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Classification of Affine Varieties Being Cones over Nonsingular Projective Varieties: Hypersurface Case

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Dedicated to Professor Salah Baouendi on the occasion of his 65th birthday.

Let X be a nonsingular projective variety in $\mathbb{C}P^n$. Then the cone over X in $\mathbb{C}^{n+1}$ is an affine variety V with an isolated singularity at the origin. It is a very natural question to ask when an affine variety with an isolated singularity at the origin is a cone over nonsingular projective variety.

In this paper we shall treat the hypersurface case. Specifically, we will prove the Yau conjecture on 3-dimensional weighted homogeneous hypersurface singularities. In particular, we have obtained sharp upper estimate of geometric genus in terms of Milnor number and multiplicity of the singularity. An important corollary that we obtain is a numerical characterization when an affine hypersurface in $\mathbb{C}^4$ with only isolated critical point at the origin is a cone over nonsingular hypersurface in $\mathbb{C}P^3$ after biholomorphic change of coordinates.

1. Introduction.

Let X be a nonsingular projective variety in $\mathbb{C}P^n$. Then the cone over X in $\mathbb{C}^{n+1}$ is an affine variety V with an isolated singularity at the origin. It is a very natural question to ask when an affine variety with an isolated singularity at the origin is a cone over nonsingular projective variety. In this paper we shall treat the hypersurface case.

This work is a natural continuation of our previous work [Xu-Ya 2]. Let $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ be the germ of a complex analytic function with an isolated critical point at the origin. Let $V = \{(x_0, x_1, \cdots, x_n) : f(x_0, x_1, \cdots, x_n) = 0\}$. The Milnor number μ of the singularity $(V, 0)$ is $dim\mathbf{C}\{x_0, x_1, \cdots, x_n\}/(f_{x_0}, f_{x_1}, \cdots, f_{x_n})$. Let $\pi : (M, A) \to (V, 0)$ be a resolution of singularity with exceptional set $A = \pi^{-1}(0)$. The geometric

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genus p_g of the singularity $(V, 0)$ is the dimension of $H^{n-1}(M, \mathcal{O})$. μ and p_g are two important numerical measures of the complexity of the singularity. In 1978, Durfee conjectured that $\mu \geq (n+1)!\, p_g$. In [Xu-Ya 2], Xu and Yau proved the following theorem.

Theorem. *Let $(V, 0)$ be a two dimensional isolated singularity defined by a weighted homogeneous polynomial $f(x_0, x_1, x_2) = 0$. Let μ be the Milnor number, p_g be the geometric genus, and ν be the multiplicity of the singularity. Then*

$$\mu - \nu + 1 \geq 6p_g$$

with equality if and only if $(V, 0)$ is defined by the homogeneous polynomial.

In view of the above theorem, the second author in 1995 made the following conjecture.

Conjecture. *Let $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, \mathbf{0})$ be a weighted homogeneous polynomial with an isolated critical point at the origin. Then*

$$\mu - h(\nu) \geq (n+1)!\, p_g$$

with equality if and only if f is a homogeneous polynomial, where $h(\nu)$ is a polynomial function on multiplicity with the properties $h(\nu) \geq 0$ and $h(\nu) = 0$ if and only if $\nu = 1$. Note that h is a polynomial function from $\mathbf{Z}_+$ to $\mathbf{Z}_+ \cup \{0\}$. In fact, $h(\nu) = (\nu - 1)^{n+1} - \nu(\nu - 1) \ldots (\nu - n)$.

Remark. The above conjecture together with the result of Saito [Sai] will give us a numerical characterization when an affine hypersurface in $\mathbb{C}^{n+1}$ is a cone over nonsingular hypersurface in $\mathbb{C}P^n$.

For two-dimensional isolated singularity the above Xu-Yau's theorem asserts that Yau conjecture is true. In fact $h(\nu) = \nu - 1$ in this case. After several years of hard work, we have proved the conjecture for 3-dimensional case.

