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A Sharp Upper Estimate of the Number of Integral Points in a 5-Dimensional Tetrahedra[1]

Ke-Pao Lin

Department of Information Management, Chang Gung Institute of Nursing, Kwei-Shan, Tao-Yuan, Taiwan
E-mail: kplin@cc.cgin.edu.tw

and

Stephen S.-T. Yau[2]

Department of MSCS (M/C 249), University of Illinois at Chicago, 851 South Morgan Street, Chicago, Illinois 60607-7045
E-mail: yau@uic.edu

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1. INTRODUCTION

The general problem of counting the number $Q_n = Q(a_1, ..., a_n)$ of non-negative integral points satisfying

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leqslant 1, \tag{1.1}$$

where $a_1, a_2, ..., a_n$ are positive integers, has been a challenging problem for many years. In 1951, Mordell [Mo] gave a formula for Q_3, expressed in terms of three Dedekind sums, in the case that a_1, a_2, a_3 are pairwise relatively prime. In 1993, Pommersheim [Po], using toric varieties, gave a formula for Q_3 for arbitrary positive integers a_1, a_2, and a_3. More generally, let Δ be a polytope of dimension n in the lattice $\mathbf{Z}^n$. Denote $\ell_\Delta(k)$

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the number of lattice points in Δ dilated by a factor k, where k is an integer:

$$\ell_\Delta(k) = \sharp(k\Delta \cap \mathbf{Z}^n), \qquad k \in \mathbf{Z}_+. \tag{1.2}$$

Ehrhart proved that $\ell_\Delta(k)$ is a polynomial in k of degree n,

$$\ell_\Delta(k) = b_n k^n + b_{n-1} k^{n-1} + \cdots + b_0, \tag{1.3}$$

where $b_n =$ volume of Δ, $b_{n-1} =$ half the sum of the volumes of $(n-1)$-dimensional faces of Δ (measured with respect to the $(n-1)$-dimensional lattice in the $(n-1)$-plane), $b_0 = 1$. In 1993, Kantor and Khovanskii [Ka-Kh] succeeded in computing b_{n-2}. In fact they gave a general formula for the number of integral points in any integral polytope in $\mathbf{R}^4$. In 1994 Cappell and Shaneson [Ca-Sh] have announced a fantastic result with which they can compute all the coefficients b_i in (1.3) explicitly. Unfortunately, in all the above mentioned results, they need to assume the vertex points of Δ are in $\mathbf{Z}^n$.

However, for the sake of applications in number theory and geometry, we are interested in the problem of estimating the number $P_n = P(a_1, ..., a_n)$ of positive integral points satisfying (1.1), where $a_1, a_2 \cdots a_n$, are positive real numbers. Of course one can deduce an estimate for Q_n once an estimate for P_n, is known and vice versa [Li-Ya 2]. In fact, let $a = 1/a_1 + \cdots + 1/a_n$. It is easy to show that

$$P(a_1, ..., a_n) = Q(a_1(1-a), ..., a_n(1-a)). \tag{1.4}$$

The novelty in our problem is that we count the lattice points in a polytope whose vertices are not necessarily integer points (or even rational points). There are at least two reasons to consider this problem.

First, we are told by Professor Granville [Gr] that this is an extremely important question in number theory; it would have many applications to current problems in analytic number theory, primality testing and in factoring. Given a set $\mathscr{P}$ of primes $p_1 < p_2 < \cdots < p_n \leqslant y$. Number theorists are interested in counting the number of integers $m \leqslant x$ where $m = p_1^{\ell_1} p_2^{\ell_2} \cdots p_n^{\ell_n}$ is composed only of primes from $\mathscr{P}$, and $x = y^u$, for u not too large (for all $u > 2$ would be nice, but for all $u > \log y$ would still be interesting). Thus they wish to count the number of $(\ell_1, ..., \ell_n) \in (\mathbf{Z}_+ \cup \{0\})^n$ such that

$$\frac{\ell_1}{a_1} + \frac{\ell_2}{a_2} + \cdots + \frac{\ell_n}{a_n} \leqslant 1, \qquad \text{where} \quad a_i = \frac{\log x}{\log p_i} \geqslant u.$$

Observe that a_i's are real numbers. This is, of course, the problem Q_n that we consider above. Perhaps the best reference for the application is Carl Pomerance's ICM 1994 lecture at Zürich [Pom 1], and his lecture notes [Pom 2].

Second, the problem has an interesting application in geometry and singularity theory. Let $f: (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be the germ of a complex analytic function with isolated critical point at the origin. Let M be a resolution of $V = \{(z_1, ..., z_n) \in \mathbf{C}^n : f(z_1, ..., z_n) = 0\}$. The Milnor number of the singularity $(V, 0)$ is $\mu = dim\, \mathbf{C}\{z_1, z_2, ..., z_n\}/(f_{z_1}, f_{z_2}, ..., f_{z_n})$. The geometric genus p_g of the singularity $(V, 0)$ is $dim\, H^{n-2}(M, \mathcal{O})$, which is an important invariant of the singularity. In 1978 Durfee [Du] made the following conjecture.

Durfee Conjecture. $n!\, p_g \leqslant \mu$ with equality only when $\mu = 0$.

The connection between Durfee conjecture and the proposed problem is as follows. A polynomial $f(z_1, ..., z_n)$ is weighted homogeneous of type $(w_1, ..., w_n)$, where $w_1, ..., w_n$ are fixed positive rational numbers, if f can be expressed as a linear combination of monomials $z_1^{i_1} \cdots z_n^{i_n}$ for which $i_1/w_1 + \cdots + i_n/w_n = 1$. If $f(z_1, ..., z_n)$ is a weighted homogeneous polynomial of type $(a_1, ..., a_n)$ with an isolated singularity at the origin, then Milnor and Orlik [Mi-Or] proved that $\mu = (a_1 - 1)(a_2 - 1) \cdots (a_n - 1)$. On the other hand, Merle and Teissier [Me-Te] showed that p_g is exactly the number P_n of positive integral points satisfying (1.1). Thus Durfee conjecture provides us a guidance for the upper estimate of P_n. Unfortunately Durfee conjecture is not sharp, see for example [Xu-Ya 2]. In 1995, the second author has made the following conjecture.

Conjecture. Let $f: (\mathbf{C}^n, 0) \to (\mathbf{C}, \mathbf{0})$ be a weighted homogeneous polynomial with an isolated critical point at the origin. Then

$$\mu - h(v) \geqslant n!\, p_g$$

and equality holds if and only if f is a homogeneous polynomial. Here $h(v)$ is a polynomial function on the multiplicity v with the properties $h(v) \geqslant 0$ and $h(v) = 0$ if and only if $v = 1$.

For $n = 3, 4$, we have given sharp upper bounds for P_n cf. [Xu-Ya 1, Xu-Ya 3]. Thus the Durfee conjecture was proven in those two cases. In fact, the second author's conjecture was also solved in these two cases cf. [Xu-Ya 2, Li-Ya 1]. The purpose of this paper is to prove the following theorem.

MAIN THEOREM. *Let $a \geqslant b \geqslant c \geqslant d \geqslant e \geqslant 4$, be real number and P_5 be the number of positive integral solutions of $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{u}{d} + \frac{v}{e} \leqslant 1$, i.e.,*

$$P_5 = \sharp\left\{(x, y, z, u, v) \in \mathbf{Z}_+^5 : \frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{u}{c}+\frac{v}{e} \leqslant 1\right\}. \tag{1.5}$$

Then

$$\begin{aligned}120P_5 \leqslant f_5(a, b, c, d, e) &:= abcde - 2(abcd+abce+abde+acde+bcde)\\ &+\frac{35}{4}(abc+abd+acd+bcd)\\ &-\frac{50}{6}(ab+ac+ad+bc+bd+cd)+6(a+b+c+d).\end{aligned} \tag{1.6}$$

Equality is attained if and only if $a=b=c=d=e=$ *integer.*

Counting the number of integral points in an n-dimensional tetrahedra with non-integral vertices is a classical subject which has attracted a lot of famous mathematicians. For $n=2$, Hardy and Littlewood [Ha-Li 2, Ha-Li 3] wrote two famous papers on the lattice points of a right-angled triangle because of its relations to their International Congress of Mathematics lecture [Ha-Li 1] in 1912 on Diophantine approximation. A general result was obtained by D. C. Spencer [Sp 1, Sp 2] in 1942 via complex function-theoretic methods, as follows. Write $a_i = x/\rho_i$, $1 \leqslant i \leqslant n$. Then

$$\begin{aligned}Q(a_1, ..., a_n) = {}&\frac{1}{n!}a_1\cdots a_n + \frac{1}{2(n-1)!}\\ &\times\left(\frac{a_n}{a_1}+\cdots+\frac{a_n}{a_{n-1}}+1\right)(a_1\cdots a_{n-1})+o(x^{n-1}).\end{aligned}$$

In 1975, Beukers [Be] used an elementary method to reprove the above formula. Unlike the sharp upper estimate in our Main Theorem, the above formula of Hardy–Littlewood–Spencer is only asymptotic.

Rosser [Ro] in 1939 obtained a lower bound for $Q(a_1, ..., a_n)$ as a polynomial $R(a_1, ..., a_n)$ which, when extended to the general tetrahedron, may be written

$$\begin{aligned}R(a_1, ..., a_n) = {}&\frac{1}{n!\,\rho_1\rho_2\cdots\rho_n}\\ &\times\left(x^n+\frac{n}{2}\sigma'_1x^{n-1}+\frac{n(n-1)}{2^2}\sigma'_2x^{n-2}+\cdots+\frac{n!}{2^{n-1}}\sigma'_{n-1}x\right),\end{aligned}$$

where σ'_k is the sum of the products k at a time of $\rho_2, \rho_3, ..., \rho_n$. Lehmer [Le] constructed two polynomials $\ell(a_1, ..., a_n)$ and $L(a_1, ..., a_n)$ which approximate $Q(a_1, ..., a_n)$ from below and above, respectively. $\ell(a_1, ..., a_n)$ and $L(a_1, ..., a_n)$ are polynomials of degree n in x with coefficients depending on $\rho_1, ..., \rho_n$. The first two coefficients of $\ell(a_1, ..., a_n)$ are seen to agree with those of $R(a_1, ..., a_n)$. For $n > 2$ and x large enough, Lehmer asserted that

$$R(a_1, ..., a_n) < \ell(a_1, ..., a_n) < Q(a_1, ..., a_n).$$

Let $(\rho_1, ..., \rho_5)$ be $(\log_{10} 2, \log_{10} 3, \log_{10} 5, \log_{10} 5, \log_{10} 7, \log_{10} 11)$. A. E. Western has prepared extensive tables of $Q(a_1, a_2, a_3, a_4, a_5)$. From these tables, he has constructed an approximating polynomial $w(a_1, a_2, ..., a_n)$ by applying the method of least squares. Unfortunately, all these polynomials are far from being sharp. In particular, they are not useful for the geometric applications mentioned above.

We remark that our sharp estimate for the number of integral points in 5-dimensional tetrahedra gives a sharp polynomial upper bound of the generalized Dedekind sum appearing in the formula of Cappell and Shaneson [Ca-Sh]. Although the idea of the proof of our theorem is very simple, our proof is quite delicate. We try to estimate the solutions of (1.5) on hyperplanes parallel to $xyzu$-plane by using our upper bounds in the 4-dimensional case, namely (2.1) and (2.2) below, and sum these estimates up. In order to avoid the negative amount difficulty in the right hand side of (2.1) which we have also faced in $n = 4$ case, we need a careful analysis on the last two hyperplanes. For this reason, Lemma 2.3, Theorem 2.4, and Theorem 2.5 are needed to deal with the problem. Our main theorem follows from a careful analysis of this sum. All the computations in this paper are done by Maple V release 5.

