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Analysis for a sharp polynomial upper estimate of the number of positive integral points in a 4-dimensional tetrahedron

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1. Introduction

Let Δ_4 be a 4-dimensional tetrahedron defined by $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{w}{d} \leqq 1$ and $x \geqq 0, \ldots, w \geqq 0$ where a, b, c, d are positive real numbers. Let

$$P_4 = \# \left\{ (x,y,z,w) \in \mathbb{Z}_+^4 \colon \frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{w}{d} \leqq 1 \right\}$$

$$= \text{number of positive integral points in } \Delta_4.$$

The general problem of counting the number P_4 has been a challenging problem for many years. For the case where the vertices are all integral points, there exists exact formula for this problem [Ca-Sh], [Ka-Kh]. But the formula is too complicated and it is very difficult to get a sharp upper estimate from this formula. Motivated from the number theory and singularities theory, we are interested in counting P_4 for positive real numbers a, b, c, d. This latter problem is much harder than the former one because the technique of toric variety from algebraic geometry is no longer applicable. Beside the applications in number theory and singularities theory, the latter problem has an interesting application in coordinate free characterization of homogeneous polynomials. Without loss of generality we can assume $a \geqq b \geqq c \geqq d$. For the case $d \geqq 3$ we get a sharp polynomial upper estimate for P_4 without any restriction [Xu-Ya 3], [Li-Ya 1]. In fact it is shown that

$$(1.1)\quad 24P_4 \leqq abcd - \frac{3}{2}(abc + abd + acd + bcd) + \frac{11}{3}(ab + ac + bc) - 2(a + b + c).$$

When $d \geqq 2$ the same upper estimate will hold under the condition $P_4 > 0$ [Li-Ya 1]. But this estimate is not true for the case $1 \leqq d < 2$. We were told by Professor Granville

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[Gr] that this problem is an extremely important question in number theory: it would have many applications to current problems in analytic number theory, primality testing and in factoring. Let $a_1 \geqq 1, \dots, a_n \geqq 1$ be real numbers for an n-dimensional tetrahedron $\Delta_{(a_1,\dots,a_n)}$: $\frac{x_1}{a_1} + \cdots + \frac{x_n}{a_n} \leqq 1$, and $x_1 \geqq 0, \dots, x_n \geqq 0$, we define

$$Q_{(a_1,\dots,a_n)} = \#\left\{(x_1,\dots,x_n) \in (\mathbb{Z}_+ \cup \{0\})^n : \sum_{i=1}^{n} \frac{x_i}{a_i} \leqq 1\right\},$$

$$P_{(a_1,\dots,a_n)} = \#\left\{(x_1,\dots,x_n) \in \mathbb{Z}_+^n : \sum_{i=1}^{n} \frac{x_i}{a_i} \leqq 1\right\}.$$

People in number theory are interested in a sharp upper estimate of $Q_{(a_1,\dots,a_n)}$ while people in singularity theory are interested in a sharp upper estimate of $P_{(a_1,\dots,a_n)}$. These two problems are related as follows. Let $x_1 = y_1 + 1, \dots, x_n = y_n + 1$. Then

$$\begin{aligned} P_{(a_1,\dots,a_n)} &= \#\left\{(x_1,\dots,x_n) \in \mathbb{Z}_+^n : \sum_{i=1}^{n} \frac{x_i}{a_i} \leqq 1\right\} \\ &= \#\left\{(y_1,\dots,y_n) \in (\mathbb{Z}_+ \cup \{0\})^n : \sum_{i=1}^{n} \frac{y_i + 1}{a_i} \leqq 1\right\} \\ &= \#\left\{(y_1,\dots,y_n) \in (\mathbb{Z}_+ \cup \{0\})^n : \sum_{i=1}^{n} \frac{y_i}{a_i\left(1 - \sum_{j=1}^{n} \frac{1}{a_j}\right)} \leqq 1\right\} \\ &= Q_{\left(a_1\left(1-\sum_{j=1}^{n}\frac{1}{a_j}\right),\dots,a_n\left(1-\sum_{j=1}^{n}\frac{1}{a_j}\right)\right)}. \end{aligned}$$

The current method in counting $Q_{(a_1,\dots,a_n)}$ is the following method:

Attach a unit cube to the right and above each lattice point of $\Delta_{(a_1,\dots,a_n)}$. Then

$$\begin{aligned} Q_{(a_1,\dots,a_n)} &= \sum_{i=1}^{Q} 1 = \sum_{i=1}^{Q} \text{volume of the unit cube attached to each lattice points} \\ &\leqq \text{volume}\left\{(x_1,\dots,x_n) \in (\mathbb{R}_+ \cup \{0\})^n : \sum_{i=1}^{n} \frac{x_i - 1}{a_i} \leqq 1\right\} \\ &= \frac{1}{n!}\left(\prod_{i=1}^{n} a_i\right)\left(1 + \sum_{j=1}^{n} \frac{1}{a_j}\right)^n. \end{aligned}$$

