Canad. Math. Bull. Vol. **46** (2), 2003 pp. 229–241

Counting the Number of Integral Points in General n-Dimensional Tetrahedra and Bernoulli Polynomials

Ke-Pao Lin and Stephen S.-T. Yau

Abstract. Recently there has been tremendous interest in counting the number of integral points in n-dimensional tetrahedra with non-integral vertices due to its applications in primality testing and factoring in number theory and in singularities theory. The purpose of this note is to formulate a conjecture on sharp upper estimate of the number of integral points in n-dimensional tetrahedra with non-integral vertices. We show that this conjecture is true for low dimensional cases as well as in the case of homogeneous n-dimensional tetrahedra. We also show that the Bernoulli polynomials play a role in this counting.

1 Introduction

Let $\Delta(a_1, \dots, a_n)$ be an n-dimensional tetrahedron with nonintegral vertices described by

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \le 1, \quad x_1 \ge 0, \dots, x_n \ge 0 \tag{1.1}$$

where $a_1 \ge a_2 \ge \cdots \ge a_n$ are positive real numbers. Let

$$Q_{(a_1,\dots,a_n)} = \#\Big\{ (x_1, \dots, x_n) \in (\mathbb{Z}_+ \cup \{0\})^n : \sum_{i=1}^{n} \frac{x_i}{a_i} \le 1 \Big\} \tag{1.2}$$

$$P_{(a_1,\dots,a_n)} = \#\Big\{ (x_1, \dots, x_n) \in \mathbb{Z}_+^n : \sum_{i=1}^{n} \frac{x_i}{a_i} \le 1 \Big\}. \tag{1.3}$$

Let $b_i = a_i(1 - \sum_{j=1}^{n} \frac{1}{a_j})$, $1 \le i \le n$. Then

$$P_{(a_1,\dots,a_n)} = Q_{(b_1,\dots,b_n)}. \tag{1.4}$$

In number theory, people are interested in sharp estimates of $Q_{(a_1,\dots,a_n)}$ for application in primality testing and factoring. Given a set $\mathcal{P}$ of primes $p_1 < p_2 < \cdots < p_n \le y$, number theorists want to count the number of integers $m \le x$ where

Received by the editors September 5, 2001; revised October 1, 2002.
The second author was supported in part by NSA. Ze-Jiang Professor of East China Normal University.
AMS subject classification: 11B75, 11H06, 11P21, 11Y99.

$m = p_1^{\ell_1} p_2^{\ell_2} \cdots p_n^{\ell_n}$, $x = y^u$ for all $u \geq 2$. This is equivalent to counting the number of $(\ell_1, \ldots, \ell_n) \in (\mathbb{Z}_+ \cup \{0\})^n$ such that $\ell_1 \log p_1 + \cdots + \ell_n \log p_n \leq \log x$, which is also equivalent to counting the number of $(\ell_1, \ldots, \ell_n) \in (\mathbb{Z}_+ \cup \{0\})^n$ such that

$$\frac{\ell_1}{a_1} + \frac{\ell_2}{a_2} + \cdots + \frac{\ell_n}{a_n} \leq 1 \quad \text{where } a_i = \frac{\log x}{\log p_i} = \frac{u \log y}{\log p_i}.$$

This is of course the problem of computing $Q_{(a_1,\ldots,a_n)}$.

Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a complex analytic function with isolated critical point at the origin. Let $V = \{z \in \mathbb{C}^n : f(z) = 0\}$. The geometric genus p_g of the singularity $(V, 0)$ is defined to be $\dim \Gamma(V - \{0\}, \Omega^{n-1})/L^2(V - \{0\}, \Omega^{n-1})$, where Ω^{n-1} is the sheaf of germs of holomorphic $(n-1)$-forms on $V - \{0\}$. It is well known that geometric genus is an important numerical invariant which measures the complexity of the singularity $(V, 0)$.

Let $f(z_1, \ldots, z_n) = \Sigma a_\lambda z^\lambda$, where $z^\lambda = z_1^{\lambda_1} \cdots z_n^{\lambda_n}$, be the power series expansion of f. The Newton boundary $\Gamma(f)$ is the union of the compact faces of $\Gamma_+(f)$ where $\Gamma_+(f)$ is the convex hull of the union of the subsets $\{\lambda + (\mathbb{R}_+)^n\}$ for λ such that $a_\lambda \neq 0$. The Newton polyhedron $\Gamma_-(f)$ of f is the cone over $\Gamma(f)$ with cone point at 0. For each closed face $\Delta \subseteq \Gamma(f)$, we define $f_\Delta(z) = \Sigma a_\lambda z^\lambda$, $\lambda \in \Delta$. f is nondegenerate if f_Δ has no critical point in $(\mathbb{C}^*)^n$ for any $\Delta \in \Gamma(f)$ where $\mathbb{C}^* = \mathbb{C} - \{0\}$.