2. SHARP ESTIMATE OF THE NUMBER OF INTEGRAL POINTS IN 3- AND 4-DIMENSIONAL TETRAHEDRA AND A PRELIMINARY FACT

The following Theorem 2.1 is proved in [Xu-Ya 3, Li-Ya 1].

THEOREM 2.1. *Let $a \geqslant b \geqslant c \geqslant d \geqslant 2$, and P_4 be the number of positive integral solutions of $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leqslant 1$, i.e., $P_4 = \sharp\{(x, y, z, w) \in \mathbf{Z}_+^4 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leqslant 1\}$ If $P_4 > 0$, then*

$$\begin{aligned} 24P_4 &\leqslant f_4(a, b, c, d) \\ &:= abcd - \frac{3}{2}(abc + abd + acd + bcd) + \frac{11}{3}(ab + ac + bc) - 2(a + b + c) \end{aligned} \tag{2.1}$$

and equality is attained if and only if $a = b = c = d =$ integer.

If $a \geqslant b \geqslant c \geqslant d \geqslant 3$ *or* $a \geqslant 8$, $b \geqslant 6$, $c \geqslant 4$, $d \geqslant 2$, *then* (2.1) *is true without the condition* $P_4 > 0$ *and equality is attained if and only if* $a = b = c = d =$ *integer.*

LEMMA 2.2. $f_4(a, b, c, d)$ *defined in* (2.1) *is nonnegative for the following two cases* (i) $a \geqslant b \geqslant c \geqslant d \geqslant 3$, (ii) $a \geqslant b \geqslant c \geqslant d$ *and* $2 \geqslant d \geqslant 1.5$. *Equality* (2.1) *is attained if and only if when* $a = b = c = d = 3$ *or* $a = b = c = d = 2$.

The above lemma is proved in [Li-Ya 1] while the following lemma follows directly from Proposition 2.3 of [Xu-Ya 3] and its proof.

LEMMA 2.3. *Let* $a \geqslant b \geqslant c \geqslant 1$ *be real number and* $P_3 = \#\{(x, y, z) \in \mathbf{Z}_+^3 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leqslant 1\}$. *Then the following statements hold*

(1) *if* $b \leqslant 2$, *then* $P_3 = 0$
(2) $(a-1)(b-1)(c-1)-(c-1) < 0$ *if and only if* $a < \frac{b}{b-1}$
(3) $(a-1)(b-1)(c-1)-(c-1) < 0$ *implies* $b < 2$ *and* $P_3 = 0$.

The following theorem, which follows directly from Theorem 2.1 and Proposition 2.3 in [Xu-Ya 3] (cf. also [Xu-Ya 1]), will be used frequently.

THEOREM 2.4. *Let* $a \geqslant b \geqslant c \geqslant 1$ *be real number. Let* P_3 *be the number of positive integral solutions of* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leqslant 1$, *i.e.,* $P_3 = \#\{(x, y, z) \in \mathbf{Z}_+^3 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leqslant 1\}$. *If* $b \geqslant 2$, *then* $6P_3 \leqslant (a-1)(b-1)(c-1) - c + 1$, *and equality is attained if and only if* $a = b = c =$ *integer.*

Theorem 2.5 is proved in [Li-Ya 1]

THEOREM 2.5. *Let* $a \geqslant b \geqslant c \geqslant d \geqslant 1$, *and* P_4 *be the number of positive integral solutions of* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leqslant 1$, *i.e.,* $P_4 = \#\{(x, y, z, w) \in \mathbf{Z}_+^4 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leqslant 1\}$. *Define* $\mu = (a-1)(b-1)(c-1)(d-1)$, *then*

$$24P_4 \leqslant \mu = abcd - (abc + abd + acd + bcd) + (ab + ac + ad + bc + bd + cd) - (a + b + c + d) + 1. \tag{2.2}$$

THEOREM 2.6. *Let* $a \geqslant b \geqslant c \geqslant d \geqslant e \geqslant 4$. *Then* f_5 *as defined in* (1.6) *is nonnegative and* $f_5 = 0$ *if and only if* $a = b = c = d = e = 4$.

Proof. First we consider the special case when $a = b = c = d = e$. Then f_5 becomes

$$e^5 - 10e^4 + 35e^3 - 50e^2 + 24e = e(e-1)(e-2)(e-3)(e-4).$$

Hence in this case $f_5 > 0$ when $e > 4$, $f_5 = 0$ when $e = 4$. Next we consider the case when $a \geqslant b \geqslant c \geqslant d \geqslant e \geqslant 4$. Let $A = \frac{a}{e}$, $B = \frac{b}{e}$, $C = \frac{c}{e}$, $D = \frac{d}{e}$, then we have $A \geqslant B \geqslant C \geqslant D \geqslant 1$, and $e \geqslant 4$. Rewrite f_5 as

$$\begin{aligned} f_5 = {} & ABCDe^5 - 2(ABCD + ABC + ABD + ACD + BCD)\, e^4 \\ & + \frac{35}{4}(ABC + ABD + ACD + BCD)\, e^3 \\ & - \frac{25}{3}(AB + AC + AD + BC + BD + CD)\, e^2 + 6(A + B + C + D)\, e. \end{aligned}$$

The idea is show that for all $e \geqslant 4$, the minimum of f_5 in $A \geqslant B \geqslant C \geqslant D \geqslant 1$ occurs at $A = B = C = D = 1$ and $f_5(1, 1, 1, 1) = e(e-1)(e-2) \times (e-3)(e-4) \geqslant 0$ for $e \geqslant 4$.

Note that f_5 is symmetric with respect to A, B, C, and D.

$$\frac{\partial^4 f_5}{\partial A\, \partial B\, \partial C\, \partial D} = e^5 - 2e^4 = e^4(e-2) > 0 \qquad \text{for} \quad e \geqslant 4.$$

It follows that $\partial^3 f_5 / \partial A\, \partial B\, \partial C$ is an increasing function of D for $e \geqslant 4$, $D \geqslant 1$ Hence the minimum $\partial^3 f_5 / \partial A\, \partial B\, \partial C$ occurs at $D = 1$,

$$\begin{aligned} \left. \frac{\partial^3 f_5}{\partial A\, \partial B\, \partial C} \right|_{D=1} & = e^5 - 4e^4 + \frac{35}{4} e^3 \\ & = e^3 \left[(e-2)^2 + \frac{19}{4} \right] > 0 \qquad \text{for} \quad e \geqslant 4. \end{aligned}$$

It follows that

$$\frac{\partial^3 f_5}{\partial A\, \partial B\, \partial C} > 0 \qquad \text{for} \quad D \geqslant 1, \quad e \geqslant 4. \tag{2.3}$$

Note that $\partial^2 f_5 / \partial A\, \partial B$ is symmetric with respect to C, D. Combining with (2.3) we deduce

$$\frac{\partial^3 f_5}{\partial A\, \partial B\, \partial D} > 0 \qquad \text{for} \quad C \geqslant 1, \quad e \geqslant 4. \tag{2.4}$$

Putting (2.3) and (2.4) together, we have $\partial^2 f_5 / \partial A\, \partial B$ is an increasing function of C and D for $C \geqslant 1$, $D \geqslant 1$ and $e \geqslant 4$. The minimum of $\partial^2 f_5 / \partial A\, \partial B$ occurs at $C = 1$ and $D = 1$

$$\left.\frac{\partial^2 f_5}{\partial A\,\partial B}\right|_{C=1,\,D=1} = e^5 - 6e^4 + \frac{35}{2}e^3 - \frac{25}{3}e^2$$

$$= \frac{1}{6}e^2(6e^3 - 36e^2 + 105e - 50) > 0 \qquad \text{for} \quad e \geqslant 4.$$

It follows that

$$\frac{\partial^2 f_5}{\partial A\,\partial B} > 0 \qquad \text{for} \quad C \geqslant 1, \quad D \geqslant 1, \quad e \geqslant 4. \tag{2.5}$$

From the property that $\partial f_5/\partial A$ is symmetric with respect to B, C, and D and (2.5), we also have

$$\frac{\partial^2 f_5}{\partial A\,\partial C} > 0 \qquad \text{for} \quad B \geqslant 1, \quad D \geqslant 1, \quad e \geqslant 4 \tag{2.6}$$

$$\frac{\partial^2 f_5}{\partial A\,\partial D} > 0 \qquad \text{for} \quad B \geqslant 1, \quad C \geqslant 1, \quad e \geqslant 4. \tag{2.7}$$

Combining (2.5), (2.6), and (2.7), we have that $\partial f_5/\partial A$ is an increasing function of B, C, and D for $B \geqslant 1$, $C \geqslant 1$, $D \geqslant 1$ and $e \geqslant 4$. Hence the minimum of $\partial f_5/\partial A$ occurs at $B = 1$, $C = 1$, and $D = 1$

$$\left.\frac{\partial f_5}{\partial A}\right|_{B=1,\,C=1,\,D=1} = e^5 - 8e^4 + \frac{105}{4}e^3 - 25e^2 + 6e$$

$$= \frac{1}{4}e(4e^4 - 32e^3 + 105e^2 + 24).$$

The largest root of $(4e^4 - 32e^3 + 105e^2 - 100e + 24)$ is approximately 0.965. It follows that

$$\frac{\partial f_5}{\partial A} > 0 \qquad \text{for} \quad B \geqslant 1, \quad C \geqslant 1, \quad D \geqslant 1, \quad e \geqslant 4. \tag{2.8}$$

By the property f_5 is symmetric with respect to A, B, C, and D and (2.8) we have the minimum of f_5 occurs at $A = B = C = D = 1$,

$$f_5|_{A=1,\,B=1,\,C=1,\,D=1} = e^5 - 10e^4 + 35e^3 - 50e^2 + 24e$$

$$= e(e-1)(e-2)(e-3)(e-4) \geqslant 0 \qquad \text{for} \quad e \geqslant 4.$$

Therefore we have $f_5 \geqslant 0$ when $a \geqslant b \geqslant c \geqslant d \geqslant e \geqslant 4$ and $f_5 = 0$ only when $a = b = c = d = e = 4$.