Hence

$$P_{(a_1,\dots,a_n)} = Q_{\left(a_1\left(1-\sum_{j=1}^{n}\frac{1}{a_j}\right),\dots,a_n\left(1-\sum_{j=1}^{n}\frac{1}{a_j}\right)\right)}$$

$$\leqq \frac{1}{n!}\prod_{i=1}^{n}\left[a_i\left(1-\sum_{j=1}^{n}\frac{1}{a_j}\right)\right]\left(1+\sum_{k=1}^{n}\frac{1}{a_k\left(1-\sum_{j=1}^{n}\frac{1}{a_j}\right)}\right)^n$$

$$= \frac{1}{n!}\left(\prod_{i=1}^{n}a_i\right).$$

Professor Granville [Gr] said that the above estimate is not good particularly when many of the a_i's are small. For $n=4$, the above estimate consists of only the first term of R.H.S. of (1.1) in the Xu-Yau estimate. Since the Xu-Yau estimate holds only for $d \geqq 3$, it is highly desirable to look for better estimate for P_4 when d is small and without any restriction. In fact in order to get a sharp estimate of P_5, we need to apply induction procedure. A good estimate of P_4 for $1 \leqq d \leqq 3$ is also needed.

Since the estimate in (1.1) is not valid for $1 \leqq d \leqq 3$, we need to search for good polynomial estimate of P_4 for $1 \leqq d \leqq 3$. It turns out that the singularity theory provides us a good candidate. Recall that on the one hand a theorem of Merle-Teissier [Me-Te] implies that computing the geometric genus of isolated singularity defined by a weighted homogeneous polynomial is equivalent to counting the number of positive integral points in a tetrahedron Δ_4. By a theorem of Milnor and Orlik [Mi-Or] for weighted homogeneous polynomial of type (a,b,c,d) with isolated singularity at the origin, the Milnor number is $\mu = (a-1)(b-1)(c-1)(d-1)$. On the other hand Durfee conjecture in singularity theory states that (4!) times geometric genus is bounded above by Milnor number. Combining with the above two theorems, Durfee conjecture suggests the following polynomial estimate: $4!P_4 \leqq (a-1)(b-1)(c-1)(d-1)$. We prove in this paper that this formula is indeed true and it is the formula which people in number theory are looking for when $1 \leqq d < 3$. The following is our main theorem.

Main Theorem. *Let $a \geqq b \geqq c \geqq d \geqq 1$ be real numbers, and P_4 be the number of positive integral solutions of $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{w}{d} \leqq 1$, i.e.*

$$P_4 = \#\left\{(x,y,z,w) \in \mathbb{Z}_+^4 : \frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{w}{d} \leqq 1\right\}.$$

Define $\mu = (a-1)(b-1)(c-1)(d-1)$, then

$$24P_4 \leqq \mu = abcd - (abc+abd+acd+bcd) + (ab+ac+ad+bc+bd+cd) - (a+b+c+d) + 1.$$

As a corollary to the proof of the main theorem, we have

Corollary. *Let $a \geqq b \geqq c \geqq d \geqq 3$ be real numbers, and P_4 be the number of positive integral solutions of $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{w}{d} \leqq 1$, i.e. $P_4 = \#\left\{(x,y,z,w) \in \mathbb{Z}_+^4 : \frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{w}{d} \leqq 1\right\}$.*

Then the upper estimate of P_4 by Xu-Yau is strictly sharper than the estimate suggested by Durfee conjecture, i.e.

$$(4!)P_4 \leqq abcd - \frac{3}{2}(abc+abd+acd+bcd) + \frac{11}{3}(ab+ac+bc) - 2(a+b+c)$$
$$\leqq (a-1)(b-1)(c-1)(d-1).$$

2. Sharp estimate of the number of integral points in a 4-dimensional tetrahedron

The following Theorem 2.1 is proved in [Xu-Ya 3] and [Li-Ya 1].

Theorem 2.1. *Let $a \geqq b \geqq c \geqq d \geqq 2$, and P_4 be the number of positive integral solutions of $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{w}{d} \leqq 1$, i.e. $P_4 = \#\left\{(x,y,z,w) \in \mathbb{Z}_+^4 : \frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{w}{d} \leqq 1\right\}$. If $P_4 > 0$, then*

$$24P_4 \leqq f_4(a,b,c,d) := abcd - \frac{3}{2}(abc+abd+acd+bcd) + \frac{11}{3}(ab+ac+bc) - 2(a+b+c) \tag{2.1}$$

and equality is attained if and only if $a=b=c=d=$ integer.

If $a \geqq b \geqq c \geqq d \geqq 3$ or $a \geqq 8$, $b \geqq 6$, $c \geqq 4$, $d \geqq 2$, then (2.1) is true without the condition $P_4 > 0$ and equality is attained if and only if $a=b=c=d=$ integer.

Lemma 2.2. *$f_4(a,b,c,d)$ defined in (2.1) is nonnegative for the following two cases: (i) $a \geqq b \geqq c \geqq d \geqq 3$, (ii) $a \geqq b \geqq c \geqq d$ and $2 \geqq d \geqq 1.5$. Equality (2.1) is attained if and only if $a=b=c=d=3$ or $a=b=c=d=2$.*

The above lemma is proved in [Li-Ya 1] while the following lemma follows directly from Proposition 2.3 of [Xu-Ya 3] and its proof.

