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Algebraic Classification of Rational CR Structures on Topological 5–Sphere with Transversal Holomorphic S^1–Action in $\mathbb{C}^4$

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Dedicated to Professor F. Hirzebruch *on his Seventy–Fifth Birthday*

Abstract. Let X be a compact connected CR manifold in $\mathbb{C}^N$. X is called a rational CR manifold if its geometric genus $p_g(X)$ is equal to zero. In this paper we classify all rational CR structures on a topological 5–sphere with transverse S^1–action in $\mathbb{C}^4$ up to algebraic equivalence.

In fact the main theorem of this paper gives a classification of all 3–dimensional isolated rational weighted homogeneous singularities with links homeomorphic to topological 5–sphere. These are the simplest kind of singularities in some sense.

1. Introduction

In view of an example of Webster [28], it is clear that the problem of studying when two given CR manifolds are analytically equivalent is very difficult. Webster's example suggests that it is difficult to study the wiggles of a CR structure (cf. Definition 1.3 and Definition 1.4 below). In our previous papers [19] and [18], we have introduced a notion of algebraic equivalence relation among CR manifolds. If a CR manifold is a wiggle of another CR manifold, then they are algebraically equivalent. In some sense, in order to understand the strata of the moduli space of embeddable CR structures which are not a wiggle of each other, we need to study algebraic equivalence among embeddable CR structures. In 1974, Boutet De Monvel [4] (cf. [12] also) proved that if X is a compact strongly pseudoconvex CR manifold of dimension $2n-1$ and $n \geq 3$, then X is CR embeddable in $\mathbb{C}^N$. H. Grauert has constructed compact 3–dimensional strongly pseudoconvex CR manifolds which are not embeddable. Such examples were also studied by H. Rossi [26] and D. Burns [5]. Recall that any

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compact strongly pseudoconvex CR manifold X in $\mathbb{C}^N$ bounds a complex variety V in $\mathbb{C}^N$ [9].

Definition 1.1. Let X_1, X_2 be two connected compact strongly pseudoconvex CR manifolds of dimension $2n-1$ which bound complex varieties V_1, V_2 of dimension n respectively in $\mathbb{C}^N$. Let $\widetilde{V}_1, \widetilde{V}_2$ be the normalization of V_1, V_2 respectively. X_1 is said to be *algebraically equivalent* to X_2 if the corresponding normal varieties $\widetilde{V}_1$ and $\widetilde{V}_2$, which are bounded by X_1 and X_2 respectively, have isomorphic singularities $\widetilde{y}_1$ and $\widetilde{y}_2$, i. e., $(\widetilde{V}_1, \widetilde{y}_1) \cong (\widetilde{V}_2, \widetilde{y}_2)$ as germs of varieties.

It was observed [19] that two CR equivalent manifolds are automatically algebraically equivalent. In [19], we also introduced some numerical invariants under algebraic equivalence for connected compact strongly pseudoconvex embeddable CR manifolds of real dimension 3. In particular, the geometric genus $p_g(X)$ of the CR 3–fold X was introduced. In this paper, we shall as usual, define the geometric genus $p_g(X)$ to be $\dim H^{n-1}(M, \mathcal{O})$ where M is a resolution of singularities of V for any strongly pseudoconvex CR manifold X of dimension $2n-1$. $p_g(X)$ measures the complexity of the CR manifold.

Definition 1.2. A real $(2n-1)$–dimensional connected compact strongly pseudoconvex CR manifold is called a *rational CR manifold* if its geometric genus vanishes. A CR structure with geometric genus zero is called a rational CR structure.

In [16], Lawson and Yau introduced the notion of transversal holomorphic S^1–action on a CR manifold. Let us first recall the definition of a real expression of the CR structure on a manifold X.

Definition 1.3. Let X be a compact connected orientable real manifold of dimension $2n-1$. A CR structure on X is an $(n-1)$–*dimensional subbundle* S *of* $\mathbb{C}T(X)$ such that

(1) $S \cap \overline{S} = \{0\}$.

(2) If L, L' are local sections of S, then so is $[L, L']$.

There is a unique subbundle $\mathcal{H}$ of $T(X)$ such that $\mathbb{C}\,\mathcal{H} = S \oplus \overline{S}$. Furthermore, there is a unique homomorphism $J : \mathcal{H} \to \mathcal{H}$ such that $J^2 = -$identity. The pair $(\mathcal{H}, J)$ is called the *real expression* of the CR structure.

Definition 1.4. Let X_1 and X_2 be connected CR manifolds of dimension $2n-1$. X_1 is said to be a *wiggle* of X_2 if $X_1 \cup X_2$ bounds a complex mainfold of dimension n.

Definition 1.5. (In [16].) With the notation in the above definition, a smooth S^1–action on X is said to be *holomorphic* if it preserves the subbundle $\mathcal{H} \subset T(X)$ and commutes with J. It is said to be *transversal* if, in addition, the vector field V which generates the action is transversal to $\mathcal{H}$ at all points of X.

For CR manifolds admitting a transversal holomorphic S^1–action, the solution of the corresponding complex Plateau problem has a very nice property (cf. Corollary 2.7 of [16]).

Theorem 1.6. ([16].) *Let $X \subseteq \mathbb{C}^{n+1}$ be a CR manifold of dimension $2n-1>1$, and suppose that X admits a transversal holomorphic S^1–action. Then after a holomorphic change of coordinates in $\mathbb{C}^{n+1}$, X is contained in an affine algebraic hypersurface $V \subseteq \mathbb{C}^{n+1}$. The hypersurface V has at most one singular point. It also has a $\mathbb{C}^*$–action and the embedding $X \subseteq V$ is S^1–equivariant.*

It is proposed to classify rational $(2n-1)$–dimensional CR manifolds with transversal holomorphic S^1–action. For any CR structure on a topological 3–sphere X with transversal holomorphic S^1–action, it can be shown that X is algebraically equivalent to the CR structure of the standard S^3 in $\mathbb{C}^2$. In particular $p_g(X)=0$. On the other hand, we can put a lot of non–rational CR structures on a topological 5–sphere with transversal holomorphic S^1–action in $\mathbb{C}^4$, e. g. $X_u = \{(x,y,z,w) \in \mathbb{C}^4 : x^3+y^4+z^4+w^u=0\} \cap S^7$, $u = 7, 11, 13, 17, 19, 23, 25, 29$ etc. Therefore it is a natural question to ask whether we can algebraically classify all rational CR structures on a topological 5–sphere with transversal S^1–action in $\mathbb{C}^4$. Obviously this is the first step toward the global classification of 5–dimensional CR manifolds.

The following is our main theorem which provides us with many explicit important examples of rational CR structures on topological 5–sphere.

Let $S^7 = \{(x,y,z,w) \in \mathbb{C}^4 : |x|^2+|y|^2+|z|^2+|w|^2=1\}$. For integers m, n, the notations $m|n$ and $m \not| n$ stand for the statements "m divides n" and "m does not divide n" respectively.

Theorem 1.7. (**Main Theorem**.) *Any rational CR structure on a topological 5–sphere with transverse S^1–action in $\mathbb{C}^4$ is algebraically equivalent to one of the following CR manifolds.*

(I) $\{(x,y,z,w) \in \mathbb{C}^4 : x^a+y^b+z^c+w^d=0\} \cap S^7$ *where (a,b,c,d) is one of the following:*

1. $(2,2,u,v)$, $u \geq 2$, $v \geq u$, *g.c.d.* $(u,v)=1$.
2. $(2,3,3,u)$, $u \geq 3$, $(2 \not| u$ *and* $3 \not| u)$.
3. $(2,3,4,u)$, $u \geq 4$, $(2 \not| u$ *and* $3 \not| u)$ *or* $(2|u$ *and* $3 \not| u$ *and* $4 \not| u)$.
4. $(2,3,5,u)$, $u \geq 5$, $(2 \not| u$ *and* $3 \not| u)$ *or* $(2 \not| u$ *and* $3|u$ *and* $5 \not| u)$ *or* $(2|u$ *and* $3 \not| u$ *and* $5 \not| u)$.
5. $(2,3,7,u)$, $u \in \{7,8,\ldots,41\} \setminus \{12,14,18,21,24,28,30,36\}$.
6. $(2,3,8,u)$, $u \in \{10,11,13,14,17,19,22,23\}$.
7. $(2,3,9,u)$, $u \in \{11,13,17\}$.
8. $(2,3,10,u)$, $u \in \{11,13,14\}$.
9. $(2,3,11,u)$, $u \in \{11,13\}$.
10. $(2,4,5,u)$, $u \in \{6,7,9,11,13,14,17,18,19\}$.
11. $(2,4,6,u)$, $u \in \{7,11\}$.
12. $(2,4,7,9)$.
13. $(2,5,5,u)$, $u \in \{7,9\}$.
14. $(2,5,6,7)$.
15. $(3,3,4,u)$, $u \in \{5,7,11\}$.
16. $(3,3,5,7)$.

17. $(3,4,4,5)$.

(II) $\{(x,y,z,w) \in \mathbb{C}^4 : x^a + y^b + z^c + zw^d = 0\} \cap S^7$ *where* (a,b,c,d) *is one of the following:*

1. (1) (a) $(2,u,2,v)$, $u \geq 3$, $v \geq u$, *g.c.d.* $(u,2v) = 1$.
   (b) $(2,u,v,2)$, $u \geq 3$, $v \geq u$, $(2 \nmid u$ *and g.c.d.* $(u,v) = 1)$.
   (2) (a) $(2,v,2,u)$, $u \geq 2$, $v \geq u+1$, *g.c.d.* $(v,2u) = 1$.
   (b) $(2,v,u,2)$, $u > 2$, $v \geq u+1$, $(2 \nmid v$ *and g.c.d.* $(u,v) = 1)$.
   (3) (a) $(3,v,2,2)$, $v \geq 3$, $(2 \nmid v$ *and* $3 \nmid v)$ *or* $(2|v$ *and* $3 \nmid v$ *and* $4 \nmid v)$.
   (b) $(5,u,2,2)$, $u \in \{6,7,9,11,13,14,17,18,19\}$.
   (c) $(6,u,2,2)$, $u \in \{7,11\}$.
   (d) $(7,9,2,2)$.
2. (1) (a) (i) $(2,3,3,v)$, $v \geq 3$, *g.c.d.* $(2,v) = 1$.
   (ii) $(2,3,v,3)$, $v \geq 4$, $($*g.c.d.* $(3,v-1) = 1$ *and* $2 \nmid v$ *and* $3|v)$
   or $($*g.c.d.* $(3,v-1) = 3$ *and* $2 \nmid v$ *and* $3 \nmid v)$.
   (b) (i) $(2,3,4,v)$, $v \geq 4$, *g.c.d.* $(3,v) = 1$.
   (ii) $(2,3,v,4)$, $v \geq 5$, $($*g.c.d.* $(4,v-1) = 1$ *and* $2|v$ *and* $3 \nmid v)$
   or $($*g.c.d.* $(4,v-1) = 4$ *and* $2 \nmid v$ *and* $3 \nmid v)$.
   (c) (i) $(2,3,5,v)$, $v \geq 5$, $($*g.c.d.* $(4,v) = 1$ *and* $3 \nmid v)$
   or $($*g.c.d.* $(4,v) = 2$ *and* $3 \nmid v)$
   or $($*g.c.d.* $(4,v) = 4$ *and* $8 \nmid v$ *and* $3 \nmid v)$.
   (ii) $(2,3,v,5)$, $v \geq 6$, $($*g.c.d.* $(5,v-1) = 1$ *and* $2 \nmid v$ *and* $3 \nmid v)$
   or $($*g.c.d.* $(5,v-1) = 1$ *and* $2 \nmid v$ *and* $3|v)$
   or $($*g.c.d.* $(5,v-1) = 1$ *and* $2|v$ *and* $3 \nmid v)$
   or $($*g.c.d.* $(5,v-1) = 5$ *and* $2 \nmid v$ *and* $3 \nmid v)$.
   (d) $(2,3,v,6)$, $v \geq 7$, *g.c.d.* $(6,v-1) = 6$.
   (e) (i) $(2,3,7,u)$, $u \in \{7,10,11,13,14,15,17,19,21,22,23,25,26,29,30,31,33,34,35\}$.
   (ii) $(2,3,u,7)$, $u \in \{9,10,11,13,14,16,17,19,20,21,23,25,26,27,28,29,31,32,33,34,35\}$.
   (f) (i) $(2,3,8,u)$, $u \in \{8,10,11,13,16,17,19,20\}$.
   (ii) $(2,3,u,8)$, $u \in \{10,14,16,17,20\}$.
   (g) (i) $(2,3,9,u)$, $u \in \{9,11,13,15\}$.
   (ii) $(2,3,15,9)$.
   (h) (i) $(2,3,10,u)$, $u \in \{10,11,13\}$.
   (ii) $(2,3,u,10)$, $u \in \{11,13\}$.
   (i) $(2,3,11,11)$.
   (2) (a) (i) $(2,5,3,v)$, $v \geq 5$, $($*g.c.d.* $(2,v) =$ *g.c.d.* $(5,v) = 1)$
   or $\left(\textit{g.c.d. } (2,v) = 2 \textit{ and g. c. d } \left(2,\frac{v}{2}\right) = \textit{g.c.d. } \left(5,\frac{v}{2}\right) = 1\right)$.
   (ii) $(2,5,v,3)$, $v \geq 5$, $($*g.c.d.* $(3,v-1) =$ *g.c.d.* $(2,v) = 1)$
   or $($*g.c.d.* $(2,v) = 2$ *and g.c.d.* $(3,v-1) =$ *g.c.d.* $(5,v) = 1)$
   or $($*g.c.d.* $(3,v-1) = 3$ *and g.c.d.* $(2,v) =$ *g.c.d.* $(5,v) = 1)$.
   (b) (i) $(2,7,3,u)$, $u \in \{9,10,11,13,15,17,18,19,22,23,25,26,27\}$.
   (ii) $(2,7,u,3)$, $u \in \{8,9,11,12,13,15,17,18,19,20,21,23,24,25,26,27\}$.

