SOME REMARKS ON THE LOCAL MODULI OF TANGENT BUNDLES OVER COMPLEX SURFACES

By WING-SUM CHEUNG, BUN WONG, and STEPHEN S.-T. YAU

Dedicated to Professor F. Hirzebruch on the occasion of his seventy-fifth birthday. His celebrated work on the Riemann-Roch formula and mathematical insights have greatly influenced our work.

Abstract. Using the Hirzebruch's Riemann-Roch formula for endomorphism bundles over a compact complex two-fold we prove that the tangent bundle of a complex surface M of general type admits a nontrivial trace-free deformation, unless M is holomorphically covered by the euclidean ball. It follows that the tangent bundle of the Mostow-Siu surface, which is a Kähler surface with a negative definite curvature tensor, does have a nontrivial trace-free moduli. Among some other results we also point out a relationship between the Kuranishi obstruction and symmetric holomorphic two tensors on a complex surface.

1. Statements of results. We denote by X a compact complex manifold with holomorphic tangent bundle $T(X)$. Let E be a holomorphic vector bundle over X. The deformation theory of Kodaira-Spencer says that the Zariski tangent vector of the local moduli space of holomorphic vector bundles at E can be interpreted as an element of $H^1(X, \mathrm{End}(E))$, where $\mathrm{End}(E)$ is the endomorphism bundle of E. Since one can always deform E holomorphically simply by tensoring with a family of line bundles, it is therefore more interesting to consider a more intrinsic deformation of E, namely the trace-free part $H^1(X, \mathrm{End}_0(E))$ of $H^1(X, \mathrm{End}(E))$, because of the canonical decomposition $H^1(X, \mathrm{End}(E)) = H^1(X, \mathrm{End}_0(E)) \oplus H^1(X, \mathcal{O}_X)$. It is well known that the Kuranishi map

$$K_{[E]}\colon\ H^1(X, \mathrm{End}(E)) \to H^2(X, \mathrm{End}(E))\ ,$$

defined by $\omega \mapsto [\omega, \omega]$ with $\omega \in H^1(X, \mathrm{End}(E))$, measures the obstruction for E to admit a nontrivial deformation in the direction along ω. One can easily see that the bracket $[\omega, \omega]$ always lies in the trace-free part $H^2(X, \mathrm{End}_0(E))$ of $H^2(X, \mathrm{End}(E))$ because of the fact that $\mathrm{tr}(AB - BA) = 0$ for any two square matrices A and B. The model for the trace-free local moduli space of E is same as the Kuranishi space

Research of the first author supported in part by the RGC of the HKSAR (Project No. HKU 7130/00P).
American Journal of Mathematics 125 (2003), 1029–1035.

$K_0(E)$ with a fixed determinant (see [F-M,1], p.300). It can also be regarded as the local moduli of the associated projective bundle $P(E)$ over X. The restriction of $K_{[E]}$ on the trace-free part will be denoted by

$$K^0_{[E]}\colon\ H^1(X, \mathrm{End}_0(E)) \to H^2(X, \mathrm{End}_0(E))\ ,$$

given by the same map $\omega \mapsto [\omega, \omega]$, where $\omega \in H^1(X, \mathrm{End}_0(E))$. In this paper we are going to study $K_0(E)$ when E is the holomorphic tangent bundle $T(M)$ over a compact complex two-fold M (i.e., a complex surface) of general type.

THEOREM 1.1. *Let M be a compact complex surface of general type. Then its holomorphic tangent bundle $T(M)$ admits a nontrivial trace-free deformation (i.e., a nontrivial element $\omega \in H^1(M, End_0(T))$ with $[\omega, \omega] = 0$) unless M is covered holomorphically by the euclidean ball (i.e., $M = B_2/\Gamma$, $B_2 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 < 1\}$, $\Gamma \subset Aut(B_2)$ acting freely on B_2).*

We define the expected dimension $\mathcal{M}_0(E)$ of $K_0(E)$ as the number

$$\dim_{\mathbb{C}} H^1(X, \mathrm{End}_0(X)) - \dim_{\mathbb{C}} H^2(X, \mathrm{End}_0(E))\ .$$

This is not the complex dimension $\dim_{\mathbb{C}} K_0(E)$ of $K_0(E)$ as a complex space. Nevertheless, the inequality $\mathcal{M}_0(E) \leq \dim_{\mathbb{C}} K_0(E)$ is always true (See [F-M,1], p. 302).