3. PROOF OF THE MAIN THEOREM

By level $v = k$, we shall mean the intersection of the tetrahedron in (1.5) with the hyperplane $v = k$. For $v = k$ points in our tetrahedron are in following 4 dimensional tetrahedron

$$\frac{x}{a\left(1-\frac{k}{e}\right)}+\frac{y}{b\left(1-\frac{k}{e}\right)}+\frac{z}{c\left(1-\frac{k}{e}\right)}+\frac{u}{d\left(1-\frac{k}{e}\right)}\leqslant 1$$

for which the number of integral points can be bounded by means of Theorem 2.1 if $k \leqslant [e]-3$. By summing these upper bounds in various levels, this leads us to consider the following function

$$\begin{aligned} &g(a, b, c, d, e, n)\\ &\quad = 5\sum_{k=1}^{n}\left[abcd\left(1-\frac{k}{e}\right)^4-\frac{3}{2}(abc+abd+acd+bcd)\left(1-\frac{k}{e}\right)^3\right.\\ &\qquad\left.+\frac{11}{3}(ab+bc+ac)\left(1-\frac{k}{e}\right)^2-(a+b+c)\left(1-\frac{k}{e}\right)\right]. \end{aligned}$$

Define $S_5 = abcde$, $S_4 = abcd$, $S_3 = abc+abd+acd+bcd$, $S_2 = ab+ac+bc$, and $S_1 = a+b+c$. Then we have

$$\begin{aligned} &g(a, b, c, d, e, n)\\ &\quad = S_5\left(\frac{n}{e}\right)^5+\left[S_4\left(\frac{1}{2e^4}-\frac{1}{e^3}\right)+\frac{3}{8}\frac{S_3}{e^3}\right]5n^4\\ &\qquad+\left[S_4\left(\frac{1}{3e^4}-\frac{2}{e^3}+\frac{2}{e^2}\right)+3S_3\left(\frac{1}{4e^3}-\frac{1}{2e^2}\right)+\frac{11}{9}\frac{S_2}{e^2}\right]5n^3\\ &\qquad+\left[S_4\left(\frac{-1}{e^3}+\frac{3}{e^2}-\frac{2}{e}\right)+3S_3\left(\frac{1}{8e^3}-\frac{3}{4e^2}+\frac{3}{4e}\right)\right.\\ &\qquad\left.+11S_2\left(\frac{1}{6e^2}-\frac{1}{3e}\right)+\frac{S_1}{e}\right]5n^2\\ &\qquad+\left[S_4\left(\frac{-1}{30e^4}+\frac{1}{e^2}-\frac{2}{e}\right)+\frac{3}{4}S_3\left(\frac{-1}{e^2}+\frac{3}{e}\right)+\frac{11}{3}S_2\left(\frac{1}{6e^2}-\frac{1}{e}\right)+\frac{S_1}{e}\right]5n\\ &\qquad+\left(5S_4-\frac{15}{2}S_3+\frac{55}{3}S_2-10S_1\right)n. \end{aligned}$$

For fixed a, b, c, d, and e then $g(a, b, c, d, e, n)$ is a function of n and we denote it by $g(n)$. Let $e = [e] + \beta$ with $0 \leqslant \beta < 1$. The strategy of our proof is roughly as follows. Let L_0, L_1 and L_2 be the number of positive integral solutions at level $v = [e]$, $[e]-1$ and $[e]-2$, respectively. Then

$$120P_5 \leqslant h := g([e]-3) + 120(L_0 + L_1 + L_2). \tag{3.1}$$

In order to prove our Main Theorem, it suffices to bound h by f_5. We use calculus (much as in the proof of Theorem 2.5, above) to prove that $f_5 - h$ is nonnegative and $f_5 = h$ if and only if $a = b = c = d = e = integer$.

We divide our proof in two cases depending on whether level $v = [e]$ the tetrahedron

$$\frac{x}{\frac{a}{e}\beta} + \frac{y}{\frac{b}{e}\beta} + \frac{z}{\frac{c}{e}\beta} + \frac{u}{\frac{d}{e}\beta} \leqslant 1 \tag{3.2}$$

has positive integral solutions.

Case (1). $\frac{a}{e}\beta \geqslant 4$, $\frac{b}{e}\beta > 3$, $\frac{c}{e}\beta > 2$, and $\frac{d}{e}\beta > 1$, where $0 < \beta < 1$.

In this case (3.2) may have positive integral solutions.

Case (2). $\frac{a}{e}\beta < 4$ or $\frac{b}{e}\beta \leqslant 3$ or $\frac{c}{e}\beta \leqslant 2$ or $\frac{d}{e}\beta \leqslant 1$.

In this case there is no positive integral solutions at level $v = [e]$.

In case (1), $h = g([e]-3) + 120(L_0 + L_1 + L_2) \leqslant g([e]-1) + 120L_0$. There are three subcases to be considered

Case (1a). $1 < \frac{d}{e}\beta < 2$, $\frac{b}{e}\beta \geqslant 4$. In this case $120L_0$ can be bounded from above by R.H.S. of (3.4) by using Theorem 2.4. We shall prove that $\Delta = f_5 - g([e]-1) - \text{R.H.S. of (3.4)} > 0$.

Case (1b). $1 < \frac{d}{e}\beta < 2$, $3 < \frac{b}{e}\beta < 4$. In this case $L_0 = 0$. We shall prove that $\Delta = f_5 - g([e]-1) > 0$.

Case (1c). $\frac{d}{e}\beta \geqslant 2$. If $L_0 = 0$, then the proof is the same as Case (1b). If $L_0 > 0$, then $120L_0$ can be bounded from above by R.H.S. of (3.21). We shall prove that $\Delta = f_5 - g([e]-1) - \text{R.H.S. of (3.21)} > 0$.

In case (2), $L_0 = 0$ and $h = g([e]-3) + 120(L_1 + L_2)$. There are five subcases to be considered.

Case (2a). $a = b = c = d = e$. In this case, $L_0 = L_1 = L_2 = L_3 = 0$ where L_3 is the number of positive integral solutions at level $v = [e]-3$. Then $120P_5 \leqslant g([e]-4)$. We shall prove that $\Delta = f_5 - g([e]-4) \geqslant 0$ and $\Delta = 0$ if and only if $a = b = c = d = e = integer$.

Case (2b). $a = b = c = d \geqslant e$. There are three subcases.

(b1) $L_1 = L_2 = 0$. If $L_3 = 0$, then $4 > \frac{a}{e}(\beta + 3)$. Since $120P_5 \leqslant g([e]-4)$, we shall prove that $\Delta = f_5 - g([e]-4) \geqslant 0$. On the other hand if $L_3 > 0$, then $a \geqslant \frac{4e}{\beta+3}$. We shall prove that $\Delta = f_5 - g([e]-3) \geqslant 0$. In all cases, $\Delta = 0$ if and only if $a = b = c = d = e = integer$.

(b2) $L_1 = 0$, $L_2 > 0$. In this case $120L_2$ can be bounded above by 5 (R.H.S. of (3.23)). We shall prove that $\Delta = f_5 - g([e]-3) - 5$ (R.H.S. of (3.23)) $\geqslant 0$ and $\Delta = 0$ if and only if $a = b = c = d = e = integer$.

(b3) $L_1 > 0$, $L_2 > 0$. In this case $120(L_1 + L_2)$ can be bounded above by 5 (R.H.S. of (3.22)+R.H.S. of (3.23)). We shall prove that $\Delta = f_5 - g([e]-3) - 5$ (R.H.S. of (3.22)+R.H.S. of (3.23)) $\geqslant 0$ and $\Delta = 0$ if and only if $a = b = c = d = e = integer$.

Case (2c). $a = b = c \geqslant d \geqslant e$.

Case (2d). $a = b \geqslant c \geqslant d \geqslant e$.

Case (2e). $a \geqslant b \geqslant c \geqslant d \geqslant e$.

In all the last three cases, we have to consider three subcases as in case (2b). However, in the subcase $L_1 = L_2 = 0$, unlike the case (2b1), the situation is simpler because we only need to prove $\Delta = g([e]-3) \geqslant 0$.

Case (1). $\frac{a}{e}\beta \geqslant 4$, $\frac{b}{e}\beta > 3$, $\frac{c}{e}\beta > 2$, and $\frac{d}{e}\beta > 1$.

Case (1a). $1 < \frac{d}{e}\beta < 2$, $\frac{b}{e}\beta \geqslant 4$.
At level $v = [e] = e - \beta$, u can only be 1 to have any positive integral solution. Rewrite (3.2) as

$$\frac{x}{\frac{a}{e}\beta} + \frac{y}{\frac{b}{e}\beta} + \frac{z}{\frac{c}{e}\beta} \leqslant 1 - \frac{1}{\frac{d}{e}\beta} < \frac{1}{2}. \tag{3.3}$$

Let L_0 be the number of positive integral solution at level $v = [e] = e - \beta$. By Theorem 2.4 we have

$$120L_0 \leqslant \frac{5}{2}\frac{\beta^3 abc}{e^3} - 5\frac{\beta^2(ab+ac+bc)}{e^2} + 10\frac{\beta(a+b)}{e}. \tag{3.4}$$

At level $v = [e] - 1 = e - \beta - 1$ we have $x/[(a/e)(\beta+1)] + y/[(b/e)(\beta+1)] + z/[(c/e)(\beta+1)] + u/[(d/e)(\beta+1)] \leqslant 1$ where $\frac{a}{e}(\beta+1) > 8$, $\frac{b}{e}(\beta+1) > 8$, $\frac{c}{e}(\beta+1) > 4$, and $\frac{d}{e}(\beta+1) > 2$. Hence by Theorem 2.1 we have $f_4 > 0$. Therefore we need to sum from level 1 to level $n = e - \beta - 1$ in $g(n)$. Let $\Delta = f_5 -$ R.H.S. of $(3.4) - g|_{n=e-\beta-1}$. Then we have

$$\Delta = \frac{1}{72}(36abcde^4 - 9abce^5 - 9abde^5 - 9acde^5 - 9bcde^5 - 120abcde^3$$
$$+ 360abce^4 + 360abde^4 - 440abe^5 + 360acde^4 - 440ace^5$$
$$+ 360bcde^4 - 440bce^5 + 135abce^3 + 135abde^3 + 60abe^4 + 135acde^3$$
$$+ 60ace^4 - 600ade^4 + 360ae^5 + 135bcde^3 + 60bce^4 - 600bde^4$$
$$+ 360be^5 - 600cde^4 + 360ce^5 + 12abcde - 220abe^3 - 220ace^3$$
$$+ 72ae^4 - 220bce^3 + 72be^4 + 72ce^4 + 432de^4 + 72\beta^5 abcd$$
$$+ 120\beta^3 abcd + 180\beta^4 abcd - 12\beta abcd - 135\beta^4 abce - 135\beta^2 abce$$
$$- 450\beta^3 abce - 135\beta^2 abde - 135\beta^4 abde - 270\beta^3 abde + 440\beta^3 abe^2$$
$$+ 220\beta abe^2 + 1020\beta^2 abe^2 - 135\beta^2 acde - 270\beta^3 acde - 135\beta^4 acde$$
$$+ 1020\beta^2 ace^2 + 220\beta ace^2 + 440\beta^3 ace^2 - 360\beta^2 ae^3 - 1080\beta ae^3$$
$$- 270\beta^3 bcde - 135\beta^4 bcde - 135\beta^2 bcde + 440\beta^3 bce^2 + 220\beta bce^2$$
$$+ 1020\beta^2 bce^2 - 1080\beta be^3 - 360\beta^2 be^3 - 360\beta ce^3 - 360\beta^2 ce^3)/e^4.$$