(c) (i) $(2, 9, 3, 11)$.
(ii) $(2, 9, 9, 3)$.
(3) (a) $(2, v, 3, 3)$, $v \geq 4$, $(2 \not| v$ *and* $3 \not| v)$ *or* $(2 \not| v$ *and* $3|v$ *and* $9 \not| v)$.
(b) (i) $(2, u, 3, 5)$, $u \in \{7, 9, 11, 13, 17, 19, 21, 23, 27, 29\}$.
(ii) $(2, u, 5, 3)$, $u \in \{7, 11, 13, 17, 19, 23, 25, 29\}$.
(c) (i) $(2, u, 3, 6)$, $u \in \{7, 11, 13, 17\}$.
(ii) $(2, u, 6, 3)$, $u \in \{7, 11, 13, 17\}$.
(d) (i) $(2, u, 3, 7)$, $u \in \{9, 11, 13\}$.
(ii) $(2, u, 7, 3)$, $u \in \{9, 11, 13\}$
(e) $(2, 11, 8, 3)$.
(4) (a) (i) $(3, 4, 2, v)$, $v \geq 4$, $(2 \not| v$ *and* $3 \not| v)$.
(ii) $(2, 4, v, 2)$, $v \geq 4$, $(g.c.d.\ (2, v-1) = 1$ *and* $4|v$ *and* $3 \not| v)$
or $(g.c.d.\ (2, v-1) = 2$ *and* $3 \not| v)$.
(b) (i) $(3, 5, 2, v)$, $v \geq 5$, $(3 \not| v$ *and* $5 \not| v)$.
(ii) $(3, 5, v, 2)$, $v \geq 5$, $(g.c.d.\ (2, v-1) = 1$ *and* $3 \not| v)$
or $(g.c.d.\ (2, v-1) = 1$ *and* $3|v$ *and* $5 \not| v)$
or $(g.c.d.\ (2, v-1) = 2$ *and* $3 \not| v$ *and* $5 \not| v)$.
(c) (i) $(3, 7, 2, u)$, $u \in \{8, 10, 11, 13, 16, 17, 19, 20\}$.
(ii) $(3, 7, u, 2)$, $u \in \{8, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20\}$.
(d) (i) $(3, 8, 2, 11)$.
(ii) $(3, 8, u, 2)$, $u \in \{8, 11\}$.
(5) (a) (i) $(3, u, 2, 4)$, $u \in \{5, 7, 10, 11, 13, 14, 17, 19, 22, 23\}$.
(ii) $(3, u, 4, 2)$, $u \in \{5, 7, 10, 11, 13, 14, 17, 19, 20, 22, 23\}$.
(b) (i) $(3, u, 2, 5)$, $u \in \{7, 8, 11, 13, 14\}$.
(ii) $(3, u, 5, 2)$, $u \in \{7, 8, 11, 13, 14\}$.
(c) $(3, u, 6, 2)$, $u \in \{7, 11\}$.
(d) (i) $(3, u, 2, 7)$, $u \in \{8, 10\}$.
(ii) $(3, u, 7, 2)$, $u \in \{8, 10\}$.
(6) (a) (i) $(4, u, 2, 3)$, $u \in \{5, 7, 11\}$.
(ii) $(4, u, 3, 2)$, $u \in \{5, 7, 11\}$.
(b) (i) $(5, 7, 2, 3)$.
(ii) $(5, 7, 3, 2)$.

3. (1) (a) $(2, 5, 4, u)$, $u \in \{7, 8, 11, 13, 14\}$.
(b) $(2, 5, u, 4)$, $u \in \{6, 8, 9, 12, 13, 14\}$.
(2) (a) $(2, u, 4, 4)$, $u \in \{5, 7, 9, 11, 13, 15\}$.
(b) (i) $(2, u, 4, 5)$, $u \in \{7, 9\}$.
(ii) $(2, u, 5, 4)$, $u \in \{7, 9\}$.
(c) $(2, 7, 6, 4)$.
(3) (a) $(4, 5, 2, u)$, $u \in \{7, 9\}$.
(b) $(4, 5, u, 2)$, $u \in \{7, 8, 9\}$.
(4) $(4, u, 4, 2)$, $u \in \{5, 7\}$.

4. (1) (a) $(2, 5, 5, u)$, $u \in \{5, 7\}$.
(b) $(2, 5, 6, 6)$.

(2) $(2, 7, 5, 5)$.

5. (1) (a) $(3, 4, 3, u)$, $u \in \{5, 7\}$.
 (b) $(3, 4, 7, 3)$.
 (2) (a) $(3, u, 3, 3)$, $u \in \{4, 5, 7, 8\}$.
 (b) $(3, 5, 4, 3)$.
 (3) $(4, 5, 3, 3)$.
6. $(3, 4, 4, 4)$.

(III) $\{(x, y, z, w) \in \mathbb{C}^4 : x^a + y^b + z^c w + z w^d = 0\} \cap S^7$ *where* (a, b, c, d) *is one of the following:*

1. (1) $(2, u, 2, v)$, $u \geq 3$, $v \geq u$, (*g.c.d.* $(u, 2v - 1) =$ *g.c.d.* $(2, u) = 1$).
 (2) $(2, v, 2, u)$, $u \geq 2$, $v \geq u + 1$, (*g.c.d.* $(v, 2u - 1) =$ *g.c.d.* $(2, v) = 1$).
 (3) (a) $(4, u, 2, 2)$, $u \in \{5, 7, 11\}$.
 (b) $(5, 7, 2, 2)$.
2. (1) (a) $(2, 3, 3, v)$, $v \geq 3$, $\big(2 \nmid \frac{3v-1}{d_1}$ *and* $3 \nmid \frac{3v-1}{d_1}$ *where* $d_1 =$ *g.c.d.* $(3v - 1, v - 1)\big)$.
 (b) $(2, 3, 4, v)$, $v \geq 4$, $\big(2 \nmid \frac{4v-1}{d_1}$ *and* $3 \nmid \frac{4v-1}{d_1}$ *where* $d_1 =$ *g.c.d.* $(4v - 1, v - 1)\big)$.
 (c) $(2, 3, 5, v)$, $v \geq 5$, $\big(2 \nmid \frac{5v-1}{d_1}$ *and* $3 \nmid \frac{5v-1}{d_1}$ *where* $d_1 =$ *g.c.d.* $(5v - 1, v - 1)\big)$.
 (d) $(2, 3, 6, v)$, $v \geq 6$, $\big(2 \nmid \frac{6v-1}{d_1}$ *and* $3 \nmid \frac{6v-1}{d_1}$ *where* $d_1 =$ *g.c.d.* $(6v - 1, v - 1)\big)$.
 (e) $(2, 3, 7, u)$, $u \in \{8, 9, 10, 12, 14, 16, 17, 18, 20, 21, 24, 25, 26, 28, 29, 30\}$.
 (f) $(2, 3, 8, u)$, $u \in \{9, 10, 12, 13, 15, 16, 18\}$.
 (g) $(2, 3, 9, u)$, $u \in \{10, 11, 12, 13, 14\}$.
 (h) $(2, 3, 10, u)$, $u \in \{10, 11, 12\}$.
 (2) (a) $(2, 5, 3, v)$, $v \geq 5$, $\big(2 \nmid \frac{3v-1}{d_1}$ *and* $5 \nmid \frac{3v-1}{d_1}$ *where* $d_1 =$ *g.c.d.* $(3v - 1, v - 1)\big)$.
 (b) $(2, 7, 3, u)$, $u \in \{8, 9, 10, 13, 14, 16, 17, 18\}$.
 (3) (a) $(2, u, 3, 4)$, $u \in \{5, 7, 9, 13, 15, 17, 19, 21\}$.
 (b) $(2, u, 3, 5)$, $u \in \{9, 11, 13\}$.
 (c) $(2, u, 3, 6)$, $u \in \{7, 9, 11\}$.
 (d) $(2, 9, 3, 8)$.
 (4) (a) $(3, 4, 2, v)$, $v \geq 4$, *g.c.d.* $(3, 2v - 1) = 1$.
 (b) $(3, 5, 2, v)$, $v \geq 5$, ($3 \nmid (2v - 1)$ *and* $5 \nmid (2v - 1)$).
 (c) $(3, 7, 2, u)$, $u \in \{7, 9, 10\}$.
 (5) (a) $(3, u, 2, 3)$, $u \in \{4, 7, 8, 11, 13, 14\}$.
 (b) $(3, u, 2, 4)$, $u \in \{5, 8, 10\}$.
 (c) $(3, u, 2, 6)$, $u \in \{7, 8\}$.
3. (1) $(2, 5, 4, u)$, $u \in \{5, 6, 7, 8, 10, 11\}$.
 (2) (a) $(2, u, 4, 4)$, $u \in \{7, 9\}$.
 (b) $(2, 7, 4, 5)$.

(3) $(4,5,2,5)$.
(4) $(4,5,2,4)$.

4. $(2,5,5,6)$.
5. (1) $(3,4,3,4)$, $u \in \{4,5\}$.
   (2) $(3,5,3,3,)$.

(IV) $\{(x,y,z,w) \in \mathbb{C}^4 : x^a + y^b + yz^c + zw^d = 0\} \cap S^7$ *where* (a,b,c,d) *is one of the following:*

1. (1) $(a,b,1,d)$, $a \geq 2$, $b \geq 2$, $d \geq 2$, *g.c.d.* $(a,bd) = 1$.
   (2) $(a,b,c,1)$, $a \geq 2$, $b \geq 2$, $c \geq 2$, *g.c.d.* $(a,b) = 1$.
   (3) $(a,b,1,1)$, $a \geq 2$, $b \geq 2$, *g.c.d.* $(a,b) = 1$.

