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BERGMAN KERNELS ON RESOLUTIONS OF ISOLATED SINGULARITIES

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In memory of S. Bergman.

1. Introduction

The Bergman kernel is a basic biholomorphic invariant on domains in $\mathbb{C}^n$. A lot of important work has been done on its explicit computation and asymptotic expansion. Generalized to complex manifolds [K], the Bergman kernel continues to play a fundamental role. However, it seems that there is little attention given to the possible role of the Bergman kernel on analytic spaces, in connection with the study of singularities.

In this paper, we give an initial step in studying the Bergman kernel on a resolution of an isolated 2-dimensional singularity. The problem is to investigate how the Bergman kernel holds information on the singularity. Our main result (Theorem 5 and Theorem 7) is that for any Gorenstein surface singularity, the exceptional set of the resolution is exactly the minimal set of the Bergman kernel. The rational singularities will be studied explicitly and a canonical holomorphic 2-form will be given. Since the exceptional set provides crucial topological information on the resolution, our result shows that the analytic definition of the Bergman kernel contains important topological information on the singularity.

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1. Bergman kernel and monoidal transformation

Let M be a complex n-dimensional manifold. We recall the definition of the Bergman kernel. Let F be the set of all holomorphic n-forms φ on M such that $\left|\int_M \varphi \wedge \overline{\varphi}\right| < \infty$. ($\varphi$ will be called L^2 or square integrable.) F is a separable complex Hilbert space under the inner product $(\varphi_1, \varphi_2) = (\sqrt{-1})^{n^2} \int_M \varphi_1 \wedge \overline{\varphi}_2$. The corresponding norm $(\varphi, \varphi)^{\frac{1}{2}}$ will be denoted by $\|\varphi\|$. Let $\{\omega_j\}$ be a complete orthonormal basis of F. Then $K(z, \overline{w}) = \sum \omega_j(z) \wedge \overline{\omega_j(w)}$ can be

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shown to converge uniformly on compact subsets to a holomorphic $2n$-form on $M \times \overline{M}$. Here, $\overline{M}$ denotes the conjugate complex manifold obtained by taking the conjugate coordinate charts of M. Further, $K(z, \overline{w})$ is independent of the choice of complete orthonormal basis of F. If each point $z \in M$ is identified with the point $(z, \overline{z}) \in M \times \overline{M}$, then $K(z) = K(z, \overline{z})$ can be regarded as a $2n$-form on M. $K(z)$ is referred to as the Bergman kernel of M. Since the Hilbert space F with its inner product is invariant under biholomorphic maps, so is the Bergman kernel.

Theorem 1. *Let M be a complex n-dimensional manifold and A be a compact submanifold of M. Let $\pi : M_1 \to M$ be the blow up of M along A. Assume that $K_M(z)$ is well-defined. Then we have $K_{M_1}(z) = \pi^* K_M(z)$.*

Proof. Let F, F_1 be the respective spaces of L^2 holomorphic n-forms on M, M_1. Since π^* pulls back L^2 holomorphic n-forms on M to L^2 holomorphic n-forms on M_1, preserving inner product, we have an injective map $\pi^* : F \to F_1$.

Consider any $\varphi_1 \in F_1$. φ_1 defines a L^2 holomorphic n-form φ on $M \backslash A$. Then φ extends to a holomorphic n-form on the complex manifold M. Clearly $\varphi \in F$ and $\pi^* \varphi = \varphi_1$. Hence $\pi^* : F \to F_1$ is surjective.

For any complete orthonormal basis $\{\omega_j\}$ of F, $\{\pi^* \omega_j\}$ is a complete orthonormal basis of F_1. Hence

$$K_{M_1} = \sum \pi^* \omega_j \wedge \overline{\pi^* \omega_j} = \pi^* \sum_j \omega_j \wedge \overline{\omega_j} = \pi^* K_M$$

□

We propose to study the Bergman kernel on a resolution of an isolated 2-dimensional singularity. More precisely, let $(\widetilde{V}, 0)$ be a normal surface singularity in $\mathbb{C}^n$ and $\widetilde{\pi} : \widetilde{M} \to \widetilde{V}$ be a resolution of $\widetilde{V}$. We choose an open ball B in $\mathbb{C}^n$ centered at 0 such that 0 is the only singularity in the closure of $V = \widetilde{V} \cap B$. We shall consider $M = \widetilde{\pi}^{-1}(V)$ and denote by $\pi : M \to V$ the restriction of $\widetilde{\pi}$ to M. It is known that for a normal surface singularity, every holomorphic function in a deleted neighborhood of the singularity extends through the singularity.

Resolutions of isolated singularities are not unique. However, it is well known [L1, Theorem 5.10] that for any two resolutions $\pi_i : M_i \to V$, $i = 1, 2$, of a normal 2-dimensional singularity, there exist a resolution $\pi : M \to V$ and factorization $\theta_i : M \to M_i$ such that $\pi = \pi_i \theta_i$, $i = 1, 2$, and θ_1, θ_2 are iterated blow ups. Hence, according to Theorem 1, the choice of resolution is not important for the study of the Bergman kernel.

In what follows, we shall assume that $(V, 0)$ is a Gorenstein surface singularity. This means that there exists a holomorphic 2-form φ_0 on $V \backslash \{0\}$ which is nowhere vanishing on $V \backslash \{0\}$, i.e. the canonical bundle on $V \backslash \{0\}$ is trivial. We remark that hypersurface and complete intersection singularities are Gorenstein singularities. Further, we shall work, in the remainder of the paper, with the minimal resolution $\pi : M \to V$.

2. The minimal set of the Bergman kernel

Let $E = \pi^{-1}(0)$ be the exceptional set in M. Let φ_0 be a 2-form on $V\backslash\{0\}$ as described above. We shall denote the pull back $\pi^*(\varphi_0)$ also by φ_0. The context will be made clear so that no confusion would arise.

The geometric genus of V is given by

$$p_g = \dim H^1(M, \mathcal{O}) = \dim \Gamma(M\backslash E, \Omega^2)/\Gamma(M, \Omega^2)$$

For a reference to the identification of $H^1(M, \mathcal{O})$ with the quotient space of global sections, see [Y, p.56]. 0 is defined to be a rational singularity if $p_g = 0$.

Proposition 2. *Under the conditions given above, $p_g = 0$ if and only if φ_0 is a holomorphic 2-form on M.*

Proof. Let ψ be any element of $\Gamma(M\backslash E, \Omega^2)$. ψ defines a holomorphic 2-form on $V\backslash\{0\}$, which we also denote by ψ. On $V\backslash\{0\}$, $\psi = h\varphi_0$ for some holomorphic function h. As noted above, h extends to a holomorphic function on V. If we compose π and the extension of h, then on $M\backslash E$, we have $\psi = h\varphi_0$, where h is now a holomorphic function on M. Thus, if φ_0 is holomorphic on M, then so is ψ, which implies that $p_g = 0$. Conversely, if φ_0 is not holomorphic on M, then $\varphi_0 \in \Gamma(M\backslash E, \Omega^2)\backslash\Gamma(M, \Omega^2)$, hence $p_g \geq 1$. □

Our situation is clarified by the following theorem of Laufer.