Lemma 2.3. *Let $a \geqq b \geqq c \geqq 1$ be real numbers and*

$$P_3 = \#\left\{(x,y,z) \in \mathbb{Z}_+^3 : \frac{x}{a}+\frac{y}{b}+\frac{z}{c} \leqq 1\right\}.$$

Then the following statements hold:

(1) *If $b \leqq 2$, then $P_3 = 0$.*

(2) $(a-1)(b-1)(c-1)-(c-1) < 0$ *if and only if* $a < \frac{b}{b-1}$.

(3) $(a-1)(b-1)(c-1)-(c-1) < 0$ *implies $b < 2$ and $P_3 = 0$.*

The following theorem, which follows directly from Theorem 2.1 and Proposition 2.3 in [Xu-Ya 3] (cf. also [Xu-Ya 1]), will be used frequently.

Theorem 2.4. *Let $a \geqq b \geqq c \geqq 1$ be real numbers. Let P_3 be the number of positive integral solutions of $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leqq 1$, i.e. $P_3 = \#\left\{(x, y, z) \in \mathbb{Z}_+^3 \colon \frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leqq 1\right\}$. If $b \geqq 2$, then*

$$6P_3 \leqq (a-1)(b-1)(c-1) - c + 1 \tag{2.2}$$

and equality is attained if and only if $a = b = c =$ integer.

The following Lemma 2.5, and Main Theorem will be used in our forthcoming paper [Li-Ya 2], which provides a sharp upper estimate of number of integral points in a 5-dimensional tetrahedron.

Lemma 2.5. *Let $a \geqq b \geqq c \geqq d \geqq 2$, and P_4 be the number of positive integral solutions of $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leqq 1$, i.e. $P_4 = \#\left\{(x, y, z, w) \in \mathbb{Z}_+^4 \colon \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leqq 1\right\}$. If*

$$a \geqq 8, \quad b \geqq 6, \quad c \geqq 4, \quad d \geqq 2,$$

then

$$24P_4 \leqq abcd - \frac{3}{2}(abc + abd + acd + bcd) + \frac{11}{3}(ab + ac + bc) - 2(a + b + c)$$

and the equality is attained if and only if $a = b = c = d =$ integer.

Proof. In view of Theorem 2.1 we only need to prove

$$f_4 > 0 \quad \text{for } a \geqq 8,\ b \geqq 6,\ c \geqq 4,\ d \geqq 2 \text{ and } a \geqq b \geqq c \geqq d \geqq 2.$$

$$\frac{\partial^3 f_4}{\partial a \partial b \partial c} = d - \frac{3}{2} > 0 \quad \text{for } d \geqq 2. \tag{2.3}$$

By the property that $\frac{\partial^2 f_4}{\partial a \partial b}$ is symmetric with respect to c and d and (2.3), we also have,

$$\frac{\partial^3 f_4}{\partial a \partial b \partial d} > 0 \quad \text{for } c \geqq 2. \tag{2.4}$$

By the property that $\frac{\partial f_4}{\partial a}$ is symmetric with respect to b and c and (2.4), we also have,

$$\frac{\partial^3 f_4}{\partial a \partial c \partial d} > 0 \quad \text{for } b \geqq 2. \tag{2.5}$$

Combine (2.3) and (2.4), we have $\frac{\partial^2 f_4}{\partial a \partial b}$ is an increasing function of c and d. $\frac{\partial^2 f_4}{\partial a \partial b}\Big|_{c=4, d=2} = \frac{8}{3} > 0$, hence

(2.6)
$$\frac{\partial^2 f_4}{\partial a \partial b} > 0 \quad \text{for } c \geqq 4, d \geqq 2.$$

Combine (2.4) and (2.5), we have $\frac{\partial^2 f_4}{\partial a \partial d}$ is an increasing function of b and c. $\frac{\partial^2 f_4}{\partial a \partial d}|_{b=6,c=4} = 9 > 0$, hence

(2.7)
$$\frac{\partial^2 f_4}{\partial a \partial d} > 0 \quad \text{for } b \geqq 6, c \geqq 4.$$

By the property that $\frac{\partial f_4}{\partial a}$ is symmetric with respect to b and c and (2.6), we also have,

(2.8)
$$\frac{\partial^2 f_4}{\partial a \partial c} > 0 \quad \text{for } b \geqq 4, d \geqq 2.$$

Combine (2.6), (2.7), and (2.8), we have $\frac{\partial f_4}{\partial a}$ is an increasing function of b, c, and d. $\frac{\partial f_4}{\partial a}|_{b=6,c=4,d=2} = \frac{50}{3} > 0$, hence

(2.9)
$$\frac{\partial f_4}{\partial a} > 0 \quad \text{for } b \geqq 6, c \geqq 4, d \geqq 2.$$

Combine the property that f_4 is symmetric with respect to a, b, and c and (2.9), we also have,

(2.10)
$$\frac{\partial f_4}{\partial b} > 0 \quad \text{for } a \geqq 6, c \geqq 4, d \geqq 2,$$