2. (1) (a) $(2,u,2,v)$, $u \geq 2$, $v \geq u$, (*g.c.d.* $(2,u-1) = 2$ *and g.c.d.* $(2,v) = 1$ *and g.c.d.* $(2v,u+1) = 2$).
   (b) $(2,v,2,u)$, $u \geq 2$, $v \geq u+1$, (*g.c.d.* $(2,v-1) = 2$ *and g.c.d.* $(2,u) = 1$ *and g.c.d.* $(2u,v+1) = 2$).
   (2) (a) (i) $(3,2,2,v)$, $v \geq 3$, *g.c.d.* $(3,v) = 1$.
   (ii) $(3,v,2,2)$, $v \geq 3$, (*g.c.d.* $(2,v-1) =$ *g.c.d.* $(3,v) = 1$) *or* (*g.c.d.* $(2,v-1) = 2$ *and g.c.d.* $\left(2, \frac{v+1}{2}\right) =$ *g.c.d.* $(3,v) = 1$).
   (b) (i) $(5,2,2,u)$, $u \in \{7,8,11,13,14\}$.
   (ii) $(5,u,2,2)$, $u \in \{6,8,9,12,13,14\}$.
   (3) (a) $(v,2,2,2)$, $v \geq 3$, *g.c.d.* $(v,8) = 1$.
   (b) (i) $(u,2,2,4)$, $u \in \{5,7,9,11,13,15\}$.
   (ii) $(u,4,2,2)$, $u \in \{5,7,9,11,13,15\}$.
   (c) (i) $(u,2,2,5)$, $u \in \{7,9\}$.
   (ii) $(u,5,2,2)$, $u \in \{7,9\}$.
   (d) $(7,6,2,2)$.
   (4) (a) $(u,2,v,2)$, $u \geq 3$, $v \geq u$, (*g.c.d.* $(2,u) =$ *g.c.d.* $(u,v) = 1$).
   (b) $(v,2,u,2)$, $u \geq 3$, $v \geq u+1$, (*g.c.d.* $(2,v) =$ *g.c.d.* $(u,v) = 1$).

3. (1) (a) (i) $(2,3,3,v)$, $v \geq 3$, (*g.c.d.* $(v,7) =$ *g.c.d.* $(v,2) = 1$).
   (ii) $(2,v,3,3)$, $v \geq 4$, (*g.c.d.* $(3,v-1) =$ *g.c.d.* $(3,2v+1) =$ *g.c.d.* $(2,v) = 1$) *or* (*g.c.d.* $(3,v-1) = 3$ *and g.c.d.* $\left(3, \frac{2v+1}{3}\right) =$ *g.c.d.* $(2,v) = 1$).
   (b) (i) $(2,5,3,u)$, $u \in \{5,7,9,13,15,17,19,21\}$.
   (ii) $(2,u,3,5)$, $u \in \{9,11,13,15,19,21\}$.
   (c) (i) $(2,7,3,u)$, $u \in \{7,9\}$.
   (ii) $(2,u,3,7)$, $u \in \{7,9\}$.
   (2) (a) (i) $(3,4,2,u)$, $u \in \{4,7,8,11,13,14\}$.
   (ii) $(3,u,2,4)$, $u \in \{5,8,10,13,14\}$.
   (b) (i) $(3,5,2,u)$, $u \in \{5,7,8\}$.
   (ii) $(3,u,2,5)$, $u \in \{5,7,8\}$.
   (3) (a) $(2,v,4,3)$, $v \geq 4$, (*g.c.d.* $(4,v-1) = 4$ *and g.c.d.* $(2,v) = 1$).
   (b) (i) $(2,3,5,u)$, $u \in \{5,7,9,11,15,17,19,21,23,25\}$.

(ii) $(2, u, 5, 3)$, $u \in \{7, 9, 13, 15, 19, 21, 25\}$.
(c) (i) $(2, 3, 6, u)$, $u \in \{7, 9, 11, 13, 15\}$.
(ii) $(2, u, 6, 3)$, $u \in \{9, 11, 13, 15\}$.
(d) (i) $(2, 3, 7, u)$, $u \in \{7, 9, 11\}$.
(ii) $(2, u, 7, 3)$, $u \in \{7, 9, 11\}$.
(e) $(2, 9, 8, 3)$.
(f) (i) $(2, 3, 9, 9)$.
(ii) $(2, 9, 9, 3)$.
(4) (a) $(4, 3, 2, u)$, $u \in \{5, 7\}$.
(b) $(4, 7, 2, 3)$.
(5) (a) $(2, 3, v, 4)$, $v \geq 4$, $(g.c.d.\ (2, v) = 1)$
or $\big(g.c.d.\ (2, v) = 2$ *and* $g.c.d.\ \big(2, \frac{v}{2}\big) = 1\big)$.
(b) (i) $(2, 3, v, 5)$, $v \geq 6$, $(g.c.d.\ (2, v) = g.c.d.\ (5, 3v - 2) = 1)$
or $\big(g.c.d.\ (2, v) = 2$ *and* $g.c.d.\ (5, 3v - 2) = g.c.d.\ \big(2, \frac{v}{2}\big) = 1\big)$.
(ii) $(2, 5, v, 3)$, $v \geq 6$, $(g.c.d.\ (4, v) = g.c.d.\ (3, 5v - 4) = 1)$
or $\big(g.c.d.\ (4, v) = 2$ *and* $g.c.d.\ (3, 5v - 4) = g.c.d.\ (2, \frac{v}{2}) = 1\big)$
or $\big(g.c.d.\ (4, v) = 4$ *and* $g.c.d.\ (3, 5v - 4) = g.c.d.\ \big(2, \frac{v}{4}\big) = 1\big)$.
(c) (i) $(2, 3, u, 7)$, $u \in \{9, 11, 13, 14, 15, 18, 19, 21, 22, 23\}$.
(ii) $(2, 7, u, 3)$, $u \in \{9, 10, 11, 13, 14, 17, 18, 19, 21, 22, 23\}$.
(d) (i) $(2, 3, 10, 9)$.
(ii) $(2, 9, 10, 3)$.
(6) (a) $(u, 3, 2, 3)$, $u \in \{4, 5, 7, 8\}$.
(b) $(5, 4, 2, 3)$.
(7) (a) $(3, v, 3, 2)$, $v \geq 3$, $g.c.d.\ (3, v - 1) = 3$.
(b) (i) $(3, 2, 4, u)$, $u \in \{4, 5, 8, 10, 11, 13, 16, 17, 19, 20\}$.
(ii) $(3, u, 4, 2)$, $u \in \{4, 7, 8, 10, 11, 14, 16, 17, 19, 20\}$.
(c) (i) $(3, 2, 5, u)$, $u \in \{5, 7, 8, 10, 11, 13\}$.
(ii) $(3, u, 5, 2)$, $u \in \{5, 7, 8, 10, 11, 13\}$.
(d) $(3, 10, 6, 2)$.
(e) (i) $(3, 2, 7, u)$, $u \in \{7, 8\}$.
(ii) $(3, u, 7, 2)$, $u \in \{7, 8\}$.
(f) (i) $(3, 2, 8, 8)$.
(ii) $(3, 8, 8, 2)$.
(8) (a) $(5, 2, 3, 6)$.
(b) $(5, 6, 3, 2)$.
(9) (a) (i) $(3, 2, v, 4)$, $v \geq 5$, $g.c.d.\ (3, v) = 1$.
(ii) $(3, 4, v, 2)$, $v \geq 5$, $(g.c.d.\ (3, v) = 1)$
or $\big(g.c.d.\ (3, v) = 3$ *and* $g.c.d.\ \big(3, \frac{v}{3}\big) = 1\big)$.
(b) (i) $(3, 2, v, 5)$, $v \geq 6$, $(g.c.d.\ (5, 2v - 1) = g.c.d.\ (3, v) = 1)$.
(ii) $(3, 5, v, 2)$, $v \geq 6$, $(g.c.d.\ (4, v) = g.c.d.\ (3, v) = 1)$
or $\big(g.c.d.\ (4, v) = 2$ *and* $g.c.d.\ \big(2, \frac{5v-4}{2}\big) = g.c.d.\ (3, v) = 1\big)$
or $\big(g.c.d.\ (4, v) = 4$ *and* $g.c.d.\ \big(2, \frac{5v-4}{4}\big) = g.c.d.\ (3, v) = 1\big)$.
(c) (i) $(3, 2, u, 7)$, $u \in \{8, 10, 13, 14, 16, 17\}$.

(ii) $(3,7,u,2)$, $u \in \{8,11,12,13,15,16,17\}$.

(d) (i) $(3,2,10,8)$.

(ii) $(3,8,10,2)$.

(10) (a) (i) $(u,2,3,3)$, $u \in \{5,7,11,13,17\}$.

(ii) $(u,3,3,2)$, $u \in \{5,7,11,13,17\}$.

(b) (i) $(u,2,3,4)$, $u \in \{5,7\}$.

(ii) $(u,4,3,2)$, $u \in \{5,7\}$.

(11) (a) (i) $(5,2,v,3)$, $v \geq 5$, *g.c.d.* $(3,2v-1) =$ *g.c.d.* $(5,v) = 1$.

(ii) $(5,3,v,2)$, $v \geq 5$, (*g.c.d.* $(2,v) =$ *g.c.d.* $(5,v) = 1$) *or* $\big($*g.c.d.* $(2,v) = 2$, *g.c.d.* $\big(2, \frac{3v-2}{2}\big) =$ *g.c.d.* $(5,v) = 1\big)$.

(b) (i) $(7,2,u,3)$, $u \in \{9,10,12,13\}$.

(ii) $(7,3,u,2)$, $u \in \{8,9,11,12,13\}$.

(12) (a) (i) $(u,2,4,3)$, $u \in \{5,7,11\}$.

(ii) $(u,3,4,2)$, $u \in \{5,7,11\}$.

(b) $(7,3,5,2)$.

(c) $(7,2,6,3)$.

4. (1) $(2,5,4,u)$, $u \in \{5,7\}$.

(2) (a) $(5,2,4,4)$.

(b) $(5,4,4,2)$.

(3) (a) $(5,2,u,4)$, $u \in \{6,7\}$.

(b) $(5,4,u,2)$, $u \in \{6,7\}$.

5. (1) $(2,5,5,5)$.

(2) $(2,5,6,5)$.

6. (1) $(3,4,3,4)$.

(2) $(u,3,3,3)$, $u \in \{4,5\}$.

(3) $(4,3,5,3)$.

(V) $\big\{(x,y,z,w) \in \mathbb{C}^4 : x^a + y^b + yz^c + yw^d + z^p w^q = 0$, *where* $\frac{p(b-1)}{bc} + \frac{q(b-1)}{bd} = 1\big\} \cap S^7$, *where* (a,b,c,d) *is one of the following:*

1. (1) $(a,b,1,d)$, $a \geq 2$, $b \geq 2$, $d \geq 2$, $\big((b-1)|d$, *and g.c.d.* $\big(a, \frac{bd}{b-1}\big) = 1\big)$.

(2) $(a,b,c,1)$, $a \geq 2$, $b \geq 2$, $c \geq 2$, $\big((b-1)|c$ *and g.c.d.* $\big(a, \frac{bc}{b-1}\big) = 1\big)$.

(3) $(a,2,1,1)$, $a \geq 2$, *g.c.d.* $(a,2) = 1$.

2. (1) (a) $(2,3,2,v)$, $v \geq 3$, *g.c.d.* $(2,v) = 1$.

(b) $(2,3,v,2)$, $v \geq 3$, *g.c.d.* $(2,v) = 1$.

(2) (a) (i) $(3,2,2,v)$, $v \geq 3$, (*g.c.d.* $(2,v) =$ *g.c.d.* $(3,v) = 1$).

(ii) $(3,2,v,2)$, $v \geq 3$ (*g.c.d.* $(2,v) =$ *g.c.d.* $(3,v) = 1$).

(b) (i) $(5,2,2,u)$, $u \in \{7,9\}$.

(ii) $(5,2,u,2)$, $u \in \{7,9\}$.

(3) (a) $(u,2,2,3)$, $u \in \{5,7,11\}$.

(b) $(u,2,3,2)$, $u \in \{5,7,11\}$.

3. (1) (a) (i) $(2,3,3,v)$, $v \geq 3$, $\big($*g.c.d.* $(2,v) = 2$ *and g.c.d.* $\big(2, \frac{v}{2}\big) =$ *g.c.d.* $(3,v) = 1\big)$.

(ii) $(2,3,v,3)$, $v \geq 3$, (*g.c.d.* $(2,v)=2$ *and g.c.d.* $\left(2,\frac{v}{2}\right)=$ *g.c.d.* $(3,v)=1$).

(b) (i) $(2,3,5,u)$, $u \in \{6,14,18\}$.

(ii) $(2,3,u,5)$, $u \in \{6,14,18\}$.

(c) (i) $(2,3,6,u)$, $u \in \{7,11\}$.

(ii) $(2,3,u,6)$, $u \in \{7,11\}$.

(2) (a) (i) $(3,2,4,u)$, $u \in \{5,7,11\}$.

(ii) $(3,2,u,4)$, $u \in \{5,7,11\}$.

(b) (i) $(3,2,5,7)$.

(ii) $(3,2,7,5)$.

(3) (a) $(2,5,3,20)$.

(b) $(2,5,20,3)$.