Theorem 3. (*Laufer* [L2], *see also* [Y]) *A normal surface singularity is rational if and only if any holomorphic 2-form on a deleted neighborhood of the singular point is square integrable.*

On the other hand, we have the following lemma.

Lemma 4. *Let $(V, 0)$ be a normal surface singularity and M be the minimal resolution of V. Let E be the exceptional set of M. Let η be a meromorphic 2-form on M which is holomorphic and nowhere zero on $M\backslash E$. If η is not holomorphic on M, then E is the pole set of η.*

Proof. Let $E = \cup_{i=1}^n A_i$ be the irreducible decomposition of E. Then $(\eta) = -\sum_{i=1}^{k_1} a_iA_i + \sum_{i=k_2+1}^{n} b_iA_i$, where $a_i > 0$, $b_i > 0$ and $k_2 \geq k_1 \geq 1$. Let K be the canonical divisor on M. Then $K = (\eta)$. Since M is the minimal resolution, we have by the adjunction formula, $A_j \cdot K \geq 0$ for all $A_j \subset E$, i.e.

$$A_j \cdot \big(\sum_{i=1}^{k_1} a_iA_i - \sum_{i=k_2+1}^{n} b_iA_i\big) \leq 0, \quad \text{for all } j.$$

We want to prove that $k_1 = n$. Suppose on the contrary that $k_1 < n$. Recall that by a theorem of Artin [A] [L2], there exists a unique minimal positive cycle $Z = \sum z_iA_i$, where all $z_i > 0$, with the property that $A_j \cdot Z \leq 0$ for all $A_j \subset E$.

Let $Y = \sum_{i=1}^{k_1} a_i A_i - \sum_{i=k_2+1}^{n} b_i A_i$. Then $A_j \cdot Y \leq 0$ for all $A_j \subset E$. Since $k_1 < n$, we can create an effective cycle $Y_1 \neq 0$ with the following properties: $A_j \cdot Y_1 \leq 0$ for all $A_j \subset E$ and for some $A_{i_0} \subset E$, the coefficient of A_{i_0} in Y_1 is zero. In fact, if $k_2 = n$ and there is no b_i, let $Y_1 = Y$. If $k_2 < n$, by taking positive linear combination of Y and Z (i.e. $c_1 Y + c_2 Z$, $c_1, c_2 > 0$), we can reduce the number of negative coefficients in Y by at least one and get at least one zero coefficient. Repeating the procedure if necessary, we get Y_1.

Y_1 contradicts the minimality of Z, as follows. Let $Y_1 = \sum y_{1i} A_i$ and consider $Y_2 = \sum \min(y_{1i}, z_i) A_i$ (i.e. $Y_2 = \min(Y_1, Z)$). Since $z_{i_0} > 0$ and $y_{1i_0} = 0$, $Z \not\leq Y_2$. To see that $A_j \cdot Y_2 \leq 0$ for any fixed $A_j \subset E$, observe that since $A_j \cdot A_i \geq 0$ for all $i \neq j$, $A_j \cdot Y_2 \leq A_j \cdot Y_1$ (respectively $A_j \cdot Z$) if $\min(y_{1j}, z_j) = y_{1j}$ (respectively z_j). □

It is now natural to divide our discussion into two cases : Case I $p_g \geq 1$ and Case II $p_g = 0$. In each case, we shall introduce an invariant notion of minimal set for the Bergman kernel on M.

Case I. When $p_g \geq 1$, pick any meromorphic 2-form η on M satisfying the following conditions:

(1) η is holomorphic and nowhere zero on $M \backslash E$.
(2) E is the pole set of η.

By Proposition 2 and Lemma 4, one such choice of η is φ_0. (Since φ_0 is not canonical, we consider all choices of η.)
Observe that since $V = \widetilde{V} \cap B$ has no singularities in its closure other than 0, (1) implies that η is L^2 on the complement of a neighborhood of E in M.

Theorem 5. *Under the above conditions of Case I,* $\dfrac{K(z,\overline{z})}{\eta(z) \wedge \overline{\eta(z)}}$ *defines a non-negative real valued function* σ *on* M*. The zero set of* σ *is exactly the exceptional set* E*. We shall refer to the zero set of* σ *as the minimal set of the Bergman kernel on* M *in this case.*

Proof. Let $\{\omega_j\}$ be a complete orthonormal basis of the space F of all L^2 holomorphic 2-forms on M. As in the proof of Proposition 2, each $\omega_j = h_j \eta$ for some holomorphic function h_j on M which vanishes on E. Then $K(z,\overline{z}) = \sum_j \omega_j(z) \wedge \overline{\omega_j(z)} = (\sum_j |h_j(z)|^2) \eta(z) \wedge \overline{\eta(z)}$. Hence, $\sigma(z) = \dfrac{K(z,\overline{z})}{\eta(z) \wedge \overline{\eta(z)}} = \sum_j |h_j(z)|^2 \geq 0$ and $\sigma = 0$ on E.

Observe that there exists $k \in \mathbb{N}$ such that each $z_1^k \eta, \ldots, z_n^k \eta(z_1, \ldots, z_n$ being the coordinates of $\mathbb{C}^n$) is a holomorphic 2-form on M, by virtue of the finiteness of p_g. Since η is L^2 on the complement of a neighborhood of E, $z_1^k \eta, \ldots, z_n^k \eta$ are L^2 on M hence belong to F. Applying the Gram-Schmidt process to these elements of F, we get orthonormal elements $g_i \eta$ with the same linear span in F, for some holomorphic functions g_i on M. Then we can extend $\{g_i\eta\}$ to a complete orthonormal basis of F, by means of which, we see that $\sigma = \dfrac{K}{\eta\wedge\overline{\eta}} \geq \sum_i |g_i|^2$.