(2.11)
$$\frac{\partial f_4}{\partial c} > 0 \quad \text{for } a \geqq 6, b \geqq 4, d \geqq 2.$$

From the symmetry of f_4 with respect to a, b, and c and (2.7) we also have,

$$\frac{\partial^2 f_4}{\partial b \partial d} > 0 \quad \text{for } a \geqq 6, c \geqq 4, \qquad \frac{\partial^2 f_4}{\partial c \partial d} > 0 \quad \text{for } a \geqq 6, b \geqq 4.$$

It follows that $\frac{\partial f_4}{\partial d}$ is an increasing function of a, b, and c. $\frac{\partial f_4}{\partial d}|_{a=8,b=6,c=4} = 36 > 0$, hence

(2.12)
$$\frac{\partial f_4}{\partial d} > 0 \quad \text{for } a \geqq 8, b \geqq 6, c \geqq 4.$$

Combine (2.9), (2.10), (2.11), and (2.12), we have f_4 is an increasing function of a, b, c, and d. $f_4|_{a=8,b=6,c=4,d=2} = \frac{388}{3} > 0$, hence

$$f_4 > 0 \quad \text{for } a \geqq 8, b \geqq 6, c \geqq 4, d \geqq 2.$$

Main Theorem. *Let $a \geqq b \geqq c \geqq d \geqq 1$ be real numbers, and P_4 be the number of positive integral solutions of $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{w}{d} \leqq 1$, i.e.*

$$P_4 = \#\left\{(x,y,z,w) \in \mathbb{Z}_+^4 : \frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{w}{d} \leqq 1\right\}.$$

Define $\mu = (a-1)(b-1)(c-1)(d-1)$, then

$$(2.13) \quad 24P_4 \leqq \mu = abcd - (abc+abd+acd+bcd) + (ab+ac+ad+bc+bd+cd) - (a+b+c+d)+1.$$

Proof. First we consider the case for $a \geqq b \geqq c \geqq d \geqq 3$. It is obvious that

$$\mu = (a-1)(b-1)(c-1)(d-1) > 0.$$

Hence we only need to show

$$\mu \geqq f_4 = abcd - \frac{3}{2}(abc+abd+acd+bcd) + \frac{11}{3}(ab+ac+bc) - 2(a+b+c).$$

Let $\Delta = \mu - f_4$, then we have

$$\Delta = \frac{1}{2}abc + \frac{1}{2}abd + \frac{1}{2}acd + \frac{1}{2}bcd - \frac{8}{3}ab - \frac{8}{3}ac + ad - \frac{8}{3}bc + bd + cd + a + b + c - d + 1.$$

Let $A = \frac{a}{d}, B = \frac{b}{d}, C = \frac{c}{d}$, then we have $A \geqq B \geqq C \geqq 1$, and $d \geqq 3$. Rewrite Δ as,

$$\Delta = \frac{1}{2}ABCd^3 + \frac{1}{2}ABd^3 + \frac{1}{2}ACd^3 + \frac{1}{2}BCd^3 - \frac{8}{3}ABd^2 - \frac{8}{3}ACd^2 - \frac{8}{3}BCd^2 + Ad^2 + Bd^2 + Cd^2 + Ad + Bd + Cd - d + 1,$$

$$\frac{\partial^2 \Delta}{\partial A \partial B} = \frac{1}{2}Cd^3 + \frac{1}{2}d^3 - \frac{8}{3}d^2, \qquad \frac{\partial^3 \Delta}{\partial A \partial B \partial C} = \frac{d^3}{2} > 0 \quad \text{for } d \geqq 3.$$

Hence we have $\frac{\partial^2 \Delta}{\partial A \partial B}$ is an increasing function of C. $\frac{\partial^2 \Delta}{\partial A \partial B}|_{C=1} = d^3 - \frac{8}{3}d^2 > 0$ for $d \geqq 3$, hence

$$(2.14) \qquad \frac{\partial^2 \Delta}{\partial A \partial B} > 0 \quad \text{for } C \geqq 1, d \geqq 3.$$

Note that Δ is symmetric with respect to B, C. Combining with (2.14) we deduce,

$$(2.15) \qquad \frac{\partial^2 \Delta}{\partial A \partial C} > 0 \quad \text{for } B \geqq 1, d \geqq 3.$$

Putting (2.14) and (2.15) together, we have $\frac{\partial\Delta}{\partial A}$ is an increasing function of B and C for $B \geqq 1$, $C \geqq 1$ and $d \geqq 3$. $\frac{\partial\Delta}{\partial A}|_{B=1,C=1} = d\left(\frac{3}{2}d^2 - \frac{13}{3}d + 1\right) > 0$ for $d \geqq 3$, hence

$$(2.16) \qquad \frac{\partial\Delta}{\partial A} > 0 \quad \text{for } B \geqq 1, C \geqq 1, d \geqq 3.$$