(4) (a) (i) $(2,5,3,4)$.

(ii) $(2,5,4,3)$.

(b) (i) $(2,9,3,8)$.

(ii) $(2,9,8,3)$.

(5) (a) $(3,4,2,15)$.

(b) $(3,4,15,2)$.

(6) (a) $(4,3,2,u)$, $u \in \{5,7\}$.

(b) $(4,3,u,2)$, $u \in \{5,7\}$.

(7) (a) (i) $(3,4,2,3)$.

(ii) $(3,4,3,2)$.

(b) (i) $(3,8,2,7)$.

(ii) $(3,8,7,2)$.

(8) (a) $(u,3,2,3)$, $u \in \{4,5,7,8\}$.

(b) $(u,3,3,2)$, $u \in \{4,5,7,8\}$.

(9) (a) $(u,4,2,3)$, $u \in \{5,7\}$.

(b) $(u,4,3,2)$, $u \in \{5,7\}$.

4. (1) $(2,5,4,u)$, $u \in \{5,7\}$.

(2) $(2,5,u,4)$, $u \in \{5,7\}$.

5. (1) $(3,4,3,4)$.

(2) $(3,4,4,3)$.

(VI) $\left\{(x,y,z,w) \in \mathbb{C}^4 : x^a + y^b w + z^c w + y w^d + y^p z^q = 0,\ (b,d) \neq (1,1),\ \frac{p(d-1)}{bd-1} + \frac{qb(d-1)}{c(bd-1)} = 1\right\} \cap S^7$, *where* (a,b,c,d) *is one of the following:*

1. $(a,b,1,d)$, $a \geq 2$, $b \geq 2$, $d \geq 2$, $\left((d-1)|(bd-1)\ \textit{and g.c.d.}\ \left(a,\frac{bd-1}{d-1}\right)=1\right)$.
2. (1) $(2,2,v,2)$, $v \geq 3$, *g.c.d.* $(2,v)=1$.

(2) $(2,u,v,2)$, $u \geq 3$, $v \geq u$, (*g.c.d.* $(u,v)=$ *g.c.d.* $(2,v)=1$).

(3) $(4,2,u,2)$, $u \in \{5,7\}$.

(4) $(2,v,u,2)$, $u \geq 3$, $v \geq u+1$, (*g. c. d.* $(u,v)=$ *g.c.d.* $(2,u)=1$).

(5) $(u,2,3,2)$, $u \in \{4,5,7,8\}$.

(6) (a) $(3,v,2,2)$, $v \geq 3$, (*g.c.d.* $(2,v)=$ *g.c.d.* $(3,2v-1)=1$).

(b) $(5,u,2,2)$, $u \in \{5,7\}$.

(7) $(u, 3, 2, 2)$, $u \in \{7, 9\}$.

3. (1) (a) $(2, 4, 3, 4)$.
 (b) $(2, 8, 3, 8)$.
 (2) $(2, 5, u, 3)$, $u \in \{7, 9\}$.
 (3) (a) $(2, v, 3, 3)$, $v \geq 4$, $\big(g.c.d.\,(2, v-1) = 2$ *and* $g.c.d.\,(3, v) = g.c.d.\,\big(2, \frac{3v-1}{2}\big) = 1\big)$.
 (b) $(2, v, 3, 4)$, $v \geq 5$, $g.c.d.\ (3, v-1) = 3$.
 (c) $(2, 17, 3, 5)$.
 (d) $(2, 11, 3, 6)$.
 (4) $(2, u, 5, 3)$, $u \in \{9, 13\}$.
 (5) (a) $(3, 3, 2, 3)$.
 (b) $(3, 7, 2, 7)$.
 (6) (a) $(3, 3, u, 2)$, $u \in \{4, 5, 7, 8\}$.
 (b) $(3, 4, 5, 2)$.
 (7) (a) $(3, v, 2, 3)$, $v \geq 4$, $g.c.d.\ (2, v-1) = 2$.
 (b) $(3, 13, 2, 4)$.
 (c) $(3, 9, 2, 5)$.
 (8) $(3, 7, 4, 2)$.
 (9) $(u, 3, 2, 3)$, $u \in \{5, 7\}$.
 (10) $(4, 4, 3, 2)$.
4. $(2, 4, u, 4)$, $u \in \{5, 7\}$.
5. $(3, 3, 4, 3)$.

(VII) $\big\{(x, y, z, w) \in \mathbb{C}^4 : x^a + y^b z + z^c w + y w^d = 0\big\} \cap S^7$, *where* (a, b, c, d) *is one of the following:*

1. (1) $(a, 1, c, d)$, $a \geq 2$, $c \geq 2$, $d \geq 2$, $g.c.d.\ (a, cd+1) = 1$.
 (2) $(a, 1, 1, d)$, $a \geq 2$, $d \geq 2$, $g.c.d.\ (a, d+1) = 1$.
 (3) $(a, 1, 1, 1)$, $a \geq 2$, $g.c.d.\ (a, 2) = 1$.
2. (1) $(2, 2, u, v)$, $u \geq 2$, $v \geq 2$, $g.c.d.\ (2uv+1, u+1) = 1$.
 (2) $(2, 2, v, u)$, $v \geq u+1$, $u \geq 3$, $g.c.d.\ (2uv+1, v+1) = 1$.
 (3) (a) $(3, 2, 2, v)$, $v \geq 3$, $g.c.d.\ (4v+1, 3) = 1$.
 (b) $(4, 2, 2, v)$, $v \geq 4$, $g.c.d.\ (4v+1, 3) = 1$.
 (c) $(5, 2, 2, u)$, $u \in \{7, 9, 10\}$.
 (d) $(6, 2, 2, 6)$.
 (4) (a) $(u, 2, 2, 3)$, $u \in \{4, 5, 6, 7, 8, 9, 10, 11, 12\}$.
 (b) $(u, 2, 2, 4)$, $u \in \{5, 6, 7, 8\}$.
3. (1) (a) $(2, 3, 3, v)$, $v \geq 3$, $(g.c.d.\ (2, 9v+1) = g.c.d.\ (7, 9v+1) = 1)$.
 (b) $(2, 3, 4, v)$, $v \geq 4$.
 (c) $(2, 3, 5, u)$, $u \in \{6, 10, 12, 14, 16, 18\}$.
 (d) $(2, 3, 6, u)$, $u \in \{6, 7, 8, 9, 10, 11\}$.
 (e) $(2, 3, 7, 8)$.
 (2) (a) $(2, 3, v, 4)$, $v \geq 5$, $g.c.d.\ (12v+1, 5) = 1$.
 (b) $(2, 3, u, 5)$, $u \in \{8, 10, 12, 14, 16, 18\}$.

(c) $(2, 3, u, 6)$, $u \in \{7, 8, 9, 10, 11\}$.
(d) $(2, 3, 8, 7)$.
(3) (a) $(3, 2, 3, v)$, $v \geq 3$.
(b) $(3, 2, 4, u)$, $u \in \{5, 6, 9, 11, 12\}$.
(c) $(3, 2, 5, u)$, $u \in \{6, 7\}$.
(d) $(3, 2, 6, 6,)$.
(4) (a) $(3, 2, v, 3)$, $v \geq 4$, *g.c.d.* $(6v + 1, 5) = 1$.
(b) $(3, 2, u, 4)$, $u \in \{5, 8, 9, 11, 12\}$.
(c) $(3, 2, u, 5)$, $u \in \{6, 7\}$.
(5) $(4, 2, 3, u)$, $u \in \{4, 5, 6\}$.
(6) $(4, 2, u, 3)$, $u \in \{5, 6\}$.
(7) $(u, 2, 3, 3)$, $u \in \{4, 5, 6\}$.
4. (1) (a) $(2, 4, 4, u)$, $u \in \{5, 6, 7, 8, 9, 10\}$.
(b) $(2, 4, 5, u)$, $u \in \{5, 6\}$.
(2) $(2, 4, 6, 5)$.
(3) $(4, 2, 4, 4)$.
5. $(3, 3, 3, u)$, $u \in \{4, 5\}$.

Corollary 1.8. *Let (V, O) be a 3–dimensional isolated rational hypersurface singularity with $\mathbb{C}^*$–action in $\mathbb{C}^4$. Suppose that the link $K := S^7 \cap V$ is homeomorphic to S^5. Then $(V, 0)$ is defined by a weighted homogeneous polynomial of one of the forms listed in the Main Theorem.*

Remark 1.9. The list in (V) of the Main Theorem may be reduced slightly by change of coordinates.

In §2, we shall recall some basic notions and results for CR manifolds which we need later. In §3, we shall recall Milnor's characteristic polynomial which will help us to determine when a link of an isolated singularity is a topological sphere. In §4, we shall give a classification (up to deformation which preserves weights) of weighted homogeneous polynomials of 4 variables with isolated singularity at the origin. This list was obtained first by Kouchnirenko [13] and Orlik–Randell [23] (see also [11]) independently. Since there is no proof in the literature, we provide a proof here. In §5, we classify rational compact strongly pseudoconvex CR 5–dimensional manifolds with transverse S^1–action in $\mathbb{C}^4$ up to algebraically equivalence. As a byproduct, we also classify all 3–dimensional isolated rational hypersurface singularities with $\mathbb{C}^*$–action. In §6, we classify all rational CR structures on a topological 5–sphere with transverse S^1–action in $\mathbb{C}^4$ up to algebraic equivalence.

2. Basic theory on global CR manifolds

In this section, we shall recall some basic notions and facts about CR manifolds that will be needed for later discussion.

Definition 2.1. With the notations in Definition 1.3, let $L_1, \dots, L_{n-1}$ be a local basis for sections of S over on open subset of X so that $\bar{L}_1, \dots, \bar{L}_{n-1}$ is a local basis for sections of $\overline{S}$. Since $S \oplus \overline{S}$ has complex codimension one in $\mathbb{C}TX$, we may choose a local section N of $\mathbb{C}TX$. We may assume that N is purely imaginary. Then the matrix $\big(c_{ij}\big)$ defined by

$$\big[L_i, \bar{L}_j\big] \;=\; \sum_k a_{ij}^k L_k + \sum_k b_{ij}^k \bar{L}_k + c_{ij} N$$

is Hermitian and is called the *Levi form.*

The Levi form is non–invariant; however its essential features are invariant.

Proposition 2.2. ([7].) *The number of nonzero eigenvalues and the absolute value of the signature of the Levi form $\big(c_{ij}\big)$ at each point are independent of the choice of $L_1, \dots, L_{n-1}, N$.*

Definition 2.3. Let X be a CR manifold. Then X is *strongly pseudoconvex* if the Levi form $\big(c_{ij}\big)$ in Definition 2.1 is always nonsingular and its eigenvalues are of the same sign.

Proposition 2.4. ([19].) *Let X_1 and X_2 be two strongly pseudoconvex compact connected CR manifolds in $\mathbb{C}^N$. If X_1 is CR equivalent to X_2, then X_1 is algebraically equivalent to X_2 in the sense of Definition* 1.1.

Proof. We may assume that X_1 and X_2 bound the normal varieties $\widetilde{V}_1$ and $\widetilde{V}_2$ in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$ respectively. Let $\widetilde{Y}_1$ and $\widetilde{Y}_2$ be the singularities of $\widetilde{V}_1$ and $\widetilde{V}_2$ respectively. Then we need to show that $\big(\widetilde{V}_1, \widetilde{Y}_1\big)$ is isomorphic to $\big(\widetilde{V}_2, \widetilde{Y}_2\big)$ as a germ. Let φ_1 be the CR isomorphism from the boundary X_1 of $\widetilde{V}_1$ to the boundary of X_2 to $\widetilde{V}_2$. By the strong pseudoconvexity of $X_1 = \partial\widetilde{V}_1$ and the normality of $\widetilde{V}_1$, it is easy to see that φ_1 extends to a holomorphic map $\overline{\varphi}_1 : \widetilde{V}_1 \to \mathbb{C}^{N_2}$. Clearly $\overline{\varphi}_1\big(\widetilde{V}_1\big)$ and $\widetilde{V}_2$ are both complex varieties in $\mathbb{C}^{N_2}$ which have the same boundary. By the uniqueness of the complex Plateau problem (see [9]), we see that $\overline{\varphi}_1\big(\widetilde{V}_1\big) = \widetilde{V}_2$. Similarly, let φ_2 be the inverse mapping of φ_1. Then φ_2 is a CR isomorphism from X_2 to X_1. The same argument as before shows that φ_2 extends to a holomorphic map $\overline{\varphi}_2 : \widetilde{V}_2 \to \mathbb{C}^{N_1}$ such that $\overline{\varphi}_2\big(\widetilde{V}_2\big) = \widetilde{V}_1$. $\overline{\varphi}_2 \circ \overline{\varphi}_1 : \widetilde{V}_1 \to \widetilde{V}_1$ is a holomorphic mapping which extends the identity map $Id : \partial\widetilde{V}_1 \to \partial\widetilde{V}_1$. By the uniqueness of the extension, we conclude that $\overline{\varphi}_2 \circ \overline{\varphi}_1 : \widetilde{V}_1 \to \widetilde{V}_1$ is the identity. Similarly $\overline{\varphi}_1 \circ \overline{\varphi}_2 : \widetilde{V}_2 \to \widetilde{V}_2$ is the identity map. □

The following theorem is due to Lawson–Yau [16].

