ASIAN J. MATH.
Vol. 8, No. 2, pp. 259–286, June 2004

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HAUSDORFF DIMENSION OF INVARIANT C-VECTOR OF M-MATRIX AND SELF-AFFINE FRACTAL *

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Dedicated to Professor Yum-Tong Siu on the occasion of his 60th birthday

1. Introduction. Let $\Phi = \{\phi_1, \cdots, \phi_T\}$ be an iterated function system (IFS) on $\mathbb{R}^d$. The attractor $E \subset \mathbb{R}^d$ of Φ is the unique compact set such that $E = \cup_{i=1}^T \phi_i(E)$. If Φ consists of contractive affine mappings, i.e. each $\phi_i(x) = B_i x + b_i$, where B_i are $d \times d$ contractive matrices in $\mathbb{R}^d$ and $b_i \in \mathbb{R}^d$, then E is called a self-affine set. In this paper, we study a class of self-affine sets with overlapping by applying the M-matrix theory([20]).

In 1984, McMullen[26] gave the Hausdorff dimension of the McMullen carpet (general Sierpinski carpet), which is the attractor of $\{B(x + b_i); i = 1, \cdots, T\}$ with $B = \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{m} \end{pmatrix}$ and $b_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$, $0 \le x_i < n$, $0 \le y_i < m$, where m, n, x_i and y_i are integers and $n \ge m > 1$. Is also is proved that $0 < H^D(E) < \infty$ for McMullen carpet E, where $D = \dim_H(E)$. In 1992, Lalley and Gatzouras[23] studied a wider class of self-affine sets in the plane and give a formula for their Hausdorff dimension, where the iterated function systems are of the form $\Phi = \{\begin{pmatrix} a_{ij} & 0 \\ 0 & b_i \end{pmatrix} x + \begin{pmatrix} c_{ij} \\ d_i \end{pmatrix} | 1 \le i \le m, 1 \le j \le n_i\}$ with certain conditions on $m, n_i, a_{ij}, b_i, c_{ij}$ and d_i. In 1994, Pollicott and Weiss[31] discussed the attractor of $\{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} x + \begin{pmatrix} c_1 \\ d_1 \end{pmatrix}, \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} x + \begin{pmatrix} c_2 \\ d_2 \end{pmatrix}\}$ with the open set condition in the plane, the Hausdorff dimension were given in several cases.

In 1988, Falconer[11] gave an formula for the upper estimation of Hausdorff dimension of self-affine set, and this upper estimation reaches the exact dimension for almost all $(b_1, \cdots, b_T) \in \mathbb{R}^{dT}$, when $B_1, \cdots, B_T$ are fixed and each $\|B_i\|_2 < \frac{1}{3}$. This estimation is called the Falconer's dimension of self-affine set. In 1992, Edgar[9] gave examples which imply that when some $\|B_i\|_2 > \frac{1}{2}(\sqrt{5} - 1)$, the Falconer's dimension may not equal to the Hausdorff dimension for any $(b_1, \cdots, b_T) \in \mathbb{R}^{dT}$. In 1995, Hueter and Lalley[16] studied the self-affine sets in the plane. They present five simple hypotheses on $B_1, \cdots, B_T$ and $(b_1, \cdots, b_T)$ which ensure that Hausdorff dimension equals to the Falconer's dimension. In 1998, Solomyak[32] proved that Falconer's dimension is still equal to the Hausdorff dimension for almost all $(b_1, \cdots, b_T) \in \mathbb{R}^{dT}$ if all $\|B_i\|_2 < \frac{1}{2}$. Furthermore, it is shown that if $B_1 = \cdots = B_T = B$, the condition $\|B\|_2 < \frac{1}{2}$ can be replaced by a weaker condition that B has no eigenvalues in the "*Mandelbrot set for the pair of transformations*". In [20], one family of examples were given with some $\|B_i\|_2 + \|B_j\|_2 > 1$ such that the Falconer's upper estimation is strictly bigger than the Hausdorff dimension for any $(b_1, \cdots, b_T) \in \mathbb{R}^{dT}$.

*Received July 2, 2003; accepted for publication November 5, 2003.
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Research partially supported by Zi-Jiang Professorship of East China Normal University.

In 1990, Bedford and Urbański[6] studied those self-affine sets which can be viewed as the graph $E = \{(x, f(x)); x \in [0,1]\}$ of a function $f : [0,1] \to \mathbb{R}$. They gave the condition when the Bowen's formula holds for the box dimension of E (see [5, 6, 7] for detail). They also gave some conditions when the box dimension coincides with the Hausdorff dimension of such self-affine set.

In 1992, Falconer[12] gave a lower bound of the Hausdorff dimension of self-affine set in the case that $\phi_i(E) \cap \phi_j(E) = \emptyset$ for all $i \neq j$ (this condition is stronger than the open set condition). In 1995, Paulsen[30] got a similar lower bound on the Hausdorff dimension under the condition that the self-affine set E is Hausdorff rectifiable for a generalized Hausdorff measure H_h (i.e. $0 < H_h(E) < \infty$), and $H_h(\phi_i(E) \cap \phi_j(E)) = 0$ for $i \neq j$, where h is a continuous, monotonically increasing function from $\mathbb{R}^+$ to $\mathbb{R}$. Although Paulsen's result is more general than those of Falconer, his condition is not very useful because it is very difficult to find such function h. In fact, for any such h, it has the property that $\lim_{r\to 0} \frac{h(ar)}{h(r)} = a^{\dim_H E}$ for any $a > 0$. In 2002, Abercrombie and Nair[1] gave the lower and upper bound of the Hausdorff dimension of plane self-affine set under the assumption that the iterated function system may contains countably infinite mappings, each affine mapping ϕ_i preserves the directions of the coordinate axes, and the intersection of the interiors of $\phi_i(E)$ and $\phi_j(E)$ is empty for all $i \neq j$.

In [21](1996), Kenyon and Peres studied the B-invariant Sofic set, a class of direct-graph construction objects, where $B = \begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}$, $m, n \geq 2$ are integers. They gave some sequences converging to the Hausdorff dimension and a formula to calculate box dimension of B-invariant Sofic set. They also gave some examples of self-affine sets which are B-invariant Sofic sets. In 2003, He, Lau and Rao[15] constructed a new direct-graph constructions to study the boundaries of self-similar sets and self-affine sets. They applied the method of Kenyon and Peres[21](1996) to calculate the Hausdorff dimensions of boundaries of some self-affine sets with overlappings on real plane. In [22](1996), Kenyon and Peres studied the self-affine set of IFS $\{B(x+b)|b \in \Xi\}$, where $B = \mathrm{diag}(n_1^{-1}, \cdots, n_d^{-1})$, $n_i > 1$ are integers and $\Xi \subseteq \{0, \cdots, n_1 - 1\} \times \{0, \cdots, n_2 - 1\} \times \cdots \times \{0, \cdots, n_d - 1\}$. They determined the Hausdorff dimension and box dimension of such self-affine sets, which are called self-affine Sierpinski sponges.

In §4 of this paper, we study the self-affine sets $A(\Phi)$ which are the attractors of the iterated function system $\Phi = \{Bx + b|b \in \Xi\}$, where $B = \mathrm{diag}(n_1^{-1}, \cdots, n_d^{-1})$, $n_i > 1$ are integers and $\Xi \subseteq \mathbb{R}^d$ is a finite set. We prove that if $\Gamma\Xi + \beta \subseteq \mathbb{Q}^d$ for some invertible matrix Γ and vector $\gamma \in \mathbb{R}^d$, then $A(\Phi)$ can be expressed as the union of some components of an invariant c-vector of a *net M-matrix* (see Theorem 4.1 and Definition 3.1 for detail). Consequently, according to the results of McMullen[26], Bedford[3], Mauldin and Williams[25], Kenyon and Peres[21, 22], and our Theorem 2.5.1, we claim that:

- 1-dimensional case (cf. [20] or Theorem 2.5.1). Let $n \geq 2$ be an integer. Let $\Xi \subseteq \mathbb{R}$ be a finite set. If there exist numbers $\Gamma, \gamma \in \mathbb{R}$, $\Gamma \neq 0$, such that $\Gamma\Xi + \gamma \subseteq \mathbb{Q}$, then the Hausdorff dimension of the attractor of IFS $\{\frac{1}{n}x + b|b \in \Xi\}$ is determined.
- 2-dimension case (cf. [3, 21, 26]). Let $m, n \geq 2$ be integers and $B = \begin{pmatrix} 1/n & 0 \\ 0 & 1/m \end{pmatrix}$. Let $\Xi \subseteq \mathbb{R}^2$ be a finite set. If there exist invertible matrix $\Gamma \in \mathbb{R}^{2\times 2}$ and vector $\gamma \in \mathbb{R}^2$ such that $\Gamma\Xi + \gamma \subseteq \mathbb{Q}^2$ and $B\Gamma = \Gamma B$, then the box dimension of the attractor $A(\Phi)$ of IFS $\Phi = \{Bx + b|b \in \Xi\}$ is

determined. For the Hausdorff dimension, there are sequences of boundaries $g_s \leq \dim_H A(\Phi) \leq G_s$ such that $\dim_H A(\Phi) = \lim_{s\to\infty} g_s = \lim_{s\to\infty} G_s$. Moreover, the value of $\dim_H A(\Phi)$ can be computed explicitly in some cases (cf. Proposition 3.4 and some examples in [21]).

- High-dimension case (cf. [22]). Let $n_1, \cdots, n_d \geq 2$ be integers and $B = \mathrm{diag}(1/n_1, \cdots, 1/n_d)$. Let $\Xi \subseteq \mathbb{R}^d$ be a finite set. If there exist invertible matrix $\Gamma \in \mathbb{R}^{d\times d}$ and vector $\gamma \in \mathbb{R}^d$ such that $\Gamma\Xi + \gamma \subseteq \{0, 1, \cdots, n_1 - 1\} \times \cdots \times \{0, 1, \cdots, n_d - 1\}$ and $B\Gamma = \Gamma B$, then the Hausdorff and box dimensions of the attractor $A(\Phi)$ of IFS $\Phi = \{Bx + b | b \in \Xi\}$ are determined.

On the other hand, in §3, in the plane case, we give an generalization of McMullen[26]'s work to the net M-matrix rather than Kenyon and Peres[21]'s work. We get lower and upper estimations of Hausdorff dimension of the components of invariant c-vector of *net M-matrix* on $\mathbb{R}^2$ (see Theorem 3.2).

We study an example of plane self-affine set (with overlaps) in §5. We find the associated net M-matrix according to the proof of Theorem 4.1, which in fact is a general method to determine the net M-matrix. Then we perform the numerical calculations of the lower and upper boundaries of this self-affine set according to Kenyon and Peres' formulas and our Theorem 3.2. It seems to us that our lower boundaries are sharper than Kenyon and Peres' but our upper boundaries are bigger. For the sake of convenience of the readers, a brief review of the M-matrix theory is given in §2.

2. Preliminaries.

In this section, we give a brief review of the M-matrix theory established in [20].

2.1. M-matrix.

Let $\phi : \mathbb{R}^d \to \mathbb{R}^d$ be a mapping on $\mathbb{R}^d$. Define

$$l_\phi = \sup\{a \in \mathbb{R} | \|\phi(x) - \phi(y)\|_2 \geq a\|x - y\|_2 \text{ for all } x, y \in \mathbb{R}^d\};$$

$$u_\phi = \inf\{a \in \mathbb{R} | \|\phi(x) - \phi(y)\|_2 \leq a\|x - y\|_2 \text{ for all } x, y \in \mathbb{R}^d\}.$$

In this paper, we only consider the mappings with $l_\phi > 0$ and $u_\phi < \infty$, i.e. the bi-Lipschitz mappings. ϕ is a contraction if $u_\phi < 1$ and ϕ is a similarity if $l_\phi = u_\phi$. We denote the sets of all finite sets of mappings, similarities, and contractions on $\mathbb{R}^d$ by $\mathfrak{X}$, $\mathfrak{S}$ and $\mathfrak{C}$ respectively. Define

$$\begin{aligned} \mathfrak{M}(T,T) &= \{(M_{ij})_{1\leq i,j\leq T}; M_{ij} \in \mathfrak{X}\}, \\ \mathfrak{M}_c(T,T) &= \{(M_{ij})_{1\leq i,j\leq T}; M_{ij} \in \mathfrak{C}\}, \\ \mathfrak{M}_s(T,T) &= \{(M_{ij})_{1\leq i,j\leq T}; M_{ij} \in \mathfrak{S}\}. \end{aligned} \tag{2.1.1}$$

We call a matrix $M \in \mathfrak{M}(T,T)$ a M-matrix of size $T \times T$. For M-matrices $(M_{ij}), (N_{ij}) \in \mathfrak{M}(T,T)$, we define $(M_{ij}) \cup (N_{ij}) = (M_{ij} \cup N_{ij})$, $(M_{ij}) \cap (N_{ij}) = (M_{ij} \cap N_{ij})$, and $(M_{ij})(N_{ij}) = (\cup_k M_{ik}N_{kj})$. Then $(\mathfrak{M}(T,T), \cup, \cap, \cdot)$ satisfies some algebraic law with $\emptyset$ as "zero" element and I as "unit" element, where $\emptyset$ is the M-matrix whose entries are all empty sets and $I = \mathrm{diag}(\{1\}, \cdots, \{1\})$, 1 is the identity mapping. We call $\mathfrak{M}(T,T)$ *M-algebra* for the time being. $\mathfrak{M}_c(T,T)$ and $\mathfrak{M}_s(T,T)$ form two M-subalgebras of $\mathfrak{M}(T,T)$.

Definition 2.1.1. *For an M-matrix $M = (M_{ij}) \in \mathfrak{M}(T,T)$ (or numerical matrix $A = (a_{ij})_{1\leq i,j\leq m}$, $a_{ij} \in \mathbb{R}$), M (or A) is* irreducible *if for any $i, j = 1, \cdots, m$,* *there exist some* $k_1, \cdots, k_s \in \{1, \cdots, m\}$ *such that* $M_{ik_1}M_{k_1k_2}\cdots M_{k_sj} \neq \emptyset$ *(or* $a_{ik_1}a_{k_1k_2}\cdots a_{k_sj} \neq 0$*).*

A permutation M-matrix is an M-matrix $P = (P_{ij}) \in \mathfrak{M}(T,T)$ such that each column and row has precisely one nonempty entry $P_{ij} = \{1\}$, where 1 is the identity mapping, all the other entries are empty set. Then we have

Theorem 2.1.1. *For any* $M \in \mathfrak{M}(T,T)$*, there exists a permutation M-matrix* $P \in \mathfrak{M}(T,T)$ *such that*

$$PMP^t = \begin{pmatrix} H_1 & B_{12} & \cdots & B_{1s} \\ \emptyset & H_2 & \cdots & B_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ \emptyset & \emptyset & \cdots & H_s \end{pmatrix}$$

where, for each $i = 1, \cdots, s$*, either* $H_i \in \mathfrak{M}(T_i, T_i)$ *is irreducible with* $T_i > 0$ *or* $H_i = (\{\emptyset\})$*.*

2.2. Space of c-vectors. Let $\mathcal{K}$ be the set of all non-empty compact subsets of $\mathbb{R}^d$: $\mathcal{K} = \{E \subset \mathbb{R}^d | E \text{ is compact}, E \neq \emptyset\}$. Let $\overline{\mathcal{K}} = \mathcal{K} \cup \{\emptyset\}$ be the set of all compact subset of $\mathbb{R}^d$. Define $\overline{\mathcal{K}}^T = \{(E_1, \cdots, E_T)^t | E_1, \cdots, E_T \in \overline{\mathcal{K}}\}$ and $\mathcal{K}^T = \{(E_1, \cdots, E_T)^t | E_1, \cdots, E_T \in \mathcal{K}\}$. $\mathcal{K}^T$ and $\overline{\mathcal{K}}^T$ are called T-dimensional c-space and extended c-space. A vector $(E_i)_{1\le i\le T} \in \overline{\mathcal{K}}^T$ is called a c-vector. Naturally, we can define union and intersection operations on $\overline{\mathcal{K}}^m$: $(E_i)_{1\le i\le T} \cup (F_i)_{1\le i\le T} = (E_i \cup F_i)_{1\le i\le T}$ and $(E_i)_{1\le i\le T} \cap (F_i)_{1\le i\le T} = (E_i \cap F_i)_{1\le i\le T}$. For any set of mappings $\Phi \in \mathfrak{X}$ and subset $E \subset \mathbb{R}^d$, denote the set $\{f(x) | f \in \Phi, x \in E\}$ by $\Phi(E)$. For any $M = (M_{ij}) \in \mathfrak{M}(T,T)$ and $S = (S_1, \cdots, S_T)^t$, $S_i \subseteq \mathbb{R}^d$, define $M(S) = (\cup_k M_{ik}(S_k))_{1\le i\le T}$.