Let $\Delta = (1/72e^4)\,\Delta_1$. The idea is to show that Δ_1 is an increasing function in a, b, c and d for $e \geqslant 4$, $a \geqslant \frac{4e}{\beta}$, $b \geqslant \frac{4e}{\beta}$, $c > \frac{2e}{\beta}$, $d > \frac{e}{\beta}$. Then we show that

$$\Delta_1 \big|_{a = \frac{4e}{\beta},\, b = \frac{4e}{\beta},\, c = \frac{2e}{\beta},\, d = \frac{e}{\beta}} = 16 \left(\frac{e}{\beta}\right)^4 \Delta_8,$$

where Δ_8 is an increasing function of e whose minimum occurs at $e = 4$. Since $\Delta_8|_{e=4} > 0$, this will prove that $\Delta_1 > 0$ and hence $\Delta > 0$ in case (1a). Note that Δ_1 is symmetric with respect to a and b and that $\partial\Delta_1/\partial a$ is symmetric with respect to b and c,

$$\frac{\partial^4 \Delta_1}{\partial a\, \partial b\, \partial c\, \partial d} = 36e^4 - 120e^3 + 12e - 12\beta + 72\beta^5 + 180\beta^4 + 120\beta^3$$
$$\frac{\partial^4 \Delta_1}{\partial a\, \partial b\, \partial c\, \partial d} > 24e^3 > 0 \qquad \text{for} \quad e \geqslant 4, \quad 0 < \beta < 1. \tag{3.5}$$

Hence $\partial^3\Delta_1/\partial a\, \partial b\, \partial d$ is an increasing function of c for $c \geqslant \frac{2e}{\beta}$ and the minimum of $\partial^3\Delta_1/\partial a\, \partial b\, \partial d$ is at $c = \frac{2e}{\beta}$. $\partial^3\Delta_1/\partial a\, \partial b\, \partial c$ is an increasing function of d for $d \geqslant \frac{e}{\beta}$ and the minimum of $\partial^3\Delta_1/\partial a\, \partial b\, \partial c$ is at $d = \frac{e}{\beta}$,

$$\left.\frac{\partial^3 \Delta_1}{\partial a\, \partial b\, \partial d}\right|_{c = \frac{2e}{\beta}} = 3e(24e^4 - 3\beta e^4 - 80e^3 + 120\beta e^3 + 45\beta e^2 + 8e - 8\beta + 3\beta^5$$
$$+ 30\beta^4 + 35\beta^3)/\beta > \frac{3e}{\beta}(4e^3) > 0 \qquad \text{for} \quad e \geqslant 4, \quad 0 < \beta < 1.$$

Hence we have

$$\frac{\partial^3 \Delta_1}{\partial a\,\partial b\,\partial d} > 0 \qquad \text{for} \quad e \geqslant 4, \quad \text{and} \quad c \geqslant \frac{2e}{\beta} \tag{3.6}$$

$$\left.\frac{\partial^3 \Delta_1}{\partial a\,\partial b\,\partial c}\right|_{d=\frac{e}{\beta}} = 3(12e^4 - 3\beta e^4 - 40e^3 + 120\beta e^3 + 45\beta e^2 + 4e - 4\beta - 21\beta^5$$

$$-90\beta^4 - 5\beta^3)/\beta = \frac{3e}{\beta}\Delta_2$$

$$\frac{\partial \Delta_2}{\partial e} = 48e^3 - 12\beta e^3 - 120e^2 + 360\beta e^2 + 90e\beta + 4$$

$$> 24e^2 > 0 \qquad \text{for} \quad e \geqslant 4, \quad 0 < \beta < 1.$$

Hence Δ_2 is an increasing function of e for $e \geqslant 4$, and the minimum of Δ_2 is at $e = 4$,

$$\Delta_2|_{e=4} = 528 + 7628\beta - 21\beta^5 - 90\beta^4 - 5\beta^3 > 0 \qquad \text{for} \quad 0 < \beta < 1.$$

Hence we have

$$\frac{\partial^3 \Delta_1}{\partial a\,\partial b\,\partial c} > 0 \qquad \text{for} \quad e \geqslant 4, \quad \text{and} \quad d \geqslant \frac{e}{\beta}. \tag{3.7}$$

By (3.6) and (3.7) we have $\partial^2 \Delta_1 / \partial a\,\partial b$ is an increasing function of c and d for $c \geqslant \frac{2e}{\beta}$, $d \geqslant \frac{e}{\beta}$ and the minimum of $\partial^2 \Delta_1 / \partial a\,\partial b$ occurs at $c = \frac{2e}{\beta}$ and $d = \frac{e}{\beta}$,

$$\left.\frac{\partial^2 \Delta_1}{\partial a\,\partial b}\right|_{c=\frac{2e}{\beta},\, d=\frac{e}{\beta}} = e^2(-27\beta e^4 + 72e^4 + 1080\beta e^3 - 240e^3 - 440\beta^2 e^3 + 60\beta^2 e^2$$

$$+405\beta e^2 - 220\beta^2 e + 24e + 55\beta^3 + 210\beta^4 + 179\beta^5 - 24\beta)/\beta^2$$

$$= \left(\frac{e}{\beta}\right)^2 \Delta_3$$

$$\frac{\partial \Delta_3}{\partial e} = -108\beta e^3 + 288e^3 + 3240\beta e^2 - 720e^2 - 1320\beta^2 e^2 + 120\beta^2 e$$

$$+810e\beta - 220\beta^2 + 24 > 24 > 0 \qquad \text{for} \quad e \geqslant 4, \quad 0 < \beta < 1.$$

Hence Δ_3 is an increasing function of e for $e \geqslant 4$, and the minimum of Δ_3 is at $e = 4$,

$$\Delta_3|_{e=4} = 68664\beta + 3168 - 28080\beta^2 + 55\beta^3 + 210\beta^4 + 179\beta^5$$

$$> 0 \qquad \text{for} \quad 1 > \beta > 0.$$

Hence we have

$$\frac{\partial^2 \Delta_1}{\partial a\, \partial b} > 0 \qquad \text{for} \quad e \geqslant 4, \quad c \geqslant \frac{2e}{\beta} \quad \text{and} \quad d \geqslant \frac{e}{\beta}. \tag{3.8}$$

By the symmetry of $\partial \Delta_1 / \partial a$ with respect to b and c, and Eq. (3.8) we also have

$$\frac{\partial^2 \Delta_1}{\partial a\, \partial c} > 0 \qquad \text{for} \quad e \geqslant 4, \quad b \geqslant \frac{2e}{\beta} \quad \text{and} \quad d \geqslant \frac{e}{\beta}. \tag{3.9}$$

By the symmetry of $\partial \Delta_1 / \partial a$ with respect to b and c, and Eq. (3.6) we also have

$$\frac{\partial^3 \Delta_1}{\partial a\, \partial c\, \partial d} > 0 \qquad \text{for} \quad e \geqslant 4, \quad b \geqslant \frac{2e}{\beta}. \tag{3.10}$$

Combining (3.6), and (3.10), we have $\partial^2 \Delta_1 / \partial a\, \partial d$ is an increasing function of b and c for $b \geqslant \frac{4e}{\beta}$ and $c \geqslant \frac{2e}{\beta}$, and the minimum of $\partial^2 \Delta_1 / \partial a\, \partial d$ occurs at $b = \frac{4e}{\beta}$ and $c = \frac{2e}{\beta}$,

$$\begin{aligned} \left.\frac{\partial^2 \Delta_1}{\partial a\, \partial d}\right|_{b = \frac{4e}{\beta},\, c = \frac{2e}{\beta}} &= 6e^2(48e^4 - 9\beta e^4 - 160e^3 + 360\beta e^3 + 135\beta e^2 - 100\beta^2 e^2 + 16e \\ &\quad - 16\beta - 30\beta^4 + 25\beta^3 - 39\beta^5)/\beta^2 = 6\left(\frac{e}{\beta}\right)^2 \Delta_4 \end{aligned}$$

$$\begin{aligned} \frac{\partial \Delta_4}{\partial e} &= 192e^3 - 36\beta e^3 - 480e^2 + 1080\beta e^2 + 270e\beta - 200\beta^2 e + 16 \\ &> 144e^2 > 0 \qquad \text{for} \quad e \geqslant 4, \quad \text{and} \quad 1 > \beta > 0. \end{aligned}$$

Hence Δ_4 is an increasing function of e and the minimum of Δ_4 occurs at $e = 4$,

$$\begin{aligned} \Delta_4|_{e=4} &= 2112 + 22880\beta - 1600\beta^2 - 30\beta^4 + 25\beta^3 - 39\beta^5 \\ &> 0 \qquad \text{for} \quad 1 > \beta > 0. \end{aligned}$$

Hence we have

$$\frac{\partial^2 \Delta_1}{\partial a\, \partial d} > 0 \qquad \text{for} \quad e \geqslant 4, \quad b \geqslant \frac{4e}{\beta}, \quad \text{and} \quad c \geqslant \frac{2e}{\beta}. \tag{3.11}$$

Combining (3.8), (3.9), and (3.11), we have $\partial \Delta_1 / \partial a$ is an increasing function of b, c, and d for $b \geqslant \frac{4e}{\beta}$, $c \geqslant \frac{2e}{\beta}$ and $d \geqslant \frac{e}{\beta}$. The minimum of $\partial \Delta_1 / \partial a$ occurs at $b = \frac{4e}{\beta}$, $c = \frac{2e}{\beta}$, and $d = \frac{e}{\beta}$,

$$\left.\frac{\partial \Delta_1}{\partial a}\right|_{b=\frac{4e}{\beta},\, c=\frac{2e}{\beta},\, d=\frac{e}{\beta}} = 6e^3(48e^4 - 21\beta e^4 - 440\beta^2 e^3 - 160e^3 + 840\beta e^3 - 40\beta^2 e^2 + 60\beta^3 e^2 + 315\beta e^2 - 220\beta^2 e + 16e + 12\beta^3 e - 16\beta + 210\beta^4 + 161\beta^5 + 65\beta^3)/\beta^3 = 6\left(\frac{e}{\beta}\right)^3 \Delta_5$$

$$\frac{\partial \Delta_5}{\partial e} = 192e^3 - 84\beta e^3 - 1320\beta^2 e^2 - 480e^2 + 2520\beta e^2 - 80\beta^2 e + 120\beta^3 e + 630 e\beta - 220\beta^2 + 16 + 12\beta^3$$

$$\frac{\partial^2 \Delta_5}{\partial e^2} = 576e^2 - 252\beta e^2 - 2640\beta^2 e - 960e + 5040e\beta - 80\beta^2 + 120\beta^3 + 630\beta > 336e > 0 \qquad \text{for} \quad e \geqslant 4, \quad 0 < \beta < 1.$$

Hence $\partial\Delta_5/\partial e$ is an increasing function of e and the minimum of $\partial\Delta_5/\partial e$ occurs at $e = 4$,

$$\left.\frac{\partial \Delta_5}{\partial e}\right|_{e=4} = 6424 + 37464\beta - 21660\beta^2 + 492\beta^3 > 0 \qquad \text{for} \quad 0 < \beta < 1.$$

Hence Δ_5 is an increasing function of e and the minimum of Δ_5 occurs at $e = 4$,

$$\Delta_5|_{e=4} = 2112 + 53408\beta - 29680\beta^2 + 1073\beta^3 + 210\beta^4 + 161\beta^5 > 0 \qquad \text{for} \quad 0 < \beta < 1.$$

Hence we have

$$\frac{\partial \Delta_1}{\partial a} > 0 \qquad \text{for} \quad e \geqslant 4, \quad b \geqslant \frac{4e}{\beta}, \quad c \geqslant \frac{2e}{\beta}, \quad \text{and} \quad d \geqslant \frac{e}{\beta}. \tag{3.12}$$

From the symmetry of Δ_1 with respect to a and b we also have

$$\frac{\partial \Delta_1}{\partial b} > 0 \qquad \text{for} \quad e \geqslant 4, \quad a \geqslant \frac{4e}{\beta}, \quad c \geqslant \frac{2e}{\beta}, \quad \text{and} \quad d \geqslant \frac{e}{\beta}. \tag{3.13}$$