From the property that Δ is symmetric with respect to A, B, and C and (2.16), we also have,

$$(2.17) \qquad \frac{\partial\Delta}{\partial B} > 0 \quad \text{for } A \geqq 1, C \geqq 1, d \geqq 3,$$

$$(2.18) \qquad \frac{\partial\Delta}{\partial C} > 0 \quad \text{for } A \geqq 1, B \geqq 1, d \geqq 3.$$

Combining (2.16), (2.17), and (2.18), we have Δ is an increasing function of A, B, and C for $A \geqq 1, B \geqq 1, C \geqq 1$ and $d \geqq 3$. $\Delta|_{A=1,B=1,C=1} = (d-1)(2d^2 - 3d - 1) > 0$ for $d \geqq 3$, hence

$$(2.19) \qquad \Delta > 0 \quad \text{for } A \geqq 1, B \geqq 1, C \geqq 1, d \geqq 3,$$

which is equivalent to $\Delta > 0$ for $a \geqq b \geqq c \geqq d \geqq 3$.

Now we consider the case when $1 < d \leqq 2$. Let $d = \frac{n}{n-1}$ where $n \in \mathbb{R}_+$ and $n \geqq 2$. Then we have, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leqq 1 - \frac{1}{d} = \frac{1}{n}$. When $P_4 = 0$, (2.13) holds because $\mu \geqq 0$. Hence we only need to consider the case when $P_4 > 0$. It follows that we have $a \geqq 3n, b > 2n, c > n$. Hence by applying Theorem 2.4 to equation $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leqq \frac{1}{n}$ we have

$$(2.20) \qquad 24P_4 \leqq \frac{4}{n^3}abc - \frac{4}{n^2}(ab + ac + bc) + \frac{4}{n}(a+b).$$

Let $\Delta_1 = (a-1)(b-1)(c-1)\left(\frac{1}{n-1}\right)$ − R.H.S. of (2.20). It suffices to prove $\Delta_1 \geqq 0$. We have,

$$\Delta_1 = abc\left(\frac{1}{n-1} - 4\frac{1}{n^3}\right) + (ab + ac + bc)\left(4\frac{1}{n^2} - \frac{1}{n-1}\right) + (a+b)\left(\frac{1}{n-1} - 4\frac{1}{n}\right) + \frac{c-1}{n-1} = \frac{1}{(n-1)n^3}\Delta_2$$

where

$$\Delta_2 = (abcn^3 - abn^3 - acn^3 - bcn^3 - 4abcn + 4abn^2 + 4acn^2 - 3an^3 + 4bcn^2 - 3bn^3 + cn^3 + 4abc - 4abn - 4acn + 4an^2 - 4bcn + 4bn^2 - n^3),$$

(2.21) $$\frac{\partial^3 \Delta_2}{\partial a \partial b \partial c} = n^3 - 4n + 4 > 0 \quad \text{for } n \geqq 2.$$

Hence we have $\frac{\partial^2 \Delta_2}{\partial a \partial b}$ is an increasing function of c for $n \geqq 2$. $\frac{\partial^2 \Delta_2}{\partial a \partial b}|_{c=1} = 4(n-1)^2 > 0$, hence

(2.22) $$\frac{\partial^2 \Delta_2}{\partial a \partial b} > 0 \quad \text{for } c \geqq 1, n \geqq 2.$$

Note that $\frac{\partial \Delta_2}{\partial a}$ is symmetric with respect to b and c. In view of (2.22), we also have,

(2.23) $$\frac{\partial^2 \Delta_2}{\partial a \partial c} > 0 \quad \text{for } b \geqq 1, n \geqq 2.$$

(2.22), and (2.23) imply $\frac{\partial \Delta_2}{\partial a}$ is an increasing function of b, and c for $b \geqq 1$, $c \geqq 1$ and $n \geqq 2$. $\frac{\partial \Delta_2}{\partial a}|_{b=2n, c=n} = n^3(2n-1)(n-1) > 0$ for $n \geqq 2$, hence

(2.24) $$\frac{\partial \Delta_2}{\partial a} > 0 \quad \text{for } b > 2n, c > n, n \geqq 2.$$

Since Δ_2 is symmetric with respect to a and b, by (2.24) and (2.23), we also have,

(2.25) $$\frac{\partial \Delta_2}{\partial b} > 0 \quad \text{for } a > 2n, c > n, n \geqq 2,$$

(2.26) $$\frac{\partial^2 \Delta_2}{\partial b \partial c} > 0 \quad \text{for } a \geqq 1, n \geqq 2.$$

Combining (2.23) and (2.26), we have $\frac{\partial \Delta_2}{\partial c}$ is an increasing function of a and b for $a \geqq 1, b \geqq 1$ and $n \geqq 2$. $\frac{\partial \Delta_2}{\partial c}|_{a=3n, b=2n} = n^2(6n^3 - 5n^2 - 3n + 4) > 0$ for $n \geqq 2$, hence

(2.27) $$\frac{\partial \Delta_2}{\partial c} > 0 \quad \text{for } a \geqq 3n, b > 2n, n \geqq 2.$$

Combining (2.24), (2.25), and (2.27), we have Δ_2 is an increasing function of a, b, and c for $a \geqq 3n, b > 2n, c > n$ and $n \geqq 2$. $\Delta_2|_{a=3n, b=2n, c=n} = n^3(n-1)(3n-1)(2n-1)$ for $n \geqq 2$, hence

(2.28) $$\Delta_1 = \frac{1}{n^3(n-1)} \Delta_2 > 0 \quad \text{for } a \geqq 3n, b > 2n, c > n, n \geqq 2.$$

It follows that (2.13) holds for the case $a \geqq b \geqq c \geqq d$ and $1 < d = \frac{n}{n-1} \leqq 2$.