Theorem 1.1 **(Merle-Teissier)** *Let $(V, 0)$ be an isolated hypersurface singularity defined by a nondegenerate holomorphic function $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. Then the geometric genus $p_g = \#\{P \in \mathbb{Z}_+^n \cap \Gamma_-(f)\}$.*

We say that $f(z_1, \ldots, z_n)$ is weighted homogeneous of type $(w_1, \ldots, w_n)$, where $w_1, \ldots, w_n$ are fixed positive rational numbers, if f can be expressed as a linear combination of monomials $z_1^{i_1} \cdots z_n^{i_n}$ for which $i_1/w_1 + \cdots + i_n/w_n = 1$. Therefore for an isolated singularity defined by a weighted homogeneous polynomial of type $(w_1, \ldots, w_n)$, computing the geometric genus is equivalent to computing the number $P_{(w_1,\ldots,w_n)}$, *i.e.*, the number of positive integral points in n-dimensional tetrahedra $\Delta_{(w_1,\ldots,w_n)}$.

Computation of $Q_{(a_1,\ldots,a_n)}$ has received attention by a lot of distinguished mathematicians. Hardy and Littlewood wrote a series of papers for $n = 2$ [Ha-Li1] [Ha-Li2] [Ha-Li3] [Ha-Li4]. D. Spencer [Sp1] [Sp2] followed up the efforts of Hardy and Littlewood and wrote two papers on this subject. In recent years, there are tremendous activities in finding the exact formula for $Q_{(a_1,\ldots,a_n)}$ or $P_{(a_1,\ldots,a_n)}$ for $a_1, \ldots, a_n$ integers, see [Mo1], [Mor] [Po],[Ca-Sh], [Br-Ve2], [Di-Ro], [Ka-Kh]. The exact formula is complicated. It involves the generalized Dedekind sum. It is difficult to tell how large $P_{(a_1,\ldots,a_n)}$ is from the exact formula. Therefore one would like to get a sharp upper estimate of $P_{(a_1,\ldots,a_n)}$ in terms of a polynomial in $a_1, \ldots, a_n$. Such a sharp upper polynomial estimate of $P_{(a_1,\ldots,a_n)}$ is important because it would have application in the following Durfee Conjecture [Du].

***Durfee Conjecture* (1978)** *Let $(V, 0)$ be an isolated hypersurface singularity defined by a holomorphic function $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. Let*

$$\mu = \dim \mathbb{C}\{z_1, z_2, \ldots, z_n\}/(f_{z_1}, f_{z_2}, \ldots, f_{z_n})$$

be the Milnor number of the singularity. Then $n!\, p_g \le \mu$ where p_g is the geometric genus of $(V, 0)$.

If $f(z)$ is a weighted homogeneous polynomial of type $(w_1, \ldots, w_n)$, then the Milnor number μ is given by $\mu = (w_1 - 1)(w_2 - 1)\cdots(w_n - 1)$. Therefore Durfee Conjecture is a special case of the following conjecture.

Conjecture 1.1 *Let $a_1, \ldots, a_n$ be positive real numbers greater than or equal to two. Then*

$$n!\, P_{(a_1,\ldots,a_n)} \le (a_1 - 1)(a_2 - 1)\cdots(a_n - 1). \tag{1.5}$$

The estimate in the above conjecture is sharper than the following polynomial estimate (1.7) provided by number theorist. Attach a unit cube to the right and above each lattice point of $\Delta(a_1, \ldots, a_n)$. Then

$$\begin{aligned} Q_{(a_1,\ldots,a_n)} &= \Sigma \text{ volume of the unit cube attached to each lattice point} \\ &\le \text{ volume of } \left\{ (x_1, \ldots, x_n) \in \mathbf{R}^n_+ : \sum_{i=1}^{n} \frac{x_i - 1}{a_i} \le 1 \right\} \\ &= \frac{1}{n!} \left(\prod_{i=1}^{n} a_i \right) \left(1 + \sum_{j=1}^{n} \frac{1}{a_j} \right)^n . \end{aligned} \tag{1.6}$$

Hence by (1.4) and (1.6), we have

$$\begin{aligned} P_{(a_1,\ldots,a_n)} &= Q_{(b_1,\ldots,b_n)} \\ &\le \frac{1}{n!} \left(\prod_{i=1}^{n} b_i \right) \left(1 + \sum_{j=1}^{n} \frac{1}{b_j} \right)^n \\ &= \frac{1}{n!} \prod_{i=1}^{n} a_i . \end{aligned} \tag{1.7}$$

The estimate of $P_{(a_1,\ldots,a_n)}$ given by (1.5) is nice. However it is not sharp enough to provide a solution of the following thirty year old problem in singularities theory.

Problem Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a complex analytic function with isolated critical point at the origin. Find an intrinsic characterization for f to be a homogeneous polynomial.