From the symmetry of Δ_1 with respect to a and b and by (3.9) we also have

$$\frac{\partial^2 \Delta_1}{\partial b\, \partial c} > 0 \qquad \text{for} \quad e \geqslant 4, \quad a \geqslant \frac{2e}{\beta} \quad \text{and} \quad d \geqslant \frac{e}{\beta}. \tag{3.14}$$

From the symmetry of Δ_1 with respect to a and b and by (3.10) we also have

$$\frac{\partial^3 \Delta_1}{\partial b\, \partial c\, \partial d} > 0 \qquad \text{for} \quad e \geqslant 4, \quad a \geqslant \frac{2e}{\beta}. \tag{3.15}$$

From the symmetry of Δ_1 with respect to a and b and by (3.11) we also have

$$\frac{\partial^2 \Delta_1}{\partial b\, \partial d} > 0 \qquad \text{for} \quad e \geqslant 4, \quad a \geqslant \frac{4e}{\beta}, \quad \text{and} \quad c \geqslant \frac{2e}{\beta}. \tag{3.16}$$

Combining (3.10), and (3.15), we have $\partial^2 \Delta_1 / \partial c\, \partial d$ is an increasing function of a and b for $a \geqslant \frac{4e}{\beta}$ and $b \geqslant \frac{4e}{\beta}$, hence the minimum of $\partial^2 \Delta_1 / \partial c\, \partial d$ occurs at $a = \frac{4e}{\beta}$ and $b = \frac{4e}{\beta}$,

$$\begin{aligned} \left.\frac{\partial^2 \Delta_1}{\partial c\, \partial d}\right|_{a = \frac{4e}{\beta},\, b = \frac{4e}{\beta}} &= 24e^2(24e^4 - 3\beta e^4 - 80e^3 + 120\beta e^3 + 45\beta e^2 - 25\beta^2 e^2 + 8e \\ &\quad - 8\beta + 30\beta^4 + 35\beta^3 + 3\beta^5)/\beta^2 \\ &> 24\left(\frac{e}{\beta}\right)^2 (4e^3) > 0 \qquad \text{for} \quad e \geqslant 4, \quad 0 < \beta < 1. \end{aligned}$$

Hence we have

$$\frac{\partial^2 \Delta_1}{\partial c\, \partial d} > 0 \qquad \text{for} \quad e \geqslant 4, \quad 1 > \beta > 0, \quad a \geqslant \frac{4e}{\beta}, \quad b \geqslant \frac{4e}{\beta}. \tag{3.17}$$

Combining (3.9), (3.14), and (3.17), we have $\partial \Delta_1 / \partial c$ is an increasing function of a, b, and d for $a \geqslant \frac{4e}{\beta}$, $b \geqslant \frac{4e}{\beta}$ and $d \geqslant \frac{e}{\beta}$. Hence the minimum of $\partial \Delta_1 / \partial c$ occurs at $a = \frac{4e}{\beta}$, $b = \frac{4e}{\beta}$, and $d = \frac{e}{\beta}$,

$$\begin{aligned} \left.\frac{\partial \Delta_1}{\partial c}\right|_{a = \frac{4e}{\beta},\, b = \frac{4e}{\beta},\, d = \frac{e}{\beta}} &= 8e^3(-27\beta e^4 + 72e^4 - 440\beta^2 e^3 + 1080\beta e^3 - 240e^3 + 45\beta^3 e^2 \\ &\quad - 15\beta^2 e^2 + 405\beta e^2 + 9\beta^3 e - 220\beta^2 e + 24e + 165\beta^4 + 134\beta^5 \\ &\quad - 24\beta + 55\beta^3)/\beta^3 = 8\left(\frac{e}{\beta}\right)^3 \Delta_6 \end{aligned}$$

$$\begin{aligned} \frac{\partial \Delta_6}{\partial e} &= -108\beta e^3 + 288e^3 - 1320\beta^2 e^2 + 3240\beta e^2 - 720e^2 + 90\beta^3 e \\ &\quad - 30\beta^2 e + 810e\beta + 9\beta^3 - 220\beta^2 + 24 > 24 \\ &> 0 \qquad \text{for} \quad e \geqslant 4, \quad 0 < \beta < 1. \end{aligned}$$

Hence Δ_6 is an increasing function of e and the minimum of Δ_6 occurs at $e=4$,

$$\begin{aligned}\Delta_6|_{e=4} &= 68664\beta+3168-29280\beta^2+811\beta^3+165\beta^4+134\beta^5\\ &>0 \qquad \text{for} \quad 0<\beta<1.\end{aligned}$$

Hence we have

$$\frac{\partial \Delta_1}{\partial c}>0 \qquad \text{for} \quad e\geqslant 4, \quad a\geqslant \frac{4e}{\beta}, \quad b\geqslant \frac{4e}{\beta}, \quad \text{and} \quad d\geqslant \frac{e}{\beta}. \tag{3.18}$$

Combining (3.11), (3.16), and (3.17), we have $\partial \Delta_1/\partial d$ is an increasing function of a, b, and c for $a\geqslant \frac{4e}{\beta}$, $b\geqslant \frac{4e}{\beta}$ and $c>\frac{2e}{\beta}$. Hence the minimum of $\partial \Delta_1/\partial d$ occurs at $a=\frac{4e}{\beta}$, $b=\frac{4e}{\beta}$, and $c=\frac{2e}{\beta}$,

$$\begin{aligned}\left.\frac{\partial \Delta_1}{\partial d}\right|_{a=\frac{4e}{\beta},\, b=\frac{4e}{\beta},\, c=\frac{2e}{\beta}} &= 48e^3(24e^4-6\beta e^4-80e^3+240\beta e^3-125\beta^2e^2+90\beta e^2+8e\\ &\quad +9\beta^3 e-8\beta-10\beta^3-42\beta^5-60\beta^4)/\beta^3 = 48\left(\frac{e}{\beta}\right)^3 \Delta_7\\ \frac{\partial \Delta_7}{\partial e} &= 96e^3-24\beta e^3-240e^2+720\beta e^2+180\beta e-250\beta^2 e+9\beta^3\\ &\quad +8>48e^2>0 \qquad \text{for} \quad e\geqslant 4, \quad 0<\beta<1.\end{aligned}$$

Hence Δ_7 is an increasing function of e and the minimum of Δ_7 occurs at $e=4$,

$$\begin{aligned}\Delta_7|_{e=4} &= -42\beta^5-60\beta^4+26\beta^3-2000\beta^2+15256\beta+1056\\ &>0 \qquad \text{for} \quad 0<\beta<1.\end{aligned}$$

It follows that

$$\frac{\partial \Delta_1}{\partial d}>0 \qquad \text{for} \quad e\geqslant 4, \quad a\geqslant \frac{4e}{\beta}, \quad b\geqslant \frac{4e}{\beta}, \quad \text{and} \quad c\geqslant \frac{2e}{\beta}. \tag{3.19}$$

Combining (3.12), (3.13), (3.18), and (3.19), we have Δ_1 is an increasing function of a, b, c, and d for $a\geqslant \frac{4e}{\beta}$, $b\geqslant \frac{4e}{\beta}$, $c>\frac{2e}{\beta}$, and $d>\frac{e}{\beta}$. Hence the minimum of Δ_1 occurs at $a=\frac{4e}{\beta}$, $b=\frac{4e}{\beta}$, $c=\frac{2e}{\beta}$, and $d=\frac{e}{\beta}$.

$$\Delta_1|_{a=\frac{4e}{\beta}, b=\frac{4e}{\beta}, c=\frac{2e}{\beta}, d=\frac{e}{\beta}} = 16e^4(72e^4 - 36\beta e^4 - 240e^3 + 1440\beta e^3 - 880\beta^2 e^3 + 225\beta^3 e^2 + 540\beta e^2 - 255\beta^2 e^2 - 440\beta^2 e + 72\beta^3 e + 24e - 24\beta + 259\beta^5 + 375\beta^4 + 140\beta^3)/\beta^4 = 16\left(\frac{e}{\beta}\right)^4 \Delta_8$$

$$\frac{\partial \Delta_8}{\partial e} = 288e^3 - 144\beta e^3 - 720e^2 + 4320\beta e^2 - 2640\beta^2 e^2 + 450\beta^3 e + 1080\beta e - 510\beta^2 e - 440\beta^2 + 72\beta^3 + 24$$

$$\frac{\partial^2 \Delta_8}{\partial e^2} = 864e^2 - 432\beta e^2 - 1440e + 8640\beta e - 5280\beta^2 e + 450\beta^3 + 1080\beta - 510\beta^2 > 288e > 0 \qquad \text{for} \quad e \geqslant 4, \quad 0 < \beta < 1.$$

Hence $\partial \Delta_8/\partial e$ is an increasing function of e and the minimum of $\partial \Delta_8/\partial e$ occurs at $e = 4$,

$$\left.\frac{\partial \Delta_8}{\partial e}\right|_{e=4} = 6936 + 64224\beta - 44720\beta^2 + 1872\beta^3 > 0 \qquad \text{for} \quad 0 < \beta < 1.$$

Hence Δ_8 is an increasing function of e and the minimum of Δ_8 occurs at $e = 4$,

$$\Delta_8|_{e=4} = 3168 + 91560\beta - 62160\beta^2 + 4028\beta^3 + 259\beta^5 + 375\beta^4 > 0 \qquad \text{for} \quad 0 < \beta < 1.$$

Hence we have

$$\Delta_1 > 0 \qquad \text{for} \quad e \geqslant 4, \quad a \geqslant \frac{4e}{\beta}, b \geqslant \frac{4e}{\beta}, c \geqslant \frac{2e}{\beta}, \quad \text{and} \quad d \geqslant \frac{e}{\beta}.$$

Finally we have $\Delta = (1/72e^4)\, \Delta_1 > 0$ for $e \geqslant 4$, $a \geqslant \frac{4e}{\beta}$, $b \geqslant \frac{4e}{\beta}$, $c \geqslant \frac{2e}{\beta}$, and $d \geqslant \frac{e}{\beta}$.

We shall write subsequent (analogous) cases out in far less detail.

Case (1b). $1 < \frac{d}{e}\beta < 2$, $3 < \frac{b}{e}\beta < 4$.

At level $v = [e] = e - \beta$, from (3.3) we have $\frac{1}{2}\frac{b}{e} < 2$. Hence by Lemma 2.3 we know that no integral points exists at level $v = [e]$. Similar to case (1a) we need to sum from level 1 to level $n = e - \beta - 1$ in $g(n)$. Define $\Delta_1 = 72e^4(f_5 - g|_{n=e-\beta-1})$, which is symmetric with respect to a, b, and c.

Similar to the proof in case (1a) we can show that Δ_1 is an increasing function of a, b, c, and d for $a \geqslant \frac{4e}{\beta}$, $b \geqslant \frac{3e}{\beta}$, $c \geqslant \frac{2e}{\beta}$ and $d \geqslant \frac{e}{\beta}$. Hence the minimum of Δ_1 occurs at $a = \frac{4e}{\beta}$, $b = \frac{3e}{\beta}$, $c = \frac{2e}{\beta}$, and $d = \frac{e}{\beta}$. Now define $\Delta_2 = \frac{1}{2}\left(\frac{\beta}{e}\right)^4 \Delta_1|_{a=4e/\beta,\, b=3e/\beta,\, c=2e/\beta,\, d=e/\beta}$. We find that $\partial^2\Delta_2/\partial e^2$ for $e \geqslant 4$, $0 < \beta < 1$ so that $\partial\Delta_2/\partial e$ is an increasing function of e and the minimum of $\partial\Delta_2/\partial e$ occurs at $e = 4$ in our range.