Now consider the case for $2 < d < 3$. Let L_1 be the number of integral points at level $w = 1$. We have the inequality $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leqq 1 - \frac{1}{d}$. To have a positive L_1, we need

$\frac{2}{3}a > a\left(1-\frac{1}{d}\right) \geqq 3$, $\frac{2}{3}b > b\left(1-\frac{1}{d}\right) \geqq 2$, and $c \geqq d > 2$. Hence if $L_1 > 0$, we have $a \geqq \frac{9}{2}, b > 3, c > 2$ and $d > 2$. Denote L_2 the number of integral points at level $w = 2$. We have the inequality $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} \leqq 1-\frac{2}{d}$. It is clear that $L_1 = 0$ implies $L_2 = 0$. Therefore $L_1 = 0$ implies $P_4 = 0$ and (2.13) holds trivially.

From now on, we shall assume without loss of generality that $L_1 > 0$. In particular, we have $a \geqq \frac{9}{2}, b > 3, c > 2$ and $d > 2$. Hence by Theorem 2.4 we have

$$(2.29)\quad 24L_1 \leqq 4abc\left(1-\frac{1}{d}\right)^3 - 4(ab+ac+bc)\left(1-\frac{1}{d}\right)^2 + 4(a+b)\left(1-\frac{1}{d}\right).$$

There are two cases to be considered.

Case (a): $L_2 > 0$. In this case Theorem 2.4 implies

$$(2.30)\quad 24L_2 \leqq 4abc\left(1-\frac{2}{d}\right)^3 - 4(ab+ac+bc)\left(1-\frac{2}{d}\right)^2 + 4(a+b)\left(1-\frac{2}{d}\right).$$

Let $\Delta_3 = \mu -$ R.H.S. of (2.29) $-$ R.H.S. of (2.30) $= \frac{1}{d^3}\Delta_4$, where

$$\begin{aligned}\Delta_4 = (&abcd^4 - 9abcd^3 - abd^4 - acd^4 - bcd^4 + 36abcd^2 + 9abd^3 + 9acd^3 \\ &+ ad^4 + 9bcd^3 + bd^4 + cd^4 - 60abcd - 24abd^2 - 24acd^2 - 9ad^3 \\ &- 24bcd^2 - 9bd^3 - cd^3 - d^4 + 36abc + 20abd + 20acd + 12ad^2 \\ &+ 20bcd + 12bd^2 + d^3),\end{aligned}$$

$$\frac{\partial^4\Delta_4}{\partial a\partial b\partial c\partial d} = 4d^3 - 27d^2 + 72d - 60 > 0 \quad \text{for } d \geqq 2.$$

It follows that $\frac{\partial^3\Delta_4}{\partial a\partial b\partial c}$ is an increasing function of d for $d \geqq 2$ and $\frac{\partial^3\Delta_4}{\partial a\partial b\partial d}$ is an increasing function of c for $c \geqq 2$. $\frac{\partial^3\Delta_4}{\partial a\partial b\partial c}|_{d=2} = 4 > 0$, $\frac{\partial^3\Delta_4}{\partial a\partial b\partial d}|_{c=2} = 4d^3 - 27d^2 + 96d - 100 > 0$ for $d \geqq 2$, hence

$$(2.31)\qquad \frac{\partial^3\Delta_4}{\partial a\partial b\partial c} > 0 \quad \text{for } d \geqq 2,$$

$$(2.32)\qquad \frac{\partial^3\Delta_4}{\partial a\partial b\partial d} > 0 \quad \text{for } c \geqq 2, d \geqq 2.$$

Since $\frac{\partial \Delta_4}{\partial a}$ is symmetric with respect to b and c, by (2.32), we also have,

$$\frac{\partial^3 \Delta_4}{\partial a \partial c \partial d} > 0 \quad \text{for } b \geqq 2, d \geqq 2. \tag{2.33}$$

Combining (2.31) and (2.32), we have that $\frac{\partial^2 \Delta_4}{\partial a \partial b}$ is an increasing function of c and d for $c \geqq 2$, $d \geqq 2$. $\frac{\partial^2 \Delta_4}{\partial a \partial b}|_{c=2,d=2} = 8 > 0$, hence

$$\frac{\partial^2 \Delta_4}{\partial a \partial b} > 0 \quad \text{for } c \geqq 2, d \geqq 2. \tag{2.34}$$