In order to solve the above problem, we need a sharp upper polynomial estimate of $P_{(a_1,\ldots,a_n)}$ such that equality holds if and only $a_1 = a_2 = \cdots = a_n =$ integer. The intrinsic characterization of homogeneous polynomial problem was solved for $n = 3$ by Xu and Yau [Xu-Ya1] and for $n = 4$ by Lin and Yau [Li-Ya1]. The purpose of this paper is to formulate a conjectural sharp polynomial upper estimatefor $P_{(a_1,\ldots,a_n)}$ such that equality holds if and only if $a_1 = a_2 = \cdots = a_n =$ integer. In Section 2, we show that our conjecture is true for $n = 3$, 4 and 5. In Section 3, we give an exact formula for $P_{(a_1,\ldots,a_n)}$ for homogeneous n-dimensional tetrahedra (*i.e.*, $a_1 = \cdots = a_n$). We show that our conjecture is also true in this case. In Section 4 we discuss the possible role of Bernoulli polynomials in this problem.

2 Sharp Polynomial Upper Estimate

Before we formulate our sharp polynomial upper estimate conjecture, it is convenient to introduce some notations. Let $a, a_1, \ldots, a_n$ be positive real numbers greater than or equal to $n-1$. We shall denote

$$S_k^{n-1} = \sum_{1\le i_1<i_2<\cdots<i_k\le n-1} i_1 i_2 \cdots i_k, \quad S_0^{n-1} = 1, \quad S_{n-1}^{n-1} = 1\cdot 2\cdots(n-1) \tag{2.1}$$

where $i_1, i_2, \ldots, i_k$ are integers. Then

$$\begin{aligned} &a(a-1)(a-2)\cdots\big(a-(n-1)\big) \\ &= a^n - \Big(\sum_{i_1=1}^{n-1} i_1\Big) a^{n-1} + (-1)^2 \sum_{1\le i_1<i_2\le n-1} i_1 i_2 a^{n-2} \\ &\quad + \cdots + (-1)^k \sum_{1\le i_1<\cdots<i_k\le n-1} i_1 i_2 \cdots i_k a^{n-k} + \cdots + (-1)^{n-1}\Big(\prod_{i=1}^{n-1} i\Big) a \\ &= a^n + (-1)S_1^{n-1}a^{n-1} + (-1)^2 S_2^{n-1} a^{n-2} + \cdots + (-1)^k S_k^{n-1} a^{n-k} \\ &\quad + \cdots + (-1)^{n-1} S_{n-1}^{n-1} a. \end{aligned} \tag{2.2}$$

We shall denote

$$A_{n-k}^n = \Big(\prod_{i=1}^n a_i\Big) \sum_{1\le i_1<i_2<\cdots<i_k\le n} \frac{1}{a_{i_1} a_{i_2} \cdots a_{i_k}} \tag{2.3}$$

$$A_n^n = \prod_{i=1}^n a_i, \quad A_0^n = 1. \tag{2.4}$$

Observe that A_{n-k}^n is a polynomial in $a_1, \ldots, a_n$ of degree $n-k$.

The following is our conjecture on sharp polynomial estimate for $P_{(a_1,\ldots,a_n)}$.

Main Conjecture *Let* $P_{(a_1,\dots,a_n)} = \#\{(x_1,\dots,x_n) \in \mathbb{Z}_+^n : \frac{x_1}{a_1} + \cdots + \frac{x_n}{a_n} \le 1\}$, *where* $a_1 \ge a_2 \ge \cdots \ge a_n \ge n-1$, $n \ge 3$. *Then*

(2.5)

$$
\begin{aligned}
n!\, P_{(a_1,\dots,a_n)} \le{} & \prod_{i=1}^{n} a_i + (-1)\frac{S_1^{n-1}}{n}\Big(\prod_{i=1}^{n} a_i\Big)\Big(\sum_{k=1}^{n}\frac{1}{a_k}\Big) \\
& + (-1)^2\frac{S_2^{n-1}}{\binom{n-1}{1}}\Big(\prod_{i=1}^{n-1} a_i\Big)\Big(\sum_{k=1}^{n-1}\frac{1}{a_k}\Big) \\
& + (-1)^3\frac{S_3^{n-1}}{\binom{n-1}{2}}\Big(\prod_{i=1}^{n-1} a_i\Big)\Big(\sum_{1\le i_1<i_2\le n-1}\frac{1}{a_{i_1}a_{i_2}}\Big) \\
& + (-1)^4\frac{S_4^{n-1}}{\binom{n-1}{3}}\Big(\prod_{i=1}^{n-1} a_i\Big)\Big(\sum_{1\le i_1<i_2<i_3\le n-1}\frac{1}{a_{i_1}a_{i_2}a_{i_3}}\Big) \\
& + \cdots + (-1)^{k+1}\frac{S_{k+1}^{n-1}}{\binom{n-1}{k}}\Big(\prod_{i=1}^{n-1} a_i\Big)\Big(\sum_{1\le i_1<\cdots<i_k\le n-1}\frac{1}{a_{i_1}a_{i_2}\cdots a_{i_k}}\Big) \\
& + \cdots + (-1)^{n-1}\frac{S_{n-1}^{n-1}}{\binom{n-1}{n-2}}(a_1 + a_2 + \cdots + a_{n-1}) \\
={} & A_n^n + (-1)\frac{S_1^{n-1}}{n}A_{n-1}^n + (-1)^2\frac{S_2^{n-1}}{\binom{n-1}{1}}A_{n-2}^{n-1} + (-1)^3\frac{S_3^{n-1}}{\binom{n-1}{2}}A_{n-3}^{n-1} \\
& + (-1)^4\frac{S_4^{n-1}}{\binom{n-1}{3}}A_{n-4}^{n-1} + \cdots + (-1)^{k+1}\frac{S_{k+1}^{n-1}}{\binom{n-1}{k}}A_{n-k-1}^{n-1} \\
& + \cdots + (-1)^{n-1}\frac{S_{n-1}^{n-1}}{\binom{n-1}{n-2}}A_1^{n-1},
\end{aligned}
$$