We then find that $(\partial\Delta_2/\partial e)|_{e=4} > 0$ for $0 < \beta < 1$, so Δ_2 is an increasing function of e and the minimum of Δ_2 occurs at $e = 4$.

We next find that $\Delta_2|_{e=4} > 0$ for $0 < \beta < 1$ so that

$$\Delta_1 > 0 \qquad \text{for} \quad e \geqslant 4, \quad a \geqslant \frac{4e}{\beta}, \quad b \geqslant \frac{3e}{\beta}, \quad c \geqslant \frac{2e}{\beta}, \quad \text{and} \quad d \geqslant \frac{e}{\beta}. \tag{3.20}$$

Finally we have $\Delta = (1/72e^4)\, \Delta_1 > 0$ for $e \geqslant 4$, $a \geqslant \frac{4e}{\beta}$, $b \geqslant \frac{3e}{\beta}$, $c \geqslant \frac{2e}{\beta}$, and $d \geqslant \frac{e}{\beta}$.

Case (1c). $\frac{d}{e}\beta \geqslant 2$.

Let L_0 be the number of integral points at level $v = [e] = e - \beta$. If $L_0 = 0$, then similar to case (1b) we have (1.6) holds in this case because we have not used the condition $\frac{b}{e}\beta < 4$ to prove that $\Delta > 0$ in case (1b). Hence we only need to consider the case when $L_0 > 0$. By Theorem 2.1 we have

$$\begin{aligned} 120L_0 \leqslant 5\Bigg(& \frac{\beta^4 abcd}{e^4} - \frac{3}{2}\frac{\beta^3 abc}{e^3} - \frac{3}{2}\frac{\beta^3 abd}{e^3} - \frac{3}{2}\frac{\beta^3 acd}{e^3} - \frac{3}{2}\frac{\beta^3 bcd}{e^3} + \frac{11}{3}\frac{\beta^2 ab}{e^2} \\ & + \frac{11}{3}\frac{\beta^2 ac}{e^2} + \frac{11}{3}\frac{\beta^2 bc}{e^2} - 2\frac{\beta a}{e} - 2\frac{\beta b}{e} - 2\frac{\beta c}{e}\Bigg). \end{aligned} \tag{3.21}$$

Define $\Delta_1 = 72e^4(f_5 - g|_{n=e-\beta-1} - (\text{R.H.S. of (3.21)}))$ which is symmetric with respect to a, b, and c. This is an increasing function of a, b, c, and d for $a \geqslant \frac{4e}{\beta}$, $b \geqslant \frac{3e}{\beta}$, $c \geqslant \frac{2e}{\beta}$, and $d \geqslant \frac{2e}{\beta}$, and we proceed analogously to case (1b) to prove that this is positive in this range with $e \geqslant 4$ and $0 < \beta < 1$.

Case (2). $\frac{a}{e}\beta < 4$ or $\frac{b}{e}\beta \leqslant 3$ or $\frac{c}{e}\beta \leqslant 2$ or $\frac{d}{e}\beta \leqslant 1$.

In this case there are no positive integral solutions at level $v = [e] = e - \beta$.

L_1, the number of positive integral solution at level $v = [e] - 1 = e - \beta - 1$, satisfies

$$24L_1 \leqslant \left(\frac{(\beta+1)\,a}{e} - 1\right)\left(\frac{(\beta+1)\,b}{e} - 1\right)\left(\frac{(\beta+1)\,c}{e} - 1\right)\left(\frac{(\beta+1)\,d}{e} - 1\right) \tag{3.22}$$

by Theorem 2.5. L_2, be the number of positive integral solution at level $v=[e]-2=e-\beta-2$, satisfies.

$$24L_2 \leqslant \frac{(\beta+2)^4\, abcd}{e^4} - \frac{3}{2}\, \frac{(\beta+2)^3\,(abc+abd+acd+bcd)}{e^3} + \frac{11}{3}\, \frac{(\beta+2)^2\,(ab+ac+bc)}{e^2} - 2\,\frac{(\beta+2)(a+b+c)}{e} \tag{3.23}$$

by Theorem 2.1, provided $L_2 > 0$.

At level $v=[e]-3=e-\beta-3$ solutions correspond to solutions of $x/[(a/e)(\beta+3)]+y/[(b/e)(\beta+3)]+z/[(c/e)(\beta+3)]+u/[(d/e)(\beta+3)] \leqslant 1$. Since $\frac{d}{e}(\beta+3) \geqslant 3$, we have $f_4 \geqslant 0$ by Lemma 2.2. So we can use f_4 as the bound for the number of positive integral points for each level from level $v=1$ to level $v=e-\beta-3$. We will consider the following subcases.

Case (2a). $a=b=c=d=e$.

In this case there is no integral point from level $v=[e]-1=e-\beta-1$ to level $v=[e]-3=e-\beta-3$. Let $\Delta = f_5 - g|_{n=e-\beta-4}$. Then

$$\Delta|_{a=b=c=d=e} = \beta(\beta+1)(\beta+2)(\beta+3)(\beta+4),$$

so that $\Delta|_{a=b=c=d=e}=0$ only when $\beta=0$ which is equivalent to $a=b=c=d=e=$ integer. Therefore if $\beta \neq 0$ which is equivalent to $a=b=c=d=e \neq$ integer, we have $\Delta|_{a=b=c=d=e} > 0$.

Case (2b). $a=b=c=d \geqslant e$.

(b1) $L_1=0$, $L_2=0$. We solve the case for $a=b=c=d=e$ in case (2a). Now we consider the case $a=b=c=d>e$. Let L_3 be the number of positive integral solution at level $v=[e]-3=e-\beta-3$.

(b1a) $L_3=0$. Then $4/[(a/e)(\beta+3)]>1$, hence $e<a<\frac{4e}{\beta+3}$. Define $\Delta_1 = 72e^4\Delta|_{a=b=c=d}$. Then

$$\frac{\partial^4 \Delta_1}{\partial a^4} > 0 \qquad \text{for} \quad e \geqslant 4, \quad 0 \leqslant \beta < 1.$$

Hence $\partial^3\Delta_1/\partial a^3$ is an increasing function of a for $a>e\geqslant 4$ and the minimum of $\partial^3\Delta_1/\partial a^3$ is at $a=e$,

$$\begin{aligned}\left.\frac{\partial^3 \Delta_1}{\partial a^3}\right|_{a=e} &= 648e^5+5760e^4+3240e^3+288e^2+164880\beta^3 e+489240\beta^2 e\\ &\quad +1728\beta^5 e+699552\beta e+27000\beta^4 e+380160e\\ &> 0 \qquad \text{for} \quad e\geqslant 4, \quad 0\leqslant\beta<1.\end{aligned}$$

It follows that

$$\frac{\partial^3 \Delta_1}{\partial a^3} > 0 \qquad \text{for} \quad a > e \geqslant 4, \quad 0 \leqslant \beta < 1.$$

Hence $\partial^2 \Delta_1 / \partial a^2$ is an increasing function of a for $a > e \geqslant 4$ and the minimum of $\partial^2 \Delta_1 / \partial a^2$ is at $a = e$,

$$\begin{aligned} \left.\frac{\partial^2 \Delta_1}{\partial a^2}\right|_{a=e} &= 216e^6 + 4560e^5 - 1176e^3 + 62400\beta^3 e^2 + 154080\beta^2 e^2 + 864\beta^5 e^2 \\ &\quad + 173976\beta e^2 + 11880\beta^4 e^2 + 67680e^2 \\ &> 0 \qquad \text{for} \quad e \geqslant 4, \quad 0 \leqslant \beta < 1. \end{aligned}$$

It follows that

$$\frac{\partial^2 \Delta_1}{\partial a^2} > 0 \qquad \text{for} \quad a > e \geqslant 4, \quad 0 \leqslant \beta < 1.$$

Hence $\partial \Delta_1 / \partial a$ is an increasing function of a for $a > e \geqslant 4$ and the minimum of $\partial \Delta_1 / \partial a$ is at $a = e$,

$$\begin{aligned} \left.\frac{\partial \Delta_1}{\partial a}\right|_{a=e} &= 36e^7 + 1200e^6 - 540e^5 - 624e^4 + 15000\beta^3 e^3 + 288\beta^5 e^3 + 3420\beta^4 e^3 \\ &\quad + 5760e^3 + 23952\beta e^3 + 29340\beta^2 e^3 > 0 \qquad \text{for} \quad e \geqslant 4, \quad 0 \leqslant \beta < 1. \end{aligned}$$

It follows that

$$\frac{\partial \Delta_1}{\partial a} > 0 \qquad \text{for} \quad a > e \geqslant 4, \quad 0 \leqslant \beta < 1.$$

Hence Δ_1 is an increasing function of a for $a > e \geqslant 4$ and the minimum of Δ_1 is at $a = e$,

$$\begin{aligned} \Delta_1|_{a=e} &= 3600\beta^2 e^4 + 1728\beta e^4 + 720\beta^4 e^4 + 2520\beta^3 e^4 + 72\beta^5 e^4 \\ &= 72\beta e^4(\beta+4)(\beta+3)(\beta+2)(\beta+1) \\ &\geqslant 0 \qquad \text{for} \quad e \geqslant 4, \quad 0 \leqslant \beta < 1. \end{aligned}$$

It follows that

$$\Delta_1 \geqslant 0 \qquad \text{for} \quad a \geqslant e \geqslant 4, \quad 0 \leqslant \beta < 1.$$

Hence $\Delta = (1/72e^4)\,\Delta_1 \geqslant 0$ and $\Delta = 0$ if and only if $a = b = c = d = e$ and $\beta = 0$, i.e., if and only if $a = b = c = d = e =$ integer.

(b1b) $L_3 > 0$. We have $a \geqslant \frac{4e}{\beta+3}$. Let $\Delta = f_5 - g|_{n=e-\beta-3}$, and define

$$\Delta_1 = 72e^4 \Delta|_{b=a,\, c=a,\, d=a}. \tag{3.24}$$

Similar to the case for $L_3 = 0$ we can show that $\partial\Delta_1/\partial a$ is an increasing function of a for $a \geqslant e \geqslant 4$ and the minimum of $\partial\Delta_1/\partial a$ is at $a = e$,

$$\begin{aligned}\left.\frac{\partial\Delta_1}{\partial a}\right|_{a=e} &= 36e^7 + 1200e^6 - 540e^5 - 624e^4 + 288\beta^5 e^3 + 4200\beta^3 e^3 + 1980\beta^2 e^3 \\ &\quad - 1968\beta e^3 - 720e^3 + 1980\beta^4 e^3 > 0 \qquad \text{for} \quad e \geqslant 4, \quad 0 \leqslant \beta < 1.\end{aligned}$$

It follows that

$$\frac{\partial\Delta_1}{\partial a} > 0 \qquad \text{for} \quad a \geqslant e \geqslant 4, \quad 0 \leqslant \beta < 1.$$

Hence Δ_1 is an increasing function of a for $a \geqslant e \geqslant 4$ and the minimum of Δ_1 is at $a = \frac{4e}{\beta+3}$,

$$\begin{aligned}\Delta_1|_{a=\frac{4e}{\beta+3}} &= 96e^4(1-\beta)(-7\beta^4 - 67\beta^3 - 227\beta^2 - 45\beta^2 e^2 - 27\beta^2 e - 160\beta e \\ &\quad - 317\beta + 220\beta e^3 - 180\beta e^2 - 135e^2 + 24e^4 - 150 - 229e \\ &\quad + 580e^3)/(\beta+3)^4 > 0 \qquad \text{for} \quad e \geqslant 4, \quad 0 \leqslant \beta < 1.\end{aligned}$$

Hence $\Delta = (1/72e^4)\,\Delta_1 > 0$ for this subcase.