Theorem 2.5. ([16].) *Let X be a strongly pseudoconvex CR manifold of dimension $2n-1 > 1$ and suppose that X admits a transversal holomorphic S^1–action. Then there exists a holomorphic equivariant embedding $X \to V$ as a hypersurface in an n–dimensional algebraic variety $V \subset \mathbb{C}^N$ with a linear $\mathbb{C}^*$–action. V has at most one singular point at the origin.*

Corollary 2.6. ([16].) *Let $X \subseteq \mathbb{C}^{n+1}$ be a strongly pseudoconvex CR manifold of dimension $2n-1 > 1$, and suppose M admits a transversal holomorphic S^1–action.*

Then after a holomorphic change of coordinates in $\mathbb{C}^{n+1}$, X is contained in an affine algebraic hypersurface $V \subseteq \mathbb{C}^{n+1}$. The hypersurface V has at most one singular point. It also has a $\mathbb{C}^*$–action and the embedding $X \subset V$ is S^1–equivariant.

Definition 2.7. Let X be a compact connected strongly pseudoconvex embeddable CR manifold of real dimension $2n-1$. Let V be the normal subvariety in $\mathbb{C}^N$ such that the boundary of V is X. Let $\pi : M \to V$ be a resolution of singularities of V. The *geometric genus* of X denoted by $p_g(X)$ is defined to be $\dim H^{n-1}(M,\mathcal{O})$.

Proposition 2.8. *Let X be a connected compact strongly pseudoconvex CR manifold of real dimension $2n-1$ and $n \geq 2$. Suppose that X bounds a normal variety $V \subseteq \mathbb{C}^N$ with isolated singularities $Y = \{q_1, \ldots, q_m\}$. Let $\pi : M \to V$ be a resolution of singularities of V. Then the geometric genus $p_g(X) := \dim H^{n-1}(M,\mathcal{O})$ is a CR invariant of X. In fact, let U be any small strongly pseudoconvex neighborhood of Y. Then*

$$p_g(X) \;=\; \dim H^0(U-Y,\Omega^n)/L^2(U-Y,\Omega^n)$$

where Ω^n is the sheaf of germs of holomorphic n–forms and $L^2(U-Y,\Omega^n)$ is the space of holomorphic n forms w on $U-Y$ which are L^2–integrable, i. e., $\int_{U-Y} w \wedge \overline{w} < \infty$.

Proof. By Lemma 5.3 of [14], $\dim H^{n-1}(M,\mathcal{O}) = \dim H^{n-1}\big(\pi^{-1}(U),\mathcal{O}\big)$. By Laufer [15], we have the exact sequence

$$\begin{aligned} 0 \to H^0_c\big(\pi^{-1}(U),\Omega^n\big) \to H^0\big(\pi^{-1}(U),\Omega^n\big) \to H^0_\infty\big(\pi^{-1}(U),\Omega^n\big) \to H^1_c\big(\pi^{-1}(U),\Omega^n\big) \\ \to H^1\big(\pi^{-1}(U),\Omega^n\big) \;\to\; \cdots . \end{aligned}$$

Take a 1–convex exhaustion function φ on $\pi^{-1}(U)$ such that $\varphi \geq 0$ on $\pi^{-1}(U)$ and $\varphi(p) = 0$ if and only if $p \in \pi^{-1}(Y)$. Put $\pi^{-1}(U)_r = \big\{p \in \pi^{-1}(U) : \varphi(p) \leq r\big\}$. Then by Laufer [15], $\varinjlim_r H^0\big(\pi^{-1}(U) - \pi^{-1}(U)_r,\Omega^n\big) \cong H^0_\infty\big(\pi^{-1}(U),\Omega^n\big)$. On the other hand by Andreotti and Grauert (Théorème 15 of [2]), $H^0\big(\pi^{-1}(U)-\pi^{-1}(Y),\Omega^n\big)$ is isomorphic to $H^0\big(\pi^{-1}(U)-\pi^{-1}(U)_r,\Omega^n\big)$. The above exact sequence becomes

$$\begin{aligned} 0 \longrightarrow H^0_c\big(\pi^{-1}(U),\Omega^n\big) \longrightarrow H^0\big(\pi^{-1}(U),\Omega^n\big) &\longrightarrow H^0\big(\pi^{-1}(U)-\pi^{-1}(Y),\Omega^n\big) \\ &\longrightarrow H^1_c\big(\pi^{-1}(U),\Omega^n\big) \\ &\longrightarrow H^1\big(\pi^{-1}(U),\Omega^n\big) \longrightarrow \cdots . \end{aligned}$$

Now $H^1_c\big(\pi^{-1}(U),\Omega^n\big)$ is Serre dual to $H^{n-1}\big(\pi^{-1}(U),\mathcal{O}\big)$ and $H^1\big(\pi^{-1}(U),\Omega^n\big) = 0$ by the Grauert–Riemenschneider vanishing theorem [8]. Hence

$$\begin{aligned} p_g(X) &= \dim H^{n-1}(M,\mathcal{O}) \\ &= \dim H^{n-1}\big(\pi^{-1}(U),\mathcal{O}\big) \\ &= \dim H^0\big(\pi^{-1}(U)-\pi^{-1}(Y),\Omega^n\big)/H^0\big(\pi^{-1}(U),\Omega^n\big) . \end{aligned}$$

By Theorem 3.1 of [15], $H^0\big(\pi^{-1}(U),\Omega^n\big) = L^2(U-Y,\Omega^n)$, and we have proved

$$p_g(X) \;=\; \dim H^0(U-Y,\Omega^n)/L^2(U-Y,\Omega^n) . \tag{2.1}$$

Now suppose that X' bounds a normal variety $V' \subseteq \mathbb{C}^{N'}$ with isolated singularities $Y' = \{q_1, \ldots, q_{m'}\}$. Then the proof of Proposition 2.4 shows that (V, Y) is biholomorphically equivalent to (V', Y'). In view of (2.1), we see immediately that $p_g(X) = p_g(X')$. Therefore $p_g(X)$ is a CR invariant. □

3. Milnor's theory on topology of isolated hypersurface singularities

In this section, we shall review the fundamental theory developed by Milnor [21] on the topology of isolated hypersurface singularities. We are particularly interested in his complete characterization when a link of a hypersurface singularity is a topological sphere in terms of characteristic polynomials. We also recall the beautiful work of Milnor and Orlik [22] on the computation of characteristic polynomials for singularities of weighted homogeneous polynomials.

Let $f(z) = f(z_0, z_1, \ldots, z_n)$ be a polynomial function in $n+1$ complex variables. The zero locus of $f(z)$, $V = \{z \in \mathbb{C}^{n+1} : f(z) = 0\}$ is called the *hypersurface* defined by $f(z)$. A point $p \in V$ is *singular* if all partial derivatives $\frac{\partial f}{\partial z_i}$ vanish at p, i. e., if f has p as critical point. A point in V is called smooth if it is not a singular point. A singularity p is called *isolated* if in some neighborhood of p there is no other singularity. From now on, we shall assume that V has an isolated singularity at the origin. The topology of V at 0 was studied in detail by Milnor [21]. Denote $B_\varepsilon = \{z \in \mathbb{C}^{n+1} : \|z\| \le \varepsilon\}$ and $S_\varepsilon = \partial B_\varepsilon = \{z \in \mathbb{C}^{n+1} : \|z\| = \varepsilon\}$.

Theorem 3.1. ([21].) *For $\varepsilon > 0$ small enough, S_ε cuts the smooth part of the hypersurface V transversally. The pairs $(S_\varepsilon, S_\varepsilon \cap V)$ for any ε small enough are diffeomorphic, and $(B_\varepsilon, B_\varepsilon \cap V)$ is homeomorphic to $(B_\varepsilon, C(S_\varepsilon \cap V))$, where $C(S_\varepsilon \cap V)$ is the cone which is the union of the real line segments joining* 0 *and points of $S_\varepsilon \cap V$.*

Definition 3.2. The *link* of the singularity $(V, 0)$ is defined to be $K := S_\varepsilon \cap V$.

Theorem 3.3. ([21].) *For $\varepsilon > 0$ small enough, the mapping $\varphi_\varepsilon : S_\varepsilon - K \to S^1$ (= unit circle) defined by $\varphi_\varepsilon(z) = f(z)/\|f(z)\|$ is a smooth fibration.*

Theorem 3.4. ([21].) *For $\varepsilon > 0$ small enough and $\varepsilon >> \eta > 0$, the mapping $\psi_{\varepsilon,\eta} : (\mathrm{Int}B_\varepsilon) \cap f^{-1}(\partial D_\eta) \to S^1$ defined by $\psi_{\varepsilon,\eta}(z) = f(z)/\|f(z)\|$, where $\partial D_\eta = \{z \in \mathbb{C} : |z| = \eta\}$, is a smooth fibration isomorphic to the fibration φ_ε in Theorem* 3.3 *by an isomorphism which preserves the arguments.*

Definition 3.5. The fibrations of Theorem 3.1 and Theorem 3.3 are called *Milnor fibrations* of f at 0.

Theorem 3.6. ([21].) *Each fiber F_θ of the Milnor fibration is a smooth parallelizable $2n$–dimensional manifold which has the homotopy type of a bouquet $S^n \vee \ldots \vee S^n$ of n–spheres, the number in this bouquet (i. e., the middle Betti number of F_θ), being* *strictly positive. Each fiber can be considered as the interior of a smooth compact manifold with boundary,*

$$\text{closure}\,(F_\theta) \;=\; F_\theta U K$$

where the link K being the common boundary is an $(n-2)$–connected manifold. (*A bouquet of spheres is the topological space which is a union of spheres having a single point in common.*)

Definition 3.7. The middle Betti number of the Milnor fiber is called the *Milnor number* of the singularity and is denoted by μ.

We obtain $S_\varepsilon - K$ from $F_\theta \times [0, 2\pi]$ by identifying $F_\theta \times \{0\}$ and $F_\theta \times \{2\pi\}$ by a homeomorphism $h : F_\theta \to F_\theta$ called the *characteristic map.* Since F_θ has the homotopy type of $S_1^n \vee \ldots \vee S_\mu^n$ of μ copies of n–spheres, the induced map

$$h_* \;:\; H_n(F_\theta, \mathbb{C}) \;\longrightarrow\; H_n(F_\theta, \mathbb{C})$$

is of particular interest.

Definition 3.8. The *characteristic polynomial* of the hypersurface singularity $(V, 0)$ is defined to be $\Delta(t) = \det\,(tI_* - h_*)$, where I is the identity map.

We shall use the following theorem of Milnor frequently.

Theorem 3.9. *For $n \neq 2$, the manifold K is a topological sphere if and only if the integer $\Delta(1) = \det\,(I_* - h_*)$ is equal to ± 1.*

Brieskorn [3] and Pham [25] computed μ and $\Delta(t)$ for varieties defined by polynomials of the form $f(z) = z_0^{a_0} + z_1^{a_1} + \ldots + z_n^{a_n}$. Milnor and Orlik [22] computed μ and $\Delta(t)$ for all weighted homogeneous polynomials.

Definition 3.10. The polynomial $f(z_0, z_1, \ldots, z_n)$ is called *weighted homogeneous of type* $(w_0, w_1, \ldots, w_n)$, where $w_0, w_1, \ldots, w_n$ are positive rational numbers, if it can be expressed as a linear combination of monomials $z_0^{i_0} z_1^{i_1} \ldots z_n^{i_n}$ for which $i_0/w_0 + i_1/w_1 + \ldots + i_n/w_n = 1$.