and the equality holds if and only if $a_1 = a_2 = \cdots = a_n =$ *integer.*

For $n = 3$, the Main Conjecture asserts that for $a_1 \ge a_2 \ge a_3 \ge 2$,

$$
\begin{aligned}
3!\, P_{(a_1,a_2,a_3)} &\le A_3^3 - \frac{S_1^2}{3}A_2^3 + \frac{S_2^2}{\binom{2}{1}}A_1^2 \\
&= a_1a_2a_3 - \frac{1+2}{3}a_1a_2a_3\Big(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\Big) + \frac{2}{2}a_1a_2\Big(\frac{1}{a_1}+\frac{1}{a_2}\Big) \\
&= a_1a_2a_3 - (a_1a_2 + a_1a_3 + a_2a_3) + (a_1 + a_2)
\end{aligned}
$$

with equality if and only if $a_1 = a_2 = a_3 =$ integer. This is the main result proved by Xu and Yau in their paper [Xu-Ya1].

For $n = 4$, the Main Conjecture asserts that for $a_1 \geq a_2 \geq a_3 \geq a_4 \geq 3$,

$$
\begin{aligned}
4!\, P_{(a_1,a_2,a_3,a_4)} &\leq A_4^4 + (-1)\frac{S_1^3}{4}A_3^4 + (-1)^2\frac{S_2^3}{\binom{3}{1}}A_2^3 + (-1)^3\frac{S_3^3}{\binom{3}{2}}A_1^3 \\
&= a_1a_2a_3a_4 - \frac{1+2+3}{4}a_1a_2a_3a_4\Big(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}\Big) \\
&\quad + \frac{1\cdot 2 + 1\cdot 3 + 2\cdot 3}{3}a_1a_2a_3\Big(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\Big) \\
&\quad - \frac{1\cdot 2\cdot 3}{3}a_1a_2a_3\Big(\frac{1}{a_1a_2}+\frac{1}{a_1a_3}+\frac{1}{a_2a_3}\Big) \\
&= a_1a_2a_3a_4 - \frac{3}{2}(a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4) \\
&\quad + \frac{11}{3}(a_1a_2 + a_1a_3 + a_2a_3) - 2(a_1 + a_2 + a_3)
\end{aligned}
$$

with equality if and only if $a_1 = a_2 = a_3 = a_4 =$ integer. Xu and Yau [Xu-Ya3] proved that if $a_1 \geq a_2 \geq a_3 \geq a_4 \geq 2$ and $P_{(a_1,a_2,a_3,a_4)} > 0$, then the above result is true. Lin and Yau [Li-Ya1] proved that the condition $P_{(a_1,a_2,a_3,a_4)} > 0$ can be removed if we assume $a_1 \geq a_2 \geq a_3 \geq a_4 \geq 3$. Therefore our Main Conjecture is true for $n = 4$.

For $n = 5$, the Main Conjecture asserts that for $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq 4$.