Now for any $p \in M\backslash E$, $\pi(p) \neq 0$. Not all $z_1, \dots, z_n$ are zero at $\pi(p)$, hence some $g_i(p) \neq 0$ and $\sigma(p) > 0$. □

Case II. When $p_g = 0$, 0 is a rational singularity. First, note that a Gorenstein rational singularity is actually a rational double point. This can be seen as follows. Since $(V, 0)$ is a Gorenstein rational singularity, the canonical divisor K in M is effective with support in $E(= \cup_i A_i)$, i.e. $K = \sum n_i A_i$, $n_i \geq 0$. By the adjunction formula, $A_i \cdot K \geq 0$ for all $A_i \subset E$. We claim that $K = 0$. Suppose on the contrary that $K = \sum_{i=k}^{n} n_i A_i$, where $n_i > 0$ and $k \leq n$. Then $-K = \sum_{i=k}^{n} (-n_i) A_i$ and $A_i \cdot (-K) \leq 0$ for all A_i. By the same argument as in the proof of Lemma 4, we get a contradiction to the minimality of Z. Hence $K = 0$ and $A_i \cdot K = 0$ for all A_i. The adjunction formula also gives $A_i^2 = -2$ for all A_i. Then it can be shown that the weighted dual graph of E is one of those of rational double points. By the tautness of rational double points, we are done.

In particular, $(V, 0)$ is a hypersurface singularity given by a defining equation $f(x, y, z) = 0$. f provides us with a useful 2-form as follows. Let $\iota : V \to \mathbb{C}^3$ be the inclusion map. Since $\iota^*(df) = 0$, we have

$$\iota^*\Big(\frac{dy\wedge dz}{f_x}\Big) = \iota^*\Big(\frac{dz\wedge dx}{f_y}\Big) = \iota^*\Big(\frac{dx\wedge dy}{f_z}\Big),$$

which defines a meromorphic 2-form φ_0 on V. Since f_x, f_y, f_z vanish simultaneously only at $0 \in V$, φ_0 is holomorphic and nowhere zero on $V\backslash\{0\}$. Hence, the pull back $\pi^*(\varphi_0)$ is a meromorphic 2-form on M, which is holomorphic and nowhere zero on $M\backslash E$. For simplicity, we shall write

$$\varphi_0 = \frac{dy\wedge dz}{f_x} = \frac{dz\wedge dx}{f_y} = \frac{dx\wedge dy}{f_z}$$

and denote the pull back $\pi^*(\varphi_0)$ also by φ_0. It turns out further that φ_0 is an L^2 nowhere vanishing holomorphic 2-form on M; this will be verified explicitly in Section 3.

Consider the subspace $F' = \{\varphi \in F : \varphi \text{ vanishes on } E\}$.

Proposition 6. *Under the above conditions of Case II,* $\dim F/F' = 1$.

Proof. Consider any $\varphi \in F$. Since φ_0 is nowhere zero on M, $\dfrac{\varphi}{\varphi_0} = h$ for some holomorphic function on M. h is constant on E, say equal to c. Then, $\lambda : F \to \mathbb{C}$ mapping φ to c is a well-defined linear functional. Since $\varphi_0 \in F$, λ is surjective. F' is the kernel of λ, and the claim follows. □

Theorem 7. *Under the above conditions of Case II, there exists a holomorphic 2-form ω_0 on M such that*

(1) *ω_0 is nowhere zero on M.*
(2) *$\omega_0 \in F$ with $\|\omega_0\| = 1$, and*
(3) *$F = \mathbb{C}\omega_0 \oplus F'$ (orthogonal sum).*

The function $\sigma : M \to \mathbb{R}$ defined by $\sigma(z) = \dfrac{K(z,\overline{z})}{\omega_0(z) \wedge \overline{\overline{\omega_0(z)}}}$ is a biholomorphic invariant. The minimal set of σ is exactly E. We shall refer to the minimal set of σ as the minimal set of the Bergman kernel in this case.

Proof. The existence of ω_0 will be verified directly in Section 3 for each case of a rational double point. By Proposition 6, ω_0 is unique up to a constant multiple $c\omega_0$ with $|c| = 1$. Hence σ is independent of the choice of ω_0. Again since F with its inner product is invariant under biholomorphic maps, σ is a biholomorphic invariant.

By (2), we may take a complete orthonormal basis $\{\omega_0, \omega_1, \omega_2, \cdots\}$ of F. Then for each $j = 1, 2, 3, \cdots, w_j \in F'$ by (3) and $\omega_j = h_j\omega_0$ for some holomorphic function h_j which vanishes on E. Further,

$$\sigma = \frac{K}{\omega_0 \wedge \overline{\omega_0}} = \frac{\omega_0 \wedge \overline{\omega_0} + \sum \omega_j \wedge \overline{\omega_j}}{\omega_0 \wedge \overline{\omega_0}} = 1 + \sum |h_j|^2.$$

Hence $\sigma \geq 1$ on M and $\sigma = 1$ on E.

To see that $\sigma > 1$ on $M \backslash E$, we consider $x\omega_0, y\omega_0, z\omega_0$. As in the previous proof, we get a complete orthonormal basis of F of the form $\{\omega_0, g_i\omega_0, \cdots\}$, where g_i are holomorphic functions on M vanishing simultaneously only on E. Hence outside E,

$$\sigma \geq 1 + \sum |g_i|^2 > 1.$$

□

Remark. It is tempting to define σ to be $\dfrac{K}{\omega \wedge \overline{\omega}}$ for arbitrary nowhere vanishing L^2 holomorphic 2-form ω on M. However, it may then be impossible to locate the exceptional set by the minimal set. For example, if we take $\omega = e^{Nx}\omega_0$ and fix a point $p \in M$ with $Re(x) > 0$ at $\pi(p)$, then for sufficiently large N,

$$\frac{K}{\omega \wedge \overline{w}} = \frac{K}{w_0 \wedge \overline{\omega_0}}\ e^{-2NRe(x)} \begin{cases} = 1 \text{ on } E \\ < 1 \text{ at } p \end{cases}$$

3. Explicit computations

In case $p_g = 0$, 0 is a rational double point. In this section, we use the explicit resolutions of $A_n, n \geq 1, D_n, n \geq 4, E_6, E_7, E_8$ (see [M]) to construct the form ω_0 in Theorem 7.

Type A_n.
Let $\widetilde{V} = \{(x, y, z) \in \mathbb{C}^3 : f(x, y, z) = xy - z^{n+1} = 0\}$.
An explicit resolution $\widetilde{\pi} : \widetilde{M} \to \widetilde{V}$ can be given in terms of coordinate charts

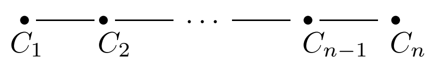
and transition functions as follows :
Coordinate charts: $\widetilde{W}_k = \mathbb{C}^2 = \{(u_k, v_k)\}, k = 0, 1, \cdots, n.$

Transition functions: $\begin{cases} u_{k+1} = \dfrac{1}{v_k} \\ \\ v_{k+1} = u_k v_k^2 \end{cases}$ or $\begin{cases} u_k = u_{k+1}^2 v_{k+1} \\ \\ v_k = \dfrac{1}{u_{k+1}} \end{cases}$