We shall also prove the following observation which will imply Theorem 1.1.

THEOREM 1.2. *Let M be a compact complex surface of general type. Then $\mathcal{M}_0(T)$ is equal to zero iff M is covered holomorphically by the enclidean ball.*

The proofs of Theorems 1.1 and 1.2 rely on a uniformization theorem of S. T. Yau [Y,1] [Y,2] and the Hirzebruch's Riemann-Roch formula for the endomorphism bundle End(E) over a complex surface.

An immediate application of Theorem 1.1 says that the tangent bundle of Mostow-Siu surface [M-S,1], which is a compact Kähler two-fold with negative sectional curvature, admits a nontrivial trace-free deformation. It was proved in the same paper that this surface is not covered by the euclidean ball through a Chern number argument. This supports a speculation that the Bochner technique needs to be modified before it could be applicable to the deformation problem of vector bundles, although it has been very successful in proving the rigidity of complex structures on Kähler manifolds with negative definite curvature [S,1].

In §5 we shall give a proof of the following result which can justify our belief even in the case of Kähler manifolds with constant negative holomorphic sectional curvature.

THEOREM 1.3. *Let $M = B_2/\Gamma$ be a complex surface covered by the euclidean ball. Then $\dim_{\mathbb{C}} H^1(M, End_0(T)) = \dim_{\mathbb{C}} H^0(M, S^2T^*)$, where S^2T^* is the symmetric* *tensor product of the cotangent bundle T^*. Suppose the 1st Betti number $b_1(M)$ of M is positive. Then* $\dim_{\mathbb{C}} H^1(M, End_0(T))$ *is also positive.*

We would like to mention here that there exists a nonarithmetic surface due to G. D. Mostow [M,1] whose 1st Betti number is positive. Thus $H^1(M, \mathrm{End}_0(T))$ is nonzero on the Mostow surface M by Theorem 1.3. On the other hand, David Mumford [Mu,1] constructed an arithmetic surface with its 1st Betti number equal to zero. Nevertheless, one can still hope for proving a vanishing theorem of $H^1(M, \mathrm{End}_0(T))$ for those $M = B_2/\Gamma$ in the case of $b_1(M) = 0$ by Bochner's method. It seems of interest to determine whether a nontrivial symmetric holomorphic two tensor on the Mumford surface exists or not.

We plan to discuss the trace-free rigidity problem of the tangent bundle over compact complex manifolds with constant negative holomorphic sectional curvature (i.e., the converse of Theorem 1.1), via a different path rather than using the Bochner type formula of harmonic forms, in a separate paper. We intend to apply the Weil-Corlette type local rigidity for discrete subgroups of a Lie group, Uhlenbeck-Yau theory of Yang-Mill connections for stable bundles, and the Lübke's criterion for projective flatness by Chern numbers.

In §4, we include a discussion of the relationship between the obstruction of the Kuranishi map and the symmetric holomorphic two tensors on a complex surface.

We have been informed that our results are related to a problem asked by Y. T. Siu in an article of a volume dedicated to Loo Keng Hua in 1991. It has also come to our attention that S. T. Yau posed a similar question in the problems section of a monograph in honor of S. S. Chern.

2. Proof of Theorem 1.2. We begin the discussion of this section by giving a proof of the following folklore result which will be used in the proof of Theorem 1.2.

LEMMA 2.1. *Let X be a complex surface with 1st Chern class negative definite. Then $T(X)$ is simple (i.e.,* $\dim_{\mathbb{C}} H^0(X, End_0(T)) = 0$*) unless X is covered by the bidisk.*

The higher dimensional analog is also true and its proof is similar.