$$
\begin{aligned}
&5!\, P_{(a_1,a_2,a_3,a_4,a_5)} \\
&\leq A_5^5 + (-1)\frac{S_1^4}{5}A_4^5 + \frac{S_2^4}{\binom{4}{1}}A_3^4 - \frac{S_3^4}{\binom{4}{2}}A_2^4 + \frac{S_4^4}{\binom{4}{3}}A_1^4 \\
&= a_1a_2a_3a_4a_5 - \frac{1+2+3+4}{5}a_1a_2a_3a_4a_5\Big(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}+\frac{1}{a_5}\Big) \\
&\quad + \frac{(1\cdot 2 + 1\cdot 3 + 1\cdot 4 + 2\cdot 3 + 2\cdot 4 + 3\cdot 4)}{4}a_1a_2a_3a_4\Big(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}\Big) \\
&\quad - \frac{(1\cdot 2\cdot 3 + 1\cdot 2\cdot 4 + 1\cdot 3\cdot 4 + 2\cdot 3\cdot 4)}{\binom{4}{2}}a_1a_2a_3a_4 \\
&\qquad \cdot \Big(\frac{1}{a_1a_2}+\frac{1}{a_1a_3}+\frac{1}{a_1a_4}+\frac{1}{a_2a_3}+\frac{1}{a_2a_4}+\frac{1}{a_3a_4}\Big) \\
&\quad + \frac{1\cdot 2\cdot 3\cdot 4}{4}a_1a_2a_3a_4\Big(\frac{1}{a_1a_2a_3}+\frac{1}{a_1a_2a_4}+\frac{1}{a_2a_3a_4}\Big) \\
&= a_1a_2a_3a_4a_5 - 2(a_1a_2a_3a_4 + a_1a_2a_3a_5 + a_1a_2a_4a_5 + a_1a_3a_4a_5 + a_2a_3a_4a_5) \\
&\quad + \frac{35}{4}(a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4) \\
&\quad - \frac{25}{3}(a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4) + 6(a_1 + a_2 + a_3 + a_4)
\end{aligned}
$$

with equality if and only if $a_1 = a_2 = a_3 = a_4 = a_5 =$ integer. This is the main result proved by Lin and Yau in their paper [Li-Ya3].

3 Number of Positive Integral Points in a Homogeneous n-Dimensional Tetrahedron

In this section we shall prove that our Main Conjecture in Section 2 holds for homogeneous n-dimensional tetrahedra $\Delta(a, a, \ldots, a)$. We first start with an elementary summation by parts lemma.

Lemma 3.1 *Let $\{u_k\}, \{v_k\}, k = 1, \ldots, n$ be two sequences of numbers. Then*

$$\sum_{k=1}^{n} u_k v_k = u_n \sum_{k=1}^{n} v_k - \sum_{r=2}^{n} \Big[(u_r - u_{r-1}) \sum_{k=1}^{r-1} v_k \Big]. \tag{3.1}$$

Proof We shall prove (3.1) by induction. (3.1) is obviously true for $n = 2$. Assume that (3.1) holds for $n - 1$. Then

$$\begin{aligned}
\sum_{k=1}^{n} u_k v_k &= \sum_{k=1}^{n-1} u_k v_k + u_n v_n \\
&= u_{n-1} \sum_{k=1}^{n-1} v_k - \sum_{r=2}^{n-1} \Big[(u_r - u_{r-1}) \sum_{k=1}^{r-1} v_k \Big] + u_n v_n \\
&= u_n \sum_{k=1}^{n} v_k - u_n \sum_{k=1}^{n-1} v_k + u_{n-1} \sum_{k=1}^{n-1} v_k - \sum_{r=2}^{n-1} \Big[(u_r - u_{r-1}) \sum_{k=1}^{r-1} v_k \Big] \\
&= u_n \sum_{k=1}^{n} v_k - (u_n - u_{n-1}) \sum_{k=1}^{n-1} v_k - \sum_{r=2}^{n-1} \Big[(u_r - u_{r-1}) \sum_{k=1}^{r-1} v_k \Big] \\
&= u_n \sum_{k=1}^{n} v_k - \sum_{r=2}^{n} \Big[(u_r - u_{r-1}) \sum_{k=1}^{r-1} v_k \Big]
\end{aligned}$$

■

Theorem 3.1 *Let $n \geq 2$ be a positive integer and $a \geq n$. Let P_n be the number of positive integral solutions of $\frac{x_1}{a} + \frac{x_2}{a} + \cdots + \frac{x_n}{a} \leq 1$, i.e., $P_n = \#\{(x_1, x_2, \ldots, x_n) \in \mathbb{Z}_+^n : \frac{x_1}{a} + \frac{x_2}{a} + \cdots + \frac{x_n}{a} \leq 1\}$. Let $[a]$ be the greatest integer less than or equal to a. Then*

$$\begin{aligned}
n!\, P_n &= [a]([a]-1)([a]-2)\cdots([a]-n+1) \\
&\leq a(a-1)(a-2)\cdots(a-n+1)
\end{aligned} \tag{3.2}$$

with equality if and only if a is an integer.

Proof For $n = 2$, we have $\frac{x_1}{a} + \frac{x_2}{a} \le 1$. From the level $x_2 = 1$ to the level $x_2 = [a] - 1$, we have the following positive integral points:

$$x_2 = 1 : (1,1), (2,1), \dots, ([a]-1, 1)$$
$$x_2 = 2 : (1,2), (2,2), \dots, ([a]-2, 2)$$
$$\vdots$$
$$x_2 = [a] - 1 : (1, [a]-1).$$

Hence $P_2 = 1 + 2 + \cdots + ([a] - 1) = \frac{[a]([a]-1)}{2}$, *i.e.*, $2!\,P_2 = [a]([a]-1)$.