Projection map : $\widetilde{\pi}(u_k, v_k) = (u_k^{k+1} v_k^k, u_k^{n-k} v_k^{n-k+1}, u_k v_k)$ or

$$(x, y, z) = (u_0, u_0^n v_0^{n+1}, u_0 v_0) = \cdots = (u_n^{n+1} v_n^n, v_n, u_n v_n)$$

Exceptional set : $E = \widetilde{\pi}^{-1}(0) = C_1 \cup \cdots \cup C_n,$ where
$C_k = \{u_{k-1} = 0\} \cup \{v_k = 0\}$



We consider $V = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}$ and $|x|^2 + |y|^2 + |z|^2 < \varepsilon_0\}$.
Then $M = \widetilde{\pi}^{-1}(V)$ is given by the coordinate charts:

$$W_k = \big\{(u_k, v_k) : |u_k|^{2(k+1)}|v_k|^{2k} + |u_k|^{2(n-k)}|v_k|^{2(n-k+1)} + |u_k|^2|v_k|^2 < \varepsilon_0\big\}, \quad k = 0, 1, \cdots, n$$

On M, $\varphi_0 = \dfrac{dx \wedge dy}{f_z} = -\dfrac{dx \wedge dy}{(n+1)z^n}$
$= -du_k \wedge dv_k, \; k = 0, \cdots, n,$ by an easy computation.

Thus, φ_0 is a nowhere vanishing holomorphic 2-form on M. Observe that under $\pi : M \to V$, $W_0 \backslash C_1$ is mapped biholomorphically onto $V \backslash y$-axis. In particular, $M \backslash W_0$ is of measure zero in the obvious sense. Hence, we may compute integrals on M using the (u_0, v_0) coordinate on the chart W_0 alone.

Proposition 8. *In the notations for A_n, let $\varphi_{\alpha\beta} = u_0^\alpha v_0^\beta du_0 \wedge dv_0$, $\alpha, \beta = 0, 1, 2, \cdots$. Then $\Big\{\dfrac{\varphi_{\alpha\beta}}{\|\varphi_{\alpha\beta}\|} : \alpha \geq \dfrac{n}{n+1}\beta\Big\}$ is a complete orthonormal basic of F.*

Proof. The transition functions imply that $u_k^\alpha v_k^\beta = u_{k+1}^{2\alpha-\beta} v_{k+1}^\alpha$, hence

$$u_0^\alpha v_0^\beta = \cdots = u_k^{(k+1)\alpha - k\beta} v_k^{k\alpha-(k-1)\beta} = \cdots = u_n^{(n+1)\alpha - n\beta} v_n^{n\alpha-(n-1)\beta}$$

Also, $du_k \wedge dv_k = du_{k+1} \wedge dv_{k+1}$. So, $u_0^\alpha v_0^\beta du_0 \wedge dv_0$ defines a holomorphic 2-form $\varphi_{\alpha\beta}$ on M if and only if $(n+1)\alpha - n\beta \geq 0$.

Next, write $u_0 = re^{i\theta}$ and $v_0 = \rho e^{i\varphi}$. Then

$$\int_M \varphi_{\alpha\beta} \wedge \overline{\varphi_{\alpha'\beta'}}$$

$$= \int_{W_0} u_0^\alpha \, \overline{u_0^{\alpha'}} v_0^\beta \overline{v_0^{\beta'}} \, du_0 \wedge dv_0 \wedge d\overline{u_0} \wedge d\overline{v_0} \quad \text{, by the preceding observation}$$

$$= 4 \iint\limits_{\substack{r^2+r^{2n}\rho^{2(n+1)}+r^2\rho^2<\varepsilon_0 \\ r,\rho\geq 0}} r^{\alpha+\alpha'+1}\rho^{\beta+\beta'+1} dr d\rho \int_0^{2\pi} e^{i(\alpha-\alpha')\theta} d\theta \cdot \int_0^{2\pi} e^{i(\beta-\beta')\varphi} d\varphi$$

$$\begin{cases} = 0 \text{ if } \alpha \neq \alpha' \text{ or } \beta \neq \beta' \\ < \infty \text{ if } \alpha = \alpha' \text{ and } \beta = \beta' \end{cases}$$

Therefore each $\varphi_{\alpha\beta}$ is L^2 and $\{\frac{\varphi_{\alpha\beta}}{\|\varphi_{\alpha\beta}\|}\}$ form an orthonormal system in F. Note: the fact that $\varphi_{\alpha\beta}$ is L^2 also follows from Theorem 3, instead of direct checking as above.

Finally, for any $\varphi \in F$, $\frac{\varphi}{\varphi_0} = h$ is a holomorphic function on M. Hence $\varphi = h(u_0, v_0) du_0 \wedge dv_0$ on W_0. Since W_0 is a Reinhardt domain, $h(u_0, v_0)$ has a convergent power series expansion $h(u_0, v_0) = \sum c_{\alpha\beta} u_0^\alpha v_0^\beta$ on W_0. This implies that $\varphi = \sum c_{\alpha\beta}\varphi_{\alpha\beta}$ on M, completing the proof. □

Observe that $\varphi_{00} = -\varphi_0$ while any other $\varphi_{\alpha\beta}$ has $\alpha(\geq \frac{n}{n+1}\beta) \geq 1$ and so vanishes on C_1 given by $u_0 = 0$. Also, $\omega_0 = \frac{\varphi_0}{\|\varphi_0\|}$ satisfies all three conditions in Theorem 7.

Type D_n, $n \geq 4$ and even.
Let $\widetilde{V} = \{(x, y, z) \in \mathbb{C}^3 : f(x, y, z) = x^2 z + y^2 - z^{n-1} = 0\}$.
An explicit resolution $\widetilde{\pi} : \widetilde{M} \to \widetilde{V}$ can be given in terms of coordinate charts and transition functions as follows.
Coordinate charts :

$$\widetilde{W_k} = \left\{(u_k, v_k) : u_k^{n-k-3} v_k^{n-k-2} \neq 1\right\}, \quad 0 \leq k \leq n-4$$
$$\widetilde{W_k} = \left\{(u_k, v_k)\right\}, \ k = n-3, n-2$$
$$\widetilde{W_k} = \left\{(u_k, v_k) : u_k^2 v_k \neq -1\right\}, \quad k = n-1, n$$

Transition functions: $\begin{cases} u_{k+1} = \dfrac{1}{v_k} \\ v_{k+1} = u_k v_k^2 \end{cases}$ or $\begin{cases} u_k = u_{k+1}^2 v_{k+1} \\ v_k = \dfrac{1}{u_{k+1}} \end{cases}$ $, 0 \leq k \leq n-3$

$$\begin{cases} u_{n-1} = \dfrac{1}{u_{n-2}v_{n-2}} \\ \\ v_{n-1} = u_{n-2}v_{n-2}^2(1-u_{n-2}) \end{cases} \quad \text{or} \quad \begin{cases} u_{n-2} = \dfrac{1}{1+u_{n-1}^2 v_{n-1}} \\ v_{n-2} = \dfrac{1+u_{n-1}^2 v_{n-1}}{u_{n-1}} \end{cases}$$