Proof. Using the decomposition $T \otimes T \cong \wedge^2 T \oplus S^2 T$ one obtains the following by tensoring both sides with the canonical bundle K,

$$T \otimes T \otimes K \cong \mathcal{O} \oplus S^2 T \otimes K \, .$$

Since $\dim_{\mathbb{C}} X = 2$, we have $T \otimes K \cong T^*$. Thus $T \otimes T^* \cong \mathcal{O} \oplus S^2 T \otimes K$. Combining the facts $\mathrm{End}(T) \cong \mathcal{O} \oplus \mathrm{End}_0(T)$ and $\mathrm{End}(T) \cong T \otimes T^*$, this gives rise to $\dim_{\mathbb{C}} H^0(X, \mathrm{End}_0(T)) = \dim_{\mathbb{C}} H^0(X, S^2 T \otimes K)$. By a result of S. Kobayashi (Corollary B.2, [K,2]), if $H^0(X, S^2 T \otimes K) \neq 0$, then X is covered by the bidsk. This finishes the proof. □

The Hirzebruch Riemann-Roch formula for End(E) over a compact complex twofold M, where E is a holomorphic vector bundle of rank r, can be written as follows ([H,1], [K,3] p. 288):

$$\begin{aligned}&\dim_{\mathbb{C}} H^0(M,\mathrm{End}(E)) - \dim_{\mathbb{C}} H^1(M,\mathrm{End}(E)) + \dim_{\mathbb{C}} H^2(M,\mathrm{End}(E))\\ &\quad = \int_M (r-1)C_1^2(E) - 2rC_2(E) + \frac{r^2}{12}(C_1^2(M) + C_2(M)).\end{aligned}$$

Letting $E = T(M)$, we obtain

$$\begin{aligned}&\dim_{\mathbb{C}} H^0(M,\mathrm{End}(T)) - \dim_{\mathbb{C}} H^1(M,\mathrm{End}(T)) + \dim_{\mathbb{C}} H^2(M,\mathrm{End}(T))\\ &\quad = \frac{4}{3}C_1^2 - \frac{11}{3}C_2,\end{aligned}$$

where $C_1^2 = \int_M C_1^2(M)$, $C_2 = \int_M C_2(M)$, which are the 1st and 2nd Chern numbers on M.

Making use of the decompositions

$$H^i(M,\mathrm{End}(T)) \cong H^i(M,\mathrm{End}_0(T)) \oplus H^i(M,\mathcal{O}_M), \quad 0 \le i \le 2,$$

and the Noëther identity $1 - q + p_g = \frac{1}{12}(C_1^2 + C_2)$, where $q = \dim_{\mathbb{C}} H^1(M,\mathcal{O}_M)$, $p_g = \dim_{\mathbb{C}} H^2(M,\mathcal{O}_M)$, we have the following Hirzebruch's formula for the trace-free $\mathrm{End}_0(T)$ over M:

$$\begin{aligned}&\dim_{\mathbb{C}} H^0(M,\mathrm{End}_0(T)) - \dim_{\mathbb{C}} H^1(M,\mathrm{End}_0(T)) + \dim_{\mathbb{C}} H^2(M,\mathrm{End}_0(T))\\ &\quad = \frac{5}{4}(C_1^2 - 3C_2).\end{aligned}$$

By the definition of expected dimension $\mathcal{M}_0(T)$, that is the same thing as saying

$$\mathcal{M}_0(T) = \dim_{\mathbb{C}} H^0(M,\mathrm{End}_0(T)) + \frac{5}{4}(3C_2 - C_1^2).$$

The proof of Theorem 1.2 proceeds as follows.

Suppose $\mathcal{M}_0(T)$ is zero. Then we have

$$\dim_{\mathbb{C}} H^0(M,\mathrm{End}_0(T)) + \frac{5}{4}(3C_2 - C_1^2) = 0\ .$$

Since M is of general type, $3C_2 - C_1^2 \ge 0$ by the Miyaoka-Yau inequality ([Mi,1], [Y,1]). Thus both terms on the left-hand side of the identity are nonnegative. Hence $3C_2 = C_1^2$ and $\dim_{\mathbb{C}} H^0(M,\mathrm{End}_0(T)) = 0$. The condition $3C_2 = C_1^2$ implies that the 1st Chern class of M is negative definite by a result of Miyaoka [Mi,2].

Finally a uniformization theorem of S. T. Yau [Y,1] concludes that M must be covered by the ball.

Conversely, suppose M is covered by the ball. A computation of the Chern numbers C_1^2 and C_2 in terms of the curvature tensor of the Bergman metric shows that $C_1^2 = 3C_2$ (see [Y,1], for example). Therefore, $\mathcal{M}_0(T) = \dim_{\mathbb{C}} H^0(M, \mathrm{End}_0(T))$. Here we recall a classical theorem of Poincaré that the ball and the bidisk are not biholomorphic to each other. The tangent bundle $T(M)$ of M is simple (i.e., $\dim_{\mathbb{C}} H^0(M, \mathrm{End}_0(T)) = 0$) because $C_1(M)$ is negative definite and M is not covered by the bidisk (Lemma 2.1). As a consequence, $\mathcal{M}_0(T)$ is equal to zero.

