho^2&\rho^2\end{pmatrix}$$

$$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&\rho^2\\\rho^2&\rho&\rho^2\\\rho&\rho^2&\rho^2\end{pmatrix},\ \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&\rho^2\\\rho^2&\rho&\rho^2\\\rho&\rho^2&\rho^2\end{pmatrix}$$

$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&\rho^2\\\rho^2&\rho&\rho^2\\\rho&\rho^2&\rho^2\end{pmatrix},\ \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&\rho^2\\\rho^2&\rho&\rho^2\\\rho&\rho^2&\rho^2\end{pmatrix}$$

with $\lambda_1, \lambda_2 \in \{1, \rho, \rho^2\}$ and the induced fractional linear transformations on the parameter space given by $s = P^k RP(t)$ if $\lambda_1\lambda_2 = \rho^{-k}$
(c)

$$\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&\rho\\\rho^2&\rho&\rho\\\rho&\rho^2&\rho\end{pmatrix},\qquad \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&\rho\\\rho^2&\rho&\rho\\\rho&\rho^2&\rho\end{pmatrix}$$

$$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&\rho\\\rho^2&\rho&\rho\\\rho&\rho^2&\rho\end{pmatrix},\ \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&\rho\\\rho^2&\rho&\rho\\\rho&\rho^2&\rho\end{pmatrix}$$

$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&\rho\\\rho^2&\rho&\rho\\\rho&\rho^2&\rho\end{pmatrix},\ \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}\begin{pmatrix}1&1&\rho\\\rho^2&\rho&\rho\\\rho&\rho^2&\rho\end{pmatrix}$$

with $\lambda_1, \lambda_2 \in \{1, \rho, \rho^2\}$ and the induced fractional linear transformations on the parameter space given by $s = P^k RP^2(t)$ if $\lambda_1\lambda_2 = \rho^{-k}$.

Corollary 5. *The group H of fractional linear transformations generated by P and R are as follows:*

$$H = \left\{ \begin{array}{llll} I(t) = t, & P(t) = \rho t, & P^2(t) = \rho^2 t, & R(t) = \frac{18-3t}{3+t}, \\ PR(t) = \frac{18\rho - 3\rho t}{3+t}, & P^2R(t) = \frac{18\rho^2 - 3\rho^2 t}{3+t}, & RP(t) = \frac{18-3\rho t}{3+\rho t} & \\ RP^2(t) = \frac{18\rho - 3t}{t+3\rho}, & PRP(t) = \frac{18-3\rho t}{t+3\rho^2}, & P^2RP(t) = \frac{18\rho^2 - 3t}{3+\rho t} & \\ PRP^2(t) = \frac{18\rho^2 - 3\rho t}{t+3\rho}, & P^2RP^2 = \frac{18-3t\rho^2}{t+3\rho} & & \end{array} \right\}$$

H is a group of order 12 isomorphic to alternating group on four letters. H is also the group of automorphisms of the parameter space which do not change the isomorphism types of the algebras A_t. For any t and s such that $t^3, s^3 \neq 216$ and -27, if A_s is isomorphic to A_t, then s must be one of those above twelve values.

Proof. Observe that

(31) $\quad PRPR = RP^2$
(32) $\quad P^2RP^2R = RP$
(33) $\quad P^3 = R^2 = I$

Since all elements in the group H generated by P and R are of the form

(34) $$\cdots P^{i_1} R^{i_2} P^{i_3} R^{i_4} P^{i_5} \cdots,$$

by using the relations (31), (32) and (33), we see that the expression (34) can be reduced to one of the following forms

(35) $$\cdots P^i R^j, \;\; R^j P^i, \;\; P^i R^l P^k \cdots$$

By using relations (35) again, we know that all possible expressions in (35) are listed in Corollary 5. The rest of Corollary 5 follows from Theorem 4 and the discussion before Lemma 1. $Q.E.D.$

Corollary 6. *The subgroup W of U (mod scalar) which consists of projective linear transformation inducing an identity on the parameter space is a normal subgroup of order 18 in U (mod scalar). The matrices in W are those matrices in Type I with $\lambda_1\lambda_2 = 1$. The quotient group U/W is isomorhpic to the group H in Corollary 1.*

Proof. This is an immediate consequence of Theorem 4. $Q.E.D.$

We are now ready to prove the following corollary.

Corollary 7. *Let $A_t = \langle 1, x, y, z, xy, yz, zx, xyz \rangle$ with multiplication rules $x^2 = -\frac{t}{3}yz$, $y^2 = -\frac{t}{3}zx$ and $z^2 = -\frac{t}{3}xy$ be a one parameter family of moduli* *algebras for $\tilde{E}_6$ singularities. Then $k(t) = \frac{t^3(t^3-216)^3}{(t^3+27)^3}$ is a modules of this one parameter family i.e. $A_s \simeq A_t$ if and only if $k(t) = k(s)$.*

Proof. In view of Theorem 4 and Corollary 5, we need to construct a modulus k which is invariant under P and R. Let $k(0) = k(6) = k(6\rho) = k(6\rho^2) = 0$. Then assume $k(\infty) = \infty$. The poles can occur only at ∞ or at those t's for which $t^3 + 27 = 0$. Writing k as a rational function of t, since it is at most 12 to 1, the numerator and denominator of the rational expression can have powers of t no greater than 12. Because of P, k is a function of t^3 which further restricts the possible powers of t which can appear. Write

$$k(t) = \frac{(t^3)^\lambda (t^3 - 216)^\mu}{(t^3 + 27)^\nu}.$$

Observe that $R(-2) = 24$ and hence $k(24) = k(-2)$, which implies

$$\frac{(-1)^\lambda 2^{3\lambda}(-1)^\mu 2^{5\mu} u^\mu}{(19)^\nu} = \frac{2^{9\lambda}3^{3\lambda}(24^3 - 6^3)^\mu}{(24^3 + 3^3)^\nu}$$

$$\Rightarrow \frac{(-1)^{\lambda+\mu} 2^{3\lambda+5\mu} 7^\mu}{(19)^\nu} = \frac{2^{9\lambda+3\mu} 3^{3\lambda+5\mu} 7^\mu}{3^{6\nu}(19)^\nu}$$

$$\Rightarrow \lambda + \mu = 4,\ 3\lambda + 5\mu - 6\nu = 0,\ 6\lambda - 2\mu = 0$$

$$\Rightarrow \lambda = 1,\ \mu = 3,\ \nu = 3.$$

A short computation shows that $k(t) = \frac{t^3(t^3-216)^3}{(t^3+27)^3}$ does have the property that $k(R(t)) = k(t)$. Thus $k(t)$ is the modulus function. *Q.E.D.*

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