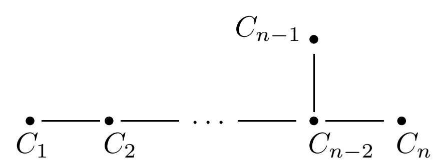
$$\begin{cases} u_{n} = \dfrac{1}{v_{n-2}(1-u_{n-2})} \\ \\ v_{n} = u_{n-2}v_{n-2}^2(1-u_{n-2}) \end{cases} \quad \text{or} \quad \begin{cases} u_{n-2} = \dfrac{u_n^2 v_n}{1+u_n^2 v_n} \\ \\ v_{n-2} = \dfrac{1+u_n^2 v_n}{u_n} \end{cases}$$

Projection map : Let $\widetilde{\pi}(u_k, v_k) = (x, y, z)$. Then

$$x = u_0(u_0^{n-3}v_0^{n-2}-1)^{\frac{n}{2}-1}(u_0^{n-3}v_0^{n-2}-2) = \cdots =$$
$$= u_{n-3}^{n-2}v_{n-3}^{n-3}(v_{n-3}-1)^{\frac{n}{2}-1}(v_{n-3}-2) = \cdots = \frac{v_n^{\frac{n}{2}-1}(1-u_n^2 v_n)}{1+u_n^2 v_n}$$
$$y = 2u_0^2 v_0(u_0^{n-3}v_0^{n-2}-1)^{\frac{n}{2}} = \cdots = 2u_{n-3}^{n-1}v_{n-3}^{n-2}(v_{n-3}-1)^{\frac{n}{2}} = \cdots = \frac{2u_n v_n^{\frac{n}{2}}}{1+u_n^2 v_n}$$
$$z = u_0^2 v_0^2(u_0^{n-3}v_0^{n-2}-1) = \cdots = u_{n-3}^2 v_{n-3}^2(v_{n-3}-1) = \cdots = v_n$$

Exceptional set : $E = \widetilde{\pi}^{-1}(0) = C_1 \cup \cdots \cup C_k$, where

$$C_k = \{u_{k-1} = 0\} \cup \{v_k = 0\} \quad 1 \le k \le n-2$$
$$C_{n-1} = \{v_{n-3} = 1\} \cup \{u_{n-2} = 1\} \cup \{v_{n-1} = 0\}$$
$$C_n = \{u_{n-2} = 0\} \cup \{v_n = 0\}$$



For $V = \{(x, y, z) \in \mathbb{C}^3 : x^2 z + y^2 - z^{n-1} = 0 \text{ and } |x|^2 + |y|^2 + |z|^2 < \varepsilon_0\}$, $M = \widetilde{\pi}^{-1}(V)$ is given by the coordinate charts:

$$W_k = \{(u_k, v_k) \in \widetilde{W}_k : |x|^2 + |y|^2 + |z|^2 < \varepsilon_0\}, \; k = 0, 1, \cdots, n.$$

On M, $\varphi_0 = \dfrac{dz \wedge dx}{f_y} = \dfrac{dz \wedge dx}{2y}$

Computation shows that

$$\varphi_0 = du_0 \wedge dv_0 = \cdots = du_{n-3} \wedge dv_{n-3} = du_{n-2} \wedge dv_{n-2}$$
$$= \frac{du_{n-1} \wedge dv_{n-1}}{1+u_{n-1}^2 v_{n-1}} = \frac{du_n \wedge dv_n}{1+u_n^2 v_n}.$$

Since $1 + u_{n-1}^2 v_{n-1} \neq 0$ on W_{n-1} and $1 + u_n^2 v_n \neq 0$ on W_n, φ_0 is a nowhere zero holomorphic 2-form on M. Observe that under $\pi : M \to V$, $W_{n-3} \backslash (C_{n-1} \cup$

$C_{n-2}\cup C_{n-3}$) is mapped holomorphically onto $V\backslash S$, where S consists of curves. Explicitly, the inverse map is given by

$$u_{n-3} = \frac{y^3(z^{n-1} - y^2 + xz^{\frac{n}{2}})}{2x(4z^{2(n-1)} - 3y^2z^{n-1} + (4z^{n-1} - y^2)xz^{\frac{n}{2}})}$$
$$v_{n-3} = \frac{2}{y^2}\left(z^{n-1} + xz^{\frac{n}{2}}\right)$$

for $(x, y, z) \in V$ such that the denominators are nonzero. In particular, $M\backslash W_{n-3}$ is of measure zero and we may compute integrals on M using the (u_{n-3}, v_{n-3}) coordinates alone.

Proposition 9. *$\varphi_0 \in F$ and $\omega_0 = \dfrac{\varphi_0}{\|\varphi_0\|}$ satisfies the three conditions in Theorem 7.*

Proof. The fact that φ_0 is L^2 follows from Theorem 3 or by direct checking. It remains only to prove that φ_0, hence ω_0, is orthogonal to $F^{'}$. Consider any $\omega \in F^{'}$. Then $\omega = h\varphi_0$ for a holomorphic function on M vanishing on E. Using the coordinate chart W_{n-3} and writing $(u, v) = (u_{n-3}, v_{n-3})$, we have $\omega = h(u, v)du \wedge dv$ on W_{n-3}. W_{n-3} is given by

$$|u|^{2(n-2)}|v|^{2(n-3)}|v-1|^{n-2}|v-2|^2+4|u|^{2(n-1)}|v|^{2(n-2)}|v-1|^n+|u|^4|v|^4|v-1|^2 < \varepsilon_0.$$

Each plane $\{v = \text{ constant }\}$ intersects W_{n-3} in a disc with center at the point $(0, v)$ which belongs to E. Hence $h(u, v)$ can be written as

$$h(u, v) = \sum_\alpha c_\alpha(v)u^\alpha \quad \text{with} \quad c_0(v) = 0$$

Then $(\varphi_0, \omega) = \int_M \varphi_0 \wedge \overline{\omega}$
$$= \int_{W_{n-3}} du \wedge dv \wedge \sum_\alpha \overline{c_\alpha(v)}\overline{u^\alpha}d\overline{u} \wedge d\overline{v}$$
$$= 4\sum_\alpha \left(\int \overline{c_\alpha(\rho, \varphi)}r^\alpha drd\rho d\varphi \int_0^{2\pi} e^{-i\alpha\theta}d\theta\right), \text{ writing } u = re^{i\theta} \text{ and } v = \rho e^{i\varphi}$$
$$= 0$$

□

For all the remaining types, we shall show that Proposition 9 also holds for the respective φ_0's.