3. Proof of Theorem 1.1. It is a known fact that $\mathcal{M}_0(T) \leq \dim_{\mathbb{C}} K_0(T)$ (see [F-M,1], p. 302, Proposition 1.21). Recall that $\mathcal{M}_0(T) = \frac{5}{4}(3C_2 - C_1^2) + \dim_{\mathbb{C}} H^0(M, \mathrm{End}_0(T))$ which is a nonnegative number because of the Miyaoka-Yau inequality. Suppose $T(M)$ does not admit any nontrivial trace-free deformation. It follows that $\dim_{\mathbb{C}} K_0(T) = 0$. Combining all the facts above we can conclude the inequalities

$$0 \leq \mathcal{M}_0(T) \leq 0.$$

This implies $\mathcal{M}_0(T) = 0$. We have proved that M is covered by the ball by Theorem 1.2.

4. Kuranishi obstruction and symmetric holomorphic two tensors on a complex surface. Let X be a complex surface. First, we know that $H^2(X, \mathrm{End}(T)) \cong H^2(X, \mathrm{End}_0(T)) \oplus H^2(X, \mathcal{O}_X)$. On the other hand, there is a canonial decomposition $T^* \otimes T^* \cong \wedge^2 T^* \oplus S^2 T^*$. Using this we have $H^0(X, T^* \otimes T^*) \cong H^0(X, \wedge^2 T^*) \oplus H^0(X, S^2 T^*)$. By Serre's duality and the fact that $T \otimes K \cong T^*$ on a complex surface, here K = canonical bundle of X, this shows $H^2(X, T \otimes T^*) \cong H^0(X, \wedge^2 T^*) \oplus H^0(X, S^2 T^*)$. By Serre's duality on a complex surface again it is well known that $H^0(X, \wedge^2 T^*) \cong H^2(X, \mathcal{O}_X)$. Thus because of the fact that End $T \cong T \otimes T^*$, we arrive at the following conclusion:

LEMMA 4.1. *Let X be a complex surface. Then* $\dim_{\mathbb{C}} H^2(X, \mathit{End}_0(T)) = \dim_{\mathbb{C}} H^0(X, S^2 T^*)$.

This supports our speculation that on a complex surface, the obstruction of the Kuranishi map for tangent bundle $T(X)$,

$$K^0_{[T]}\colon H^1(X, \mathrm{End}_0(T)) \to H^2(X, \mathrm{End}_0(T)),$$

actually lies in the space of symmetric holomorphic two tensors (i.e., $H^0(X, S^2 T^*)$).

The following result is an immediate consequence of the above observation.

LEMMA 4.2. (a) *Let X be a complex surface with a positive definite 1st Chern class. Then $K^0_{[T]}$ is unobstructed.*

(b) *Let X be a simply connected Kähler surface with vanishing 1st Chern class. Then* $K^0_{[T]}$ *is unobstructed.*

In both cases the Kuranishi space $K_0(T)$ *is a complex manifold if* $\dim_{\mathbb{C}} H^1(X, End_0(T))$ *is positive.*

The proof follows easily from Yau's theorem [Y,1], and a result due to S. Kobayashi ([K,1], p. 329), which says that under either condition (a) or (b), there is no nontrivial symmetric holomorphic two tensor on X. The rest of the proof is merely a standard fact in the Kodaira-Spencer-Kuranishi theory of deformation.

In the case of complex surfaces with negative definite Chern class, apparently there are many such examples with symmetric holomorphic two tensor. On the other hand, it can be proved that any smooth complex hypersurface in $\mathbb{CP}^3$, including those with high degree whose 1st Chern classes are negative definite, admit no symmetric holomorphic two tensor at all [G-G,1]. Hence their Kuranishi maps receive no obstruction. We should notice that the two examples in Lemma 4.2 and the hypersurfaces in $\mathbb{CP}^3$ are all simply connected. They have no holomorphic one form. Thus one has $K(T) = K_0(T)$ and $K_{[T]} = K^0_{[T]}$ in these cases.