Now assume that (3.2) holds for any integer n. Consider a homogeneous $(n+1)$-dimensional tetrahedron $\frac{x_1}{a} + \frac{x_2}{a} + \cdot + \frac{x_{n+1}}{a} \le 1$.

From the level $x_{n+1} = 1$ to the level $x_{n+1} = [a] - n$, we have the following homogeneous n-dimensional tetrahedron:

$$x_{n+1} = 1 : \frac{x_1}{a-1} + \frac{x_2}{a-1} + \cdots + \frac{x_n}{a-1} \le 1$$
$$x_{n+1} = 2 : \frac{x_1}{a-2} + \frac{x_2}{a-2} + \cdots + \frac{x_n}{a-2} \le 1$$
$$\vdots$$
$$x_{n+1} = [a] - n : \frac{x_1}{a-[a]+n} + \frac{x_2}{a-[a]+n} + \cdots + \frac{x_n}{a-[a]+n} \le 1.$$

Hence by induction hypothesis we have

$$\begin{aligned} n!\,P_{n+1} &= \sum_{k=n}^{[a]-1} k(k-1)\cdots\big(k-(n-1)\big) \\ &= \sum_{k=1}^{[a]-1} k(k-1)\cdots\big(k-(n-1)\big). \end{aligned}$$

Now we apply Lemma 3.1 with $u_k = (k-1)(k-2)\cdots\big(k-(n-1)\big)$, $v_k = k$. Then $\sum_{k=1}^{r-1} v_k = \sum_{k=1}^{r-1} k = \frac{r(r-1)}{2}$

$$\begin{aligned} u_r - u_{r-1} &= (r-1)(r-2)\cdots\big(r-(n-1)\big) - (r-2)(r-3)\cdots(r-n) \\ &= (r-2)(r-3)\cdots(r-n+1)\big((r-1)-(r-n)\big) \\ &= (n-1)(r-2)(r-3)\cdots(r-n+1). \end{aligned}$$

From (3.1), we have

$$\sum_{k=1}^{[a]-1} k(k-1)\cdots(k-n+1)$$

$$= u_{[a]-1}\frac{[a]([a]-1)}{2} - \sum_{r=2}^{[a]-1}(n-1)(r-2)(r-3)\cdots(r-n+1)\frac{r(r-1)}{2}$$

$$= ([a]-2)([a]-3)\cdots([a]-n)\frac{[a]([a]-1)}{2}$$

$$- \frac{n-1}{2}\sum_{k=1}^{[a]-1} k(k-1)(k-2)\cdots(k-n+1).$$

Hence we have

$$\left(1+\frac{n-1}{2}\right)\sum_{k=1}^{[a]-1} k(k-1)(k-2)\cdots(k-n+1)$$

$$= \frac{1}{2}[a]([a]-1)([a]-2)\cdots([a]-n),$$

which implies

$$(n+1)\sum_{k=1}^{[a]-1} k(k-1)(k-2)\cdots(k-n+1) = [a]([a]-1)([a]-2)\cdots([a]-n).$$

Therefore we have

$$n!\,P_{n+1} = \sum_{k=1}^{[a]-1} k(k-1)(k-2)\cdots(k-n+1)$$

$$= \frac{1}{n+1}[a]([a]-1)([a]-2)\cdots([a]-n)$$

i.e., $(n+1)!\,P_{n+1} = [a]([a]-1)([a]-2)\cdots([a]-n)$. ∎

Corollary 3.1 *The Main Conjecture in Section 2 is true for n-dimensional homogeneous tetrahedra* $\Delta(a, a, \ldots, a)$.

Proof Set $a_1 = a_2 = \cdots = a_n = a$ in the Main Conjecture. We get

$$n!\,P_{(a,\ldots,a)} \le a^n + (-1)S_1^{n-1}a^{n-1} + (-1)^2S_2^{n-1}a^{n-2} + (-1)^3S_3^{n-1}a^{n-3}$$

$$+ \cdots + (-1)^{k+1}S_{k+1}^{n-1}a^{n-k-1} + \cdots + (-1)^{n-1}S_{n-1}^{n-1}a$$

$$= a(a-1)(a-2)\cdots\big(a-(n-1)\big) \quad \text{by (2.2)}$$

This is exactly Theorem 3.1. ∎

4 Bernoulli Polynomials and Counting the Number of Integral Points in an n-Dimensional Homogeneous Tetrahedron

Let us define the Bernoulli polynomials, $B_0(x), B_1(x), B_2(x), \ldots$, by the following equation

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n. \tag{4.1}$$

Then (4.1) implies

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{B_n(x+1)}{n!} t^n &= \frac{te^{t(x+1)}}{e^t - 1} = \frac{te^{tx}}{e^t - 1}[(e^t - 1) + 1] \\ &\Rightarrow \sum_{n=0}^{\infty} \frac{B_n(x+1) - B_n(x)}{n!} t^n = te^{tx} = \sum_{n=0}^{\infty} \frac{x^n}{n!} t^{n+1} \\ &\Rightarrow \frac{B_n(x+1) - B_n(x)}{n!} = \frac{x^{n-1}}{(n-1)!} \\ &\Rightarrow B_n(x+1) - B_n(x) = nx^{n-1}. \end{aligned} \tag{4.2}$$