Combining (2.32) and (2.33), we have $\frac{\partial^2 \Delta_4}{\partial a \partial d}$ is an increasing function of b and c for $b \geqq 3$, $c \geqq 2$ and $d \geqq 2$. $\frac{\partial^2 \Delta_4}{\partial a \partial d}|_{b=3,c=2} = 8d^3 - 54d^2 + 216d - 260 > 0$ for $d \geqq 2$, hence

$$\frac{\partial^2 \Delta_4}{\partial a \partial d} > 0 \quad \text{for } b \geqq 3, c \geqq 2, d \geqq 2. \tag{2.35}$$

By (2.34) and the symmetry of $\frac{\partial \Delta_4}{\partial a}$ with respect to b and c, we also have,

$$\frac{\partial^2 \Delta_4}{\partial a \partial c} > 0 \quad \text{for } b \geqq 2, d \geqq 2. \tag{2.36}$$

Combining (2.34), (2.35), and (2.36), we have $\frac{\partial \Delta_4}{\partial a}$ is an increasing function of b, c, and d, $\frac{\partial \Delta_4}{\partial a}|_{b=3,c=2,d=2} = 16 > 0$, hence

$$\frac{\partial \Delta_4}{\partial a} > 0 \quad \text{for } b \geqq 3, c \geqq 2, d \geqq 2. \tag{2.37}$$

In view of (2.37) and the symmetry of Δ_4 with respect to a, and b, we also have,

$$\frac{\partial \Delta_4}{\partial b} > 0 \quad \text{for } a \geqq 3, c \geqq 2, d \geqq 2. \tag{2.38}$$

Combining (2.36) and the symmetry of Δ_4 with respect to a, and b, we also have,

$$\frac{\partial^2 \Delta_4}{\partial b \partial c} > 0 \quad \text{for } a \geqq 2, d \geqq 2. \tag{2.39}$$

(2.36) and (2.39) imply that $\frac{\partial \Delta_4}{\partial c}$ is an increasing function of a, and b for $a \geqq \frac{9}{2}$, $b \geqq 3$, $d \geqq 2$, $\frac{\partial \Delta_4}{\partial c}|_{a=\frac{9}{2},b=3} = 7d^4 - 55d^3 + 306d^2 - 660d + 486 > 0$ for $d \geqq 2$, hence

$$\frac{\partial \Delta_4}{\partial c} > 0 \quad \text{for } a \geqq \frac{9}{2}, b \geqq 3, d \geqq 2. \tag{2.40}$$

Combining (2.37), (2.38), and (2.40), we have that Δ_4 is an increasing function of a, b, and c. $\Delta_4|_{a=\frac{9}{2},b=3,c=2} = 7d^4 - 55d^3 + 378d^2 - 1050d + 972 > 0$ for $d \geqq 2$, hence

$$\Delta_4 > 0 \quad \text{for } a \geqq \frac{9}{2}, b \geqq 3, c \geqq 2, d \geqq 2. \tag{2.41}$$

Case (b): $L_2 = 0$. In this case, we let $\Delta_5 = \mu -$ R.H.S. of (2.29) $= \dfrac{d-1}{d^3}\Delta_6$, where

$$\begin{aligned}\Delta_6 = (&abcd^3 - 4abcd^2 - abd^3 - acd^3 - bcd^3 + 8abcd + 4abd^2\\ &+ 4acd^2 + ad^3 + 4bcd^2 + bd^3 + cd^3 - 4abc - 4abd\\ &- 4acd - 4ad^2 - 4bcd - 4bd^2 - d^3),\end{aligned}$$

$$\frac{\partial^4 \Delta_6}{\partial a \partial b \partial c \partial d} = 3d^2 - 8d + 8 = 3\left(d - \frac{4}{3}\right)^2 + \frac{8}{3} > 0.$$

It follows that $\dfrac{\partial^3 \Delta_6}{\partial a \partial b \partial c}$ is an increasing function of d for $d \geqq 2$, and $\dfrac{\partial^3 \Delta_6}{\partial a \partial b \partial d}$ is an increasing function of c for $c \geqq 2$. One can show that,

$$\frac{\partial^3 \Delta_6}{\partial a \partial b \partial c} > 0 \quad \text{for } d \geqq 2, \tag{2.42}$$

$$\frac{\partial^3 \Delta_6}{\partial a \partial b \partial d} > 0 \quad \text{for } c \geqq 2. \tag{2.43}$$

By (2.43) and the symmetry of $\dfrac{\partial \Delta_6}{\partial a}$ with respect to b and c, we also have,

$$\frac{\partial^3 \Delta_6}{\partial a \partial c \partial d} > 0 \quad \text{for } b \geqq 2. \tag{2.44}$$

Combining (2.42) and (2.43), we have $\dfrac{\partial^2 \Delta_6}{\partial a \partial b}$ is an increasing function of c and d for $c \geqq 2, d \geqq 2$. $\dfrac{\partial^2 \Delta_6}{\partial a \partial b}|_{c=2,d=2} = 8 > 0$, hence

$$\frac{\partial^2 \Delta_6}{\partial a \partial b} > 0 \quad \text{for } c \geqq 2, d \geqq 2. \tag{2.45}$$