Definition 2.2.1. *Let* $M \in \mathfrak{M}(T,T)$*. A c-vector* $E \in \overline{\mathcal{K}}^T$ *is said to be invariant under* M *if* $M(E) = E$*.*

Definition 2.2.2. *Let* $M \in \mathfrak{M}(T,T)$*. An invariant c-vector* $E \in \overline{\mathcal{K}}^T$ *of* M *is called maximal if for any invariant c-vector* $F \in \overline{\mathcal{K}}^T$ *of* M*,* $F \subseteq E$*. We denote the maximal* M*-invariant c-vector by* $A(M)$*.*

Theorem 2.2.1. *Let* $M \in \mathfrak{M}(T,T)$*. Suppose there exists an integer* $k > 0$ *such that* $M^k \in \mathfrak{M}_c(T,T)$*. Then there exists an unique invariant c-vector* $\mathcal{E} \in \overline{\mathcal{K}}^T$ *of* M *such that*

$$\begin{cases} \overline{\cup_{i\ge1} M^i(F)} = \cup_{i\ge1} M^i(F) \cup \mathcal{E} \in \overline{\mathcal{K}}^T, & \forall F \in \mathcal{K}^T, \\ \cap_{k\ge1} \overline{\cup_{i\ge k} M^i(F)} = \mathcal{E}, & \forall F \in \mathcal{K}^T, \\ \overline{\cup_{i\ge1} M^i(F)} \subseteq \cup_{i\ge1} M^i(F) \cup \mathcal{E} \in \overline{\mathcal{K}}^T, & \forall F \in \overline{\mathcal{K}}^T, \\ \cap_{k\ge1} \overline{\cup_{i\ge k} M^i(F)} \subseteq \mathcal{E}, & \forall F \in \overline{\mathcal{K}}^T. \end{cases} \tag{2.2.1}$$

Furthermore, $\mathcal{E} = A(M)$*.*

Theorem 2.2.1 indicates that the maximal invariant c-vector $A(M)$ exists for an M-matrix $M \in \mathfrak{M}(T,T)$ if $M^k \in \mathfrak{M}_c(T,T)$ for some k. Furthermore, $A(M)$ has some "*attractive*" properties. The second formula of (2.2.1) states that $A(M)$ can be obtained by iterating the action of M on any c-vector $F \in \mathcal{K}^T$. The following

Theorem completely determines the invariant c-vector of an M-matrix M with the assumption that $M^k \in \mathfrak{M}_c(T,T)$ for some k.

THEOREM 2.2.2. *Let $M \in \mathfrak{M}(T,T)$. Suppose there exists an integer $k > 0$ such that $M^k \in \mathfrak{M}_c(T,T)$.*

(1) If $E \in \mathcal{K}^T$ and $M(E) = E$ then $A(M) = E$.
(2) For the empty M-matrix $\emptyset$, $A(\emptyset) = \emptyset$.
(3) If M is irreducible, then $A(M)$ is the only non-trivial invariant c-vector of M and $A(M) \in \mathcal{K}^T$.
(4) Suppose P is a permutation M-matrix. Then E is an invariant c-vector of M if and only if $P(E)$ is invariant under PMP^t. In particular, $A(PMP^t) = P(A(M))$.
(5) Suppose $M = \begin{pmatrix} H_1 & \emptyset \\ \emptyset & H_2 \end{pmatrix}$. Then any invariant c-vector E of M must has the form $\begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$, E_1 and E_2 are invariant c-vectors of H_1 and H_2 respectively. In particular, $A(M) = \begin{pmatrix} A(H_1) \\ A(H_2) \end{pmatrix}$
(6) Suppose $M = \begin{pmatrix} H_1 & B \\ \emptyset & H_2 \end{pmatrix} \in \mathfrak{M}_c(T,T)$, and $B \neq \emptyset$. Then any invariant c-vector E of M must has the form

$$\begin{pmatrix} E_1 \cup \overline{(\cup_{i=0}^{\infty} H_1^i B(E_2))} \\ E_2 \end{pmatrix},$$

where E_1 and E_2 are invariant c-vectors of H_1 and H_2 respectively. In particular,

$$A(M) = \begin{pmatrix} A(H_1) \cup (\cup_{i=0}^{\infty} H_1^i B(A(H_2))) \\ A(H_2) \end{pmatrix}.$$

Furthermore, if $H_2 = \emptyset$ then

$$A(M) = \begin{pmatrix} A(H_1) \\ \emptyset \end{pmatrix}.$$

REMARK 2.2.1. *In [20], we have proved that several different kinds of fractals can be described uniformly by the maximal invariant c-vectors of certain M-matrices, such as the attractors of iterated function systems (IFS, Barnsley[2], Falconer[10], Hutchinson[17], Mandelbrot[24]), recurrent sets (Dekking[8], Bedford[4]), graph directed construction objects (Mauldin and Williams[25]), Julia sets and limit sets of some Kleinian groups which have been studied by limit sets of conformal iterated function schemes (Jenkinson and Pollcott[19]) and Markov partitions for conformal dynamical systems equipped with invariant densities (McMullen[27, 28, 29]).*

2.3. Code Space. Now we begin to establish the code space associated with an M-matrix $M = (M_{ij}) \in \mathfrak{M}_c(T,T)$. Suppose M_{ij} has t_{ij} mappings, $M_{ij} = \{\phi_{ijk} | 1 \leq k \leq t_{ij}\}$ and the maximal invariant c-vector $A(M) = (E_1, \cdots, E_T)^t$. Define

$$\mathfrak{D}_{j_0}(M) = \{\underbrace{(j_0(j_1k_1)\cdots(j_sk_s)\cdots)}_{\text{infinite sequence}} | 1 \leq j_s \leq T, M_{j_{s-1}j_s} \neq \emptyset, \text{ and } 1 \leq k_s \leq t_{j_{s-1}j_s}\}.$$

Let $\mathfrak{D}(M) = \cup_j \mathfrak{D}_j(M)$. We call $\mathfrak{D}(M)$ the code space of M. For $\alpha = (j_0(j_1k_1)(j_2k_2)\cdots) \in \mathfrak{D}$, let $\alpha_s = (j_0(j_1k_1)(j_2k_2)\cdots(j_sk_s))$, $s = 0,1,2,\cdots$. Let $\mathfrak{D}^0_j(M) = \{\alpha_s | \alpha \in \mathfrak{D}_j(M), s = 0,1,2,\cdots\}$. Define $\mathfrak{D}^0(M) = \cup_{1\le j\le T}\mathfrak{D}^0_j(M)$. $\mathfrak{D}^0(M)$ is called the index space of M. For convenience, we shall frequently write $\mathfrak{D}(M), \mathfrak{D}^0(M), \mathfrak{D}_i(M), \mathfrak{D}^0_i(M)$ as $\mathfrak{D}, \mathfrak{D}^0, \mathfrak{D}_i, \mathfrak{D}^0_i$ respectively.

For $\alpha = (j_0(j_1k_1)\cdots(j_sk_s)) \in \mathfrak{D}^0$, write $|\alpha|$ for s, the length of α. For $\beta = (j_s(t_1k'_1)\cdots) \in \mathfrak{D}($ or $\in \mathfrak{D}^0)$, define $\alpha\beta = (j_0(j_1k_1)\cdots(j_sk_s)(t_1k'_1)(t_2k'_2)\cdots) \in \mathfrak{D}(M)($ or $\in \mathfrak{D}^0)$. We get a map $\sigma_\alpha : \beta \mapsto \alpha\beta$ from $\mathfrak{D}_{j_s}$ to $\mathfrak{D}_{j_0}$ or from $\mathfrak{D}^0_{j_s}$ to $\mathfrak{D}^0_{j_0}$. For $\beta \in \mathfrak{D}$ or $\mathfrak{D}^0$, we write $\alpha < \beta$ if there has a γ such that $\beta = \alpha\gamma$. Let $\mathfrak{D}_\alpha = \{\beta \in \mathfrak{D} | \alpha < \beta\}$. Then

$$\begin{cases} \mathfrak{D}_\alpha = \sigma_\alpha(\mathfrak{D}_{j_s}), \\ \mathfrak{D}_{\alpha\gamma} = \sigma_\alpha(\mathfrak{D}_\gamma) \quad \text{for } \gamma \in \mathfrak{D}^0_{j_s}. \end{cases} \tag{2.3.1}$$

Obviously

$$\begin{cases} \mathfrak{D}_i = \cup_{\beta\in\mathfrak{D}^0_i, |\beta|=k} \mathfrak{D}_\beta, \\ \mathfrak{D}_\alpha = \cup_{\beta\in\mathfrak{D}^0, |\beta|=|\alpha|+k, \alpha<\beta} \mathfrak{D}_\beta, \quad k = 1,2,\cdots, \end{cases} \tag{2.3.2}$$

where the union are disjoint.

If $\alpha = (j_0(j_1k_1)\cdots(j_sk_s)\cdots) \in \mathfrak{D}$, we have a sequence of subsets containing α:

$$\begin{cases} \mathfrak{D}_{j_0} = \mathfrak{D}_{\alpha_0} \supseteq \mathfrak{D}_{\alpha_1} \supseteq \cdots \supseteq \{\alpha\} \\ \cap_{s=0}^\infty \mathfrak{D}_{\alpha_s} = \{\alpha\} \end{cases} \tag{2.3.3}$$

Now we can define a topology on $\mathfrak{D}$ by setting a sub-basis of open sets to be $\mathfrak{D}_\alpha$, $\alpha \in \mathfrak{D}^0$. This is a complete, Hausdorff and compact topology on $\mathfrak{D}$ according to (2.3.2) and (2.3.3).

For $\alpha = (j_0(j_1k_1)\cdots(j_sk_s)) \in \mathfrak{D}^0$, denote $\phi_{j_0j_1k_1}\phi_{j_1j_2k_2}\cdots\phi_{j_{s-1}j_sk_s}$ by ϕ_α and denote $\phi_\alpha(E_{j_s})$ by E_α. Then, similar to (2.3.2) and (2.3.3), we have

$$\begin{cases} E_i = \cup_{\beta\in\mathfrak{D}^0_i, |\beta|=k} E_\beta, \\ E_\alpha = \cup_{\beta\in\mathfrak{D}^0, |\beta|=|\alpha|+k, \alpha<\beta} E_\beta, & \text{for } \alpha \in \mathfrak{D}^0, \\ E_{\alpha_0} \supseteq E_{\alpha_1} \supseteq \cdots \supseteq E_{\alpha_s} \supseteq \cdots & \text{for } \alpha \in \mathfrak{D}, \\ \cap_{s=1}^\infty E_{\alpha_s} = \{a_\alpha\} \text{ for certain } a_\alpha \in E_{\alpha_0}, \end{cases} \tag{2.3.4}$$

where $k = 1,2,\cdots$. Define mapping

$$\pi : \mathfrak{D} \to X, \pi(\alpha) = a_\alpha.$$

Then π is continuous, $\pi(\mathfrak{D}) = \cup_{i=1}^T E_i$, and for $\alpha \in \mathfrak{D}^0$,

$$\begin{cases} \pi(\mathfrak{D}_\alpha) = E_\alpha \\ \pi \circ \sigma_\alpha = \phi_\alpha \circ \pi. \end{cases} \tag{2.3.5}$$

Denote the restriction of π on $\mathfrak{D}_i$ by π_i. Then π_i is also continuous and for $\alpha = (i\cdots(jk)) \in \mathfrak{D}^0_i$,

$$\begin{cases} \pi_i \circ \sigma_\alpha = \phi_\alpha \circ \pi_j, \\ \pi_i(\mathfrak{D}_\alpha) = E_\alpha. \end{cases} \tag{2.3.6}$$

2.4. Invariant measure vector. For a measure μ on $\mathbb{R}^d$ and $x \in \mathbb{R}^d$, we define $\theta_*^k(\mu, x) = \liminf_{r\to 0} \frac{\mu(B(x,r))}{\varpi_k r^k}$, $\theta^{*k}(\mu, x) = \limsup_{r\to 0} \frac{\mu(B(x,r))}{\varpi_k r^k}$ and $\theta^k(\mu, x)$ to be their common value if both are equal, where ϖ_k is a suitable normalizing constant(cf. [13] p.171. If k is an integer, $\varpi_k = \mathcal{L}^k\{x \in \mathbb{R}^k : |x| \le 1\}$ is the volume of unit ball, where $\mathcal{L}^k$ is the k-dimensional Lebesgue measure. For arbitrary k we set $\varpi_k = \Gamma(1/2)^k/\Gamma((k/2)+1)$). Then(cf. [13] p.181) we have the following relations between μ and Hausdorff measure:

$$\text{If } \theta^{*k}(\mu, x) \ge \lambda \text{ for all } x \in E, \text{ then } H^k(E) \le \lambda^{-1}\mu(E). \tag{2.4.1}$$

$$\text{If } \theta^{*k}(\mu, x) \le \lambda \text{ for all } x \in E, \text{ then } H^k(E) \ge 2^{-k}\lambda^{-1}\mu(E). \tag{2.4.2}$$

Suppose $M = (M_{ij}) \in \mathfrak{M}_c(T, T)$ is irreducible, $M_{ij} = \{\phi_{ijk} | k = 1, \cdots, t_{ij}\}$, and $A(M) = (E_1, \cdots, E_T)^t$. Choose real numbers $\omega_{ijk} \in (0, 1]$ for $1 \le i, j \le T$ and $1 \le k \le t_{ij}$, such that $\sum_k \omega_{ijk} = 1$ for every i, j with $M_{ij} \ne \emptyset$. Let $\Lambda = (\lambda_{ij})$ be a real matrix with each $\lambda_{ij} > 0$ if $M_{ij} \ne \emptyset$, $\lambda_{ij} = 0$ if $M_{ij} = \emptyset$ and $\sum_j \lambda_{ij} = 1$ for $i = 1, \cdots, T$. For $\alpha = (j_0(j_1k_1)\cdots(j_sk_s)) \in \mathfrak{D}^0$ with $s \ge 1$, denote $\Pi_{i=1}^s \lambda_{j_{i-1}j_i}\omega_{j_{i-1}j_ik_i}$ by $\triangle_\alpha$. Then we can define measure τ on $\mathfrak{D}$ by setting

$$\tau(\mathfrak{D}_\alpha) = \triangle_\alpha. \tag{2.4.3}$$

Denote τ_i for the restriction of τ on $\mathfrak{D}_i$. Define $\mu_i = \pi_i^*(\tau_i)$, where $\pi_i^*(\tau_i)(A) = \tau_i(\pi_i^{-1}(A))$. Then μ_i is a Borel regular measure on $\mathbb{R}^d$ with support E_i and mass 1. We can prove that $(\mu_1, \cdots, \mu_T)$ satisfies the following *invariant property*:

$$\mu_i = \sum_j \lambda_{ij} \sum_{k=1}^{t_{ij}} \omega_{ijk}\phi_{ijk}^*(\mu_j), \tag{2.4.4}$$

where $\phi_{ijk}^*(\mu_j)(A) = \mu_j(\phi_{ijk}^{-1}(A))$. We call $(\mu_1, \cdots, \mu_T)$ the invariant measure vector of M on $A(M)$ and τ the invariant measure of M on $\mathfrak{D}$ associated with (Λ, ω) respectively.

For $\alpha \in \mathfrak{D}_i^0$ let τ_α be the restriction of τ on $\mathfrak{D}_\alpha$. Set $\mu_\alpha = \tau_\alpha \circ \pi_i^{-1}$. Then

$$\begin{cases} \text{supp}(\mu_\alpha) = E_\alpha, \\ \mu_\alpha(E_\alpha) = \triangle_\alpha \\ \mu_i = \sum_{\beta\in\mathfrak{D}_i^0, |\beta|=k} \mu_\beta, \\ \mu_\alpha = \sum_{\beta\in\mathfrak{D}^0, |\beta|=|\alpha|+k, \alpha<\beta} \mu_\beta \end{cases} \tag{2.4.5}$$

for any $k = 1, 2, \cdots$.

2.5. Hausdorff dimension of invariant c-vector. For any set of mappings $\Phi \in \mathfrak{X}$, define $f(\Phi, x) = \sum_{\phi\in\Phi} u_\phi^x$ $(x \ge 0)$. For $M = (M_{ij}) \in \mathfrak{M}(T, T)$, define matrices $F(M, x) = (f(M_{ij}, x))$, $x \ge 0$. Then the eigenvalue of $F(M, x)$ will be used to estimate the Hausdorff dimension of the components of invariant c-vector of M. By Perron-Frobenius Theorem(cf. [14]), $F(M, x)$ has a positive real eigenvalue equal to its spectral radius and this maximal real eigenvalue will decrease when x is increasing.

The following definition generalizes the open set condition for iterated function systems to M-matrices.

Definition 2.5.1. *Let $M = (M_{ij}) \in \mathfrak{M}(T, T)$. M satisfies the open set condition if there exist non-empty bounded open sets $U_i \subseteq \mathbb{R}^d$, $i = 1, \cdots, T$, such that*

1. $M(U) \subset U$, *where* $U = (U_1, \cdots, U_T)^t$,
2. $M_{ij}U_j \cap M_{ik}U_k = \emptyset$ *if* $j \neq k$,
3. $\phi(U_j) \cap \psi(U_j) = \emptyset$ *if* $\phi, \psi \in M_{ij}$ *and* $\phi \neq \psi$.