Main Theorem. *Let $(V, 0)$ be a three dimensional isolated singularity defined by a weighted homogeneous polynomial $f(x, y, z, w) = 0$. Let p_g be the geometric genus, ν be the multiplicity and μ be the Milnor number of the singularity. Suppose $p_g > 0$. Then we have*

$$\mu - (2\nu^3 - 5\nu^2 + 2\nu + 1) \geq 4!\, p_g$$

with equality if and only if $(V, 0)$ is defined by the homogeneous polynomial.

Main Theorem is related to the classical question of counting the number of integral points in a tetrahedron. Let $f(x_0, \cdots, x_n)$ be a germ of an analytic function at the origin such that $f(0) = 0$. Suppose f has an isolated critical point at the origin, f can be developed in a convergent Taylor series $f(x_0, \cdots, x_n) = \sum a_\lambda x^\lambda$ where $x^\lambda = x_0^{\lambda_0} \cdots x_n^{\lambda_n}$. Recall that Newton boundary $\Gamma(f)$ is the union of the compact faces of $\Gamma_+(f)$ where $\Gamma_+(f)$ is the convex hull of the union of the subsets $\{\lambda + (\mathbf{R}^+)^{n+1}\}$ for λ such that $a_\lambda \neq 0$. Finally, let $\Gamma_-(f)$, the Newton polyhedron of f, be the cone over $\Gamma(f)$ with cone point at 0. For any closed face Δ of $\Gamma(f)$, we associate the polynomial $f_\Delta(x) = \sum_{\lambda \in \Delta} a_\lambda x^\lambda$. We say that f is nondegenerate if f_Δ has no critical point in $(\mathbf{C}^*)^{\mathbf{n+1}}$ for any $\Delta \in \Gamma(f)$ where $\mathbf{C}^* = \mathbf{C} - \{0\}$. We say that a point p of the integral lattice $\mathbf{Z}^{\mathbf{n+1}}$ in $\mathbf{R}^{\mathbf{n+1}}$ is positive if all the coordinates of p are positive. The following beautiful theorem is due to Merle-Teissier.

Theorem (Merle-Teissier). *Let $(V, 0)$ be an isolated hypersurface singularity defined by a nondegenerate holomorphic function $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, \mathbf{0})$. Then the geometric genus $p_g = \sharp\{p \in \mathbf{Z}^{n+1} \cap \Gamma_-(f) \ : p$ is positive $\}$.*

A polynomial $f(x_0, x_1, \cdots, x_n)$ is weighted homogeneous of type $(w_0, w_1, \cdots, w_n)$, where $w_0, w_1, \cdots, w_n$ are fixed positive rational numbers, if it can be expressed as a linear combination of monomials $x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n}$ for which $i_0/w_0 + i_1/w_1 + \cdots + i_n/w_n = 1$. As a consequence of the theorem of Merle-Teissier, we know that in case of isolated singularity defined by a weighted homogeneous polynomial, computing the geometric genus is equivalent to counting the number of positive integral points in a tetrahedron. The latter problem has attracted a lot of attention recently, see for example [Br-Ve], [Ca-Sh], [Po],[Di-Ro]. In all these papers, the vertice of the tetrahedra are assumed to be integral. Our problem here is different in that the coordinates of the vertice of the tetrahedra may be non-integral rational number. This is of course the main difficulty of the problem. We also need the following result.