(b2) $L_1 = 0$, $L_2 > 0$. The condition $L_2 > 0$ implies $4/[(a/e)(\beta+2)] \leqslant 1$. Hence $a \geqslant \frac{4e}{\beta+2} > \frac{4}{3}e$. Let $\Delta = f_5 - g|_{n=e-\beta-3} - 5$ (R.H.S. of (3.23)) and $\Delta_1 = 72e^4 \Delta|_{b=a,\, c=a,\, d=a}$.

Similar to subcase (b1) we can show that Δ_1 is an increasing function of a for $a \geqslant e \geqslant 4$ and the minimum of Δ_1 is at $a = \frac{4}{3}e$,

$$\begin{aligned}\Delta_1|_{a=\frac{4}{3}e} &= -160e^6 + \frac{18560}{27}e^7 + \frac{256}{9}e^8 - \frac{2080}{3}\beta^2 e^4 + \frac{2336}{27}\beta e^4 + \frac{1280}{3}\beta^4 e^4 \\ &\quad - \frac{10880}{27}\beta^3 e^4 + \frac{2048}{9}\beta^5 e^4 + \frac{1600}{9}e^4 - \frac{7328}{27}e^5 \\ &= \frac{32}{27}e^4(-135e^2 + 580e^3 + 24e^4 - 585\beta^2 + 73\beta + 360\beta^4 - 340\beta^3 \\ &\quad + 192\beta^5 + 150 - 229e) > 0 \qquad \text{for} \quad e \geqslant 4, \quad 0 \leqslant \beta < 1.\end{aligned}$$

Hence $\Delta = (1/72e^4)\,\Delta_1 > 0$ for this subcase.

(b3) $L_1 > 0, L_2 > 0$. The condition $L_1 > 0$ implies $4/[(a/e)(\beta+1)] \leqslant 1$. Hence $a \geqslant \frac{4e}{\beta+1} > 2e$. Let $\Delta = f_5 - g|_{n=e-\beta-3} - 5$ (R.H.S. of (3.22) + R.H.S. of (3.23)) and $\Delta_1 = 72e^4\Delta|_{b=a,\,c=a,\,d=a}$.

Similar to subcase (b1) we can show that Δ_1 is an increasing function of a for $a \geqslant e \geqslant 4$ and the minimum of Δ_1 is at $a = 2e$,

$$\begin{aligned}\Delta_1|_{a=2e} &= 4320e^7 + 288e^8 - 8640\beta^2e^4 - 4032\beta e^4 - 1440\beta^4e^4 - 7200\beta^3e^4 \\ &\quad + 1152\beta^5e^4 - 360e^4 - 1152e^5 \\ &= 72e^4(60e^3 + 4e^4 - 120\beta^2 - 56\beta - 20\beta^4 - 100\beta^3 + 16\beta^5 - 5 - 16e) \\ &> 0 \qquad \text{for} \quad e \geqslant 4, \quad 0 \leqslant \beta < 1.\end{aligned}$$

Hence $\Delta = (1/72e^4)\,\Delta_1 > 0$ for this subcase.

Combining subcases (b1), (b2), and (b3) we have for case (2b) $\Delta|_{a=b=c=d\geqslant e} \geqslant 0$ for $a=b=c=d \geqslant e \geqslant 4$, and $\Delta|_{a=b=c=d} > 0$ for $a = b = c = d > e$. From the results of case (2a) and case (2b) we have that $\Delta|_{a=b=c=d=e} \geqslant 0$ on $a=b=c=d=e$ and $\Delta|_{a=b=c=d=e} = 0$ only when $a=b=c=d=e=$ integer. Hence $\Delta|_{a=b=c=d} \geqslant 0$ on $a=b=c=d \geqslant e$. Actually $\Delta > 0$ as long as $a=b=c=d>e$ or $a=b=c=d=e$ not an integer.

Case (2c). $a=b=c \geqslant d \geqslant e$.

(c1) $L_1 = 0, L_2 = 0$. Let $\Delta = f_5 - g|_{n=e-\beta-3}$ and $\Delta_1 = 72e^4\Delta|_{b=a,\,c=a}$.

Similar to subcase (b2) we can show that $\partial\Delta_1/\partial a$ is an increasing function of a and d for $a \geqslant d \geqslant e \geqslant 4$ and the minimum of $\partial\Delta_1/\partial a$ is at $a = d = e$,

$$\begin{aligned}\left.\frac{\partial\Delta_1}{\partial a}\right|_{a=d=e} &= 27e^7 + 240e^6 + 855e^5 - 1068e^4 + 3810\beta^3e^3 + 216\beta^5e^3 + 9384\beta e^3 \\ &\quad + 6165\beta^2e^3 + 7740e^3 + 1485\beta^4e^3 > 0 \qquad \text{for} \quad e \geqslant 4, \quad 0 \leqslant \beta < 1.\end{aligned}$$

It follows that

$$\frac{\partial\Delta_1}{\partial a} > 0 \qquad \text{for} \quad a \geqslant d \geqslant e \geqslant 4, \quad 0 \leqslant \beta < 1. \tag{3.25}$$

(c2) $L_1 = 0, L_2 > 0$. Let $\Delta = f_5 - g|_{n=e-\beta-3} - 5$ (R.H.S. of (3.23)) and $\Delta_1 = 72e^4\Delta|_{b=a,\,c=a}$.

Similar to subcase (b2) we can show that $\partial\Delta_1/\partial a$ is an increasing function of a and d for $a \geqslant d \geqslant e \geqslant 4$ and the minimum of $\partial\Delta_1/\partial a$ is at $a = d = e$,

$$\left.\frac{\partial \Delta_1}{\partial a}\right|_{a=d=e} = 27e^7+240e^6+855e^5-1068e^4+30\beta^3e^3+216\beta^5e^3+3624\beta e^3$$

$$+1485\beta^2e^3+1980e^3+405\beta^4e^3>0 \qquad \text{for} \quad e\geqslant 4, \quad 0\leqslant\beta<1.$$

It follows that

$$\frac{\partial \Delta_1}{\partial a}>0 \qquad \text{for} \quad a\geqslant d\geqslant e\geqslant 4, \quad 0\leqslant\beta<1. \tag{3.26}$$

(c3) $L_1>0$, $L_2>0$. Let $\Delta=f_5-g|_{n=e-\beta-3}-5$ (R.H.S. of (3.22)+ R.H.S. of (3.23)) and $\Delta_1=72e^4\Delta|_{b=a,\,c=a}$.

Similar to subcase (b3) we can show that $\partial\Delta_1/\partial a$ is an increasing function of a and d for $a\geqslant d\geqslant e\geqslant 4$ and the minimum of $\partial\Delta_1/\partial a$ is at $a=d=e$,

$$\left.\frac{\partial \Delta_1}{\partial a}\right|_{a=d=e} = 27e^7+240e^6+855e^5-1068e^4-1050\beta^3e^3+216\beta^5e^3+3624\beta e^3$$

$$+1485\beta^2e^3+1980e^3-675\beta^4e^3>0 \qquad \text{for} \quad e\geqslant 4, \quad 0\leqslant\beta<1.$$

It follows that

$$\frac{\partial \Delta_1}{\partial a}>0 \qquad \text{for} \quad a\geqslant d\geqslant e\geqslant 4, \quad 0\leqslant\beta<1. \tag{3.27}$$

Combining subcases (c1), (c2), and (c3) we have for case (2c) $\partial\Delta|_{a=b=c}/\partial a=(1/72e^4)(\partial\Delta_1/\partial a)>0$ for $a=b=c\geqslant d\geqslant e\geqslant 4$. Therefore the minimum value of $\Delta|_{a=b=c}$ is attained at $a=b=c=d$. In case (2b) we saw that $\Delta|_{a=b=c=d}\geqslant 0$ on $a=b=c=d\geqslant e$. Hence $\Delta|_{a=b=c}\geqslant 0$ on $a=b=c\geqslant d\geqslant e$. Actually $\Delta|_{a=b=c}>0$ as long as we are not at the case $a=b=c=d=e=\text{integer}$.

Case (2d). $a=b\geqslant c\geqslant d\geqslant e$.

(d1) $L_1=0$, $L_2=0$. Let $\Delta=f_5-g|_{n=e-\beta-3}$ and $\Delta_1=72e^4\Delta|_{b=a}$.

Note that $\partial\Delta_1/\partial a$ is symmetric with respect to a and c.

Similar to subcase (b2) we can show that $\partial\Delta_1/\partial a$ is an increasing function of a, c and d for $a\geqslant c\geqslant d\geqslant e\geqslant 4$ and the minimum of $\partial\Delta_1/\partial a$ is at $a=c=d=e$,

$$\left.\frac{\partial \Delta_1}{\partial a}\right|_{a=c=d=e} = 18e^7+160e^6+570e^5-712e^4+2540\beta^3e^3+144\beta^5e^3+6256\beta e^3$$

$$+4110\beta^2e^3+5160e^3+990\beta^4e^3$$

$$>0 \qquad \text{for} \quad e\geqslant 4, \quad 0\leqslant\beta<1.$$

It follows that

$$\frac{\partial \Delta_1}{\partial a} > 0 \qquad \text{for} \quad a \geqslant c \geqslant d \geqslant e \geqslant 4, \quad 0 \leqslant \beta < 1. \tag{3.28}$$

(d2) $L_1 = 0$, $L_2 > 0$. Let $\Delta = f_5 - g|_{n=e-\beta-3} - 5$ (R.H.S. of (3.23)) and $\Delta_1 = 72e^4 \Delta|_{b=a}$.

Note that $\partial \Delta_1 / \partial a$ is symmetric with respect to a and c.

Similar to subcase (b2) we can show that $\partial \Delta_1 / \partial a$ is an increasing function of a, c and d for $a \geqslant c \geqslant d \geqslant e \geqslant 4$ and the minimum of $\partial \Delta_1 / \partial a$ is at $a = c = d = e$,

$$\begin{aligned} \left.\frac{\partial \Delta_1}{\partial a}\right|_{a=c=d=e} &= 18e^7 + 160e^6 + 570e^5 - 712e^4 + 1320e^3 + 2416\beta e^3 + 990\beta^2 e^3 \\ &\quad + 20\beta^3 e^3 + 270\beta^4 e^3 + 144\beta^5 e^3 > 0 \qquad \text{for} \quad e \geqslant 4, \quad 0 \leqslant \beta < 1. \end{aligned}$$

It follows that

$$\frac{\partial \Delta_1}{\partial a} > 0 \qquad \text{for} \quad a \geqslant c \geqslant d \geqslant e \geqslant 4, \quad 0 \leqslant \beta < 1. \tag{3.29}$$

(d3) $L_1 > 0$, $L_2 > 0$. Let $\Delta = f_5 - g|_{n=e-\beta-3} - 5$ (R.H.S. of (3.22)+ R.H.S. of (3.23)) and $\Delta_1 = 72e^4 \Delta|_{b=a}$.