Combining (2.43) and (2.44), we have $\dfrac{\partial^2 \Delta_6}{\partial a \partial d}$ is an increasing function of b and c for $b \geqq 3, c \geqq 2$. $\dfrac{\partial^2 \Delta_6}{\partial a \partial d}|_{b=3,c=2} = 6\left(d - \dfrac{4}{3}\right)^2 + \dfrac{52}{3} > 0$, hence

$$\frac{\partial^2 \Delta_6}{\partial a \partial d} > 0 \quad \text{for } b \geqq 3, c \geqq 2. \tag{2.46}$$

By (2.45) and the symmetry of $\frac{\partial \Delta_6}{\partial a}$ with respect to b and c, we also have

$$\frac{\partial^2 \Delta_6}{\partial a \partial c} > 0 \quad \text{for } b \geqq 2, d \geqq 2. \tag{2.47}$$

Combining (2.45), (2.46), and (2.47), we have that $\frac{\partial \Delta_6}{\partial a}$ is an increasing function of b, c, and d for $b \geqq 3, c \geqq 2, d \geqq 2$. $\frac{\partial \Delta_6}{\partial a}|_{b=3,c=2,d=2} = 16 > 0$, hence

$$\frac{\partial \Delta_6}{\partial a} > 0 \quad \text{for } b \geqq 3, c \geqq 2, d \geqq 2. \tag{2.48}$$

Combining (2.48) and the property that Δ_6 is symmetric with respect to a, and b, we also have

$$\frac{\partial \Delta_6}{\partial b} > 0 \quad \text{for } a \geqq 3, c \geqq 2, d \geqq 2. \tag{2.49}$$

By (2.47), (2.44) and the symmetry of Δ_6 with respect to a, and b, we also have,

$$\frac{\partial^2 \Delta_6}{\partial b \partial c} > 0 \quad \text{for } a \geqq 2, d \geqq 2, \tag{2.50}$$

$$\frac{\partial^3 \Delta_6}{\partial b \partial c \partial d} > 0 \quad \text{for } a \geqq 2. \tag{2.51}$$

Combining (2.44) and (2.51), we have that $\frac{\partial^2 \Delta_6}{\partial c \partial d}$ is an increasing function of a and b for $a \geqq \frac{9}{2}$ and $b \geqq 3$. $\frac{\partial^2 \Delta_6}{\partial c \partial d}|_{a=\frac{9}{2},b=3} = 21\left(d - \frac{8}{7}\right)^2 + \frac{354}{7} > 0$, hence

$$\frac{\partial^2 \Delta_6}{\partial c \partial d} > 0 \quad \text{for } a \geqq \frac{9}{2}, b \geqq 3. \tag{2.52}$$

Combining (2.47), (2.50), and (2.52), we have that $\frac{\partial \Delta_6}{\partial c}$ is an increasing function of a, b, and d for $a \geqq \frac{9}{2}, b \geqq 3, d \geqq 2$. $\frac{\partial \Delta_6}{\partial c}|_{a=\frac{9}{2},b=3,d=2} = 62 > 0$, hence

$$\frac{\partial \Delta_6}{\partial c} > 0 \quad \text{for } a \geqq \frac{9}{2}, b \geqq 3, d \geqq 2. \tag{2.53}$$

From the symmetry of Δ_6 with respect to a and b and (2.46) we also have,

$$\frac{\partial^2 \Delta_6}{\partial b \partial d} > 0 \quad \text{for } a \geqq 3, c \geqq 2. \tag{2.54}$$

Combining (2.46), (2.52), and (2.54), we have $\frac{\partial \Delta_6}{\partial d}$ is an increasing function of a, b, and c for $a \geqq \frac{9}{2}, b \geqq 3, c \geqq 2$. $\frac{\partial \Delta_6}{\partial d}|_{a=\frac{9}{2}, b=3, c=2} = 21d^2 - 48d + 102 > 21d^2 - 48d + 78 > 0$, hence

$$\frac{\partial \Delta_6}{\partial d} > 0 \quad \text{for } a \geqq \frac{9}{2}, b \geqq 3, c \geqq 2. \tag{2.55}$$

Combining (2.48), (2.49), (2.53), and (2.55), we have Δ_6 is an increasing function of a, b, c, and d for $a \geqq \frac{9}{2}, b \geqq 3, c \geqq 2, d \geqq 2$. $\Delta_6|_{a=\frac{9}{2}, b=3, c=2, d=2} = 56 > 0$, hence

$$\Delta_5 = \frac{d-1}{d^3}\Delta_6 > 0 \quad \text{for } a \geqq \frac{9}{2}, b \geqq 3, c \geqq 2, d \geqq 2. \tag{2.56}$$

Corollary 2.7. *Let $a \geq b \geq c \geq d \geq 1$ be real numbers, and P_4 be the number of positive integral solutions of $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leqq 1$, i.e.*

$$P_4 = \# \left\{ (x, y, z, w) \in \mathbb{Z}_+^4 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leqq 1 \right\}.$$

Then the upper estimate of P_4 by Xu-Yau is strictly sharper than the estimate suggested by Durfee conjecture for $a \geq b \geq c \geq d \geq 3$, i.e.

$$4!P_4 \leqq f_4 < (a-1)(b-1)(c-1)(d-1).$$

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