When M is irreducible, we have shown in [20] that each component of $A(M)$ has the same Hausdorff dimension. So, for convenience, we can write $\dim_H A(M)$ for each $\dim_H E_i$. The equality $\dim_H A(H_i) = D_i$ in (3) of the following theorem was proved by Mauldin and Williams[25].

THEOREM 2.5.1. *Let* $M \in \mathfrak{M}(T,T)$ *and* $M^k \in \mathfrak{M}_c(T,T)$ *for some* k. *Suppose* P *is a permutation M-matrix such that*

$$PMP^t = \begin{pmatrix} H_1 & B_{12} & \cdots & B_{1s} \\ \emptyset & H_2 & \cdots & B_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ \emptyset & \emptyset & \cdots & H_s \end{pmatrix}$$

where $H_i \in \mathfrak{M}(T_i, T_i)$, $i = 1, \cdots, s$, *are either irreducible or empty with size* 1×1.

(1) $A(M)$, $A(PMP^t)$ *and* $A(H_i)$, $i = 1, \cdots, s$, *exist.*

$$A(M) = P^t A(PMP^t) = \cup_{i=0}^{\infty} M^i P^t \begin{pmatrix} A(H_1) \\ \cdot \\ \cdot \\ \cdot \\ A(H_s) \end{pmatrix}$$

(2) Write $A(PMP^t) = P(A(M))$ *as* $(\mathcal{E}^{(1)}, \cdots, \mathcal{E}^{(s)})^t$, *where* $\mathcal{E}^{(i)} \in \mathcal{K}^{T_i}$, $i = 1, \cdots, s$. *Then for any fixed i, each component of* $\mathcal{E}^{(i)}$ *has the same Hausdorff dimension. Write this dimension as* $\dim_H \mathcal{E}^{(i)}$, *then*

$$\dim_H \mathcal{E}^{(i)} = \max\{\dim_H A(H_i), \dim_H \mathcal{E}^{(j)} | B_{ij} \neq \emptyset\},$$

(3) If $H_i \neq \emptyset$, *then* $\dim_H A(H_i) \leq D_i$, *where* D_i *is the unique number such that the biggest real eigenvalue of* $F(H_i, D_i)$ *is* 1. *In particular, if* $H_i \in \mathfrak{M}_s(T_i, T_i)$ *and the open set condition holds for* H_i, *then* $\dim_H A(H_i) = D_i$.

3. Net M-matrix and the Hausdorff dimensions of its invariant c-vectors. In this section, we shall define a special kind of M-matrices which are called "net M-matrix". We shall give an estimation of the Hausdorff dimensions of their invariant c-vectors in plane case. In the next section, we shall prove that certain self-affine sets can be expressed as the union of components of the maximal invariant c-vector of such M-matrix.

DEFINITION 3.1. *Let* $B = diag(1/n_1, 1/n_2, \cdots, 1/n_d)$ *be a* $d \times d$ *diagonal matrix, where* $n_i \geq 2$ *are integers. An M-matrix* $M = (M_{ij}) \in \mathfrak{M}(T,T)$ *is called a* net M-matrix *on* $\mathbb{R}^d$ *(with respect to B) if*

(1) For any $\phi \in \cup M_{ij}$, $\phi(x) = B(x + b)$ *and* $b = (\eta_1, \cdots, \eta_d)^t$ *for some integers* $0 \leq \eta_i \leq n_i - 1$.

(2) Suppose $M_{ij} = \{\phi_{ijk} | 1 \leq k \leq t_{ij}\}$, $1 \leq i, j \leq T$. *If* $\phi_{ijk} = \phi_{ij'k'}$, *then* $j = j'$ *and hence* $k = k'$.

From the definition, it is easy to prove the following result by direct computation.

THEOREM 3.1. *Let* $B = diag(1/n_1, 1/n_2, \cdots, 1/n_d)$ *be a* $d \times d$ *diagonal matrix, where* $n_i \geq 2$ *are integers. Suppose* M *is a net M-matrix with respect to* B. *Then*

(1) For any permutation M-matrix P, PMP^t is a net M-matrix with respect to B.
(2) For any subset $\{j_1, \cdots, j_L\} \subseteq \{1, \cdots, T\}$, the M-matrix $(M_{j_s j_t})_{1 \le s,t \le L}$ is a net M-matrix with respect to B.
(3) M^k is a net M-matrix with respect to B^k for each $k = 1, 2, 3, \cdots$.
(4) Let $U = \{(x_1, \cdots, x_d)^t \in \mathbb{R}^d | 0 < x_i < 1\}$ and $I = \overline{U}$. Then M satisfies open set condition with respect to the open sets $U_1 = \cdots = U_T = U$. Consequently, if $A(M) = (E_1, \cdots, E_T)^t$ is the maximal invariant c-vector of M, then $E_i \subseteq I$.

In the 1-dimension case, all mappings in a net M-matrix on $\mathbb{R}$ are contractive similarities and the open set condition holds. So Theorem 2.5.1 can be applied to determined the Hausdorff dimensions of the invariant c-vectors. In the 2-dimension case, if the size of a net M-matrix M is 1×1, i.e. $T = 1$, then the maximal invariant c-vector $A(M) = E_1$ is a McMullen carpet (cf. [26]). Conversely, a McMullen carpet can be represented as the maximal invariant c-vector of a 1×1 net M-matrix M. In general, a B^{-1}-invariant Sofic set defined by Kenyon and Peres in [21] is equivalent to the set $\cup_{i=1}^T E_i$, where $(E_1, \cdots, E_T)^t$ is the maximal invariant c-vector of a $T \times T$ net M-matrix M. Hence, the results of McMullen[26] and Kenyon and Peres[21] about the dimensions can be applied to the invariant c-vectors of M on plane $\mathbb{R}^2$. In the rest of this section, we shall consider the net M-matrix on $\mathbb{R}^2$.

Given $n \ge m \ge 2$, let $B = \text{diag}(1/n, 1/m)$. Let $M = (M_{ij})$ be a net M-matrix with respect to B. Suppose M_{ij} consists of t_{ij} mappings,

$$M_{ij} = \{\phi_{ijt} | \phi_{ijt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x + x_{ijt}}{n} \\ \frac{y + y_{ijt}}{m} \end{pmatrix}, t = 1, \cdots, t_{ij}\}. \tag{3.1}$$

By (1) of Definition 3.1, x_{ijt} and y_{ijt} are integers, $0 \le x_{ijt} < n$ and $0 \le y_{ijt} < m$. For each $0 \le k < m$ and $1 \le i, j \le T$, let

$$\begin{cases} M_{ijk} = \{\phi_{ijt} \in M_{ij} | y_{ijt} = k\}, \\ a_{ijk} = \# M_{ijk}, \end{cases} \tag{3.2}$$

where $\# M_{ijk}$ denotes the cardinality of M_{ijk}. Define matrices

$$A_k = (a_{ijk})_{1 \le i,j \le T}, k = 0, \cdots, m-1. \tag{3.3}$$

Give positive numbers b_{ijk}, $1 \le i, j \le T$, $0 \le k < m$. Define matrices $B_k = (a_{ijk} b_{ijk})_{1 \le i,j \le T}, k = 0, \cdots, m-1$, and $H = \sum_{0 \le k < m} B_k$. According to the Perron-Frobenius Theorem (cf. [14]), there exists an unique maximal real eigenvalue λ of H with positive eigenvector $(e_1, \cdots, e_m)^t$. Let

$$D = \frac{\ln \lambda}{\ln m}. \tag{3.4}$$

Denote the (i,j)-th entry of $m^{-D} H$ as $h_{ij}(D)$, i.e. $h_{ij}(D) = m^{-D} \sum_{0 \le k < m} a_{ijk} b_{ijk}$.

Let

$$\Lambda = (\lambda_{ij}) = (e_i^{-1} e_j h_{ij}(D)). \tag{3.5}$$

Then it is easy to check $\sum_j \lambda_{ij} = 1$. For each $\phi_{ijt} \in M_{ij}$, we associate to it a number

$$\omega_{ijt} = m^{-D} b_{ijt'} / h_{ij}(D), \tag{3.6}$$

where $t' = y_{ijt}$. Then $\sum_{t=1}^{t_{ij}} \omega_{ijt} = 1$ for any i, j when $M_{ij} \neq \emptyset$. For this (Λ, ω), we get the invariant measure vector $\mu = (\mu_1, \cdots, \mu_T)^t$ of M on $A(M)$ and the invariant measure τ of M on $\mathfrak{D}$ respectively. Then for any $\alpha = (j_0(j_1t_1)(j_2t_2)\cdots(j_st_s)) \in \mathfrak{D}^0$,

$$\begin{aligned}
&\mu_\alpha(E_\alpha) = \triangle_\alpha \\
&= \textstyle\prod_{i=1}^{s} \lambda_{j_{i-1}j_i}\omega_{j_{i-1}j_it_i} \\
&= e_{j_0}^{-1} e_{j_1} h_{j_0j_1}(D) \cdot m^{-D} b_{j_0j_1t'_1}/h_{j_0j_1}(D) \\
&\quad \cdot e_{j_1}^{-1} e_{j_2} h_{j_1j_2}(D) \cdot m^{-D} b_{j_1j_2t'_2}/h_{j_1j_2}(D) \\
&\quad \cdots \\
&\quad \cdot e_{j_{s-1}}^{-1} e_{j_s} h_{j_{s-1}j_s}(D) \cdot m^{-D} b_{j_{s-1}j_st'_s}/h_{j_{s-1}j_s}(D) \\
&= e_{j_0}^{-1} e_{j_s} m^{-sD} b_{j_0j_1t'_1} b_{j_1j_2t'_2} \cdots b_{j_{s-1}j_st'_s},
\end{aligned} \tag{3.7}$$

where $t'_i = y_{j_{i-1}j_it_i}$.

In view of Theorem 2.5.1 and (1) of Theorem 3.1, if we can get the Hausdorff dimension of the maximal invariant c-vector of irreducible net M-matrix, then we can get the Hausdorff dimension of the maximal invariant c-vector of any net M-matrix. The following theorem only consider the irreducible net M-matrix.

THEOREM 3.2. *Let $M = (M_{ij}) \in \mathfrak{M}(T,T)$ be an irreducible net M-matrix on $\mathbb{R}^2$ with respect to $B = diag\ (1/n, 1/m)$, $n \geq m$. Let $\mathcal{E} = (E_1, \cdots, E_T)^t = A(M)$ be the maximal invariant c-vector of M.*

Give positive numbers b_{ijk}, $1 \leq i, j \leq T$, $0 \leq k < m$. Define matrices A_k, B_k, H, Λ, numbers D, λ, ω_{ijk}, and measures μ_i and τ as above. Define functions f_k on $\mathfrak{D}$ by

$$f_k(j_0(j_1t_1)(j_2t_2)\cdots) = (b_{j_0j_1t'_1} b_{j_1j_2t'_2} \cdots b_{j_{l-1}j_lt'_l} \|(B_{t'_{l+1}} \cdots B_{t'_k})_{j_l}\|_\infty)^{\frac{1}{k}}, \quad k = 1, 2, \cdots,$$

where $l = \lfloor k \log_n m \rfloor$ is the maximal integer not bigger than $k \log_n m$, $t'_i = y_{j_{i-1}j_it_i}$, $(B_{t'_{l+1}} \cdots B_{t'_k})_{j_l}$ is the j_l-th row vector of the matrix $B_{t'_{l+1}} \cdots B_{t'_k}$.

a) If $\overline{\lim} f_k(\alpha) \geq 1$ for all $\alpha \in \mathfrak{D}$, then each $\dim_H E_i \leq D$ and $H^D(E_i) < \infty$.

b) If there exist $\mathcal{F}_i \subseteq \mathfrak{D}_i$, $i = 1, \cdots, T$, such that $\tau(\mathcal{F}_i) > 0$ and $\overline{\lim} f_k(\alpha) \leq 1$ for all $\alpha \in \cup \mathcal{F}_i$, then each $\dim_H E_i \geq D$ and $H^D(E_i) > 0$.

Proof. We firstly construct a class of rectangles which will be used to cover E_i. For any positive integers k, p, q, let

$$R_k(p,q) = [\frac{p}{n^l}, \frac{p+1}{n^l}] \times [\frac{q}{m^k} \frac{q+1}{m^k}], \tag{3.8}$$

where $l = \lfloor k \log_n m \rfloor$ is the maximal integer not bigger than $k \log_n m$, i.e., $n^{-l} \geq m^{-k} > n^{-l-1}$. (In the sequel l will always be related to k in this way.)

For each $\alpha = (j_0(j_1t_1)(j_2t_2)\cdots(j_st_s)) \in \mathfrak{D}^0$, by (3.1) we have

$$\phi_\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \phi_{j_0j_1t_1} \circ \cdots \circ \phi_{j_{s-1}j_st_s} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x + x_\alpha}{n^s} \\ \frac{y + y_\alpha}{m^s} \end{pmatrix}.$$

where

$$\left\{ \begin{array}{l} x_\alpha = n^{s-1} x_{j_0j_1t_1} + \cdots + n x_{j_{s-2}j_{s-1}t_{s-1}} + x_{j_{s-1}j_st_s} \\ y_\alpha = m^{s-1} y_{j_0j_1t_1} + \cdots + m y_{j_{s-2}j_{s-1}t_{s-1}} + y_{j_{s-1}j_st_s} \end{array} \right. \tag{3.9}$$

So

$$E_\alpha = \phi_\alpha(E_{j_s}) \subseteq \phi_\alpha(I) = [\frac{x_\alpha}{n^s}, \frac{x_\alpha + 1}{n^s}] \times [\frac{y_\alpha}{m^s}, \frac{y_\alpha + 1}{m^s}],$$

where $I = [0,1] \times [0,1]$ (see (4) of Theorem 3.1). If $k \le s$, then we have $\alpha_k = (j_0(j_1t_1)(j_2t_2)\cdots(j_kt_k)) \in \mathfrak{D}^0$ and

$$E_\alpha = \phi_\alpha(E_{j_s}) \subseteq \phi_{\alpha_k}(I) = [\frac{x_{\alpha_k}}{n^k}, \frac{x_{\alpha_k}+1}{n^k}] \times [\frac{y_{\alpha_k}}{m^k}, \frac{y_{\alpha_k}+1}{m^k}].$$
$$\subseteq [\frac{x_{\alpha_l}}{n^l}, \frac{x_{\alpha_l}+1}{n^l}] \times [\frac{y_{\alpha_k}}{m^k}, \frac{y_{\alpha_k}+1}{m^k}] = R_k(x_{\alpha_l}, y_{\alpha_k}).$$

For $i = 1, \cdots, T$, define

$$\mathfrak{R}_k^{(i)}(p,q) = \{\beta \in \mathfrak{D}_i^0 | |\beta| = k, x_{\beta_l} = p, y_{\beta_k} = q\}. \tag{3.10}$$

Then it is clear that

$$\alpha \in \mathfrak{R}_k^{(i)}(p,q) \Rightarrow E_\alpha \subseteq R_k(p,q), \tag{3.11}$$

For $\alpha = (j_0(j_1t_1)\cdots(j_kt_k)) \in \mathfrak{D}^0$, let $t'_i = y_{j_{i-1}j_it_i}$, $i = 1, \cdots, t$. Then according to (3.10) and (3.9), $\beta = (j_0(i_1s_1)\cdots(i_ks_k)) \in \mathfrak{R}_k^{(j_0)}(x_{\alpha_l}, y_{\alpha_k}) \Leftrightarrow (x_{\beta_l} = x_{\alpha_l}$ and $y_{\beta_k} = y_{\alpha_k}) \Leftrightarrow (x_{j_0i_1s_1} = x_{j_0j_1t_1},\ y_{j_0i_1s_1} = y_{j_0j_1t_1} = t'_1,\ x_{i_{u-1}i_us_u} = x_{j_{u-1}j_ut_u}$ for $1 \le u \le l$ and $y_{i_{v-1}i_vs_v} = y_{j_{v-1}j_vt_v} = t'_v$ for $1 \le v \le k)$. By (2) of Definition 3.1, we have $i_u = j_u$, $s_u = t_u$ for $1 \le u \le l$. By (3.2), we have $\phi_{i_{v-1}i_vs_v} \in M_{i_{v-1}i_vt'_v}$ for $l+1 \le v \le k$. Therefore

$$\begin{aligned}\mathfrak{R}_k^{(j_0)}(x_{\alpha_l}, y_{\alpha_k}) = \{&(j_0(j_1t_1)\cdots(j_lt_l)(i_{l+1}s_{l+1})\cdots(i_ks_k))| \\ &1 \le i_{l+1}, \cdots, i_k \le T, \phi_{j_li_{l+1}s_{l+1}} \in M_{j_li_{l+1}t'_{l+1}}, \\ &\phi_{i_{l+1}i_{l+2}s_{l+2}} \in M_{i_{l+1}i_{l+2}t'_{l+2}}, \cdots, \phi_{i_{k-1}i_ks_k} \in M_{i_{k-1}i_kt'_k}\}.\end{aligned}$$