Theorem (Milnor-Orlik). *Let $f(x_0, x_1, \cdots, x_n)$ be a weighted homogeneous polynomial of type $(w_0, w_1, \cdots, w_n)$ with isolated singularity at the origin. Then the Milnor number $\mu = (w_0 - 1)(w_1 - 1) \cdots (w_n - 1)$*

Thus our Main Theorem is related to the Main Theorem of [Xu-Ya 3] (Theorem 2.1 in section 2). However it does not follow from Theorem 2.1 because the minimal weight of the variables x_i may not be an integer and we also need to analyze the case when the geometric genus vanishes. It is quite easy to see that the multiplicity ν is given by $inf\{n \in \mathbf{Z}_+ : \ n \geq inf\{w_0, w_1, w_2, w_3\}$ where w_i is the weight of $x_i\}$, see for example [Sae]. We observe that if $w_0 \geq w_1 \geq w_2 \geq w_3$ and w_3 is not an integer, then $w_3 = [w_3] + \beta$, $0 < \beta < 1$ and β is either $\frac{w_3}{w_0}$, or $\frac{w_3}{w_1}$ or $\frac{w_3}{w_2}$. We then get an even sharper estimate in these three particular cases in Theorem 2.7 and Theorem 2.8 than those obtained in the Main Theorem of [Xu-Ya 3]. Unlike the surface singularities treated in [Xu-Ya 2], we still need to handle the case when the geometric genus is equal to zero, which will be treated in our subsequent paper. Our Main Theorem is substantially harder to prove than the corresponding theorem in [Xu-Ya 2].

We remark that the validity of Yau conjecture for hypersurface singularities gives a necessary condition for a singularity to be hypersurface. Given a function f with an isolated singularity at the origin, it is important to know whether f is a weighted homogeneous polynomial or a homogeneous polynomial after a biholomorphic change of variables. The former question was answered in a celebrated paper [Sai] by Saito in 1971. However, the latter question has remained open until Xu-Yau solved it for f with three variables [Xu-Ya 2]. The following Corollary gives a solution to the homogeneity question for function with four variables.

Corollary. *Let $(V,0)$ be a three dimensional isolated hypersurface singularity defined by $f(x,y,z,w) = 0$. Let μ be the Milnor number, p_g be the geometric genus, ν be the multiplicity of the singularity and τ=dimension of the semi-universal deformation space of $(V,0) = dim\mathbf{C}\{x,y,z,w\}/(f,f_x,f_y,f_z,f_w)$. Then after a biholomorphic change of coordinate f is a homogeneous polynomial if and only if $\mu - (2\nu^3 - 5\nu^2 + 2\nu + 1) = 24p_g$ and $\mu = \tau$.*

2. Sharp upper estimate of number of integral points in a 4-dimensional tetrahedra.

As mentioned in section 1 about isolated singularity defined by a weighted homogeneous polynomial, computing the geometric genus is equivalent to counting the number of positive integral points in a tetrahedron. For a weighted homogeneous polynomial $f(x,y,z,w)$ with weights $a \geq b \geq c \geq d \geq 2$, the number of positive integral solutions of the tetrahedron $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{w}{d} \leq 1$ is exactly the geometric genus we want to estimate. Theorem 3.1 uses results from Theorem 2.1 to Theorem 2.5 to count the geometric genus when the smallest weight d is an integer. But for the case the smallest weight d is not an integer we need even sharper estimate than Theorem 2.1 to Theorem 2.5. A sharper estimate can be gotten when the nonintegral part of d is the ratio of d and other weights. In Lemma 3.2 we prove that for a weighted homogeneous polynomial $f(x,y,z,w)$ with weights $a \geq b \geq c \geq d \geq 2$ the nonintegral part of d is the ratio of d and other weights. Hence we can use our sharper estimate Theorem 2.7, Theorem 2.8, and Corollary 2.9 to count the geometric genus when d is not an integer. Unfortunately when $d = 2 + \frac{d}{a}$ and $P_4 = 0$, we don't have this sharper estimate in this case we need to prove our Main Theorem with the results of [Ka] through a case by case analysis.