Differentiating (4.1) with respect to x, we get

$$\begin{aligned} \frac{t^2 e^{tx}}{e^t - 1} &= \sum_{n=0}^{\infty} \frac{B_n'(x)}{n!} t^n \\ &\Rightarrow t \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n = \sum_{n=0}^{\infty} \frac{B_n'(x)}{n!} t^n \\ &\Rightarrow \frac{B_{n-1}(x)}{(n-1)!} = \frac{B_n'(x)}{n!} \\ &\Rightarrow B_n'(x) = nB_{n-1}(x). \end{aligned} \tag{4.3}$$

Therefore if we denote $B_n^{(j)}(x)$ the j-th derivative of $B_n(x)$, then we have

$$B_n^{(j)}(x) = nB_{n-1}^{(j-1)}(x). \tag{4.4}$$

Theorem 4.1 *Let $n \geq 2$ be a positive integer and $a \geq n-1$. Let P_n be the number of positive integral solution of $\frac{x_1}{a} + \frac{x_2}{a} + \cdots + \frac{x_n}{a} \leq 1$, i.e., $P_n = \#\{(x_1, x_2, \ldots, x_n) \in \mathbb{Z}_+^n : \frac{x_1}{a} + \frac{x_2}{a} + \cdots + \frac{x_n}{a} \leq 1\}$. Let $[a]$ be the greatest integer less than or equal to a. then*

$$\begin{aligned} (n)!\, P_n &= \sum_{k=1}^{n-1} \frac{(-1)^{n-k-1} S_{n-k-1}^{n-2} n}{k+1} \left(B_{k+1}([a]) - B_{k+1}(1) \right) \\ &= \sum_{k=1}^{n-1} \frac{(-1)^{n-k-1} S_{n-k-1}^{n-2} k!}{(n-1)!} \left(B_n^{(n-k-1)}([a]) - B_n^{(n-k-1)}(1) \right). \end{aligned}$$

Proof By (4.2), we have

$$\frac{1}{n+1}\big(B_{n+1}([a]) - B_{n+1}([a]-1)\big) = ([a]-1)^n$$
$$\frac{1}{n+1}\big(B_{n+1}([a]-1) - B_{n+1}([a]-2)\big) = ([a]-2)^n$$
$$\vdots$$
$$\frac{1}{n+1}\big(B_{n+1}(3) - B_{n+1}(2)\big) = 2^n$$
$$\frac{1}{n+1}\big(B_{n+1}(2) - B_{n+1}(1)\big) = 1^n.$$

Summing the above equations, we have

$$\frac{1}{n+1}\big(B_{n+1}([a]) - B_{n+1}(1)\big) = 1^n + 2^n + \cdots + ([a]-1)^n = \sum_{x=1}^{[a]-1} x^n. \tag{4.5}$$

Similarly by (4.2) and (4.4), we have

$$\frac{1}{n}\big(B_n([a]) - B_n(1)\big) = \frac{(n-1)!}{(n+1)!}\big(B'_{n+1}([a]) - B'_{n+1}(1)\big)$$
$$\frac{1}{n-1}\big(B_{n-1}([a]) - B_{n-1}(1)\big) = \frac{(n-2)!}{(n+1)!}\big(B''_{n+1}([a]) - B''_{n+1}(1)\big)$$
$$\vdots$$
$$\frac{1}{k}\big(B_k([a]) - B_k(1)\big) = \frac{(k-1)!}{(n+1)!}\big(B^{(n-k+1)}_{n+1}([a]) - B^{(n-k+1)}_{n+1}(1)\big) \tag{4.6}$$
$$\vdots$$
$$\frac{1}{2}\big(B_2([a]) - B_2(1)\big) = \frac{1}{(n+1)!}\big(B^{(n-1)}_{n+1}([a]) - B^{(n-1)}_{n+1}(1)\big).$$

By the proof of Theorem 3.1, we have

$$n!\,P_{n+1} = \sum_{x=1}^{[a]-1} x(x-1)(x-2)\cdots(x-n+1)$$
$$= \sum_{x=1}^{[a]-1}\sum_{k=1}^{n}(-1)^{n-k}S^{n-1}_{n-k}x^k$$

$$= \sum_{k=1}^{n} \sum_{x=1}^{[a]-1} (-1)^{n-k} S_{n-k}^{n-1} x^k$$

$$= \sum_{k=1}^{n} \frac{(-1)^{n-k} S_{n-k}^{n-1}}{k+1} \big(B_{k+1}([a]) - B_{k+1}(1) \big) \quad \text{by (4.5)}$$

$$= \sum_{k=1}^{n} \frac{(-1)^{n-k} S_{n-k}^{n-1} \cdot k!}{(n+1)!} \big(B_{n+1}^{n-k}([a]) - B_{n+1}^{n-k}(1) \big) \quad \text{by (4.6)}$$

References

[Br-Ve1] M. Brion and M. Vergne, *An equivariant Riemann-Roch theorem for complete, simplicial toric varieties*. J. Reine Angew. Math. **482**(1997), 67–92.