Hence, by (3.7),

$$\begin{aligned}&\sum\nolimits_{\beta=(j_0(i_1s_1)\cdots(i_ks_k))\in\mathfrak{R}_k^{(j_0)}(x_{\alpha_l},y_{\alpha_k})} \Delta_\beta \\ &= \sum\nolimits_{(j_0(i_1s_1)\cdots(i_ks_k))\in\mathfrak{R}_k^{(j_0)}(x_{\alpha_l},y_{\alpha_k})} e_{j_0}^{-1} e_{i_k} m^{-kD} b_{j_0i_1s'_1} b_{i_1i_2s'_2} \cdots b_{i_{k-1}i_ks'_k} \\ &= \sum\nolimits_{(j_0(j_1t_1)\cdots(j_lt_l)(i_{l+1}s_{l+1})\cdots(i_ks_k))\in\mathfrak{R}_k^{(j_0)}(x_{\alpha_l},y_{\alpha_k})} \\ &\qquad e_{j_0}^{-1} e_{i_k} m^{-kD} b_{j_0j_1s'_1} \cdots b_{j_{l-1}j_ls'_l} b_{j_li_{l+1}s'_{l+1}} b_{i_{l+1}i_{l+2}s'_{l+2}} \cdots b_{i_{k-1}i_ks'_k},\end{aligned} \tag{3.12}$$

where $s'_1 = y_{j_0i_1s_1}, s'_2 = y_{i_1i_2s_2}, \cdots, s'_k = y_{i_{k-1}i_ks_k}$, and $l = \lfloor k \log_n m \rfloor$. Now,

$$\phi_{j_li_{l+1}s_{l+1}} \in M_{j_li_{l+1}t'_{l+1}} \Rightarrow y_{j_li_{l+1}s_{l+1}} = t'_{l+1} \Rightarrow s'_{l+1} = t'_{l+1},$$

$$\phi_{i_{l+1}i_{l+2}s_{l+2}} \in M_{i_{l+1}i_{l+2}t'_{l+2}} \Rightarrow y_{i_{l+1}i_{l+2}s_{l+2}} = t'_{l+2} \Rightarrow s'_{l+2} = t'_{l+2},$$
$$\cdots$$
$$\phi_{i_{k-1}i_ks_k} \in M_{i_{k-1}i_kt'_k} \Rightarrow y_{i_{k-1}i_ks_k} = t'_k \Rightarrow s'_k = t'_k,$$

So, (3.12) becomes

$$\begin{aligned}&\sum\nolimits_{\beta=(j_0(i_1s_1)\cdots(i_ks_k))\in\mathfrak{R}_k^{(j_0)}(x_{\alpha_l},y_{\alpha_k})} \Delta_\beta \\ &= \sum\nolimits_{1\le i_{l+1},\cdots,i_k\le T} \sum\nolimits_{\phi_{j_li_{l+1}s_{l+1}}\in M_{j_li_{l+1}t'_{l+1}}} \sum\nolimits_{\phi_{i_{l+1}i_{l+2}s_{l+2}}\in M_{i_{l+1}i_{l+2}t'_{l+2}}} \cdots \\ &\qquad \sum\nolimits_{\phi_{i_{k-1}i_ks_k}\in M_{i_{k-1}i_kt'_k}} e_{j_0}^{-1} e_{i_k} m^{-kD} b_{j_0j_1t'_1} \cdots b_{j_{l-1}j_lt'_l} b_{j_lj_{l+1}t'_{l+1}} \cdots b_{i_{k-1}j_kt'_k},\end{aligned}$$

$$= e_{j_0}^{-1} m^{-kD} b_{j_0 j_1 t'_1} \cdots b_{j_{l-1} j_l t'_l} \cdot$$
$$\{\sum_{\substack{1\le i_{l+1}\le T\\ \phi_{j_l i_{l+1} s_{l+1}}\in M_{j_l i_{l+1} t'_{l+1}}}} b_{j_l i_{l+1} t'_{l+1}} [\sum_{\substack{1\le i_{l+2}\le T\\ \phi_{i_{l+1} i_{l+2} s_{l+2}}\in M_{i_{l+1} i_{l+2} t'_{l+2}}}} b_{i_{l+1} i_{l+2} t'_{l+2}} \cdot$$
$$(\sum_{\substack{1\le i_{l+3}\le T\\ \phi_{i_{l+2} i_{l+3} s_{l+3}}\in M_{i_{l+2} i_{l+3} t'_{l+3}}}} \cdots \sum_{\substack{1\le i_{k-1}\le T\\ \phi_{i_{k-2} i_{k-1} s_{k-1}}\in M_{i_{k-2} i_{k-1} t'_{k-1}}}} b^{\log_n m-1}_{i_{k-2} i_{k-1} t'_{k-1}} \cdot$$
$$(\sum_{\substack{1\le i_k\le T\\ \phi_{i_{k-1} i_k s_k}\in M_{i_{k-1} i_k t'_k}}} b_{i_{k-1} i_k t'_k} e_{i_k}) \cdots)]\}, \tag{3.13}$$

We denote the (i,j)-th entry of the matrix B_k by $B_{ij}^{(k)}$, i.e., $B_k = (B_{ij}^{(k)})_{1\le i,j\le T}$ and $B_{ij}^{(k)} = a_{ijk} b_{ijk}$. Then

$$\sum_{\beta=(j_0(i_1 s_1)\cdots(i_k s_k))\in \mathfrak{R}_k^{(j_0)}(x_{\alpha_l}, y_{\alpha_k})} \Delta_\beta$$
$$= e_{j_0}^{-1} m^{-kD} b_{j_0 t'_1} \cdots b_{j_{l-1} t'_l} \cdot$$
$$\{\sum_{\substack{1\le i_{l+1}\le T\\ \phi_{j_l i_{l+1} s_{l+1}}\in M_{j_l i_{l+1} t'_{l+1}}}} b_{j_l i_{j+1} t'_{l+1}} [\sum_{\substack{1\le i_{l+2}\le T\\ \phi_{i_{l+1} i_{l+2} s_{l+2}}\in M_{i_{l+1} i_{l+2} t'_{l+2}}}} b_{i_{l+1} i_{l+2} t'_{l+2}} \cdot$$
$$(\sum_{\substack{1\le i_{l+3}\le T\\ \phi_{i_{l+2} i_{l+3} s_{l+3}}\in M_{i_{l+2} i_{l+3} t'_{l+3}}}} \cdots \sum_{\substack{1\le i_{k-1}\le T\\ \phi_{i_{k-2} i_{k-1} s_{k-1}}\in M_{i_{k-2} i_{k-1} t'_{k-1}}}} b_{i_{k-2} i_{k-1} t'_{k-1}} \cdot$$
$$(\sum_{1\le i_k\le T} a_{i_{k-1} i_k t'_k} b_{i_{k-1} i_k t'_k} e_{i_k}))]\}$$
$$= e_{j_0}^{-1} m^{-kD} b_{j_0 j_1 t'_1} \cdots b_{j_{l-1} j_l t'_l} \cdot$$
$$\{\sum_{1\le i_{l+1}\le T} a_{j_l i_{l+1} t'_{l+1}} b_{j_l i_{i+1} t'_{l+1}} [\sum_{1\le i_{l+2}\le T} a_{i_{l+1} i_{l+2} t'_{l+2}} b_{i_{l+1} i_{l+2} t'_{l+2}} \cdot$$
$$(\sum_{1\le i_{l+3}\le T} \cdots \sum_{1\le i_{k-1}\le T} a_{i_{k-2} i_{k-1} t'_{k-1}} b_{i_{k-2} i_{k-1} t'_{k-1}} \cdot$$
$$(\sum_{1\le i_k\le T} a_{i_{k-1} i_k t'_k} b_{i_{k-1} i_k t'_k} e_{i_k}))]\}$$
$$= e_{j_0}^{-1} m^{-kD} b_{j_0 j_1 t'_1} \cdots b_{j_{l-1} j_l t'_l} \cdot$$
$$\{\sum_{1\le i_{l+1}\le T} B^{(t'_{l+1})}_{j_l i_{l+1}} [\sum_{1\le i_{l+2}\le T} B^{(t'_{l+2})}_{i_{l+1} i_{l+2}} \cdot$$
$$(\sum_{1\le i_{l+3}\le T} \cdots \sum_{1\le i_{k-1}\le T} B^{(t'_{k-1})}_{i_{k-2} i_{k-1}} \cdot$$
$$(\sum_{1\le i_k\le T} B^{(t'_k)}_{i_{k-1} i_k} e_{i_k}))]\}$$
$$= e_{j_0}^{-1} m^{-kD} b_{j_0 j_1 t'_1} \cdots b_{j_{l-1} j_l t'_l} \cdot (\text{ the } j_l\text{-th entry of } B_{t'_{l+1}} \cdots B_{t'_k} (e_1, \cdots, e_T)^t)$$
$$= e_{j_0}^{-1} m^{-kD} b_{j_0 j_1 t'_1} \cdots b_{j_{l-1} j_l t'_l} \cdot (B_{t'_{l+1}} \cdots B_{t'_k})_{j_l} (e_1, \cdots, e_T)^t \tag{3.14}$$

For any $a \in E_{j_0}$, there is an element $\alpha = (j_0(j_1t_1)(j_2t_2)\cdots) \in \mathfrak{D}_{j_0}$ such that $a = \pi(\alpha) \in E_{j_0}$. Let $B(a,\rho)$ be the closed ball with center a and ratio ρ. Choose the least k such that $R_k(x_{\alpha_l}, y_{\alpha_k}) \subseteq B(a,\rho)$, where $l = \lfloor k \log_n m \rfloor$. Then

$$\sqrt{(m^{-k+1})^2 + (n^{-l+1})^2} \geq \rho.$$

But $m^{-k} > n^{-l-1}$, so

$$m^{-k} > (\sqrt{m^2+n^4})^{-1}\rho. \tag{3.15}$$

Let $t_i' = y_{j_{i-1}j_it_i}$. Then

$$\begin{aligned}
&\mu_{j_0}(B(a,\rho)) \geq \mu_{j_0}(R_k(x_{\alpha_l}, y_{\alpha_k}))\\
&\geq \textstyle\sum_{\beta\in\mathfrak{R}_k(\alpha_l,\alpha_k)} \mu_\beta(R_k(x_{\alpha_l}, y_{\alpha_k})) \quad \text{(by the third formula of (2.4.5))}\\
&= \textstyle\sum_{\beta\in\mathfrak{R}_k(\alpha_l,\alpha_k)} \mu_\beta(E_\beta) \quad \text{(by the first formula of (2.4.5) and (3.11))}\\
&= \textstyle\sum_{\beta\in\mathfrak{R}_k(\alpha_l,\alpha_k)} \triangle_\beta\\
&= e_{j_0}^{-1} m^{-kD} b_{j_0j_1t_1'} \cdots b_{j_{l-1}j_lt_l'} (B_{t_{l+1}'} \cdots B_{t_k'})_{j_l} (e_1, \cdots, e_T)^t\\
&> e_{j_0}^{-1} (\sqrt{m^2+n^4})^{-D} \rho^D b_{j_0j_1t_1'} \cdots b_{j_{l-1}j_lt_l'} (B_{t_{l+1}'} \cdots B_{t_k'})_{j_l} (e_1, \cdots, e_T)^t\\
&\geq e_{j_0}^{-1} (\sqrt{m^2+n^4})^{-D} \rho^D b_{j_0j_1t_1'} \cdots b_{j_{l-1}j_lt_l'} \cdot (\min_{1\leq i\leq T} e_i)(B_{t_{l+1}'} \cdots B_{t_k'})_{j_l} (1, \cdots, 1)^t\\
&\geq e_{j_0}^{-1} (\sqrt{m^2+n^4})^{-D} \rho^D b_{j_0j_1t_1'} \cdots b_{j_{l-1}j_lt_l'} \cdot (\min_{1\leq i\leq T} e_i) \|(B_{t_{l+1}'} \cdots B_{t_k'})_{j_l}\|_\infty\\
&= e_{j_0}^{-1} (\sqrt{m^2+n^4})^{-D} \rho^D \cdot (\min_{1\leq i\leq T} e_i) \cdot (f_k(\alpha))^k
\end{aligned} \tag{3.16}$$

Hence

$$\begin{aligned}
&\theta^{*D}(\mu_{j_0}, a) = \overline{\lim}_{\rho\to 0} \tfrac{\mu_{j_0}(B(a,\rho))}{\varpi_D \rho^D}\\
&\geq (\varpi_D e_{j_0})^{-1} (\sqrt{m^2+n^4})^{-D} \cdot (\min_i e_i) \cdot \overline{\lim}_{k\to\infty} (f_k(\alpha))^k
\end{aligned} \tag{3.17}$$

Under the condition of (a),

$$\theta^{*D}(\mu_{j_0}, a) \geq (\varpi_D e_{j_0})^{-1} (\sqrt{m^2+n^4})^{-D} \cdot (\min_i e_i). \tag{3.18}$$

by (2.4.1), notice that $\mu_{j_0}(E_{j_0}) = 1$, (3.18) implies $H^D(E_{j_0}) < \infty$,. So $\dim_H E_{j_0} \leq D$. (a) is proved.

Now we begin to prove (b). Fixed an integer $j_0 = 1, \cdots, T$. By the condition of (b), $\overline{\lim} f_k(\alpha) \leq 1$ for all $\alpha \in \mathcal{F}_{j_0} \subseteq \mathfrak{D}_{j_0}$. Let $F_{j_0} = \pi_{j_0}(\mathcal{F}_{j_0}) \subset E_{j_0}$. Then $\mu_{j_0}(F_{j_0}) = \tau_{j_0}(\pi_{j_0}^{-1}(F_{j_0})) \geq \tau_{j_0}(\mathcal{F}_{j_0}) > 0$. Let $\hat{\mu} = \mu_{j_0}|_{F_{j_0}}$ be the restriction of the measure μ_{j_0} on F_{j_0}.

For any $a \in F_{j_0}$, there is an element $\alpha \in \mathcal{F}_{j_0} \subseteq \mathfrak{D}_{j_0}$ such that $a = \pi(\alpha)$. Consider the closed ball $B(a,\rho)$. Suppose ρ is small enough, for example, $\rho < m^{-2}$. This time we choose the maximal integer k such that $m^{-k} > 2\rho$. Then $m^{-k-1} \leq 2\rho$, so

$$m^{-k}/2 > \rho \geq m^{-k-1}/2. \tag{3.19}$$

It is easy to see that

$$B(a,\rho) \subset \cup_{p,q=-1,0,1} R_k(x_{\alpha_l} + p, y_{\alpha_k} + q). \tag{3.20}$$

Let $\Theta = \{\beta \in \mathfrak{D}^0_{j_0} | |\beta| = k\}$. Then $\Theta = \cup_{p,q\in\mathbb{Z}} \mathfrak{R}^{(j_0)}_k(p,q)$. Now $\mathrm{supp}(\mu_\beta) = E_\beta$ and $\mu_{j_0} = \sum_{\beta\in\Theta} \mu_\beta$ (see (2.4.5)). Hence

$$\begin{aligned}
&\hat{\mu}(B(a,\rho)) = \mu_{j_0}(B(a,\rho) \cap F_{j_0}) \\
&\quad = \textstyle\sum_{\beta\in\Theta} \mu_\beta(B(a,\rho) \cap F_{j_0}) \\
&\quad = \textstyle\sum_{\beta\in\Theta} \mu_\beta(B(a,\rho) \cap E_\beta \cap F_{j_0}) \\
&\quad \le \textstyle\sum_{\beta\in\Theta, E_\beta\cap B(a,\rho)\cap F_{j_0}\neq\emptyset} \mu_\beta(E_\beta) \\
&\quad \le \textstyle\sum_{p,q\in\mathbb{Z}, \beta\in\mathfrak{R}^{(j_0)}_k(p,q), R_k(p,q)\cap B(a,\rho)\cap F_{j_0}\neq\emptyset} \mu_\beta(E_\beta) \quad \text{(by (3.11))} \\
&\quad \le \textstyle\sum_{p,q=0,\pm1, \beta\in\mathfrak{R}^{(j_0)}_k(x_{\alpha_l}+p, y_{\alpha_k}+q), R_k(x_{\alpha_l}+p, y_{\alpha_k}+q)\cap F_{j_0}\neq\emptyset} \mu_\beta(E_\beta) \quad \text{(by (3.20))} \\
&\quad = \textstyle\sum_{p,q=0,\pm1, R_k(x_{\alpha_l}+p, y_{\alpha_k}+q)\cap F_{j_0}\neq\emptyset} \sum_{\beta\in\mathfrak{R}^{(j_0)}_k(x_{\alpha_l}+p, y_{\alpha_k}+q)} \mu_\beta(E_\beta)
\end{aligned} \tag{3.21}$$