By level $w = k$, we mean the intersection of the tetrahedron $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leq 1$ with the hyperplane $w = k$. For $w = k$ positive integral points in our tetrahedron are in the following 3 dimensional tetrahedron

$$\frac{x}{a(1-\frac{k}{d})} + \frac{y}{b(1-\frac{k}{d})} + \frac{z}{c(1-\frac{k}{d})} \leq 1$$

for which the number of integral points can be bounded by means of Theorem 2.5. Lemma2.4 tell us when $b(1-\frac{k}{d}) < 2$ this 3 dimensional tetrahedron has no integral point which we need to separate the cases.

The following Theorem 2.1 and proposition 2.6 are proved in [Xu-Ya 3].

Theorem 2.1. *Let $a \geq b \geq c \geq d \geq 2$, and P_4 be the number of positive integral solutions of $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leq 1$, i.e $P_4 = \sharp\{(x, y, z, w) \in Z_+^4 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leq 1\}$. If $P_4 > 0$, then*

$$24P_4 \leq f(a, b, c, d) := abcd - \frac{3}{2}(abc + abd + acd + bcd) + \frac{11}{3}(ab + ac + bc) - 2(a + b + c) \qquad (2.1)$$

and equality is attained if and only if $a = b = c = d =$ integer.

Note that when $a \geq b \geq c \geq d \geq 2$ and $P_4 > 0$ inequality (2.1) holds and $f(a, b, c, d) > 0$. If $a \geq b \geq c \geq d \geq 2$ and $P_4 = 0$, $f(a, b, c, d)$ may become negative for certain combination of a ,b ,c and d. For the case $P_4 = 0$ we can find special combination of a ,b ,c and d to make $f(a, b, c, d) \geq 0$, then inequality (2.1) still holds. Lemma 2.2 below shows that $f(a, b, c, d) \geq 0$ under the constraint $a \geq b \geq c \geq d \geq 3$ or under the constraint $a \geq b \geq c \geq d$ and $2 \geq d \geq 1.5$, $f(a, b, c, d) = 0$ if and only if when $a = b = c = d = 3$ or $a = b = c = d = 2$ for the range considered. Hence instead of Theorem 2.1 we get Theorem 2.3 which replaces $a \geq b \geq c \geq d \geq 2$ by $a \geq b \geq c \geq d \geq 3$ without the condition $P_4 > 0$.

Lemma 2.2. *$f(a, b, c, d)$ defined in (2.1) is nonnegative for the following two cases (i) $a \geq b \geq c \geq d \geq 3$ and (ii) $a \geq b \geq c \geq d$ and $2 \geq d \geq 1.5$. $f(a, b, c, d) = 0$ if and only if $a = b = c = d = 3$ for case (i) and $a = b = c = d = 2$ for case (ii).*

Proof. First we consider the special case when $a = b = c = d$, then $f(a, b, c, d)$ becomes

$$d^4 - 6d^3 + 11d^2 - 6d = d(d-1)(d-2)(d-3)$$

Hence $f\mid_{a=b=c=d} = 0$ when $d = 2$ or $d = 3$, and $f\mid_{a=b=c=d} > 0$ when $d > 3$ or $1 < d < 2$. Next for the case $a \geq b \geq c \geq d \geq 1.5$. Let $A = \frac{a}{d}, B = \frac{b}{d}, C = \frac{c}{d}$, then $A \geq B \geq C \geq 1$, and $d \geq 1.5$. Rewrite $f(a, b, c, d)$ as

$$\begin{aligned} f(A,B,C,d) = \ & ABCd^4 - \tfrac{3}{2}(ABC + AB + AC + BC)d^3 \\ & + \tfrac{11}{3}(AB + BC + AC)d^2 - 2(A + B + C)d \end{aligned}$$