Remark 3.11. Since the origin is assumed to be a critical point, it is clear that $w_0 > 1$, $w_1 > 1, \ldots, w_n > 1$. In fact by a result of Saito [27], we can assume that $w_i \geq 2$, $1 \leq i \leq n$.

In Section 4, we shall classify the weighted homogeneous polynomals in 4 variables which have an isolated critical point at the origin.

Theorem 3.12. ([22].) *Let $f(z_0, z_1, \ldots, z_n)$ be a weighted homogenous polynomial of type $(w_0, w_1, \ldots, w_n)$ having an isolated critical point at the origin. Then the Milnor number is $\mu = (w_0 - 1)(w_1 - 1) \ldots (w_n - 1)$.*

In order to state the beautiful result of Milnor and Orlik on the characteristic polynomial $\Delta(t)$, we need the following definitions and notations.

To each monic polynomial $(t-\alpha_1)(t-\alpha_2)\dots(t-\alpha_k)$ with $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{C}^*$, the multiplicative group of nonzero complex numbers, assign the divisor

$$\text{divisor}((t-\alpha_1)(t-\alpha_2)\dots(t-\alpha_k)) \;=\; \langle\alpha_1\rangle + \langle\alpha_2\rangle + \dots + \langle\alpha_k\rangle$$

thought of as an element of the integral group ring $\mathbb{Z}\mathbb{C}^*$. Introduce the special notation

$$\Lambda_m \;=\; \text{divisor}(t^m - 1) \;=\; \langle 1\rangle + \langle\xi\rangle + \dots + \left\langle\xi^{m-1}\right\rangle$$

where $\xi = \exp(2\pi i/m)$. It will also be convenient to introduce the idempotent element

$$E_m \;=\; m^{-1}\Lambda_m$$

which belongs to the rational group ring $\mathbb{Q}\mathbb{C}^*$. The ring identity element $\Lambda_1 = E_1 = \langle 1\rangle$ will be written briefly as 1.

Given the integers $a_1, a_2, \dots, a_k$ denote by $[a_1, \dots, a_k]$ their least common multiple and by $(a_1, \dots, a_k)$ their greatest common divisor. Note the multiplication rules

$$\Lambda_a\Lambda_b \;=\; (a,b)\Lambda_{[a,b]} \quad \text{and} \quad E_aE_b \;=\; E_{[a,b]}\,.$$

Theorem 3.13. ([22].) *Let $f(z_0, z_1, \dots, z_n)$ be a weighted homogeneous polynomial of type $(w_0, w_1, \dots, w_n)$ having an isolated critical point at the origin. Express the weights in irreducible form as $w_i = u_i/v_i$, $i = 0, 1, \dots, n$. Then the characteristic polynomial $\Delta(t)$ of the linear transformation $h_* : H_n(F_\theta; \mathbb{C}) \to H_n(F_\theta; \mathbb{C})$ is determined by*

$$\begin{aligned}\text{divisor}(\Delta) \;&=\; (w_0E_{u_0} - 1)(w_1E_{u_1} - 1)\dots(w_nE_{u_n} - 1)\\ &=\; \left(v_0^{-1}\Lambda_{u_0} - 1\right)\left(v_1^{-1}\Lambda_{u_1} - 1\right)\dots\left(v_n^{-1}\Lambda_{u_n} - 1\right).\end{aligned}$$

Note that Theorem 3.13 can be reformulated as follows.

Corollary 3.14. *With the notation as in Theorem* 3.13,

$$\begin{aligned}\text{divisor}(\Delta) \;&=\; \sum(-1)^{n-s+1}w_{i_1}\dots w_{i_s}E_{[u_{i_1},\dots,u_{i_s}]}\\ &=\; \sum(-1)^{n-s+1}\frac{w_{i_1}\dots w_{i_s}}{[u_{i_1},\dots,u_{i_s}]}\Lambda_{[u_{i_1},\dots,u_{i_s}]}\end{aligned}$$

to be summed over all the 2^{n+1} subsets $\{i_1, \dots, i_s\}$ of $\{0, 1, \dots, n\}$.

Lemma 3.15. ([22].) *Let $f(z_0, z_1, \dots, z_n)$ be a weighted homogeneous polynomial of type $(w_0, w_1, \dots, w_n)$ having an isolated critical point at the origin. Suppose that the divisor of the characteristic polynomial $\Delta(t)$ is the divisor $\Delta = a_1\Lambda_1 + \dots + a_s\Lambda_s$, i. e., $\Delta(t) = \prod_{i=1}^{s}(t^i - 1)^{a_i}$. Let $\kappa = a_1 + \dots + a_s$ and $\rho = 2^{a_2}3^{a_3}\dots s^{a_s}$. Then κ and ρ are non–negative integers, and*

$$\begin{aligned}\Delta(1) \;&=\; \rho \quad \text{if} \quad \kappa \;=\; 0\,,\\ \Delta(1) \;&=\; 0 \quad \text{if} \quad \kappa \;>\; 0\,.\end{aligned}$$

It follows that $\Delta(1) \geq 0$.

4. Classification of weighted homogeneous polynomials in four variables with isolated singularity at the origin

Recall that two isolated hypersurface singularities $(V, 0)$, $(W, 0)$ in $\mathbb{C}^{n+1}$ are said to have the same topological type if $\big(\mathbb{C}^{n+1}, V, 0\big)$ is homeomorphic to $\big(\mathbb{C}^{n+1}, W, 0\big)$ (cf. [31]).

Orlik and Wagreich [24] and Arnold [1] showed that if $h(z_0, z_1, z_2)$ is a weighted homogeneous polynomial in $\mathbb{C}^3$ and $V = \big\{z \in \mathbb{C}^3 : h(z) = 0\big\}$ has an isolated singularity at the origin, then V can be deformed into one of the following seven classes of weighted homogeneous singularities below while keeping the differential structure of the link $K_V := S^5 \cap V$ constant. Let $(w_0, w_1, w_2) = (wt(z_0), wt(z_1), wt(z_2))$ be the weight type and μ be the Milnor number.

Class I:
$$\big\{z_0^{a_0} + z_1^{a_1} + z_2^{a_2} = 0\big\},$$
$$(w_0, w_1, w_2) = (a_0, a_1, a_2),$$
$$\mu = (a_0 - 1)(a_1 - 1)(a_2 - 1).$$

Class II:
$$\big\{z_0^{a_0} + z_1^{a_1} + z_1 z_2^{a_2} = 0\big\},$$
$$(w_0, w_1, w_2) = \left(a_0, a_1, \frac{a_1 a_2}{a_1 - 1}\right),$$
$$\mu = (a_0 - 1)(a_1 a_2 - a_1 + 1).$$

Class III:
$$\big\{z_0^{a_0} + z_1^{a_1} z_2 + z_1 z_2^{a_2} = 0\big\},$$
$$(w_0, w_1, w_2) = \left(a_0, \frac{a_1 a_2 - 1}{a_2 - 1}, \frac{a_1 a_2 - 1}{a_1 - 1}\right),$$
$$\mu = (a_0 - 1) a_1 a_2.$$

Class IV:
$$\big\{z_0^{a_0} + z_1^{a_1} z_2 + z_0 z_2^{a_2} = 0\big\},$$
$$(w_0, w_1, w_2) = \left(a_0, \frac{a_0 a_1 a_2}{a_0 a_2 - a_0 + 1}, \frac{a_0 a_2}{a_0 - 1}\right),$$
$$\mu = a_0 a_2 (a_1 - 1) + a_0 - 1.$$

Class V:
$$\big\{z_0^{a_0} z_1 + z_1^{a_1} z_2 + z_0 z_2^{a_2} = 0\big\},$$
$$(w_0, w_1, w_2) = \left(\frac{a_0 a_1 a_2 + 1}{a_1 a_2 - a_2 + 1}, \frac{a_0 a_1 a_2 + 1}{a_0 a_2 - a_0 + 1}, \frac{a_0 a_1 a_2 + 1}{a_0 a_1 - a_1 + 1}\right),$$
$$\mu = a_0 a_1 a_2.$$

Class VI:
$$\big\{z_0^{a_0} + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2} = 0\big\},$$
where $(a_0 - 1)(a_1 b_2 + a_2 b_1) = a_0 a_1 a_2$,
$$(w_0, w_1, w_2) = \left(a_0, \frac{a_0 a_1}{a_0 - 1}, \frac{a_0 a_2}{a_0 - 1}\right),$$
$$\mu = \frac{(a_0 a_1 - a_0 + 1)(a_0 a_2 - a_0 + 1)}{a_0 - 1}.$$

Class VII:
$$\big\{z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2} = 0\big\},$$
where $(a_0 - 1)(a_1 b_2 + a_2 b_1) = a_2 (a_0 a_1 - 1)$,

$$(w_0, w_1, w_2) = \left(\frac{a_0a_1 - 1}{a_1 - 1}, \frac{a_0a_1 - 1}{a_0 - 1}, \frac{a_2(a_0a_1 - 1)}{a_1(a_0 - 1)}\right),$$
$$\mu = \frac{a_0(a_0a_1a_2 - a_0a_1 + a_1 - a_2)}{a_0 - 1}.$$

In [29], we prove that the above deformation of Orlik and Wagreich is actually a topological type constant deformation without changing weights. Therefore any weighted homogeneous singularity has the same topological type of one of the seven classes above.

If $h(z_0, z_1, z_2, z_3)$ is a weighted homogeneous polynomial in $\mathbb{C}^4$ and $V = \{z \in \mathbb{C}^4 : h(z) = 0\}$ has an isolated singularity at the origin, then Kouchnirenko [13] and Orlik and Randell [23] observed that V can be deformed into one of the following nineteen classes of weighted homogeneous singularities below while keeping the differential structure of the link $K_V := S^7 \cap V$ constant.

Type I: $\left\{z_0^{a_0} + z_1^{a_1} + z_2^{a_2} + z_3^{a_3} = 0\right\}$,
$$(w_0, w_1, w_2, w_3) = (a_0, a_1, a_2, a_3),$$
$$\mu = (a_0 - 1)(a_1 - 1)(a_2 - 1)(a_3 - 1).$$

Type II: $\left\{z_0^{a_0} + z_1^{a_1} + z_2^{a_2} + z_2z_3^{a_3} = 0\right\}$,
$$(w_0, w_1, w_2, w_3) = \left(a_0, a_1, a_2, \frac{a_2a_3}{a_2 - 1}\right),$$
$$\mu = (a_0 - 1)(a_1 - 1)[a_2(a_3 - 1) + 1].$$

Type III: $\left\{z_0^{a_0} + z_1^{a_1} + z_2^{a_3}z_3 + z_2z_3^{a_4} = 0\right\}$,
$$(w_0, w_1, w_2, w_3) = \left(a_0, a_1, \frac{a_2a_3 - 1}{a_3 - 1}, \frac{a_2a_3 - 1}{a_2 - 1}\right),$$
$$\mu = (a_0 - 1)(a_1 - 1)a_2a_3.$$

Type IV: $\left\{z_0^{a_0} + z_0z_1^{a_1} + z_2^{a_2} + z_2z_3^{d} = 0\right\}$,
$$(w_0, w_1, w_2, w_3) = \left(a_0, \frac{a_0a_1}{a_0 - 1}, a_2, \frac{a_2a_3}{a_2 - 1}\right),$$
$$\mu = [a_0(a_1 - 1) + 1][a_2(a_3 - 1) + 1].$$

Type V: $\left\{z_0^{a_0}z_1 + z_0z_1^{a_1} + z_2^{a_2} + z_2z_3^{a_3} = 0\right\}$,
$$(w_0, w_1, w_2, w_3) = \left(\frac{a_0a_1 - 1}{a_1 - 1}, \frac{a_0a_1 - 1}{a_0 - 1}, a_2, \frac{a_2a_3}{a_2 - 1}\right),$$
$$\mu = a_0a_1[a_2(a_3 - 1) + 1].$$

Type VI: $\left\{z_0^{a_0}z_1 + z_0z_1^{a_1} + z_2^{a_2}z_3 + z_2z_3^{a_3} = 0\right\}$,
$$(w_0, w_1, w_2, w_3) = \left(\frac{a_0a_1 - 1}{a_1 - 1}, \frac{a_0a_1 - 1}{a_0 - 1}, \frac{a_2a_3 - 1}{a_3 - 1}, \frac{a_2a_3 - 1}{a_2 - 1}\right),$$
$$\mu = a_0a_1a_2a_3.$$