Type $D_n, n \geq 5$ and odd.
Let $\widetilde{V} = \{(x, y, z) \in \mathbb{C}^3 : f(x, y, z) = x^2 + y^2z - z^{n-1} = 0\}$.
An explicit resolution $\widetilde{\pi} : \widetilde{M} \to \widetilde{V}$ can be given as follows. The coordinate charts and transition functions are of the same forms as those for the even n type. The projection map is different, namely

$$x = u_0^2 v_0 (u_0^{n-3} v_0^{n-2} - 1)^{\frac{n-1}{2}} (u_0^{n-3} v_0^{n-2} - 2) = \cdots$$
$$= u_{n-3}^{n-1} v_{n-3}^{n-2} (v_{n-3} - 1)^{\frac{n-1}{2}} (v_{n-3} - 2) = \cdots = \frac{v_n^{\frac{n-1}{2}} (1 - u_n^2 v_n)}{1 + u_n^2 v_n}$$

$$y = 2u_0 (u_0^{n-3} v_0^{n-2} - 1)^{\frac{n-1}{2}} = \cdots = 2u_{n-3}^{n-2} v_{n-3}^{n-3} (v_{n-3} - 1)^{\frac{n-1}{2}} = \cdots =$$
$$= \frac{2u_n v_n^{\frac{n-1}{2}}}{1 + u_n^2 v_n}$$

$$z = u_0^2 v_0^2 (u_0^{n-3} v_0^{n-2} - 1) = \cdots = u_{n-3}^2 v_{n-3}^2 (v_{n-3} - 1) = \cdots = v_n$$

The exceptional set is also given in the same way as for the even n type.
For $V = \{(x, y, z) \in \mathbb{C}^3 : x^2 + y^2 z - z^{n-1} = 0 \text{ and } |x|^2 + |y|^2 + |z|^2 < \varepsilon_0\}$, $M = \widetilde{\pi}^{-1}(V)$ is given by the coordinate charts:

$$W_k = \{(u_k, v_k) \in \widetilde{W}_k : |x|^2 + |y|^2 + |z|^2 < \varepsilon_0\}, \qquad k = 0, 1, \cdots, n.$$

On M, $\varphi_0 = \dfrac{dy \wedge dz}{f_x} = \frac{dz \wedge dz}{2x}$.

Computation shows that

$$\varphi_0 = du_0 \wedge dv_0 = \cdots = du_{n-3} \wedge dv_{n-3} = du_{n-2} \wedge dv_{n-2}$$
$$= \frac{du_{n-1} \wedge dv_{n-1}}{1 + u_{n-1}^2 v_{n-1}} = \frac{du_n \wedge dv_n}{1 + u_n^2 v_n}$$

φ_0 is a nowhere zero holomorphic 2-form on M as before. Again, we may compute integrals on M using the (u_{n-3}, v_{n-3}) coordinates on W_{n-3} alone. W_{n-3} is given by

$$|u|^{2(n-1)}|v|^{2(n-2)}|v-1|^{n-1}|v-2|^2 + 4|u|^{2(n-2)}|v|^{2(n-3)}|v-1|^{n-1} + |u|^4|v|^4|v-1|^2 < \varepsilon_0,$$

where $(u, v) = (u_{n-3}, v_{n-3})$. Since each plane $\{v_{n-3} = \text{constant}\}$ intersects W_{n-3} in a disc with center belonging to E, the proof of Proposition 9 goes through. Hence Proposition 9 also holds for the odd n case.

Type E_6.
In this case, $\widetilde{V} = \{(x, y, z) \in \mathbb{C}^3 : f(x, y, z) = x^2 - y^3 - z^4 = 0\}$ and $\widetilde{\pi} : \widetilde{M} \to \widetilde{V}$ is given as follows.

Coordinate charts: $\widetilde{W}_k = \{(u_k, v_k) : u_k^{2-k} v_k^{3-k} \neq 1\}, \qquad k = 0, 1$

$\widetilde{W}_k = \{(u_k, v_k)\}, \qquad k = 2, 3$

$\widetilde{W}_4 = \{u_4, v_4) : u_4^2 v_4 \neq -1\}$

$\widetilde{W}_k = \{(u_k, v_k) : u_k^{k-3} v_k^{k-4} \neq -1\}, \qquad k = 5, 6$

Transition functions: $\begin{cases} u_{k+1} = \frac{1}{v_k} \\ v_{k+1} = u_k v_k^2 \end{cases}$ or $\begin{cases} u_k = u_{k+1}^2 v_{k+1} \\ v_k = \frac{1}{u_{k+1}} \end{cases}$, $k = 0, 1, 2, 5$

$$\begin{cases} u_4 = \frac{1}{u_3 v_3} \\ v_4 = u_3 v_3^2 (1 - u_3) \end{cases} \quad \text{or} \quad \begin{cases} u_3 = \frac{1}{1 + u_4^2 v_4} \\ v_3 = \frac{1 + u_4^2 v_4}{u_4} \end{cases}$$

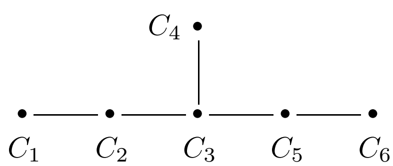
$$\begin{cases} u_5 = \frac{1}{v_3 (1 - u_3)} \\ v_5 = u_3 v_3^2 (1 - u_3) \end{cases} \quad \text{or} \quad \begin{cases} u_3 = \frac{u_5^2 v_5}{1 + u_5^2 v_5} \\ v_3 = \frac{1 + u_5^2 v_5}{u_5} \end{cases}$$

Projection map:

$$x = 4u_0^2 (u_0^2 v_0^3 - 1)^3 (u_0^2 v_0^3 + 1) = \cdots =$$
$$= 4u_2^6 v_2^4 (v_2 - 1)^3 (v_2 + 1) = \cdots = \frac{4v_6^2 (1 + 2u_6^3 v_6^2)}{(1 + u_6^3 v_6^2)^2}$$
$$y = 4u_0^2 v_0 (u_0^2 v_0^3 - 1)^2 = \cdots = 4u_2^4 v_2^3 (v_2 - 1)^2 = \cdots = \frac{4u_6 v_6^2}{1 + u_6^3 v_6^2}$$
$$z = 2u_0 (u_0^2 v_0^3 - 1)^2 = \cdots = 2u_2^3 v_2^2 (v_2 - 1)^2 = \cdots = \frac{2v_6}{1 + u_6^3 v_6^2}$$

Exceptional set: $E = C_1 \cup \cdots \cup C_6$, where

$$C_k = \{u_{k-1} = 0\} \cup \{v_k = 0\}, \quad k = 0, 1, 2, 3, 6$$
$$C_4 = \{v_2 = 1\} \cup \{u_3 = 1\} \cup \{v_4 = 0\}$$
$$C_5 = \{u_3 = 0\} \cup \{v_5 = 0\}$$



For $V = \{(x, y, z) \in \mathbb{C}^3 : x^2 - y^3 - z^4 = 0 \text{ and } |x|^2 + |y|^2 + |z|^2 < \varepsilon_0\}$, $M = \widetilde{\pi}^{-1}(V)$ is given by the coordinate charts

$$W_k = \{(u_k, v_k) \in \widetilde{W}_k : |x|^2 + |y|^2 + |z|^2 < \varepsilon_0\}, \qquad k = 0, 1, \cdots, 6$$

On M, $\varphi_0 = \dfrac{dy \wedge dz}{f_x}$.