[Br-Ve2] ______, *Lattice points in simple polytopes*. J. Amer. Math. Soc. **10**(1997), 371–392.

[Ca-Sh] S. E. Cappell and J. L. Shaneson, *Genera of algebraic varieties and counting lattice points*. Bull. Amer. Math. Soc. **30**(1994), 62–69.

[Di-Ro] R. Diaz and S. Robbin, *The Ehrhart polynomial of a lattice polytope*. Ann. of Math. **135**(1997), 503–518.

[Du] A. H. Durfee, *The signature of smoothings of complex surface singularities*. Math. Ann. **232**(1978), 85–98.

[Eh] E. Ehrhart, *Sur un probleme de geometrie diophantienne lineaire II*. J. Reine Angrew Math. **227**(1967), 25–49.

[Ha-Li1] G. H. Hardy and J. E. Littlewood, *Some problems of diophantine approximation*. Proc. 5th Int. Congress of Mathematics, (1912), 223–229.

[Ha-Li2] ______, *The lattice points of a right-angled triangle*. Proc. London Math. Soc. (2) **20**(1921), 15–36.

[Ha-Li3] ______, *The lattice points of a right-angled triangle (second memoir)*. Hamburg Math. Abh. **1**(1922), 212–49.

[Ha-Li4] ______, *A series of cosecants*. Bull. Calcutta Math. Soc. **20**(1930), 251–66.

[Hi-Za] F. Hirzebruch and D. Zagier, *The Atiyah-Singer Index Theorem and Elementary Number Theory*. Publish or Perish, Inc., Boston, Massachusetts, 1974.

[Ka-Kh] J. M. Kanter and A. Khovanskii, *Une application du Théoréme de Riemann-Roch combinatoire au polynôme d' Ehrhart des polytopes intier de* $\mathbb{R}^d$. C. R. Acad. Sci. Paris I **317**(1993), 501–507.

[Li-Ya1] K.-P. Lin, and S. S.-T. Yau, *Sharp upper estimate of geometric genus in terms of Milnor number and multiplicty with application on coordinate free characterization of 3-dimensional homogeneous singularities*. (preprint).

[Li-Ya2] ______, *Analysis of sharp polynomial upper estimate of number of positive integral points in 4-dimensional tetrahedra*. J. Reine Angew. Math. **547**(2002), 191–205.

[Li-Ya3] ______, *A sharp upper estimate of the number of integral points in 5-dimensional tetrahedra*. J. Number Theory **93**(2002), 207–234.

[Me-Te] M. Merle and B. Teissier, *Conditions d'adjonction d'aprés Du Val*. Sěminaire sur les singularities des surfaces (center de Math, de l'Ecole Polytechnique, 1976-1977), Lecture Notes in Math. **777**, Springer, Berlin, 1980, 229–245.

[Mo1] L. J. Mordell, *Lattice points in a tetrahedron and Dedekind sums*. J. Indian Math. **15**(1951), 41–46.

[Mor] R. Morelli, *Pick's theorem and the Todd class of tori variety*. Adv. in Math. **100**(1993), 183–231.

[Po] J. Pommersheim, *Toric varieties, lattice points and Dedekind sums*. Math. Ann. **295**(1993), 1–24.

[Sp1] D. C. Spencer, *On a Hardy-Littlewood problem of Diophantine approximation*. Proc. Cambridge Philos. Soc. **XXXV**(1939), 527–547.

[Sp2] ______, *The lattice points of tetrahedron*. J. Math. Phys. (3) **XXI**(1942), 189–197.

[Xu-Ya1] Y.-J. Xu and S. S.-T. Yau, *Sharp estimate of number of integral points in a tetrahedron*. J. Reine Angew. Math. **423**(1992), 199–219.

[Xu-Ya2] ———, *Durfee conjecture and coordinate free characterization of homogeneous singularities*. J. Differential Geom. **37**(1993), 375–396.

[Xu-Ya3] ———, *Sharp estimate of numbers of integral points in a 4-dimensional tetrahedron*. J. Reine Angew. Math. **473**(1996), 1–23.

Department of Information Management
Chang Gung Institute of Technology
261 Wen-Hwa 1 Road
Kwei-Shan, Tao-Yuan
Taiwan 333-03
Republic of China
e-mail: kplin@cc.cgin.edu.tw

Department of Mathematics, Statistics
and Computer Science (M/C 249)
University of Illinois at Chicago
851 South Morgan Street
Chicago, Illinois 60607-7045
U.S.A.
e-mail: yau@uic.edu