Type VII: $\left\{z_0^{a_0} + z_1^{a_1} + z_1z_2^{a_2} + z_2z_3^{a_3} = 0\right\}$,
$$(w_0, w_1, w_2, w_3) = \left(a_0, a_1, \frac{a_1a_2}{a_1 - 1}, \frac{a_1a_2a_3}{a_1(a_2 - 1) + 1}\right),$$
$$\mu = (a_0 - 1)[a_1a_2(a_3 - 1) + a_1 - 1].$$

Type VIII: $\left\{z_0^{a_0}+z_1^{a_1}+z_1z_2^{a_2}+z_1z_3^{a_3}+z_2^pz_3^q=0,\ \frac{p(a_1-1)}{a_1a_2}+\frac{q(a_1-1)}{a_1a_2}=1\right\}$,

$$(w_0,w_1,w_2,w_3) = \left(a_0, a_1, \frac{a_1a_2}{a_1-1}, \frac{a_1a_3}{a_1-1}\right),$$

$$\mu = \frac{(a_0-1)[a_1(a_2-1)+1][a_1(a_3-1)+1]}{a_1-1}.$$

Type IX: $\left\{z_0^{a_0}+z_1^{a_1}z_3+z_2^{a_2}z_3+z_1z_3^{a_3}+z_1^pz_2^q=0,\ \frac{p(a_3-1)}{a_1a_3-1}+\frac{qa_1(a_3-1)}{a_2(a_1a_3-1)}=1\right\}$,

$$(w_0,w_1,w_2,w_3) = \left(a_0, \frac{a_1a_3-1}{a_3-1}, \frac{a_2(a_1a_3-1)}{a_1(a_3-1)}, \frac{a_1a_3-1}{a_1-1}\right),$$

$$\mu = \frac{(a_0-1)a_3[a_2(a_1a_3-1)-a_1(a_3-1)]}{a_3-1}.$$

Type X: $\left\{z_0^{a_0}+z_1^{a_1}z_2+z_2^{a_2}z_3+z_1z_3^{a_3}=0\right\}$,

$$(w_0,w_1,w_2,w_3) = \left(a_0, \frac{a_1a_2a_3+1}{a_3(a_2-1)+1}, \frac{a_1a_2a_3+1}{a_1(a_3-1)+1}, \frac{a_1a_2a_3+1}{a_2(a_1-1)+1}\right),$$

$$\mu = (a_0-1)a_1a_2a_3.$$

Type XI: $\left\{z_0^{a_0}+z_0z_1^{a_1}+z_1z_2^{a_2}+z_2z_3^{a_3}=0\right\}$,

$$(w_0,w_1,w_2,w_3) = \left(a_0, \frac{a_0a_1}{a_0-1}, \frac{a_0a_1a_2}{a_0(a_1-1)+1}, \frac{a_0a_1a_2a_3}{a_0a_1(a_2-1)+(a_0-1)}\right),$$

$$\mu = a_0a_1a_2(a_3-1)+a_0(a_1-1)+1.$$

Type XII: $\left\{z_0^{a_0}+z_0z_1^{a_1}+z_0z_2^{a_2}+z_1z_3^{a_3}+z_1^pz_2^q=0,\ \frac{p(a_0-1)}{a_0a_1}+\frac{q(a_0-1)}{a_0a_2}=1\right\}$,

$$(w_0,w_1,w_2,w_3) = \left(a_0, \frac{a_0a_1}{a_0-1}, \frac{a_0a_2}{a_0-1}, \frac{a_0a_1a_3}{a_0(a_1-1)+1}\right),$$

$$\mu = \frac{(a_0(a_2-1)+1)(a_0a_1(a_3-1)+a_0-1)}{a_0-1}.$$

Type XIII: $\Big\{z_0^{a_0}+z_0z_1^{a_1}+z_1z_2^{a_2}+z_1z_3^{a_3}+z_2^pz_3^q=0,$

$\frac{p[a_0(a_1-1)+1]}{a_0a_1a_2}+\frac{q[a_0(a_1-1)+1]}{a_0a_1a_3}=1\Big\}$,

$$(w_0,w_1,w_2,w_3) = \left(a_0, \frac{a_0a_1}{a_0-1}, \frac{a_0a_1a_1}{a_0(a_1-1)+1}, \frac{a_0a_1a_3}{a_0(a_1-1)+1}\right),$$

$$\mu = \frac{[a_0a_1(a_2-1)+a_0-1][a_0a_1(a_3-1)+a_0-1]}{a_0(a_1-1)+1}.$$

Type XIV: $\Big\{z_0^{a_0}+z_0z_1^{a_1}+z_0z_2^{a_2}+z_0z_3^{a_3}+z_1^pz_2^q+z_2^rz_3^s=0,$

$\frac{p(a_0-1)}{a_0a_1}+\frac{q(a_0-1)}{a_0a_2}=1$ and $\frac{r(a_0-1)}{a_0a_2}+\frac{s(a_0-1)}{a_0a_3}=1\Big\}$,

$$(w_0,w_1,w_2,w_3) = \left(a_0, \frac{a_0a_1}{a_0-1}, \frac{a_0a_2}{a_0-1}, \frac{a_0a_3}{a_0-1}\right),$$

$$\mu = \frac{[a_0(a_1-1)+1][a_0(a_2-1)+1][a_0(a_3-1)+1]}{(a_0-1)^2}.$$

Type XV: $\left\{z_0^{a_0}z_1 + z_0z_1^{a_1} + z_0z_2^{a_2} + z_2z_3^{a_3} + z_1^pz_2^q = 0,\right.$

$$\left.\frac{p(a_0-1)}{a_0a_1-1} + \frac{qa_1(a_0-1)}{a_2(a_0a_1-1)} = 1\right\},$$

$$(w_0,w_1,w_2,w_3) = \left(\frac{a_0a_1-1}{a_1-1}, \frac{a_0a_1-1}{a_0-1}, \frac{a_2(a_0a_1-1)}{a_1(a_0-1)}, \frac{a_2a_3(a_0a_1-1)}{a_2(a_0a_1-1)-a_1(a_0-1)}\right),$$

$$\mu = \frac{a_0[a_2(a_3-1)(a_0a_1-1)+a_1(a_0-1)]}{a_0-1}.$$

Type XVI: $\left\{z_0^{a_0}z_1 + z_0z_1^{a_1} + z_0z_2^{a_2} + z_0z_3^{a_3} + z_1^pz_2^q + z_2^rz_3^s = 0,\right.$

$$\left.\frac{p(a_0-1)}{a_0a_1-1} + \frac{qa_1(a_0-1)}{a_2(a_0a_1-1)} = 1 \text{ and } \frac{ra_1(a_0-1)}{a_2(a_0a_1-1)} + \frac{sa_1(a_0-1)}{a_3(a_0a_1-1)} = 1\right\},$$

$$(w_0,w_1,w_2,w_3) = \left(\frac{a_0a_1-1}{a_1-1}, \frac{a_0a_1-1}{a_0-1}, \frac{a_2(a_0a_1-1)}{a_1(a_0-1)}, \frac{a_3(a_0a_1-1)}{a_1(a_0-1)}\right),$$

$$\mu = \frac{a_0[a_2(a_0a_1-1)-a_1(a_0-1)][a_3(a_0a_1-1)-a_1(a_0-1)]}{a_1(a_0-1)^2}.$$

Type XVII: $\left\{z_0^{a_0}z_1 + z_0z_1^{a_1} + z_1z_2^{a_2} + z_0z_3^{a_3} + z_1^pz_3^q + z_0^rz_2^s = 0,\right.$

$$\left.\frac{p(a_0-1)}{a_0a_1-1} + \frac{qa_1(a_0-1)}{a_3(a_0a_1-1)} = 1 \text{ and } \frac{r(a_1-1)}{a_0a_1-1} + \frac{sa_0(a_1-1)}{a_2(a_0a_1-1)} = 1\right\},$$

$$(w_0,w_1,w_2,w_3) = \left(\frac{a_0a_1-1}{a_1-1}, \frac{a_0a_1-1}{a_0-1}, \frac{a_2(a_0a_1-1)}{a_0(a_1-1)}, \frac{a_3(a_0a_1-1)}{a_2(a_0-1)}\right),$$

$$\mu = \frac{[a_2(a_0a_1-1)-a_0(a_1-1)][a_3(a_0a_1-1)-a_1(a_0-1)]}{(a_0-1)(a_1-1)}.$$

Type XVIII: $\left\{z_0^{a_0}z_2 + z_0z_1^{a_1} + z_1z_2^{a_2} + z_1z_3^{a_3} + z_2^pz_3^q = 0,\right.$

$$\left.\frac{p[a_0(a_1-1)+1]}{a_0a_1a_2+1} + \frac{qa_2[a_0(a_1-1)+1]}{a_3(a_0a_1a_2+1)} = 1\right\},$$

$$(w_0,w_1,w_2,w_3) = \left(\frac{a_0a_1a_2+1}{a_1(a_2-1)+1}, \frac{a_0a_1a_2+1}{a_2(a_0-1)+1}, \frac{a_0a_1a_2+1}{a_0(a_1-1)+1}, \frac{a_3(a_0a_1a_2+1)}{a_2[a_0(a_1-1)+1]}\right),$$

$$\mu = \frac{a_0a_1[a_0a_1a_2(a_3-1)+a_2(a_0-1)+a_3]}{a_0(a_1-1)+1}.$$

Type XIX: $\left\{z_0^{a_0}z_3 + z_0z_1^{a_1} + z_2^{a_2}z_3 + z_2z_3^{a_3} = 0\right\},$

$$(w_0,w_1,w_2,w_3) = \left(\frac{a_0a_1a_2a_3-1}{a_1[a_3(a_2-1)+1]-1}, \frac{a_0a_1a_2a_3-1}{a_3[a_2(a_0-1)+1]-1}, \frac{a_0a_1a_2a_3-1}{a_0[a_1(a_3-1)+1]-1}, \frac{a_0a_1a_2a_3-1}{a_2[a_0(a_1-1)+1]-1}\right),$$

$$\mu = a_0a_1a_2a_3.$$

Theorem 4.1. *Suppose* $h(z_0, z_1, z_2, z_3)$ *is a polynomial and* $V_h = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 : h(z_0, z_1, z_2, z_3) = 0\}$ *has an isolated singularity at* 0. *Then* $h(z_0, z_1, z_2, z_3) = f(z_0, z_1, z_2, z_3) + g(z_0, z_1, z_2, z_3)$ *where* f *is one of the* 19 *classes above with only isolated singularity at* 0 *and* f *and* g *have no monomial in common. If* h *is weighted homogeneous of type* (w_0, w_1, w_2, w_3), *then so are* f *and* g. *Let* $V_f = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 : f(z_0, z_1, z_2, z_3) = 0\}$ *and let*

$$K_f = V_f \cap S^7 , \quad K_h = V_h \cap S^7 .$$

Then K_f *is equivariantly diffeomorphic to* K_h.

Proof. If none of the monomials in $\{z_0^{a_0}, z_0^{a_0} z_1, z_0^{a_0} z_2, z_0^{a_0} z_3\}$ appears in $h(z_0, z_1, z_2, z_3)$, then $\frac{\partial h}{\partial z_j}(z_0, 0, 0, 0) = 0$, $0 \le j \le 3$. This contradicts the fact that h has an isolated singularity at 0. Therefore, one of the monomials in $\{z_0^{a_0}, z_0^{a_0} z_1, z_0^{a_0} z_2, z_0^{a_0} z_3\}$ appears in h. Similarly one of the monomials in each of the following sets appears in h : $\{z_0 z_1^{a_1}, z_1^{a_1}, z_1^{a_1} z_2, z_1^{a_1} z_3\}$, $\{z_0 z_2^{a_2}, z_1 z_2^{a_2}, z_2^{a_2}, z_2^{a_2} z_3\}$, $\{z_0 z_3^{a_3}, z_1 z_3^{a_3}, z_2 z_3^{a_3}, z_3^{a_3}\}$. Taking a monomial from each of the 4 sets above, we get 256 polynomials. One can check that these 256 polynomials are equivalent to one of the 19 classes above up to permutation of coordinates. Notice that in Type VIII, for example, the monomial $z_2^p z_3^q$ is added to make sure that f has an isolated singularity at 0. Obviously if h is weighted homogeneous of type (w_0, w_1, w_2, w_3), then so are f and g.