Observe that$f(A, B, C, d)$ is an increasing function of A ,B and C for $A \geq 1, B \geq 1,\ C \geq 1$ and $d \geq \frac{3}{2}$. $f\mid_{A=B=C=1} > 0$ ie $f\mid_{a=b=c=d} > 0$ for either $d > 3$ or $2 > d \geq 1.5$ and when $d = 2$ or $d = 3$ $f\mid_{A=B=C=1} = 0$. Finally we conclude that $f(A, B, C, d) > 0$ for $A \geq 1,\ B \geq 1,\ C \geq 1$ and $d > 3$ or $2 > d \geq 1.5$. In particular, $f(a, b, c, d) \geq 0$ for $a \geq b \geq c \geq d \geq 3$ or $a \geq b \geq c \geq d$ and $2 \geq d \geq 1.5$. $f(a, b, c, d) = 0$ if and only if $a = b = c = d = 3$ for $a \geq b \geq c \geq d \geq 3$. For $a \geq b \geq c \geq d$ and $2 \geq d \geq 1.5$ $f(a, b, c, d) = 0$ if and only if $a = b = c = d = 2$. Q.E.D.

Theorem 2.3. *Let $a \geq b \geq c \geq d \geq 3$, and P_4 be the number of positive integral solutions of $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leq 1$, i.e $P_4 = \sharp\{(x, y, z, w) \in Z_+^4 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leq 1\}$. Then*

$$24P_4 \leq abcd - \frac{3}{2}(abc + abd + acd + bcd) + \frac{11}{3}(ab + ac + bc) - 2(a + b + c)$$

and equality is attained if and only if $a = b = c = d = integer$.

Lemma 2.4 follows directly from Proposition 2.3 of [Xu-Ya 3] and its proof

Lemma 2.4. *Let $a \geq b \geq c \geq 1$ be real number and $P_3 = \sharp\{(x, y, z) \in Z_+^3 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq 1\}$. Then the following statements hold*

(1) if $b \leq 2$, then $P_3 = 0$

(2) $(a-1)(b-1)(c-1) - (c-1) < 0$ if and only if $a < \frac{b}{b-1}$

(3) $(a-1)(b-1)(c-1) - (c-1) < 0$ implies $b < 2$ and $P_3 = 0$

Theorem 2.5 follows directly from Theorem 2.1 and Proposition 2.3 in [Xu-Ya 3] (cf. also [Xu-Ya 1]).

Theorem 2.5. *Let $a \geq b \geq c \geq 1$ be real number. Let P_3 be the number of positive integral solutions of $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} \leq 1$, i.e. $P_3 = \sharp\{(x,y,z) \in Z^3_+ : \frac{x}{a}+\frac{y}{b}+\frac{z}{c} \leq 1\}$. If $b \geq 2$, then $6P_3 \leq (a-1)(b-1)(c-1)-c+1$ and equality is attained if and only if $a=b=c=$ integer.*

Proposition 2.6. *(cf. p.6 and p.11-p.12 in [Xu-Ya 3]) Let $a \geq b \geq c \geq d \geq 2$ be real number. Consider $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{w}{d} \leq 1$ Let P_4 be the number of positive integral solutions of the above equation; i.e $P_4 = \sharp\{(x,y,z,w) \in Z^4_+ : \frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{w}{d} \leq 1\}$. Suppose d is not an integer and $d=[d]+\beta$ where β satisfies either $\frac{a}{d}\beta < 3$ or $\frac{b}{d}\beta \leq 2$ or $\frac{c}{d}\beta \leq 1$. Then*

$$\begin{aligned}24P_4 \ \leq \ & abcd - 2abc - \tfrac{4}{3}(abd+acd+bcd) + 2(ab+ac+bc+ad+bd) \\ & -2(a+b) + \tfrac{abc}{d} - \tfrac{2}{3}\big(\tfrac{ab}{d}+\tfrac{ac}{d}+\tfrac{bc}{d}\big) \\ & +(-\beta^4-2\beta^3-\beta^2)\tfrac{abc}{d^3} + \big(\tfrac{4}{3}\beta^3+2\beta^2+\tfrac{2}{3}\beta\big)\big(\tfrac{ab}{d^2}+\tfrac{ac}{d^2}+\tfrac{bc}{d^2}\big) \\ & -2(\beta+\beta^2)\big(\tfrac{a}{d}+\tfrac{b}{d}\big)+\delta \end{aligned} \tag{2.2}$$