It seems that the Kuranishi obstruction is always reflected in the pseudoconvexity of the tangent bundle. This can be seen from the following fact.

LEMMA 4.3. *Let X be a complex surface. Suppose* $K_{[T]}$ *is obstructed. Then there exists a smooth plurisubharmonic psuedo-complex Finsler metric on* $T(X)$.

Proof. From Lemma 4.1, one observes that there is a nontrivial symmetric holomorphic two tensor $\theta \in S^2T^*(X)$ if $K_{[T]}$ is obstructed. Then $|\theta|^{\frac{1}{2}}$ defines a plurisubharmonic, possibly degenerate somewhere, complex Finler metric on $T(X)$. □

5. Proof of Theorem 1.3. The first statement follows from the Hirzebruch's formula, Lemma 4.1, as well as the facts $C_1^2 = 3C_2$ and $H^0(M, \mathrm{End}_0(T)) = 0$. This proves the identity $\dim_{\mathbb{C}} H^1(M, \mathrm{End}_0(T)) = \dim_{\mathbb{C}} H^0(M, S^2T^*)$, as what is demanded.

For the second statement, one simply observes that there exists a nonzero holomorphic one form ω on M under the hypothesis $b_1(M) > 0$. Apparently $\omega \otimes \omega$ is a nonzero element of $H^0(M, S^2T^*)$. Thus $\dim_{\mathbb{C}} H^1(M, \mathrm{End}_0(T)) > 0$.

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF HONG KONG, HONG KONG
E-mail: wscheung@hku.hk

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, RIVERSIDE, CA 92521-0135
E-mail: wong@math.ucr.edu

DEPARTMENT OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE, UNIVERSITY OF ILLINOIS AT CHICAGO, CHICAGO, IL 60607-7045 (concurrently a Ze-Jiang Professor at East China Normal University).
E-mail: yau@uic.edu

REFERENCES

[F-M,1] R. Friedman and J. W. Morgan, *Smooth Four-manifolds and Complex Surfaces*, *Ergeb. Math. Grenzgeb. 3*, Band 27, Springer-Verlag, Berlin, 1994.

[G-G,1] M. Green and P. Griffiths, Two applications of algebraic geometry to entire holomorphic mappings, *The Chern Symposium 1979 (Proc. Inter. Sympos., Berkeley, CA, 1979)*, Springer-Verlag, New York, 1980, pp. 41–74.

[H,1] F. Hirzebruch, *Topological Methods in Algebraic Geometry*, *Grundlehren Math. Wiss.*, vol. 131, Springer-Verlag, New York, 1966.

[K,1] S. Kobayashi, The first Chern class and holomorphic symmetric tensor fields, *J. Math. Soc. Japan* **32** (1980), 325–329.

[K,2] ———, First Chern class and holomorphic tensor fields, *Nagoya Math. J.* **77** (1980), 5–11.

[K,3] ———, *Differential Geometry of Complex Vector Bundles*, Iwanami Shoten Publishers and Princeton University Press, 1987.

[Mi,1] Y. Miyaoka, On the Chern numbers of surfaces of general type, *Invent. Math.* **42** (1977), 225–237.

[Mi,2] ———, The maximal number of quotient singularities on surfaces with given numerical invariants, *Math. Ann.* **268** (1984), 159–171.

[M,1] G. D. Mostow, On a remarkable class of polyhedra in complex hyperbolic space, *Pacific J. Math.* **86** (1980), 171–276.

[M-S,1] G. D. Mostow and Y. T. Siu, A compact Kähler surface of negative curvature not covered by the ball, *Ann. of Math.* **112** (1980), 321–360.

[Mu,1] D. Mumford, An algebraic surface with K ample, $K^2 = 9$, $p_g = q = 0$, *Amer. J. Math.* **101** (1979), 233–244.

[S,1] Y. T. Siu, Complex-analyticity of harmonic maps, vanishing and Lefschetz theorems, *J. Differential Geom.* **17** (1982), 55–138.

[Y,1] S. T. Yau, Calabi's conjecture and some new results in algebraic geometry, *Proc. Natl. Acad. Sci. USA* **74** (1977), 1798–1799.

[Y,2] ———, On the Ricci curature of a compact Kähler manifold and complex Monge-Ampère equation, I, *Comm. Pure Appl. Math.* **31** (1978), 339–411.