The proof of Theorem 3.1.4 in [24] shows that K_f is equivariantly diffeomorphic to K_h. □

We shall use the theory developed in [29] and [30] to show that $(V_f, 0)$ and $(V_h, 0)$ have the same topological type.

Definition 4.2. Given a real manifold B of dimension m, and a family $\{(M_t, N_t) : t \in B_t,\ N_t$ is a closed submanifold of a compact differentiable manifold $M_t\}$, we say that (M_t, N_t) *depends* C^∞ *on* t and that $\{(M_t, N_t) : t \in B\}$ is a C^∞ *family of compact manifolds with submanifolds,* if there is a C^∞ mainfold $\mathcal{M}$, a closed submanifold $\mathcal{N}$ and a C^∞ map w from $\mathcal{M}$ onto B such that $\overline{w} := w|\mathcal{N}$ is also a C^∞ map from $\mathcal{N}$ onto B satisfying the following conditions

(i) $M_t = w^{-1}(t) \supseteq N_t = \overline{w}^{-1}(t)$

(ii) The rank of the Jacobian of w (respectively $\overline{w}$) is equal to m at every point of $\mathcal{M}$ (respectively $\mathcal{N}$).

Theorem 4.3. ([29].) *Let* $((\mathcal{M}, \mathcal{N}), (w, \overline{w}))$ *be a* C^∞ *family of compact manifolds with submanifolds, with* B *connected. Then* $(M_t, N_t) = (w^{-1}(t), \overline{w}^{-1}(t))$ *is diffeomorphic to* (M_{t_0}, N_{t_0}) *for any* $t, t_0 \in B$.

Now we fix weights $(w_0, \dots, w_n)$ with $w_i \ge 2$. Suppose that there is a weighted homogeneous polynomial $f(z_0, \dots, z_n)$ with the weights $(w_0, \dots, w_n)$ such that f has an isolated singularity at the origin. Let Δ be the intersection of the plane $\sum_{i=0}^n \frac{x_i}{w_i} = 1$ with the first quadrant of $\mathbb{R}^{n+1}$. Let $\mathbb{C}[\Delta] = \{f \in \mathbb{C}[z_0, \dots, z_n] : \text{supp } f \subset \Delta\}$ where supp $f = \{(d_0, \dots, d_n) \in \mathbb{R}^{n+1} : z_0^{d_0} z_1^{d_1} \cdots z_n^{d_n}$ occurs in $f\}$. Let N be the number of the integer points which are in Δ. There is a canonical correspondence between $\mathbb{C}[\Delta]$ and $\mathbb{C}^N$. Thus we may introduce a Zariski topology on $\mathbb{C}[\Delta]$.

Theorem 4.4. *Notation as above. Let*

$$U = \{f \in \mathbb{C}[\Delta] : f \text{ has an isolated singularity at the origin}\}.$$

Then U is a nonempty Zariski open set of $\mathbb{C}[\Delta]$.

The proof of the previous theorem as well as the following theorem is the same as those of Theorem 3.4 and Theorem 3.5 in [29] respectively.

Theorem 4.5. *Suppose that $f(z_0, \dots, z_n)$ and $g(z_0, \dots, z_n)$ are weighted homogeneous polynomials with the same weights $(w_0, w_1, \dots, w_n)$. If the variety V of f and the variety W of g have an isolated singularity at the origin, then $(\mathbb{C}^{n+1}, V, 0)$ is homeomorphically equivalent to $(\mathbb{C}^{n+1}, W, 0)$.*

Corollary 4.6. *Suppose that $h(z_0, z_1, z_2, z_3)$ is a weighted homogeneous polynomial with weights (w_0, w_1, w_2, w_3) and the variety $h^{-1}(0)$ has an isolated singularity at the origin. Then $h = f + g$ where f and g have no monomials in common, f is one of the nineteen classes above and f and g are weighted homogeneous of type (w_0, w_1, w_2, w_3). Moreover $h^{-1}(0)$ and $f^{-1}(0)$ have the same topological type.*

5. Algebraic classification of 5–dimensional rational CR manifolds in $\mathbb{C}^4$ with transversal holomorphic S^1–action

Definition 5.1. Let $(V, 0)$ be an n–dimensional variety with isolated singularity at 0. The geometric genus $p_g(V, 0)$ of the singularity is defined to be $\dim H^{n-1}(M, \mathcal{O})$ where M is a resolution of the singularity $(V, 0)$. $(V, 0)$ is called a *rational singularity* if $p_g(V, 0) = 0$.

Proposition 5.2. ([24].) *Suppose $V \subseteq \mathbb{C}^{n+1}$ is an irreducible analytic variety, σ is a $\mathbb{C}^*$–action leaving V invariant,*

$$\sigma(t, (z_0, \dots, z_n)) = (t^{q_0} z_0, \dots, t^{q_n} z_n)$$

and $q_i > 0$ for all i. Then V is algebraic and the ideal of polynomials in $\mathbb{C}[z_0, \dots, z_n]$ vanishing on V is generated by weighted homogeneous polynomials.

Let $f(z_0, \dots, z_n)$ be a germ of an analytic function at the origin such that $f(0) = 0$. Suppose that f has an isolated critical point at the origin. f can be developed in a convergent Taylor series $\sum_\lambda a_\lambda z^\lambda$ where $z^\lambda = z_0^{\lambda_0} \dots z_n^{\lambda_n}$. Recall that the Newton boundary $\Gamma(f)$ of f is the union of compact faces of $\Gamma_+(f)$ where $\Gamma_+(f)$ is the convex hull of the union of the subsets $\{\lambda + (\mathbb{R}^+)^{n+1}\}$ for λ such that $a_\lambda \neq 0$. Finally, let $\Gamma_-(f)$, the Newton polyhedron of f, be the cone over $\Gamma(f)$ with vertex at 0. For any closed face Δ of $\Gamma(f)$, we associate the polynomial $f_\Delta(z) = \sum_{\lambda \in \Delta} a_\lambda z^\lambda$. We say that f is nondegenerate if f_Δ has no critical point in $(\mathbb{C}^*)^{n+1}$ for any $\Delta \in \Gamma(f)$ where $\mathbb{C}^* = \mathbb{C} - \{0\}$. The following theorem was proved by Merle and Teissier.

Theorem 5.3. ([20].) *Let $(V, 0)$ be an isolated hypersurface singularity defined by a nondegenerate holomorphic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. Then the geometric* *genus* $p_g(V,0) = \#\{p \in \mathbb{Z}^{n+1} \cap \Gamma_-(f) : p \text{ is positive}\}$.

Now we are ready to give the classification of 3–dimensional isolated rational hypersurface singularities with $\mathbb{C}^*$–action.

Theorem 5.4. *Let* $(V,0)$ *be a* 3*–dimensional isolated rational hypersurface singularity with* $\mathbb{C}^*$*–action. Then* $(V,0)$ *is defined by a weighted homogeneous polynomial of one of the form which is listed in the second author's homepage.*
(*http://www.math.ncku.edu.tw/english/yyu/alg.htm*)

Proof of Theorem 5.4. In view of Corollary 4.6 and Theorem 5.3, it is clear that an isolated hypersurface rational singularity with $\mathbb{C}^*$–action is defined by one of the 19 types in Section 4 with $p_g = 0$. The equations of the Γ_- hyperplanes of these 19 types are respectively given by $\alpha(x,y,z,w) = 1$ where $\alpha(x,y,z,w) = \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d}$, $a = wt(z_0)$, $b = wt(z_1)$, $c = wt(z_2)$, $d = wt(z_3)$.

In order to find all hypersurfaces among these 19 types with $p_g = 0$, we only need to find all solutions of $\alpha(1,1,1,1) > 1$ among these 19 types. We have used the MAPLE program [6] to perform the computations. □

Now we are ready to prove the main theorem in this section which gives the algebraic classification of 5–dimensional rational CR manifolds in $\mathbb{C}^4$ with transversal holomorphic S^1–action.

Theorem 5.5. *Let* X *be a* 5*–dimensional rational strongly pseudoconvex CR manifold in* $\mathbb{C}^4$ *with transversal holomorphic* S^1*–action. Then* X *is algebraically equivalent to the intersection of* $S^7 = \{(x,y,z,w) \in \mathbb{C}^4 : |x|^2 + |y|^2 + |z|^2 + |w|^2 = 1\}$ *with one of the rational hypersurface singularities listed in Theorem* 5.4.

Proof. In view of Corollary 2.6, after a change of coordinates, we can assume that V is contained in an affine algebraic hypersurface $V \subseteq \mathbb{C}^4$. The hypersurface V has at most one singular point which can be assumed to be the origin. V has a $\mathbb{C}^*$–action and the embedding $X \subset V$ is S^1–equivariant. By Proposition 2.8 and a theorem of Yau [32], we know that $(V,0)$ is a rational singularity. Therefore V is defined by one of the 19 types of weighted homogeneous polynomials listed in Theorem 5.4. □

6. Algebraic classification of rational CR structures on topological 5–sphere with transversal holomorphic S^1–action in $\mathbb{C}^4$

The purpose of this section is to give a proof of our main theorem stated in Section 1. In order to give an algebraic classification of rational CR structures on the topological 5–sphere with transversal holomorphic S^1–action in $\mathbb{C}^4$, it suffices to find out all the CR manifolds listed in Theorem 5.5 which are topologically a 5–sphere. By Theorem 3.9 and Lemma 3.15, we need to find out all the CR manifolds listed in Theorem 5.5 with $\Delta(1) = 1$. We shall use Theorem 3.13 and Lemma 3.15 to do this computation.

We have used the REDUCE [10] program to perform the computation of $\Delta(1)$ and identify those CR manifolds listed in Theorem 5.5 with $\Delta(1) = 1$. In what follows, we shall provide the proofs for those infinite cases which cannot be handled by computer. Please notice that we are using the notations in Section 3 and in the statement of Theorem 5.5. In order to save space, we shall only treat some cases in Type I in Theorem 5.5. The rest of the proof is similar.

(I) 1. $(a,b,c,d) = (2,2,x,y) =$ weight type, $x \geq 2$, $y \geq x$.

$$\Delta = (\Lambda_2 - 1)^2(\Lambda_x - 1)(\Lambda_y - 1) = (x,y)\Lambda_{[x,y]} - \Lambda_x - \Lambda_y + 1\,.$$

By Lemma 3.15, if X is a topological 5–sphere, then

$$\begin{cases} \kappa = (x,y) - 1\,, \\ \Delta(1) = [x,y]^{(x,y)}x^{-1}y^{-1}\,. \end{cases}$$

It follows that $\Delta(1) = 1$ if and only if $(x,y) = 1$.

(I) 2. $(a,b,c,d) = (2,3,3,x) =$ weight type, $x \geq 3$.

$$\begin{aligned} \Delta &= (\Lambda_2 - 1)(\Lambda_3 - 1)^2(\Lambda_x - 1) \\ &= (6,x)\Lambda_{[6,x]} + (2,x)\Lambda_{[2,x]} - (3,x)\Lambda_{[3,x]} - \Lambda_x - \Lambda_6 - \Lambda_2 + \Lambda_3 + 1\,. \end{aligned}$$

By Lemma 3.15,

$$\begin{cases} \kappa = (6,x) + (2,x) - (3,x) - 3 + 2\,, \\ \Delta(1) = [6,x]^{(6,x)}[2,x]^{(2,x)}[3,x]^{-(3,x)}x^{-1}6^{-1}2^{-1}3\,. \end{cases}$$

Case 1. $2 \not| x$ and $3 \not| x$.

Then $\kappa = 0$ and $\Delta(1) = 1$ and X is a topological 5–sphere.

Case 2. $2 \not| x$ and $3|x$.

$$\Delta(1) = (2x)^3(2x) \cdot x^{-3} \cdot x^{-1} \cdot 4^{-1} = 4 \neq 1\,.$$

X is *not* a topological 5–sphere.

Case 3. $2|x$ and $3 \not| x$.

$$\kappa = 2 + 2 - 1 - 3 + 2 = 2 \neq 0\,.$$

X is *not* a topological 5–sphere.

Case 4. $2|x$ and $3|x$.

$$\kappa = 6 + 2 - 3 - 3 + 2 = 4 \neq 0\,.$$

X is *not* a topological 5–sphere.

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