Inequality (2.2) is gotten from the three dimensional estimate Theorem 2.5 [Xu-Ya 1] by summing from level $w=1$ to level $w=d-\beta-1$. At level $w=d-\beta-1$ the inequality $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{w}{d} \leq 1$ becomes $\frac{x}{\frac{a}{d}(\beta+1)}+\frac{y}{\frac{b}{d}(\beta+1)}+\frac{z}{\frac{c}{d}(\beta+1)} \leq 1$. When $\frac{b}{d}(\beta+1) < 2$ by Theorem 2.5 no integral points at level $w=d-\beta-1$ and by Lemma 2.4 the estimate $[\frac{a}{d}(\beta+1)-1][\frac{b}{d}(\beta+1)-1][\frac{c}{d}(\beta+1)-1]-\frac{c}{d}(\beta+1)+1$ may become negative, so we need to add the absolute value of this negative amount on the right hand side of inequality (2.2). Here $\delta \geq 0$ is the adjusting term needed when $\frac{b}{d}(\beta+1) < 2$ due to the negative amount added at level $w=d-\beta-1$.
Note the relationship between Theorem 2.7 and Main Theorem.

Theorem 2.7. *Let $a \geq b \geq c \geq d \geq 3$ be real number. Consider $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{w}{d} \leq 1$. Let P_4 be the number of positive integral solutions of the above equation; i.e $P_4 = \sharp\{(x,y,z,w) \in Z^4_+ : \frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{w}{d} \leq 1\}$ Suppose d is not an integer and $d=[d]+\beta$ where β is either $\frac{d}{c}$ or $\frac{d}{b}$ or $\frac{d}{a}$. Define $\mu=(a-1)(b-1)(c-1)(d-1)$. Then*

$$\begin{aligned}24P_4 < \ & \mu - (2\nu^3-5\nu^2+2\nu+1)\,|_{\nu=d-\beta+1} \\ = \ & abcd-(abc+abd+acd+bcd)+(ab+ac+ad+bc+bd+cd) \\ & -(a+b+c)-(2d^3+d^2-d-1)+2\beta^3-\beta^2(6d+1) \\ & +\beta(6d^2+2d-2) \end{aligned} \tag{2.3}$$

Proof. We will consider the following three cases:
Case(a) $\beta=\frac{d}{c}$, Case(b) $\beta=\frac{d}{b}$, Case(c) $\beta=\frac{d}{a}$.

For these three cases β satisfies the condition of Proposition 2.6 hence $24P_4$ is bound by the right hand side of inequality (2.2). Let $\Delta = R.H.S.\ of\ (2.3) - R.H.S.\ of\ (2.2)$.

$$\begin{aligned}\Delta = \ & abc + \tfrac{1}{3}(abd+acd+bcd) - (ab+ac+ad+bc+bd) + cd \\ & +a+b-c-\tfrac{abc}{d} - (2d^3+d^2-d-1) + \tfrac{2}{3}(\tfrac{ab}{d}+\tfrac{ac}{d}+\tfrac{bc}{d}) \\ & +2\beta^3 - \beta^2(6d+1) + \beta(6d^2+2d-2) + \beta^2(\beta+1)^2\tfrac{abc}{d^3} \\ & -\tfrac{2}{3}(2\beta+1)(\beta+1)\beta(\tfrac{ab}{d^2}+\tfrac{ac}{d^2}+\tfrac{bc}{d^2}) + 2(\beta+\beta^2)(\tfrac{a}{d}+\tfrac{b}{d}) - \delta\end{aligned} \tag{2.4}$$

We will show that $\Delta > 0$ for these cases.