Computation shows that

$$\varphi_0 = du_0 \wedge dv_0 = du_1 \wedge dv_1 = du_2 \wedge dv_2 = du_3 \wedge dv_3$$
$$= \frac{du_4 \wedge dv_4}{1 + u_4^2 v_4} = \frac{du_5 \wedge dv_5}{1 + u_5^2 v_5} = \frac{du_6 \wedge dv_6}{1 + u_6^3 v_6^2}.$$

φ_0 is a nowhere zero holomorphic 2-form on M since the denominators are nonzero on the respective coordinate charts. $\pi : M \to V$ maps $W_2 \backslash (C_2 \cup C_3 \cup C_4)$

biholomorphically onto $V\backslash S$, where S consists of curves. Explicitly, the inverse map is given by

$$u_2 = \frac{(x-z^2)^2}{2y^2z} \qquad v_2 = \frac{(x+z^2)^2}{y^3},$$

for $(x,y,z) \in V$ such that the denominators are nonzero. In particular we may compute integrals on M using the (u_2, v_2) coordinates alone. W_2 is given by $16|u|^{12}|v|^8|v-1|^6|v+1|^2 + 16|u|^8|v|^6|v-1|^4 + 4|u|^6|v|^4|v-1|^4 < \varepsilon_0$, where $(u,v) = (u_2, v_2)$. Again each plane $\{v_2 = \text{ constant}\}$ intersects W_2 in a disc with center belonging to E. Hence, Proposition 9 also holds for E_6.

Type $\boldsymbol{E}_7$.
In this case, $\widetilde{V} = \{(x,y,z) \in \mathbb{C}^3 : f(x,y,z) = x^2 + y^3 - yz^3 = 0\}$ and $\widetilde{\pi} : \widetilde{M} \to \widetilde{V}$ is given as follows.

Coordinate charts: $\widetilde{W}_k, \quad k = 0, \cdots, 6,$ as for E_6,

$$\widetilde{W}_7 = \{(u_7, v_7) : u_7^4 v_7^3 \neq -1\}$$

Transition functions: As for E_6, with additional

$$\begin{cases} u_7 = \dfrac{1}{v_6} \\ v_7 = u_6 v_6^2 \end{cases}$$

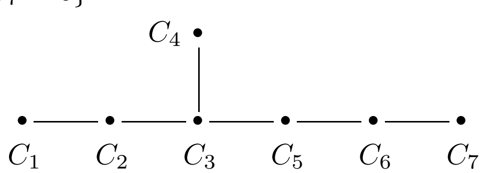
Projection map:

$$x = u_0^3(u_0^2v_0^3 - 1)^5 = \cdots = u_2^9 v_2^6 (v_2 - 1)^5 = \cdots = \frac{u_7 v_7^3}{(1 + u_7^4 v_7^3)^3}$$

$$y = u_0^2(u_0^2v_0^3 - 1)^3 = \cdots = u_2^6 v_2^4 (v_2 - 1)^3 = \cdots = \frac{u_7^2 v_7^3}{(1 + u_7^4 v_7^3)^3}$$

$$z = u_0^2 v_0 (u_0^2v_0^3 - 1)^2 = \cdots = u_2^4 v_2^3 (v_2 - 1)^2 = \cdots = \frac{v_7}{(1 + u_7^4 v_7^3)^3}$$

Exceptional set: $E = C_1 \cup \cdots \cup C_7$, where $C_1, \cdots, C_6$ are given for E_6, and $C_7 = \{u_6 = 0\} \cup \{v_7 = 0\}$.



For $V = \{(x,y,z) \in \mathbb{C}^3 : x^2 + y^3 - yz^3 = 0$ and $|x|^2 + |y|^2 + |z|^2 < \varepsilon_0\}$, $M = \widetilde{\pi}^{-1}(V)$ is given by the coordinate charts

$$W_k = \{(u_k, v_k) \in \widetilde{W}_k : |x|^2 + |y|^2 + |z|^2 < \varepsilon_0\}, \quad k = 0, 1, \cdots, 7$$

On M, $\varphi_0 = \dfrac{dy \wedge dz}{f_x}$.
Computation shows that

$$\begin{aligned}\varphi_0 &= du_0 \wedge dv_0 = \cdots = du_2 \wedge dv_2 = du_3 \wedge dv_3 \\ &= \cdots = \frac{du_7 \wedge dv_7}{1 + u_7^4 v_7^3}\end{aligned}$$

φ_0 is a nowhere zero holomorphic 2-form on M. $\pi : M \to V$ maps $W_2 \backslash (C_2 \cup C_3 \cup C_4)$ biholomorhoically onto $V \backslash S$, where S consists of curves. Explicitly, the inverse map is given by

$$u_2 = \frac{y^3}{xz^2} \qquad v_2 = \frac{z^3}{y^2}$$

for $(x, y, z) \in V$ such that the denominators are nonzero. In particular we may compute integrals on M using (u_2, v_2) on W_2. Again, each plane $\{v_2 = \text{constant}\}$ intersects W_2 in a disc with center belonging to E. Hence, Proposition 9 holds for E_7.

Type E_8.
In this case, $\widetilde{V} = \{(x, y, z) \in \mathbb{C}^3 : f(x, y, z) = x^2 - y^3 + z^5 = 0\}$ and $\widetilde{\pi} : \widetilde{M} \to \widetilde{V}$ is given as follows.