When $R_k(x_{\alpha_l} + p, y_{\alpha_k} + q) \cap F_{j_0} \neq \emptyset$, we can choose an element $\gamma(p,q) \in \mathcal{F}_{j_0}$ such that $\pi_{j_0}(\gamma(p,q)) \in R_k(x_{\alpha_l} + p, y_{\alpha_k} + q)$. So $x_{\gamma(p,q)_l} = x_{\alpha_l} + p$ and $y_{\gamma(p,q)_k} = y_{\alpha_k} + q$. Therefore $\gamma(p,q) \in \mathfrak{R}^{(j_0)}_k(x_{\alpha_l} + p, y_{\alpha_k} + q)$ and $\mathfrak{R}^{(j_0)}_k(x_{\alpha_l} + p, y_{\alpha_k} + q) = \mathfrak{R}^{(j_0)}_k(x_{\gamma(p,q)_l}, y_{\gamma(p,q)_k})$. Suppose $\gamma(p,q) = (j_0(j^{(p,q)}_1 t^{(p,q)}_1) \cdots (j^{(p,q)}_k t^{(p,q)}_k))$. Let $r^{(p,q)}_i = y_{j^{(p,q)}_{i-1} j^{(p,q)}_i t^{(p,q)}_i}$. Then by (3.14),

$$\begin{aligned}
&\textstyle\sum_{\beta\in\mathfrak{R}^{(j_0)}_k(x_{\alpha_l}+p, y_{\alpha_k}+q)} \mu_\beta(E_\beta) \\
&= \textstyle\sum_{\beta\in\mathfrak{R}^{(j_0)}_k(x_{\gamma(p,q)_l}, y_{\gamma(p,q)_k})} \Delta_\beta \\
&= e^{-1}_{j_0} m^{-kD} b_{j_0 j^{(p,q)}_1 r^{(p,q)}_1} \cdots b_{j^{(p,q)}_{l-1} j^{(p,q)}_l r^{(p,q)}_l} \cdot (B_{r^{(p,q)}_{l+1}} \cdots B_{r^{(p,q)}_k})_{j^{(p,q)}_l} (e_1, \cdots, e_T)^t \\
&\le e^{-1}_{j_0} m^{-kD} b_{j_0 j^{(p,q)}_1 r^{(p,q)}_1} \cdots b_{j^{(p,q)}_{l-1} j^{(p,q)}_l r^{(p,q)}_l} \cdot (\max_i e_i) \cdot \\
&\qquad (B_{r^{(p,q)}_{l+1}} \cdots B_{r^{(p,q)}_k})_{j^{(p,q)}_l} (1, \cdots, 1)^t \\
&\le e^{-1}_{j_0} m^{-kD} b_{j_0 j^{(p,q)}_1 r^{(p,q)}_1} \cdots b_{j^{(p,q)}_{l-1} j^{(p,q)}_l r^{(p,q)}_l} \cdot (\max_i e_i) \cdot T \| (B_{r^{(p,q)}_{l+1}} \cdots B_{r^{(p,q)}_k})_{j^{(p,q)}_l} \|_\infty \\
&= e^{-1}_{j_0} m^{-kD} (\max_i e_i) T [f_k(\gamma(p,q))]^k
\end{aligned} \tag{3.22}$$

So

$$\hat{\mu}(B(a,\rho)) \le e^{-1}_{j_0} m^{-kD} (\max_i e_i) T \textstyle\sum_{p,q=0,\pm1, R_k(x_{\alpha_l}+p, y_{\alpha_k}+q)\cap F_{j_0}\neq\emptyset} [f_k(\gamma(p,q))]^k$$

Hence

$$\begin{aligned}
&\theta^{*D}(\hat{\mu}, a) = \overline{\lim}_{\rho\to 0} \tfrac{\hat{\mu}(B(a,\rho))}{\varpi_D \rho^D} \\
&\le \overline{\lim}_{\rho\to 0} [e^{-1}_{j_0} m^{-kD} (\max_i e_i) T \\
&\qquad \textstyle\sum_{p,q=0,\pm1, R_k(x_{\alpha_l}+p, y_{\alpha_k}+q)\cap F_{j_0}\neq\emptyset} [f_k(\gamma(p,q))]^k / (\varpi_D \rho^D)] \\
&\le \overline{\lim}_{k\to\infty} [e^{-1}_{j_0} m^{-kD} (\max_i e_i) T \\
&\qquad \textstyle\sum_{p,q=0,\pm1, R_k(x_{\alpha_l}+p, y_{\alpha_k}+q)\cap F_{j_0}\neq\emptyset} [f_k(\gamma(p,q))]^k / (\varpi_D (m^{-k-1}/2)^D)] \\
&\le e^{-1}_{j_0} (\max_i e_i) T (2m)^D [\overline{\lim}_{k\to\infty} \textstyle\sum_{p,q=0,\pm1, R_k(x_{\alpha_l}+p, y_{\alpha_k}+q)\cap F_{j_0}\neq\emptyset} [f_k(\gamma(p,q))]^k] / \varpi_D \\
&\le e^{-1}_{j_0} (\max_i e_i) T (2m)^D [\textstyle\sum_{p,q=0,\pm1, R_k(x_{\alpha_l}+p, y_{\alpha_k}+q)\cap F_{j_0}\neq\emptyset} \overline{\lim}_{k\to\infty} [f_k(\gamma(p,q))]^k] / \varpi_D \\
&\le e^{-1}_{j_0} (\max_i e_i) T (2m)^D 9 / \varpi_D
\end{aligned} \tag{3.23}$$

By (2.4.2), notice that $\hat{\mu}(F_{j_0}) > 0$, we have $H^D(F_{j_0}) > 0$. So $H^D(E_{j_0}) \ge H^D(F_{j_0}) > 0$. Consequently $\dim_H E_{j_0} \ge D$. The theorem is proved. □

Corollary and Definition 3.1. *Under the condition of Theorem 3.2, set* $b_{ijk} = a^{\log_n m - 1}_{ijk}$. *Then each* $\dim_H E_i \le D$ *and* $H^D(E_i) < \infty$. *We denote this upper bound of Hausdorff dimension as* $D(M)$.

Proof. In this case, $B_k = (a_{ijk}^{\log_n m})$. We only need to prove that for any $\alpha = (j_0(j_1t_1)\cdots) \in \mathfrak{D}$, $\overline{\lim}_{k\to\infty} f_k(\alpha) \geq 1$.

Let

$$\xi_k(\alpha) = (a_{j_0j_1t_1}\cdots a_{j_{k-1}j_kt_k})^{\frac{\log_n m}{k}},$$

$$\eta_k(\alpha) = (a_{j_0j_1t_1}\cdots a_{j_{l-1}j_lt_l})^{\frac{\log_n m}{l}-\frac{1}{k}},$$

$$\zeta_k(\alpha) = \frac{\|[B_{t'_{l+1}}\cdots B_{t'_k}]_{j_l}\|_\infty}{(a_{j_lj_{l+1}t_{l+1}}\cdots a_{j_{k-1}j_kt_k})^{\log_n m}}.$$

Then

$$f_k(\alpha) = \frac{\xi_k(\alpha)}{\xi_l(\alpha)}\eta_k(\alpha)(\zeta_k(\alpha))^{\frac{1}{k}}. \tag{3.24}$$

Obviously

$$\zeta_k(\alpha) \geq 1. \tag{3.25}$$

Now $\alpha \in \mathfrak{D}$ implies $\phi_{j_{i-1}j_it_i} \in M_{j_{i-1}j_it'_i}$, $M_{j_{i-1}j_it'_i} \neq \emptyset$. So $a_{j_{i-1}j_it'_i} \geq 1$, $\xi_k(\alpha) \geq 1 > 0$. So

$$\overline{\lim}_k \frac{\xi_k(\alpha)}{\xi_l(\alpha)} \geq 1. \tag{3.26}$$

On the other hand, $a_{j_{i-1}j_it'_i} \leq n$. So $1 \leq \eta_k(\alpha) \leq (n^l)^{\frac{\log_n m}{l}-\frac{1}{k}} = n^{\log_n m - \frac{l}{k}}$. But $\log_n m - \frac{l}{k} = (k\log_n m - \lfloor k\log_n m\rfloor)/k \to 0$ as $k\to\infty$. Hence

$$\lim_k \eta_k(\alpha) = 1. \tag{3.27}$$

Using (3.25), (3.26) and (3.27) in (3.24), we have

$$\overline{\lim}_k f_k(\alpha) \geq 1.$$

□

Corollary and Definition 3.2. *Under the condition of Theorem 3.2, set* $b_{ijk} = \|A_k\|_\infty^{\log_n m-1}$. *Then each* $\dim_H E_i \geq D$ *and* $H^D(E_i) > 0$. *We denote this lower bound of Hausdorff dimension as* $d(M)$.

Proof. In this time, $B_k = \|A_k\|_\infty^{\log_n m-1}A_k$. Similar with the proof of previous corollary, let

$$\xi_k(\alpha) = (\|A_{t_1}\|_\infty\cdots\|A_{t_k}\|_\infty)^{\frac{\log_n m}{k}},$$

$$\eta_k(\alpha) = (\|A_{t_1}\|_\infty\cdots\|A_{t_l}\|_\infty)^{\frac{\log_n m}{l}-\frac{1}{k}},$$

$$\zeta_k(\alpha) = \frac{\|[A_{t'_{l+1}}\cdots A_{t'_k}]_{j_l}\|_\infty}{\|A_{t_{l+1}}\|_\infty\cdots\|A_{t_k}\|_\infty}.$$

Again, we have

$$f_k(\alpha) = \frac{\xi_k(\alpha)}{\xi_l(\alpha)}\eta_k(\alpha)(\zeta_k(\alpha))^{\frac{1}{k}}, \tag{3.28}$$

$$\lim \eta_k(\alpha) = 1. \tag{3.29}$$

And

$$\zeta_k(\alpha) \le 1 \tag{3.30}$$

is also obviously. Let $j_0 \in \{1, \cdots, T\}$. As $\tau_{j_0}(\mathfrak{D}_{j_0}) = 1$, we can consider τ_{j_0} a probability measure on $\mathfrak{D}_{j_0}$. For $k = 1, 2, 3, \cdots$, define random variables on $\mathfrak{D}_{j_0}$ by

$$\begin{cases} X_k : (j_0(j_1t_1)\cdots) \to j_k, \\ Y_k : (j_0(j_1t_1)\cdots) \to y_{j_{k-1}j_kt_k} = t'_k, \end{cases}$$

Define function $\mathcal{L}$ on $\{0, 1, \cdots, m-1\}$ by

$$\mathcal{L}(t) = \|A_t\|_\infty^{\log_n m}.$$

Then it is easy to verify that

- $\{X_k, k = 1, 2, \cdots\}$ is an irreducible Markov chain with transition probability matrix Λ, see (3.5) for the definition of Λ.
- $P\{Y_k = t | X_{k-1} = r, X_k = s\} = \omega_{rst}$, $r, s = 1, \cdots, T$, $t = 0, \cdots, m-1$. see (3.6) for the definition of ω_{rst}.
- Consider ξ_k as a random variable on $\mathfrak{D}_{j_0}$, then $\xi_k = (\mathcal{L}(Y_1)\cdots\mathcal{L}(Y_k))^{\frac{1}{k}}$. Equivalently, $\ln(\xi_k) = \frac{1}{k}(\ln(\mathcal{L}(Y_1)) + \cdots + \ln(\mathcal{L}(Y_k)))$.
- $1 \le \xi_k \le \max_{0\le t<m} \|A_t\|_\infty^{\log_n m}$.

By the limit theory of Markov process(cf. [18], §3f), we know that there exists a subset $\mathcal{F}_{j_0} \subseteq \mathfrak{D}_{j_0}$ such that $\tau(\mathcal{F}_{j_0}) > 0$ and the sequence $\xi_k(\alpha)$, $k = 1, 2, 3, \cdots$, converges for every $\alpha \in \mathcal{F}_{j_0}$. Thus

$$\lim_{k\to\infty} \frac{\xi_k(\alpha)}{\xi_l(\alpha)} = 1 \text{ for all } \alpha \in \mathcal{F}_{j_0}. \tag{3.31}$$

Using (3.29), (3.30) and (3.31) in (3.28), we have $\lim_k f_k(\alpha) \le 1$ for all $\alpha \in \mathcal{F}_{j_0}$. So the condition of (b) in Theorem 3.2 holds. Hence $\dim_H \mathcal{E} \ge D$. □

Remark 3.1. *If the size of a net M-matrix M is 1×1, then $A_k = (a_{11k})$ and $A(M)$ is the McMullen carpet[26]. In this case, the upper bound $D(M)$ of the Hausdorff dimension of $A(M)$ in Corollary 3.1 coincides with the lower bound $d(M)$ in Corollary 3.2. Moreover,*

$$\dim_H A(M) = \log_m(\sum_k \|A_k\|_\infty^{\log_n m}). \tag{3.32}$$

This result coincide with the result of McMullen[26]. So our Theorem 3.2 is a generalization of McMullen[26]'s result to the net M-matrix.

Remark 3.2. *If one can find numbers b_{ijk} such that both the conditions (a) and (b) of Theorem 3.2 are satisfied, the exact Hausdorff dimension of invariant c-vectors of net M-matrix will be determined. And the Hausdorff measures of E_i will* *satisfy the relation* $0 < H^D(E_i) < \infty$*, same as that has been proved by McMullen for McMullen carpet, where* $D = \dim_H(E_i)$*. On the other hand, Corollary 3.1 and Corollary 3.2 give an upper bound* $D(M)$ *and a lower bound* $d(M)$ *of the Hausdorff dimension respectively. As* $\mathcal{E}$ *is invariant under* M^s *for any* s *and* M^s *are all net M-matrices, so we have*

$$d(M^s) \leq \dim_H \mathcal{E} \leq D(M^s). \tag{3.33}$$

One will hope to prove

$$\lim_{s\to\infty} d(M^s) = \lim_{s\to\infty} D(M^s). \tag{3.34}$$

We can prove that $d(M^s)$ *is increasing and* $D(M^s)$ *is decreasing while* s *is increasing. So* $\lim_{s\to\infty} d(M^s)$ *and* $\lim_{s\to\infty} D(M^s)$ *exist. There are some cases that* $d(M) = D(M)$*, but it is not easy to prove (3.34) in general. In order to prove (3.34), we need to understand the behaviour of the sequence* $\{B_{i_1}\cdots B_{i_k} : k = 1,2,3,\cdots,$ *and* $1 \leq i_1,\cdots,i_k \leq m\}$*, where* $B_1,\cdots,B_m$ *are matrices with non-negative entries and* $\sum B_k$ *is irreducible.*

REMARK 3.3. *For* $s = 1,2,3,\cdots$ *and* $v = 1,\cdots,T$*, define*

$$\begin{cases} G_s(M) = \frac{1}{s}\log_m(\sum_{0\leq i_1,\cdots,i_s\leq m-1} \|A_{i_s}A_{i_{s-1}}\cdots A_{i_1}\|_\infty^{\log_n m}), \\ g_{s,v}(M) = \frac{1}{s}\log_m(\frac{1}{T}\sum_{0\leq i_1,\cdots,i_s\leq m-1}([A_{i_s}A_{i_{s-1}}\cdots A_{i_1}]_{vv})^{\log_n m}), \end{cases} \tag{3.35}$$

where $[A_{i_s}A_{i_{s-1}}\cdots A_{i_1}]_{vv}$ *means the* (v,v)*-entry of the matrix* $A_{i_s}A_{i_{s-1}}\cdots A_{i_1}$*. Kenyon and Peres[21] proved that*

$$g_{s,v}(M) \leq \dim_H(\cup_i E_i) \leq G_s(M) \tag{3.36}$$

for all $v = 1,\cdots,T$ *and* $s = 1,2,\cdots$*. They also proved that there exists at least one* $v \in 1,\cdots,T$ *such that*

$$\lim_{s\to\infty} g_{s,v}(M) = \lim_{s\to\infty} G_s(M). \tag{3.37}$$

Consequently they obtained

$$\dim_H(\cup_i E_i) = \lim_{s\to\infty} G_s(M). \tag{3.38}$$

Because M *is irreducible, each component of* $\mathcal{E} = A(M)$ *has the same Hausdorff dimension. So* $\dim_H E_i = \dim_H(\cup_i E_i)$*. In actual computation of* $\dim_H(\cup_i E_i)$ *using (3.38), one can only compute a finite number of* $G_s(M)$*. Therefore a sequence of upper bound of* $\dim_H(\cup_i E_i)$ *will be obtained. In §5, we shall study an example and give the numerical computational results for* $g_{s,v}(M), G_s(M), d(M^s)$ *and* $D(M^s)$*. It seems that* $g_{s,v}(M) \leq d(M^s) \leq G_s(M) \leq D(M^s)$ *always hold.*

For the sake of calculating $d(M^s)$ and $D(M^s)$ from M, we give the following result without proof.

PROPOSITION 3.1. *Let* M *be a net M-matrix with respect to* $B = \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{m} \end{pmatrix}$*, where* $n \geq m \geq 2$*. Then* M^s *is a net M-matrix with respect to* $B^s = \begin{pmatrix} \frac{1}{n^s} & 0 \\ 0 & \frac{1}{m^s} \end{pmatrix}$*.*

Suppose $M^s = (M_{ij}^{(s)})$*. Similar with (3.2), we define*

$$M_{ijk}^{(s)} = \{\phi \in M_{ij}^{(s)} |\ \text{the } y\text{-coordinate of } \phi(0) \text{ is } \frac{k}{m^s}\},$$

for $k = 0, 1, \cdots, m^s - 1$. *Then*

$$\sum_{1 \le t \le T} \# M_{itk}^{(s)} \cdot \# M_{tjk'}^{(s')} = \# M_{ijK}^{(s+s')}, \tag{3.39}$$

where $K = k + k'm^s$, $1 \le i, j \le T$, $0 \le k < m^s$, $0 \le k' < m^{s'}$, $s, s' = 1, 2, 3, \cdots$.