Case(a): $\beta = \frac{d}{c}$

At level $w = d-\beta-1$, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{d-\beta-1}{d} \le 1$, i.e. $\frac{x}{\frac{a}{d}(\beta+1)}+\frac{y}{\frac{b}{d}(\beta+1)}+\frac{z}{\frac{c}{d}(\beta+1)} \le 1$. Since $\frac{b}{d}(\beta+1) = \frac{b}{c}+\frac{b}{d} > 2$, by Lemma 2.4 $\delta = 0$. If $c = d$, then $\beta = 1$ and $d = [d]+\beta$ is an integer. Hence $a \ge b \ge c > d$. Let $A = \frac{a}{d}, B = \frac{b}{d}, C = \frac{c}{d}$, then $\beta = \frac{d}{c} = \frac{1}{C}$, and $A \ge B \ge C > 1$. Then Δ can be written as

$$\begin{aligned}\Delta = \ & ABCd^3 + \tfrac{d^3}{3}(AB+AC+BC) - d^2(AB+AC+BC+A+B) \\ & +Cd^2 + (A+B)d - Cd - ABCd^2 - (2d^3+d^2-d-1) \\ & +\tfrac{2d}{3}(AB+AC+BC) + \tfrac{2}{C^3} - \tfrac{1}{C^2}(6d+1) + \tfrac{1}{C}(6d^2+2d-2) \\ & +\tfrac{(C+1)^2}{C^3}AB - \tfrac{2}{3}\tfrac{1}{C^3}(2+C)(1+C)(AB+AC+BC) + 2(A+B)(\tfrac{1+C}{C^2})\end{aligned} \tag{2.5}$$

$$\begin{aligned}\Delta\,|_{A=B=C=1} = \ & 2d^3 - 5d^2 + d^2 + d - d^2 - 2d^3 - d^2 + d + 1 + 2d + 2 \\ & -6d - 1 + 6d^2 + 2d - 2 + 4 - 12 + 8 = 0\end{aligned} \tag{2.6}$$

Δ is an increasing function of A , B and C for $A \ge B \ge C \ge 1$ and $d \ge 3$. Combining this result with (2.6) we conclude that $\Delta > 0 \quad for\ a \ge b \ge c \ge d > 3$.

Case(b): $\beta = \frac{d}{b}$

If $b = d$, then $\beta = 1$ and $d = [d]+\beta$ is an integer. Hence $a \ge b > d$. Similar to Case(a), $\frac{b}{d}(\beta+1) = 1+\frac{b}{d} > 2$. By Lemma 2.4 $\delta = 0$ in (2.4). Let $A = \frac{a}{d}, B = \frac{b}{d}, C = \frac{c}{d}$, then $\beta = \frac{d}{b} = \frac{1}{B}$. From (2.4) Δ can be written as

$$\begin{aligned}\Delta = \ & ABCd^3 + \tfrac{d^3}{3}(AB+AC+BC) - d^2(AB+AC+BC+A+B) + Cd^2 \\ & +(A+B)d - Cd - ABCd^2 - (2d^3+d^2-d-1) + \tfrac{2d}{3}(AB+AC+BC) \\ & +\tfrac{2}{B^3} - \tfrac{1}{B^2}(6d+1) + \tfrac{1}{B}(6d^2+2d-2) + \tfrac{(B+1)^2}{B^3}AC \\ & -\tfrac{2}{3}\tfrac{1}{B^3}(2+B)(1+B)(AB+AC+BC) + 2(A+B)(\tfrac{1+B}{B^2})\end{aligned} \tag{2.7}$$

$$\begin{aligned}\Delta\,|_{A=B=C=1} = \ & d^3 + d^3 - 5d^2 + d^2 + 2d - d - d^2 - 2d^3 - d^2 + d + 1 \\ & +2d + 2 - 6d - 1 + 6d^2 + 2d - 2 + 4 - 12 + 8 = 0\end{aligned} \tag{2.8}$$