Coordinate charts: $\widetilde{W}_k$, $\quad k = 0, 1, \cdots, 7$ as for E_7,

$$\widetilde{W}_8 = \{(u_8, v_8) : u_8^5 v_8^4 \neq -1\}$$

Transition functions: As for E_7 with additional

$$\begin{cases} u_8 = \frac{1}{v_7} \\ v_8 = u_7 v_7^2 \end{cases}$$

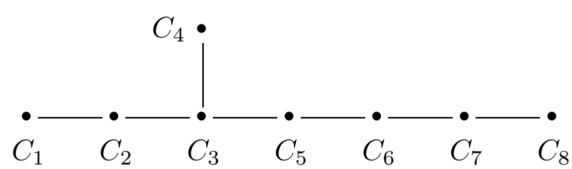
Projection map:

$$x = u_0^5 (u_0^2 v_0^3 - 1)^8 = \cdots = u_2^{15} v_2^{10} (v_2 - 1)^8 = \cdots = \frac{v_8^3}{(1 + u_8^5 v_8^4)^5}$$

$$y = u_0^4 v_0 (u_0^2 v_0^3 - 1)^5 = \cdots = u_2^{10} v_2^7 (v_2 - 1)^5 = \cdots = \frac{v_8^2}{(1 + u_8^5 v_8^4)^3}$$

$$z = u_0^2 (u_0^2 v_0^3 - 1)^3 = \cdots = u_2^6 v_2^4 (v_2 - 1)^3 = \cdots = \frac{u_8 v_8^2}{(1 + u_8^5 v_8^4)^2}$$

Exceptional set: $E = C_1 \cup \cdots \cup C_8$, where $C_1, \cdots, C_7$ are given as for E_7, and $C_8 = \{u_7 = 0\} \cup \{v_8 = 0\}$



For $V = \{(x,y,z) \in \mathbb{C}^3 : x^2 - y^3 + z^5 = 0 \text{ and } |x|^2 + |y|^2 + |z|^2 < \varepsilon_0\}$, $M = \widetilde{\pi}^{-1}(V)$ is given by the coordinate charts

$$W_k = \{(u_K, v_k) \in \widetilde{W}_k : |x|^2 + |y|^2 + |z|^2 < \varepsilon_0\}, \qquad k = 0, 1, \cdots, 8.$$

On M, $\varphi_0 = \dfrac{dy \wedge dz}{f_x}$

Computation shows that

$$\varphi_0 = -du_0 \wedge dv_0 = \cdots = -du_2 \wedge dv_2 = \cdots = -\frac{du_8 \wedge dv_8}{1 + u_8^5 v_8^4}$$

φ_0 is a nowhere zero holomorphic 2-form on M. $\pi : M \to V$ maps $W_2 \backslash (C_2 \cup C_3 \cup C_4)$ biholomorphically onto $V \backslash S$, where S consists of curves. Explicitly, the inverse map is given by

$$u = \frac{z^6}{xy^2} \qquad v = \frac{y^3}{z^5},$$

for $(x,y,z) \in V$ such that the denominators are nonzero. Thus we may compute integrals on M using (u_2, v_2) on W_2. Again, each plane $\{v_2 = \text{ constant}\}$ intersects W_2 in a disc with center belonging to E. Hence, Proposition 9 holds for E_8.

Remark. For type D_n, we can also use the coordinate chart W_{n-2} which intersects each plane $\{u_{n-2} = \text{ constant }\}$ in a disc with center belonging to E. For type $E_n (n = 6, 7, 8)$, we can also use the coordinate chart W_3 which intersects each plane $\{u_3 = \text{ constant }\}$ in a similar disc. The orthogonality of φ_0 to F' is not so easy to see in the other coordinate charts for the lack of such an obvious intersection property.

4. Appendix

In response to the request of one of the referees, we include this appendix for the convenience of the readers.

Theorem 10. *Let $(V, 0)$ be a normal isolated singularity of dimension $n \geq 2$. Suppose that h is a holomorphic function on $V \backslash \{0\}$. Then h is a holomorphic function on V.*

Proof. Let $\pi : M \longrightarrow V$ be a resolution of singularity with $E = \pi^{-1}(0)$ as an exceptional set. Following Laufer [L2], we consider the sheaf cohomology with compact support and sheaf cohomology with support at infinity. The following sequence is exact.

$$0 \longrightarrow H_c^0(M, \mathcal{O}) \longrightarrow H^0(M, \mathcal{O}) \longrightarrow H_\infty^0(M, \mathcal{O}) \longrightarrow H_c^1(M, \mathcal{O}) \longrightarrow \dots .$$

Observe that $H_c^1(M, \mathcal{O})$ is dual to $H^{n-1}(M, \Omega^n)$, where Ω^n is the sheaf of germs of holomorphic n-forms, by Serre duality. On the other hand, Grauert-Riemenschneider vanishing theorem [G-R] asserts that $H^i(M, \Omega^n) = 0$ for $i \geq 1$. Therefore the restriction mapping $H^0(M, \mathcal{O}) \longrightarrow H_\infty^0(M, \mathcal{O})$ is surjective. It follows that the pull back $\pi^* h$ of h on $M \backslash E$ can be extended holomorphically on M. As a result, h can be extended as a continuous function at the singular point 0. By normality of $(V,0)$, h can be extended as an holomorphic function at the singular point 0. □

Theorem 11. *Let $(V,0)$ be a normal isolated singularity of dimension $n \geq 2$. Let $\pi : M \longrightarrow V$ be a resolution of singularity with $E = \pi^{-1}(0)$ as exceptional set. Then any holomorphic n-form on $M \backslash E$ can be extended as a meromorphic n-form on M.*

Proof. Let $E = \cup A_i$, $1 \leq i \leq m$, be an irreducible decomposition of E. Without loss of generality, we may assume that A_i's are nonsingular divisors with normal crossing by applying Hironaka Theorem on resolution of singularities. Let ω be a holomorphic n-form on $M \backslash E$. It suffices to prove that ω has a pole of finite order on each A_i. Suppose on the contrary that ω has a pole of infinite order along A_i. Let z_1 be the coordinate function which vanishes at 0. Clearly $\pi^*(z_1)$ vanishes along A_i with finite order. It follows that for any positive integer k, $(\pi^*(z_1))^k\omega$ cannot be extended holomorphically across A_i. Therefore $\dim H^0(M\backslash E, \Omega^n)\big/H^0(M, \Omega^n) = \infty$, where Ω^n is the sheaf of germs of holomorphic n-forms.

On the other hand, we can consider the following exact sequence which relates the sheaf cohomology with compact support and sheaf cohomology with support at infinity

$$0 \to H^0_c(M, \Omega^n) \longrightarrow H^0(M, \Omega^n) \to H^0_\infty(M, \Omega^n) \to H^1_c(M, \Omega^n) \to H^1(M, \Omega^n).$$

Obviously $H^0_c(M, \Omega^n) = 0$. Also $H^1(M, \Omega^n) = 0$ by Grauert-Riemenschneider vanishing theorem. Observe that $H^0_\infty(M, \Omega^n) = H^0(M\backslash E, \Omega^n)$ by Andreotti and Grauert [A-G]. Therefore

$$\dim H^0(M\backslash E, \Omega^n)\big/H^0(M, \Omega^n) = \dim H^1_c(M, \Omega^n)$$

$H^1_c(M, \Omega^n)$ is Serre dual to $H^{n-1}(M, \mathcal{O})$ which is finite dimensional by Andreotti and Grauert [A-G]. This leads to a contradiction. □

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