Note that $\partial \Delta_1 / \partial a$ is symmetric with respect to a and c.

Similar to subcase (b3) we can show that $\partial \Delta_1 / \partial a$ is an increasing function of a, c and d for $a \geqslant c \geqslant d \geqslant e \geqslant 4$ and the minimum of $\partial \Delta_1 / \partial a$ is at $a = c = d = e$,

$$\begin{aligned} \left.\frac{\partial \Delta_1}{\partial a}\right|_{a=c=d=e} &= 18e^7 + 160e^6 + 570e^5 - 712e^4 + 1320e^3 + 2416\beta e^3 + 990\beta^2 e^3 \\ &\quad - 700\beta^3 e^3 - 450\beta^4 e^3 + 144\beta^5 e^3 \\ &> 0 \qquad \text{for} \quad e \geqslant 4, \quad 0 \leqslant \beta < 1. \end{aligned}$$

It follows that

$$\frac{\partial \Delta_1}{\partial a} > 0 \qquad \text{for} \quad a \geqslant c \geqslant d \geqslant e \geqslant 4, \quad 0 \leqslant \beta < 1. \tag{3.30}$$

Combining subcases (d1), (d2), and (d3) we have for case (2d) $\partial\Delta|_{a=b}/\partial a=(1/72e^4)(\partial\Delta_1/\partial a)>0$ for $a=b\geqslant c\geqslant d\geqslant e\geqslant 4$. Therefore the minimum value of $\Delta|_{a=b}$ is attained at $a=b=c\geqslant d\geqslant e$. In case (2c) we saw that $\Delta|_{a=b=c}\geqslant 0$ on $a=b=c\geqslant d\geqslant e$. Hence $\Delta|_{a=b}\geqslant 0$ on $a=b\geqslant c\geqslant d\geqslant e$. Actually $\Delta|_{a=b}>0$ as long as we are not at the case $a=b=c=d=e=$ integer.

Case (2e). $a\geqslant b\geqslant c\geqslant d\geqslant e$.

(e1) $L_1=0$, $L_2=0$. Let $\Delta=f_5-g|_{n=e-\beta-3}$ and $\Delta_1=72e^4\Delta$.
Note that $\partial\Delta_1/\partial a$ is symmetric with respect to b and c.

Similar to subcase (b1) we can show that $\partial\Delta_1/\partial a$ is an increasing function of b, c, and d for $b\geqslant c\geqslant d\geqslant e\geqslant 4$. Hence the minimum of $\partial\Delta_1/\partial a$ occurs at $b=c=d=e$,

$$\begin{aligned}\left.\frac{\partial\Delta_1}{\partial a}\right|_{b=e,\,c=e,\,d=e} &= 9e^7+80e^6+285e^5-356e^4+3128\beta e^3+2580e^3+1270\beta^3e^3\\ &\quad+72\beta^5e^3+495\beta^4e^3+2055\beta^2e^3\\ &>0 \qquad \text{for} \quad e\geqslant 4, \quad 0\leqslant\beta<1.\end{aligned}$$

It follows that

$$\frac{\partial\Delta_1}{\partial a}>0 \qquad \text{for} \quad a\geqslant b\geqslant c\geqslant d\geqslant e\geqslant 4, \quad 0\leqslant\beta<1. \tag{3.31}$$

(e2) $L_1=0$, $L_2>0$. Let $\Delta=f_5-g|_{n=e-\beta-3}-5$ (R.H.S. of (3.23)) and $\Delta_1=72e^4\Delta$.

Note that $\partial\Delta_1/\partial a$ is symmetric with respect to b and c.

Similar to subcase (b2) we can show that $\partial\Delta_1/\partial a$ is an increasing function of b, c, and d for $b\geqslant c\geqslant d\geqslant e\geqslant 4$. Hence the minimum of $\partial\Delta_1/\partial a$ occurs at $b=c=d=e$,

$$\begin{aligned}\left.\frac{\partial\Delta_1}{\partial a}\right|_{b=e,\,c=e,\,d=e} &= 9e^7+80e^6+285e^5-356e^4+1208\beta e^3+660e^3+10\beta^3e^3\\ &\quad+72\beta^5e^3+135\beta^4e^3+495\beta^2e^3\\ &>0 \qquad \text{for} \quad e\geqslant 4, \quad 0\leqslant\beta<1.\end{aligned}$$

It follows that

$$\frac{\partial\Delta_1}{\partial a}>0 \qquad \text{for} \quad a\geqslant b\geqslant c\geqslant d\geqslant e\geqslant 4, \quad 0\leqslant\beta<1. \tag{3.32}$$

(e3) $L_1>0$, $L_2>0$. Let $\Delta=f_5-g|_{n=e-\beta-3}-5$ (R.H.S. of (3.22)+ R.H.S. of (3.23)) and $\Delta_1=72e^4\Delta$.

Note that $\partial\Delta_1/\partial a$ is symmetric with respect to b and c.

Similar to subcase (b3) we can show that $\partial\Delta_1/\partial a$ is an increasing function of b, c, and d for $b \geqslant c \geqslant d \geqslant e \geqslant 4$. Hence the minimum of $\partial\Delta_1/\partial a$ occurs at $b = c = d = e$,

$$\left.\frac{\partial\Delta_1}{\partial a}\right|_{b=e,\,c=e,\,d=e} = 9e^7 + 80e^6 + 285e^5 - 356e^4 + 1208\beta e^3 + 660e^3 - 350\beta^3 e^3 + 72\beta^5 e^3 - 225\beta^4 e^3 + 495\beta^2 e^3 > 0 \quad \text{for} \quad e \geqslant 4, \quad 0 \leqslant \beta < 1.$$

It follows that

$$\frac{\partial\Delta_1}{\partial a} > 0 \qquad \text{for} \quad a \geqslant b \geqslant c \geqslant d \geqslant e \geqslant 4, \quad 0 \leqslant \beta < 1. \tag{3.33}$$

Combining subcases (el), (e2), and (e3) we have for case (2e) $\partial\Delta|_{a \geqslant b \geqslant c \geqslant d \geqslant e \geqslant 4}/\partial a = (1/72e^4)(\partial\Delta_1/\partial a) > 0$ for $a \geqslant b \geqslant c \geqslant d \geqslant e \geqslant 4$. Therefore the minimum value of $\Delta|_{a \geqslant b \geqslant c \geqslant d \geqslant e \geqslant 4}$ is attained at $a = b$. In case (2d) we saw that $\Delta|_{a=b} \geqslant 0$ on $a = b \geqslant c \geqslant d \geqslant e \geqslant 4$. Hence $\Delta|_{a \geqslant b \geqslant c \geqslant d \geqslant e \geqslant 4} \geqslant 0$ on $a \geqslant b \geqslant c \geqslant d \geqslant e \geqslant 4$. Actually $\Delta|_{a \geqslant b \geqslant c \geqslant d \geqslant e \geqslant 4} > 0$ as long as we are not at the case $a = b = c = d = e =$ integer. Q.E.D.

REFERENCES

[Be] F. Beukers, The lattice-points of a n-dimensional tetrahedra, *Indag. Math.* **37**, No. 5 (1975), 365–372.

[Ca-Sh] S. E. Cappell and J. L. Shaneson, Genera of algebraic varieties and counting lattice points, *Bull. Amer. Math Soc. (N.S.)* **30** (1994), 62–69.

[Du] A. H. Durfee, The signature of smoothings of complex surface singularities, *Math. Ann.* **232** (1978), 85–98.

[Eh] E. Ehrhart, Sur un problème de géométrie diophantienne linéaire, II, *J. Reine Angew. Math.* **227** (1967), 25–49.

[Gr] A. Granville, letter to Y.-J. Xu.

[Ha-Li 1] G. H. Hardy and J. E. Littlewood, Some problems of diophantine approximation, *in* "Proc. 5th Int. Congress of Mathematics, 1912," pp. 223–229.

[Ha-Li 2] G. H. Hardy and J. E. Littlewood, The lattice points of a right-angled triangle, *Proc. London Math. Soc. (2)* **20** (1921), 15–36.

[Ha-Li 3] G. H. Hardy and J. E. Littlewood, The lattice points of a right-angled triangle (second memoir), *Hamburg Math. Abh.* **1** (1922), 212–249.

[Ka-Kh] J. M. Kantor and A. Khovanskii, Une application du Théorème de Riemann–Roch combinatoire au polynôme d'Ehrhart des polytopes entier de $\mathbf{R}^d$, *C.R. Acad. Sci. Paris Sér. I* **317** (1993), 501–507.

[Le] D. H. Lehmer, The lattice points of an n-dimensional tetrahedron, *Duke Math. J.* **7** (1940), 341–353.

[Li-Ya 1] K.-P. Lin and S. S.-T. Yau, Sharp upper estimate of geometric genus in terms of Milnor number and multiplicity with application on coordinate free characterization of 3-dimensional homogeneous singularities, preprint.

[Li-Ya 2] K.-P. Lin and S. S.-T. Yau, Analysis of sharp polynomial upper estimate of number of positive integral points in a 4-dimensional tetrahedra, *J. Reine Angew. Math.*, to appear.

[Me-Te] M. Merle and B. Teissier, Conditions d'adjonction d'après Du Val, Séminaire sur les singularités des surfaces, Centre de Math. de l'Ecole Polytechnique, 1976–1977, *in* "Lecture Notes in Math.," Vol. 777, pp. 229–245, Springer-Verlag, Berlin, 1980.

[Mi-Or] J. Milnor and P. Orlik, Isolated singularities defined by weighted homogeneous polynomials, *Topology* **9** (1970), 385–393.

[Mo] L. J. Mordell, Lattice points in tetrahedron and generalized Dedekind sums, *J. Indian Math. Soc.* **15** (1951), 41–46.

[Po] J. Pommershim, Toric variety, lattice points and Dedekind sums, *Math. Ann.* **295** (1993), 1–24.

[Pom 1] C. Pomerance, The role of smooth numbers in number theoretic algorithms, pp. 411–422, ICM, Zürich, 1994.

[Pom 2] C. Pomerance, Lecture notes on primality testing and factoring : A short course, Kent State University, 1984.

[Ro] J. B. Rosser, On the first case of Fermat's last theorem, *Bull. Amer. Math. Soc.* **45** (1939), 636–640.

[Sp 1] D. C. Spencer, On a Hardy–Littlewood problem of diophantine approximation, *Math. Proc. Cambridge Philos. Soc.* **35** (1939), 527–547.

[Sp 2] D. C. Spencer, The lattice points of tetrahedra, *J. Math. Phys.* **21** (1942), 189–197.

[Xu-Ya 1] Y.-J. Xu and S. S.-T. Yau, Sharp estimate of number of integral points in a tetrahedron, *J. Reine Angew. Math.* **423** (1992), 199–219.

[Xu-Ya 2] Y.-J. Xu and S. S.-T. Yau, Durfee conjecture and coordinate free characterization of homogeneous singularities, *J. Differential Geom.* **37** (1993), 375–396.

[Xu-Ya 3] Y.-J. Xu and S. S.-T. Yau, Sharp estimate of number of integral points in a 4-dimensional tetrahedron, *J. Reine Angew. Math.* **473** (1996), 1–23.