4. Self-affine fractals and net M-matrix. In this section we study a class of self-affine sets. We do not need the assumption that the open set condition holds. We shall prove such self-affine sets can be expressed as unions of some components of the invariant c-vectors of some net M-matrices. Then we can use McMullen's formula (3.32), Kenyon and Peres' formula (3.36) and our Theorem 2.5.1, Corollary 3.1, Corollary 3.2 to estimate their Hausdorff dimension.

For $d \times d$ real matrix A and vector $b \in \mathbb{R}^d$, we use $\varphi_{A,b}$ to denote the affine mapping $x \mapsto Ax + b$ on $\mathbb{R}^d$ and use T_b to denote the translation $x \mapsto x + b$ on $\mathbb{R}^d$. Then we can check that

$$\begin{cases} \varphi_{A,b} \circ T_c = \varphi_{A,b+Ac}, \\ T_c \circ \varphi_{A,b} = \varphi_{A,b+c}, \\ \varphi_{A,b}\varphi_{A_1,c} = \varphi_{AA_1,Ac+b}, \\ \varphi_{A,b}^{-1} = \varphi_{A^{-1},-A^{-1}b}, \text{ if } A \text{ is invertible}, \end{cases} \tag{4.1}$$

where A, A_1 are matrix and b, c are vectors.

Theorem 4.1. *Let $n_1, \cdots, n_d$ be positive integers bigger than 1 and $B =$diag$(1/n_1,\ 1/n_2, \cdots, 1/n_d)$. Let $\Phi = \{\varphi_{B,u_i} | 1 \le i \le L\}$. Let $E = A(\Phi)$ be the attractor of Φ. Suppose there exist an invertible matrix Γ and a vector $\beta \in \mathbb{R}^d$ such that $\{\Gamma u_i + \beta | 1 \le i \le L\} \subset \mathbb{Q}^d$ and $\Gamma B = B\Gamma$. Then there exists a net M-matrix M with respect to B such that E can be expressed by some components of $A(M)$ as follows:*

$$A(\Phi) = \cup_{i=1}^{l} \psi_i(A(M)_{t_i}), \tag{4.2}$$

where $1 \le t_1, \cdots, t_l \le T$, $T \times T$ is the size of M, $A(M)_{t_i}$ means the t_i-th components of $A(M)$, and $\psi_1, \cdots, \psi_l$ are invertible affine mappings.

Proof. Let $E = A(\Phi)$ be the attractor of Φ. We shall prove the theorem in 6 steps.

(1) We can assume that every $u_i \in \mathbb{Q}^d$.

Proof of (1). Let $\gamma(x) = \Gamma x + (I - B)^{-1}\beta$, where I is the unit matrix. It is easy to check that $\gamma\phi_{B,u_i}\gamma^{-1}(x) = Bx + \Gamma u_i + \beta$ for each i. We know $\{\Gamma u_i + \beta | 1 \le i \le L\} \subset \mathbb{Q}^d$. As the attractor of $\gamma\Phi\gamma^{-1}$ is $\gamma(E)$, so we can assume that every $u_i \in \mathbb{Q}^d$ in the rest of the proof.

(2) Suppose that $(I - B)^{-1}u_i = (v_1^{(i)}, \cdots, v_d^{(i)})^t$. Let $v_j^{(min)} = min_i v_j^{(i)}$ and $v_j^{(max)} = max_i v_j^{(i)}$. Then we can assume that each $v_j^{(min)} = 0$ and each $v_j^{(max)}$ is either 0 or 1.

Proof of (2). Let $v = (v_1^{(min)}, \cdots, v_d^{(min)})^t$. For $j = 1 \cdots, d$, let

$$l_j = \begin{cases} 1 & \text{if } v_j^{(max)} = v_j^{(min)}, \\ v_j^{(max)} - v_j^{(min)} & \text{if } v_j^{(max)} \ne v_j^{(min)}. \end{cases}$$

Define $\gamma(x) =$diag$(l_1, \cdots, l_d)^{-1}(x - v)$. Then, after replacing Φ by $\gamma\Phi\gamma^{-1}$, we can have that each $v_j^{(min)} = 0$ and each $v_j^{(max)}$ is either 0 or 1.

(3). Let $U = \{(x_1, \cdots, x_d)^t | 0 < x_i < 1\}$. Then $\Phi^2(U) \subseteq \Phi(U) \subseteq U$ and hence $E \subseteq \Phi(\overline{U}) \subseteq \overline{U}$.
Proof of (3). This can be checked easily by some direct calculation and use Theorem 2.2.1.
(4). In this step we shall construct a special c-vector $F = (F_1, \cdots, F_T)^t$ with each component non-empty, such that

$$E = \cup_i \psi_i(F_{t_i}) \tag{4.3}$$

for some invertible affine mappings ψ_i.
Proof of (4). Suppose $u_i = (l_1^{(i)}/k_1^{(i)}, l_2^{(i)}/k_2^{(i)}, \cdots, l_d^{(i)}/k_d^{(i)})^t$, where $l_j^{(i)}, k_j^{(i)} \in \mathbb{Z}$ and $k_j^{(i)} > 0$. Let w_j be the minimal common multiple of $\{n_j, k_j^{(i)} | i = 1, \cdots, L\}$, $j = 1, \cdots, d$. For $\mathfrak{p} = (i_1, \cdots, i_d)^t \in \mathbb{Z}^d \subseteq \mathbb{R}^d$, define

$$I_{\mathfrak{p}} = [\frac{i_1}{w_1}, \frac{i_1+1}{w_1}] \times [\frac{i_2}{w_2}, \frac{i_2+1}{w_2}] \times \cdots \times [\frac{i_d}{w_d}, \frac{i_d+1}{w_d}], \tag{4.4}$$

Let

$$\Omega = \{\mathfrak{p} \in \mathbb{Z}^d | \overset{o}{I}_{\mathfrak{p}} \cap \Phi(U) \neq \emptyset\}. \tag{4.5}$$

Then $\Omega \subseteq \{\mathfrak{p} | I_{\mathfrak{p}} \cap U \neq \emptyset\} \subseteq \{(i_1, \cdots, i_d)^t \in \mathbb{Z}^d | 0 \leq i_j < w_j\}$. For any $e \in E$, there exists at least one $\mathfrak{p} \in \mathbb{Z}^d$ such that $e \in I_{\mathfrak{p}}$. If $e \in \overset{o}{I}_{\mathfrak{p}}$, then $\overset{o}{I}_{\mathfrak{p}} \cap E \neq \emptyset$. By step (3), $\overset{o}{I}_{\mathfrak{p}} \cap \Phi(\overline{U}) \neq \emptyset$. So $\overset{o}{I}_{\mathfrak{p}} \cap \Phi(U) \neq \emptyset$. Hence $\mathfrak{p} \in \Omega$. On the other hand, suppose $e \in \partial I_{\mathfrak{p}}$. By step (3), we have $e \in \Phi(\overline{U})$. If $e \in \Phi(U)$, then $I_{\mathfrak{p}} \cap \Phi(U) \neq \emptyset$. So $\overset{o}{I}_{\mathfrak{p}} \cap \Phi(U) \neq \emptyset$. Hence $\mathfrak{p} \in \Omega$. If $e \in \partial \Phi(U)$, then there have two case: (1). $\overset{o}{I}_{\mathfrak{p}} \cap \Phi(U) \neq \emptyset$. Hence $\mathfrak{p} \in \Omega$. (2). $\overset{o}{I}_{\mathfrak{p}} \cap \Phi(U) = \emptyset$. In case (2), $e \in \partial(\Phi(U)) \cap \partial \overset{o}{I}_{\mathfrak{p}}$. It is obviously that we can find a $\mathfrak{p}' \in \mathbb{Z}^d$ such that $e \in I_{\mathfrak{p}'}$ and $\overset{o}{I}_{\mathfrak{p}'} \cap \Phi(U) \neq \emptyset$. Thus $\mathfrak{p}' \in \Omega$. So we have $E \subseteq \cup_{\mathfrak{p} \in \Omega} I_{\mathfrak{p}}$. Hence

$$E = \cup_{\mathfrak{p} \in \Omega} E \cap I_{\mathfrak{p}}. \tag{4.6}$$

Let

$$W = \begin{pmatrix} w_1 & & 0 \\ & \ddots & \\ 0 & & w_d \end{pmatrix}.$$

For any non-empty subset $\Lambda \subseteq \Omega$, define

$$E_\Lambda = \bigcup_{\mathfrak{p} \in \Lambda} \varphi_{W, -\mathfrak{p}}(E \cap I_{\mathfrak{p}}). \tag{4.7}$$

It is clearly that for any $\Lambda_1, \Lambda_2 \subseteq \Omega$,

$$E_{\Lambda_1 \cup \Lambda_2} = E_{\Lambda_1} \cup E_{\Lambda_2}. \tag{4.8}$$

Notice that $\phi_{W,-\mathfrak{p}}(I_{\mathfrak{p}}) = \overline{U}$, we have

$$E_\Lambda = (\bigcup_{\mathfrak{p} \in \Lambda} \varphi_{W,-\mathfrak{p}}(E)) \cap \overline{U}.$$

Set $E_\emptyset = \emptyset$. Then

$$\begin{cases} E_\Lambda \text{ is nonempty compact set if } \Lambda \neq \emptyset, \\ E_\Lambda \subseteq \overline{U}, \\ E = \bigcup_{\{\mathfrak{p}\}\subseteq\Omega} \varphi_{W^{-1},W^{-1}\mathfrak{p}}(E_{\{\mathfrak{p}\}}). \end{cases} \tag{4.9}$$

So $\mathcal{E} = (E_\Lambda | \Lambda \subseteq \Omega, \Lambda \neq \emptyset)$ is the c-vector such that (4.3) holds.

(5) There exists an M-matrix M such that $(E_\Lambda | \Lambda \subseteq \Omega, \Lambda \neq \emptyset) = A(M)$. Consequently, (4.3) is same with (4.2).

Proof of (5). Because $E = \Phi(E) = \cup_{i=1}^{L} \varphi_{B,u_i}(E)$, so for any $\mathfrak{p} = (i_1, \cdots, i_d)^t \in \Omega$,

$$\begin{aligned} &E \cap I_{\mathfrak{p}} \\ &= \textstyle\bigcup_{i=1}^{L} \varphi_{B,u_i}(E) \cap I_{\mathfrak{p}} \\ &= \textstyle\bigcup_{i=1}^{L} \varphi_{B,u_i}(E \cap \varphi_{B^{-1},-B^{-1}u_i}(I_{\mathfrak{p}})) \end{aligned} \tag{4.10}$$

It is easy to see that

$$\begin{aligned} &\varphi_{B^{-1},-B^{-1}u_i}(I_{\mathfrak{p}}) = [\tfrac{n_1 i_1}{w_1} - n_1 u_1^{(i)}, \tfrac{n_1(i_1+1)}{w_1} - n_1 u_1^{(i)}] \times \\ &\cdots \times [\tfrac{n_d i_d}{w_d} - n_d u_d^{(i)}, \tfrac{n_d(i_d+1)}{w_d} - n_d u_d^{(i)}]. \end{aligned} \tag{4.11}$$

Let

$$\begin{aligned} \xi_{\mathfrak{p}i} &= (n_1 i_1 - n_1 w_1 u_1^{(i)}, \cdots, n_d i_d - n_d w_d u_d^{(i)})^t \\ &= (n_1 i_1 - n_1 w_1 l_1^{(i)}/k_1^{(i)}, \cdots, n_d i_d - n_d w_d l_d^{(i)}/k_d^{(i)})^t \in \mathbb{Z}^d. \end{aligned} \tag{4.12}$$

Then

$$B\xi_{\mathfrak{p}i} = \mathfrak{p} - W u_i \tag{4.13}$$

and

$$\varphi_{B^{-1},-B^{-1}u_i}(I_{\mathfrak{p}}) = \bigcup_{0\le t_1<n_1,\cdots,0\le t_d<n_d} I_{\xi_{\mathfrak{p}i}+(t_1,\cdots,t_d)^t}. \tag{4.14}$$

So

$$\begin{aligned} &E \cap I_{\mathfrak{p}} \\ &= \textstyle\bigcup_{i=1}^{L} \varphi_{B,u_i}(E \cap \varphi_{B^{-1},-B^{-1}u_i}(I_{\mathfrak{p}})) \\ &= \textstyle\bigcup_{i=1}^{L} \varphi_{B,u_i}[E \cap (\bigcup_{0\le t_1<n_1,\cdots,0\le t_d<n_d} I_{\xi_{\mathfrak{p}i}+(t_1,\cdots,t_d)^t})]. \\ &= \textstyle\bigcup_{i=1}^{L} \varphi_{B,u_i}[E \cap (\bigcup_{\substack{0\le t_1<n_1,\cdots,0\le t_d<n_d \\ \xi_{\mathfrak{p}i}+(t_1,\cdots,t_d)^t\in\Omega}} I_{\xi_{\mathfrak{p}i}+(t_1,\cdots,t_d)^t})]. \\ &= \textstyle\bigcup_{i=1}^{L} \bigcup_{\substack{0\le t_1<n_1,\cdots,0\le t_d<n_d \\ \xi_{\mathfrak{p}i}+(t_1,\cdots,t_d)^t\in\Omega}} \varphi_{B,u_i}(E \cap I_{\xi_{\mathfrak{p}i}+(t_1,\cdots,t_d)^t}). \end{aligned} \tag{4.15}$$

Hence for any nonempty $\Lambda \subseteq \Omega$,

$$\begin{aligned}
E_\Lambda &= \bigcup_{\mathfrak{p}\in\Lambda} \varphi_{W,-\mathfrak{p}}(E\cap I_{\mathfrak{p}}) \\
&= \bigcup_{\mathfrak{p}\in\Lambda} \varphi_{W,-\mathfrak{p}}\Big(\bigcup_{i=1}^{L} \bigcup_{\substack{0\le t_1<n_1,\cdots,0\le t_d<n_d \\ \xi_{\mathfrak{p}_i}+(t_1,\cdots,t_d)^t\in\Omega}} \varphi_{B,u_i}(E\cap I_{\xi_{\mathfrak{p}_i}+(t_1,\cdots,t_d)^t})\Big) \\
&= \bigcup_{\mathfrak{p}\in\Lambda} \bigcup_{i=1}^{L} \bigcup_{\substack{0\le t_1<n_1,\cdots,0\le t_d<n_d \\ \xi_{\mathfrak{p}_i}+(t_1,\cdots,t_d)^t\in\Omega}} [\varphi_{W,-\mathfrak{p}}\circ\varphi_{B,u_i}(E\cap I_{\xi_{\mathfrak{p}_i}+(t_1,\cdots,t_d)^t})] \\
&= \bigcup_{\mathfrak{p}\in\Lambda} \bigcup_{i=1}^{L} \bigcup_{\substack{0\le t_1<n_1,\cdots,0\le t_d<n_d \\ \xi_{\mathfrak{p}_i}+(t_1,\cdots,t_d)^t\in\Omega}} \{\varphi_{B,B\xi_{\mathfrak{p}_i}+B(t_1,\cdots,t_d)^t+Wu_i-\mathfrak{p}} \\
&\qquad [\varphi_{W,-\xi_{\mathfrak{p}_i}-(t_1,\cdots,t_d)^t}(E\cap I_{\xi_{\mathfrak{p}_i}+(t_1,\cdots,t_d)^t})]\} \\
&= \bigcup_{\mathfrak{p}\in\Lambda} \bigcup_{i=1}^{L} \bigcup_{\substack{0\le t_1<n_1,\cdots,0\le t_d<n_d \\ \xi_{\mathfrak{p}_i}+(t_1,\cdots,t_d)^t\in\Omega}} \{\varphi_{B,B(t_1,\cdots,t_d)^t} \\
&\qquad [\varphi_{W,-\xi_{\mathfrak{p}_i}-(t_1,\cdots,t_d)^t}(E\cap I_{\xi_{\mathfrak{p}_i}+(t_1,\cdots,t_d)^t})]\} \\
&= \bigcup_{0\le t_1<n_1,\cdots,0\le t_d<n_d} \bigcup_{i=1}^{L} \bigcup_{\mathfrak{p}\in\Lambda,\xi_{\mathfrak{p}_i}+(t_1,\cdots,t_d)^t\in\Omega} \{\varphi_{B,B(t_1,\cdots,t_d)^t} \\
&\qquad [\varphi_{W,-\xi_{\mathfrak{p}_i}-(t_1,\cdots,t_d)^t}(E\cap I_{\xi_{\mathfrak{p}_i}+(t_1,\cdots,t_d)^t})]\} \\
&= \bigcup_{0\le t_1<n_1,\cdots,0\le t_d<n_d} \{\varphi_{B,B(t_1,\cdots,t_d)^t}[\bigcup_{i=1}^{L} \bigcup_{\mathfrak{p}\in\Lambda,\xi_{\mathfrak{p}_i}+(t_1,\cdots,t_d)^t\in\Omega} \\
&\qquad (\varphi_{W,-\xi_{\mathfrak{p}_i}+(t_1,\cdots,t_d)^t}(E\cap I_{\xi_{\mathfrak{p}_i}+(t_1,\cdots,t_d)^t}))]\}.
\end{aligned} \tag{4.16}$$

For $t = (t_1, \cdots, t_d)^t \in \mathbb{Z}^t$, let

$$\Omega_{\Lambda,t} = \{\xi_{\mathfrak{p}i} + t | \mathfrak{p} \in \Lambda, 1 \le i \le L\} \cap \Omega. \tag{4.17}$$

Then (4.16) becomes

$$E_\Lambda = \bigcup_{0\le t_1<n_1,\cdots,0\le t_d<n_d,\Omega_{\Lambda,t}\ne\emptyset} \varphi_{B,B(t_1,\cdots,t_d)^t}(E_{\Omega_{\Lambda,t}}) \tag{4.18}$$

Hence $\mathcal{E} = (E_\Lambda; \Lambda \subseteq \Omega, \Lambda \ne \emptyset\ )$ is invariant under the M-matrix

$$\begin{cases} M = (M_{\Lambda_1\Lambda_2})_{\Lambda_1,\Lambda_2\subseteq\Omega\setminus\emptyset}, \qquad \text{where} \\ M_{\Lambda_1\Lambda_2} = \{\varphi_{B,Bt} | t = (t_1,\cdots,t_d)^t, 0 \le t_j < n_j, 1 \le j \le d, \Omega_{\Lambda_1,t} = \Lambda_2\}. \end{cases} \tag{4.19}$$

By (4.7), each components of $\mathcal{E}$ is non-empty. So $\mathcal{E} = A(M)$ by (1) of Theorem 2.2.2.
(6) The M-matrix M defined in (4.19) is a net Matrix with respect to B.
Proof of (6). It is clear that M satisfies the condition (1) of Definition 3.1. So, we only need to prove that the condition (2) of Definition 3.1 holds.

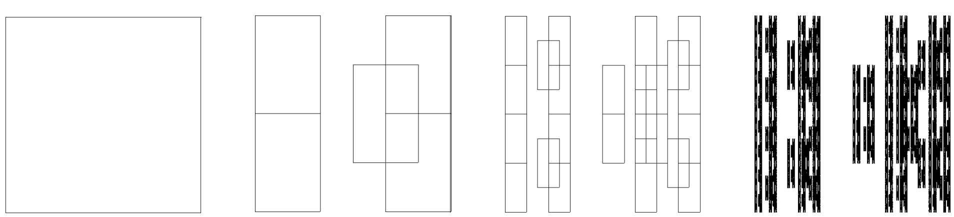
For any $\Lambda, \Lambda_1, \Lambda_2 \subseteq \Omega \setminus \{\emptyset\}$, suppose $\varphi_{B,Bt} \in M_{\Lambda\Lambda_1}$ and $\varphi_{B,B\hat{t}} \in M_{\Lambda\Lambda_2}$. If $\varphi_{B,Bt} = \varphi_{B,B\hat{t}}$, then $t = \hat{t}$. So $\Lambda_1 = \Omega_{\Lambda,t} = \Omega_{\Lambda,\hat{t}} = \Lambda_2$. This is just the condition (2) of Definition 3.1. The Theorem is proved. □

As consequences of Theorem 4.1, according the results of McMullen[26], Bedford[3], Mauldin and Williams[25], Kenyon and Peres[21, 22], and our Theorem 2.5.1, we claim that:

- 1-dimensional case (cf. [20, 25] or Theorem 2.5.1). Let $n \geq 2$ be an integer. Let $\Xi \subseteq \mathbb{R}$ be a finite set. If there exist numbers $\Gamma, \gamma \in \mathbb{R}$, $\Gamma \neq 0$, such that $\Gamma\Xi + \gamma \subseteq \mathbb{Q}$, then the Hausdorff dimension of the attractor of IFS $\{\frac{1}{n}x + b | b \in \Xi\}$ is determined.
- 2-dimension case (cf. [3, 21, 26]). Let $m, n \geq 2$ be integers and $B = \begin{pmatrix} 1/n & 0 \\ 0 & 1/m \end{pmatrix}$. Let $\Xi \subseteq \mathbb{R}^2$ be a finite set. If there exist invertible matrix $\Gamma \in \mathbb{R}^{2\times 2}$ and vector $\gamma \in \mathbb{R}^2$ such that $\Gamma\Xi + \gamma \subseteq \mathbb{Q}^2$ and $B\Gamma = \Gamma B$, then the box dimension of the attractor $A(\Phi)$ of IFS $\Phi = \{Bx + b | b \in \Xi\}$ is determined. For the Hausdorff dimension, there are sequences of boundaries $g_s \leq \dim_H A(\Phi) \leq G_s$ such that $\dim_H A(\Phi) = \lim_{s\to\infty} g_s = \lim_{s\to\infty} G_s$. Moreover, the velure of $\dim_H A(\Phi)$ can be computed explicitly in some cases (cf. Proposition 3.4 and some examples in [21]).
- High-dimension case (cf. [22]). Let $n_1, \cdots, n_d \geq 2$ be integers and B =diag$(1/n_1, \cdots, 1/n_d)$. Let $\Xi \subseteq \mathbb{R}^d$ be a finite set. If there exist invertible matrix $\Gamma \in \mathbb{R}^{d\times d}$ and vector $\gamma \in \mathbb{R}^d$ such that $\Gamma\Xi + \gamma \subseteq \{0, 1, \cdots, n_1 - 1\} \times \cdots \times \{0, 1, \cdots, n_d - 1\}$ and $B\Gamma = \Gamma B$, then the Hausdorff and box dimensions of the attractor $A(\Phi)$ of IFS $\Phi = \{Bx + b | b \in \Xi\}$ are determined.

5. Example. In this section, we study an example of self-affine set on the plane. The self-affine set in this example does not satisfy the open set condition. Following the method given in the proof of Theorem 4.1, we shall find a net Matrix M such that this self-affine set equals to the union of some components of $A(M)$. Then we get estimations of its Hausdorff dimension.

Let $B = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. Define iterated function system $\Phi = \{\phi_i | i = 1, \cdots, 5\}$, where $\phi_i(x) = Bx + b_i$, $b_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $b_2 = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$, $b_3 = \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}$, $b_4 = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{2} \end{pmatrix}$ and $b_5 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}$. Let E be the attractor of Φ. Then E is a self-affine set and Φ does not satisfies open set condition, see Figure 1, where $I = [0, 1] \times [0, 1]$.



(a) I (b) $\Phi(I)$ (c) $\Phi^2(I)$ (d) E

FIG. 1. *Self-affine set $E = A(\Phi)$ which has overlapping*

It is clearly that Φ satisfies the condition of Theorem 4.1. So we can find a net M-matrix M such that E can be expressed as an union of some components of $A(M)$. The Proof of Theorem 4.1 in fact is a method to determine M.

Following the step (4) in the proof Theorem 4.1, we have the minimal common multiple numbers $w_1 = 6$ and $w_2 = 4$, hence $W = \begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix}$. So for each $\mathfrak{p} = \begin{pmatrix} i \\ j \end{pmatrix} \in \mathbb{Z}^2$, $I_{\mathfrak{p}} = [\frac{i}{6}, \frac{i+1}{6}] \times [\frac{j}{4}, \frac{j+1}{4}]$ (see (4.4)).

Let $U = (0,1) \times (0,1)$. Then it is easy to check that (see (4.5))

$$\begin{aligned}\Omega &= \{\mathfrak{p} \in \mathbb{Z}^2 | \overset{o}{I}_{\mathfrak{p}} \cap \Phi(U) \neq \emptyset\} \\ &= \{\tbinom{0}{0}, \tbinom{1}{0}, \tbinom{4}{0}, \tbinom{5}{0}, \tbinom{0}{1}, \tbinom{1}{1}, \tbinom{3}{1}, \tbinom{4}{1}, \tbinom{5}{1}, \tbinom{0}{2}, \tbinom{1}{2}, \tbinom{3}{2}, \tbinom{4}{2}, \tbinom{5}{2}, \tbinom{0}{3}, \tbinom{1}{3}, \tbinom{4}{3}, \tbinom{5}{3}\}.\end{aligned}$$

So, according to the step (5) in the proof of Theorem 4.1, the M-matrix defined by (4.19) is the required net M-matrix, and the c-vector $(E_\Lambda | \Lambda \subseteq \Omega, \Lambda \neq \emptyset)$, where E_Λ is defined by (4.7), is the associated invariant c-vector. But the size of this M-matrix is too big — it is $2^{18} - 1$. So we shall try to find a small one.

Applying (4.18) to each $\mathfrak{p} \in \Omega$, we can obtain

$$\left\{\begin{aligned}
&E_{\{\binom{0}{0}\}} = E_{\{\binom{4}{0}\}} = E_{\{\binom{0}{2}\}} = E_{\{\binom{3}{1}\}} = \\
&\qquad \varphi_{B,B\binom{0}{0}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{1}{0}}(E_{\{\binom{1}{0}\}}) \cup \varphi_{B,B\binom{0}{1}}(E_{\{\binom{0}{1}\}}) \cup \varphi_{B,B\binom{1}{1}}(E_{\{\binom{1}{1}\}}),\\
&E_{\{\binom{1}{0}\}} = E_{\{\binom{5}{0}\}} = E_{\{\binom{1}{2}\}} = E_{\{\binom{5}{2}\}} = \\
&\qquad \varphi_{B,B\binom{1}{0}}(E_{\{\binom{4}{0}\}}) \cup \varphi_{B,B\binom{2}{0}}(E_{\{\binom{5}{0}\}}) \cup \varphi_{B,B\binom{0}{1}}(E_{\{\binom{3}{1}\}})\\
&\qquad \cup \varphi_{B,B\binom{1}{1}}(E_{\{\binom{4}{1}\}}) \cup \varphi_{B,B\binom{2}{1}}(E_{\{\binom{5}{1}\}}),\\
&E_{\{\binom{0}{1}\}} = E_{\{\binom{3}{2}\}} = E_{\{\binom{0}{3}\}} = E_{\{\binom{4}{3}\}} = \\
&\qquad \varphi_{B,B\binom{0}{0}}(E_{\{\binom{0}{2}\}}) \cup \varphi_{B,B\binom{1}{0}}(E_{\{\binom{1}{2}\}}) \cup \varphi_{B,B\binom{0}{1}}(E_{\{\binom{0}{3}\}}) \cup \varphi_{B,B\binom{1}{1}}(E_{\{\binom{1}{3}\}}),\\
&E_{\{\binom{1}{1}\}} = E_{\{\binom{5}{1}\}} = E_{\{\binom{1}{3}\}} = E_{\{\binom{5}{3}\}} = \\
&\qquad \varphi_{B,B\binom{0}{0}}(E_{\{\binom{3}{2}\}}) \cup \varphi_{B,B\binom{1}{0}}(E_{\{\binom{4}{2}\}}) \cup \varphi_{B,B\binom{2}{0}}(E_{\{\binom{5}{2}\}})\\
&\qquad \cup \varphi_{B,B\binom{1}{1}}(E_{\{\binom{4}{3}\}}) \cup \varphi_{B,B\binom{2}{1}}(E_{\{\binom{5}{3}\}}),\\
&E_{\{\binom{4}{1}\}} = \varphi_{B,B\binom{0}{0}}(E_{\{\binom{0}{2}\}}) \cup \varphi_{B,B\binom{1}{0}}(E_{\{\binom{1}{2},\binom{4}{0}\}}) \cup \varphi_{B,B\binom{2}{0}}(E_{\{\binom{5}{0}\}})\\
&\qquad \cup \varphi_{B,B\binom{0}{1}}(E_{\{\binom{0}{3},\binom{3}{1}\}}) \cup \varphi_{B,B\binom{1}{1}}(E_{\{\binom{1}{3},\binom{4}{1}\}}) \cup \varphi_{B,B\binom{2}{1}}(E_{\{\binom{5}{1}\}}),\\
&E_{\{\binom{4}{2}\}} = \varphi_{B,B\binom{0}{0}}(E_{\{\binom{0}{0},\binom{3}{2}\}}) \cup \varphi_{B,B\binom{1}{0}}(E_{\{\binom{1}{0},\binom{4}{2}\}}) \cup \varphi_{B,B\binom{2}{0}}(E_{\{\binom{5}{2}\}})\\
&\qquad \cup \varphi_{B,B\binom{0}{1}}(E_{\{\binom{0}{1}\}}) \cup \varphi_{B,B\binom{1}{1}}(E_{\{\binom{1}{1},\binom{4}{3}\}}) \cup \varphi_{B,B\binom{2}{1}}(E_{\{\binom{5}{3}\}}),
\end{aligned}\right. \tag{5.1}$$

Now, in the subscripts of E in (5.1), by replacing all $\binom{4}{0}$,$\binom{0}{2}$ and $\binom{3}{1}$ by $\binom{0}{0}$, replacing all $\binom{5}{0}$,$\binom{1}{2}$ and $\binom{5}{2}$ by $\binom{1}{0}$, replacing all $\binom{3}{2}$,$\binom{0}{3}$ and $\binom{4}{3}$ by $\binom{0}{1}$, and replacing all $\binom{5}{1}$,$\binom{1}{3}$ and $\binom{5}{3}$ by $\binom{1}{1}$, we obtain

$$\left\{\begin{aligned}
&E_{\{\binom{0}{0}\}} = \varphi_{B,B\binom{0}{0}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{1}{0}}(E_{\{\binom{1}{0}\}}) \cup \varphi_{B,B\binom{0}{1}}(E_{\{\binom{0}{1}\}}) \cup \varphi_{B,B\binom{1}{1}}(E_{\{\binom{1}{1}\}}),\\
&E_{\{\binom{1}{0}\}} = \varphi_{B,B\binom{1}{0}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{2}{0}}(E_{\{\binom{1}{0}\}}) \cup \varphi_{B,B\binom{0}{1}}(E_{\{\binom{0}{0}\}})\\
&\qquad \cup \varphi_{B,B\binom{1}{1}}(E_{\{\binom{4}{1}\}}) \cup \varphi_{B,B\binom{2}{1}}(E_{\{\binom{1}{1}\}}),\\
&E_{\{\binom{0}{1}\}} = \varphi_{B,B\binom{0}{0}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{1}{0}}(E_{\{\binom{1}{0}\}}) \cup \varphi_{B,B\binom{0}{1}}(E_{\{\binom{0}{1}\}}) \cup \varphi_{B,B\binom{1}{1}}(E_{\{\binom{1}{1}\}}),\\
&E_{\{\binom{1}{1}\}} = \varphi_{B,B\binom{0}{0}}(E_{\{\binom{0}{1}\}}) \cup \varphi_{B,B\binom{1}{0}}(E_{\{\binom{4}{2}\}}) \cup \varphi_{B,B\binom{2}{0}}(E_{\{\binom{1}{0}\}})\\
&\qquad \cup \varphi_{B,B\binom{1}{1}}(E_{\{\binom{0}{1}\}}) \cup \varphi_{B,B\binom{2}{1}}(E_{\{\binom{1}{1}\}}),\\
&E_{\{\binom{4}{1}\}} = \varphi_{B,B\binom{0}{0}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{1}{0}}(E_{\{\binom{1}{0},\binom{0}{0}\}}) \cup \varphi_{B,B\binom{2}{0}}(E_{\{\binom{1}{0}\}})\\
&\qquad \cup \varphi_{B,B\binom{0}{1}}(E_{\{\binom{0}{1},\binom{0}{0}\}}) \cup \varphi_{B,B\binom{1}{1}}(E_{\{\binom{1}{1},\binom{4}{1}\}}) \cup \varphi_{B,B\binom{2}{1}}(E_{\{\binom{1}{1}\}}),\\
&E_{\{\binom{4}{2}\}} = \varphi_{B,B\binom{0}{0}}(E_{\{\binom{0}{0},\binom{0}{1}\}}) \cup \varphi_{B,B\binom{1}{0}}(E_{\{\binom{1}{0},\binom{4}{2}\}}) \cup \varphi_{B,B\binom{2}{0}}(E_{\{\binom{1}{0}\}})\\
&\qquad \cup \varphi_{B,B\binom{0}{1}}(E_{\{\binom{0}{1}\}}) \cup \varphi_{B,B\binom{1}{1}}(E_{\{\binom{1}{1},\binom{0}{1}\}}) \cup \varphi_{B,B\binom{2}{1}}(E_{\{\binom{1}{1}\}}).
\end{aligned}\right. \tag{5.2}$$

Comparing the first and third equation in (5.2), we have $E_{\{\binom{0}{0}\}} = E_{\{\binom{0}{1}\}}$. By replacing all the $E_{\{\binom{0}{1}\}}$ in (5.2) by $E_{\{\binom{0}{0}\}}$, we get

$$\begin{cases} E_{\{\binom{0}{0}\}} = \varphi_{B,B\binom{0}{0}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{1}{0}}(E_{\{\binom{1}{0}\}}) \cup \varphi_{B,B\binom{0}{1}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{1}{1}}(E_{\{\binom{1}{1}\}}), \\ E_{\{\binom{1}{0}\}} = \varphi_{B,B\binom{1}{0}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{2}{0}}(E_{\{\binom{1}{0}\}}) \cup \varphi_{B,B\binom{0}{1}}(E_{\{\binom{0}{0}\}}) \\ \qquad \cup \varphi_{B,B\binom{1}{1}}(E_{\{\binom{4}{1}\}}) \cup \varphi_{B,B\binom{2}{1}}(E_{\{\binom{1}{1}\}}), \\ E_{\{\binom{1}{1}\}} = \varphi_{B,B\binom{0}{0}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{1}{0}}(E_{\{\binom{4}{2}\}}) \cup \varphi_{B,B\binom{2}{0}}(E_{\{\binom{1}{0}\}}) \\ \qquad \cup \varphi_{B,B\binom{1}{1}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{2}{1}}(E_{\{\binom{1}{1}\}}), \\ E_{\{\binom{4}{1}\}} = \varphi_{B,B\binom{0}{0}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{1}{0}}(E_{\{\binom{1}{0},\binom{0}{0}\}}) \cup \varphi_{B,B\binom{2}{0}}(E_{\{\binom{1}{0}\}}) \\ \qquad \cup \varphi_{B,B\binom{0}{1}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{1}{1}}(E_{\{\binom{1}{1},\binom{4}{1}\}}) \cup \varphi_{B,B\binom{2}{1}}(E_{\{\binom{1}{1}\}}), \\ E_{\{\binom{4}{2}\}} = \varphi_{B,B\binom{0}{0}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{1}{0}}(E_{\{\binom{1}{0},\binom{4}{2}\}}) \cup \varphi_{B,B\binom{2}{0}}(E_{\{\binom{1}{0}\}}) \\ \qquad \cup \varphi_{B,B\binom{0}{1}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{1}{1}}(E_{\{\binom{1}{1},\binom{0}{0}\}}) \cup \varphi_{B,B\binom{2}{1}}(E_{\{\binom{1}{1}\}}). \end{cases} \tag{5.3}$$

Comparing the forth equation with the union of the first and second equation in (5.3), applying the formula (4.8), we get $E_{\{\binom{0}{0},\binom{1}{0}\}} = E_{\{\binom{4}{1}\}}$. Similarly, from the first, third and fifth equations in (5.3), we get $E_{\{\binom{0}{0},\binom{1}{1}\}} = E_{\{\binom{4}{2}\}}$. So

$$E_{\{\binom{0}{0},\binom{1}{0},\binom{1}{1}\}} = E_{\{\binom{4}{1},\binom{1}{1}\}} = E_{\{\binom{4}{2},\binom{1}{0}\}} = E_{\{\binom{4}{1},\binom{4}{2}\}}.$$

Thus (5.3) becomes

$$\begin{cases} E_{\{\binom{0}{0}\}} = \varphi_{B,B\binom{0}{0}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{1}{0}}(E_{\{\binom{1}{0}\}}) \cup \varphi_{B,B\binom{0}{1}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{1}{1}}(E_{\{\binom{1}{1}\}}), \\ E_{\{\binom{1}{0}\}} = \varphi_{B,B\binom{1}{0}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{2}{0}}(E_{\{\binom{1}{0}\}}) \cup \varphi_{B,B\binom{0}{1}}(E_{\{\binom{0}{0}\}}) \\ \qquad \cup \varphi_{B,B\binom{1}{1}}(E_{\{\binom{4}{1}\}}) \cup \varphi_{B,B\binom{2}{1}}(E_{\{\binom{1}{1}\}}), \\ E_{\{\binom{1}{1}\}} = \varphi_{B,B\binom{0}{0}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{1}{0}}(E_{\{\binom{4}{2}\}}) \cup \varphi_{B,B\binom{2}{0}}(E_{\{\binom{1}{0}\}}) \\ \qquad \cup \varphi_{B,B\binom{1}{1}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{2}{1}}(E_{\{\binom{1}{1}\}}), \\ E_{\{\binom{4}{1}\}} = \varphi_{B,B\binom{0}{0}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{1}{0}}(E_{\{\binom{4}{1}\}}) \cup \varphi_{B,B\binom{2}{0}}(E_{\{\binom{1}{0}\}}) \\ \qquad \cup \varphi_{B,B\binom{0}{1}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{1}{1}}(E_{\{\binom{0}{0},\binom{1}{0},\binom{1}{1}\}}) \cup \varphi_{B,B\binom{2}{1}}(E_{\{\binom{1}{1}\}}), \\ E_{\{\binom{4}{2}\}} = \varphi_{B,B\binom{0}{0}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{1}{0}}(E_{\{\binom{0}{0},\binom{1}{0},\binom{1}{1}\}}) \cup \varphi_{B,B\binom{2}{0}}(E_{\{\binom{1}{0}\}}) \\ \qquad \cup \varphi_{B,B\binom{0}{1}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{1}{1}}(E_{\{\binom{4}{2}\}}) \cup \varphi_{B,B\binom{2}{1}}(E_{\{\binom{1}{1}\}}). \end{cases} \tag{5.4}$$

Taking union of the first, second and third equation of (5.4), we obtain

$$\begin{aligned} E_{\{\binom{0}{0},\binom{1}{0},\binom{1}{1}\}} = {} & \varphi_{B,B\binom{0}{0}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{1}{0}}(E_{\{\binom{0}{0},\binom{1}{0},\binom{1}{1}\}}) \cup \varphi_{B,B\binom{2}{0}}(E_{\{\binom{1}{0}\}}) \\ & \cup \varphi_{B,B\binom{0}{1}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{1}{1}}(E_{\{\binom{0}{0},\binom{1}{0},\binom{1}{1}\}}) \cup \varphi_{B,B\binom{2}{1}}(E_{\{\binom{1}{1}\}}). \end{aligned} \tag{5.5}$$

Compare the last equation in (5.4) with (5.5). Suppose $p \in E_{\{\binom{4}{2}\}}$. Then it is obviously that if $p \notin \varphi_{B,B\binom{1}{1}}(E_{\{\binom{4}{2}\}})$, then $p \in E_{\{\binom{0}{0},\binom{1}{0},\binom{1}{1}\}}$. Consequently, if $p \in \varphi_{B,B\binom{1}{1}}(E_{\{\binom{4}{2}\}})$ and $p \notin \varphi^2_{B,B\binom{1}{1}}(E_{\{\binom{4}{2}\}})$, then $p \in \varphi_{B,B\binom{1}{1}}(E_{\{\binom{0}{0},\binom{1}{0},\binom{1}{1}\}}) \subseteq E_{\{\binom{0}{0},\binom{1}{0},\binom{1}{1}\}}$. Repeating this argument, we have that if $p \notin \varphi^k_{B,B\binom{1}{1}}(E_{\{\binom{4}{2}\}})$ for some k, then $p \in E_{\{\binom{0}{0},\binom{1}{0},\binom{1}{1}\}}$. So if $p \notin \cap_{k\geq 1}\varphi^k_{B,B\binom{1}{1}}(E_{\{\binom{4}{2}\}})$, then $p \in E_{\{\binom{0}{0},\binom{1}{0},\binom{1}{1}\}}$. But $\cap_{k\geq 1}\varphi^k_{B,B\binom{1}{1}}(E_{\{\binom{4}{2}\}}) = \{\binom{\frac{1}{2}}{1}\} = \cap_{k\geq 1}\varphi^k_{B,B\binom{1}{1}}(E_{\{\binom{0}{0},\binom{1}{0},\binom{1}{1}\}}) \subseteq E_{\{\binom{0}{0},\binom{1}{0},\binom{1}{1}\}}$. So we have proved that $p \in E_{\{\binom{4}{2}\}}$ implies $p \in E_{\{\binom{0}{0},\binom{1}{0},\binom{1}{1}\}}$. Similarly, we can prove that $p \in E_{\{\binom{0}{0},\binom{1}{0},\binom{1}{1}\}}$ implies $p \in E_{\{\binom{4}{2}\}}$. Thus $E_{\{\binom{4}{2}\}} = E_{\{\binom{0}{0},\binom{1}{0},\binom{1}{1}\}}$. Using the same argument, we can also prove that $E_{\{\binom{4}{1}\}} = E_{\{\binom{0}{0},\binom{1}{0},\binom{1}{1}\}}$. So (5.4) becomes

$$\begin{cases} E_{\{\binom{0}{0}\}} = \varphi_{B,B\binom{0}{0}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{1}{0}}(E_{\{\binom{1}{0}\}}) \cup \varphi_{B,B\binom{0}{1}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{1}{1}}(E_{\{\binom{1}{1}\}}), \\ E_{\{\binom{1}{0}\}} = \varphi_{B,B\binom{1}{0}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{2}{0}}(E_{\{\binom{1}{0}\}}) \cup \varphi_{B,B\binom{0}{1}}(E_{\{\binom{0}{0}\}}) \\ \qquad \cup \varphi_{B,B\binom{1}{1}}(E_{\{\binom{4}{1}\}}) \cup \varphi_{B,B\binom{2}{1}}(E_{\{\binom{1}{1}\}}), \\ E_{\{\binom{1}{1}\}} = \varphi_{B,B\binom{0}{0}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{1}{0}}(E_{\{\binom{4}{1}\}}) \cup \varphi_{B,B\binom{2}{0}}(E_{\{\binom{1}{0}\}}) \\ \qquad \cup \varphi_{B,B\binom{1}{1}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{2}{1}}(E_{\{\binom{1}{1}\}}), \\ E_{\{\binom{4}{1}\}} = \varphi_{B,B\binom{0}{0}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{1}{0}}(E_{\{\binom{4}{1}\}}) \cup \varphi_{B,B\binom{2}{0}}(E_{\{\binom{1}{0}\}}) \\ \qquad \cup \varphi_{B,B\binom{0}{1}}(E_{\{\binom{0}{0}\}}) \cup \varphi_{B,B\binom{1}{1}}(E_{\{\binom{4}{1}\}}) \cup \varphi_{B,B\binom{2}{1}}(E_{\{\binom{1}{1}\}}), \end{cases} \tag{5.6}$$

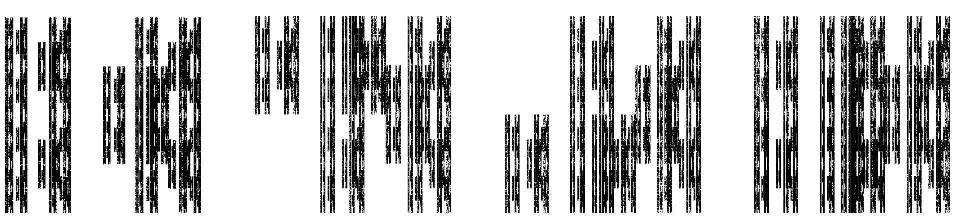
Let $\mathcal{E} = (E_{\{\binom{0}{0}\}}, E_{\{\binom{1}{0}\}}, E_{\{\binom{1}{1}\}}, E_{\{\binom{4}{1}\}})^t$ (see Figure 2). Define

$$M = \begin{pmatrix} \{\varphi_{B,B\binom{0}{0}}, \varphi_{B,B\binom{0}{1}}\} & \{\varphi_{B,B\binom{1}{0}}\} & \{\varphi_{B,B\binom{1}{1}}\} & \emptyset \\ \{\varphi_{B,B\binom{1}{0}}, \varphi_{B,B\binom{0}{1}}\} & \{\varphi_{B,B\binom{2}{0}}\} & \{\varphi_{B,B\binom{2}{1}}\} & \{\varphi_{B,B\binom{1}{1}}\} \\ \{\varphi_{B,B\binom{0}{0}}, \varphi_{B,B\binom{1}{1}}\} & \{\varphi_{B,B\binom{2}{0}}\} & \{\varphi_{B,B\binom{2}{1}}\} & \{\varphi_{B,B\binom{1}{0}}\} \\ \{\varphi_{B,B\binom{0}{0}}, \varphi_{B,B\binom{0}{1}}\} & \{\varphi_{B,B\binom{2}{0}}\} & \{\varphi_{B,B\binom{2}{1}}\} & \{\varphi_{B,B\binom{1}{0}}, \varphi_{B,B\binom{1}{1}}\} \end{pmatrix}. \tag{5.7}$$

Then (5.6) is equivalent to

$$M(\mathcal{E}) = \mathcal{E}.$$

It is easy to check that M is an irreducible net M-matrix. So $\mathcal{E} = A(M)$ and each component of $\mathcal{E}$ has the same Hausdorff dimension, we denote this dimension as $\dim_H \mathcal{E}$. According to the third equation in (4.9), $\dim_H E = \dim_H \mathcal{E}$.



(a) $E_{\{\binom{0}{0}\}}$ (b) $E_{\{\binom{1}{0}\}}$ (c) $E_{\{\binom{1}{1}\}}$ (d) $E_{\{\binom{4}{1}\}}$

FIG. 2. *c-vector* $\mathcal{E} = A(M)$

Now we apply Corollary 3.1 and Corollary 3.2 to get estimates of $\dim_H \mathcal{E}$:

$$d(M^s) \le \dim_H \mathcal{E} \le D(M^s) \text{ for } s = 1, 2, 3, \cdots .$$

From Theorem 3.2 and Proposition 3.1, it is easy to see that $d(M^s)$ and $D(M^s)$ can be calculated from the matrices defined in (3.3). Here, these matrices are

$$A_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}. \tag{5.8}$$

By [21], the sequences $g_{s,v}(M)$ and $G_s(M)$ (see Remark 3.3) also give estimations of the Hausdorff dimension. We perform numerical computation for $1 \le s \le 25$ (The formula (3.39) of Proposition 3.1 is used here to write the program). The results are listed below:

s	$g_{s,1}(M)$	$g_{s,2}(M)$	$g_{s,3}(M)$	$g_{s,4}(M)$	$d(M^s)$	$G_s(M)$	$D(M^s)$
1	−1.0000000	−2.0000000	−2.0000000	−1.0000000	1.6575031	2.0000000	2.2424656
2	0.3154649	−0.0863826	−0.0863826	0.1748419	1.7046807	1.9166481	2.0443969
3	0.7889960	0.5396336	0.5396336	0.6487352	1.7275150	1.8761924	1.9619427
4	1.0338196	0.8507275	0.8507275	0.9149821	1.7403752	1.8534427	1.9178629
5	1.1825780	1.0369919	1.0369919	1.0836720	1.7484232	1.8392345	1.8907989
6	1.2821922	1.1610793	1.1610793	1.1987008	1.7538699	1.8296319	1.8726104
7	1.3534524	1.2496921	1.2496921	1.2815894	1.7577812	1.8227412	1.8595825
8	1.4069242	1.3161464	1.3161464	1.3439602	1.7607201	1.8175655	1.8498023
9	1.4485200	1.3678318	1.3678318	1.3925283	1.7630073	1.8135379	1.8421931
10	1.4817984	1.4091798	1.4091798	1.4313992	1.7648374	1.8103153	1.8361051
11	1.5090266	1.4430099	1.4430099	1.4632072	1.7663349	1.8076786	1.8311238
12	1.5317169	1.4712016	1.4712016	1.4897153	1.7675829	1.8054813	1.8269727
13	1.5509164	1.4950561	1.4950561	1.5121455	1.7686388	1.8036220	1.8234602
14	1.5673731	1.5155029	1.5155029	1.5313715	1.7695439	1.8020283	1.8204495
15	1.5816356	1.5332234	1.5332234	1.5480341	1.7703284	1.8006471	1.8178403
16	1.5941153	1.5487288	1.5487288	1.5626139	1.7710148	1.7994386	1.8155572
17	1.6051267	1.5624101	1.5624101	1.5754784	1.7716204	1.7983722	1.8135426
18	1.6149147	1.5745712	1.5745712	1.5869135	1.7721587	1.7974243	1.8117520
19	1.6236724	1.5854522	1.5854522	1.5971449	1.7726404	1.7965762	1.8101498
20	1.6315543	1.5952451	1.5952451	1.6063532	1.7730739	1.7958129	1.8087078
21	1.6386855	1.6041054	1.6041054	1.6146845	1.7734661	1.7951223	1.8074032
22	1.6451685	1.6121602	1.6121602	1.6222584	1.7738227	1.7944945	1.8062171
23	1.6510877	1.6195145	1.6195145	1.6291737	1.7741482	1.7939213	1.8051342
24	1.6565137	1.6262560	1.6262560	1.6355127	1.7744467	1.7933959	1.8041416
25	1.6615055	1.6324582	1.6324582	1.6413446	1.7747212	1.7929124	1.8032283

Thus we have $1.7747212 \le \dim_H E \le 1.7929124$. It seems that $g_{s,v}(M) \le d(M^s) \le \dim_H E \le G_s(M) \le D(M^s)$ is always true.

Acknowledgment. We would like to thank Y. Peres and K.-S. Lau for providing useful informations and helpful suggestions.

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