Δ is an increasing function of A and C for $A \geq B \geq C \geq 1$ and $d \geq 3$. We can also prove that Δ is an increasing function of B for $A \geq B \geq C \geq 1$ and $d > 4$ but this is not true for $A \geq B \geq C \geq 1$ and $3 < d < 4$. Hence we need to consider the following two subcases:
Case(b1): $3 < d < 4$: We will show directly that (2.3) holds in this subcase.
Case(b2): $d > 4$: $\frac{\partial \Delta}{\partial B} > 0$ in this subcase.
Case(b1): $3 < d < 4$
In this case $d = 3+\beta, 0 < \beta = \frac{d}{b} < 1$. Also note that $b\beta = 3+\beta,\ \mathit{and}\ b = 1+\frac{3}{\beta} = \frac{d}{d-3} > 4$. Hence we need to consider only $a \geq b > 4, c \geq d > 3, d = 3+\frac{d}{b}$. It is easy to see that for this subcase we need to consider only the level $w = 1$ and $w = 2$ in computing P_4. L_1, the number of positive integral solutions at level $w = 1$ satisfies

$$6L_1 \leq abc(\frac{2+\beta}{3+\beta})^3 - (ab+ac+bc)(\frac{2+\beta}{3+\beta})^2 + (a+b)(\frac{2+\beta}{3+\beta})$$

by Theorem 2.5. L_2, the number of positive integral solutions at level $w = 2$ satisfies

$$6L_2 \leq abc(\frac{1+\beta}{3+\beta})^3 - (ab+ac+bc)(\frac{1+\beta}{3+\beta})^2 + (a+b)(\frac{1+\beta}{3+\beta})$$

At the level $w = 3$ the inequality becomes

$$\frac{x}{a(\frac{\beta}{3+\beta})} + \frac{y}{b(\frac{\beta}{3+\beta})} + \frac{z}{c(\frac{\beta}{3+\beta})} \leq 1$$

Note that $b(\frac{\beta}{3+\beta}) = \frac{b\beta}{d} = 1 < 2$, hence by Lemma 2.4 the number of positive integral solution of the above inequality equals zero. On the other hand for $d = 3+\beta$, $\nu = d - \beta + 1 = 4$, and $b\beta = 3+\beta$ in this subsubcase. Therefore the right hand side of (2.3) becomes

$$\begin{aligned} \mu - (2\nu^3 - 5\nu^2 + 2\nu + 1) &= (a-1)(b-1)(c-1)(2+\beta) - 57 \\ &= 2abc - 2ab - 2bc + ac - a - c + 2b - 56 \end{aligned}$$

Define Δ_1 as following:

$$\begin{aligned} \Delta_1 = \ & [2abc - 2ab - 2bc + ac - a - c + 2b - 56] \\ & -4\big[abc(\tfrac{2+\beta}{3+\beta})^3 - (ab+ac+bc)(\tfrac{2+\beta}{3+\beta})^2 + (a+b)(\tfrac{2+\beta}{3+\beta})\big] \\ & -4\big[abc(\tfrac{1+\beta}{3+\beta})^3 - (ab+ac+bc)(\tfrac{1+\beta}{3+\beta})^2 + (a+b)(\tfrac{1+\beta}{3+\beta})\big] \end{aligned}$$

Note that $\beta = \frac{3}{b-1}, \beta + 3 = \frac{3b}{b-1}$. Therefore Δ_1 becomes

$$\begin{aligned} \Delta_1 &= \tfrac{2}{3b^2}ac(b^3 - 4b^2 - 4b - 2) + \tfrac{2}{9}(\tfrac{a}{b} + \tfrac{c}{b})(b^2 + 16b + 10) \\ & \quad + \tfrac{ac}{9b^2}(29b^2 + 32b + 20) - \tfrac{a}{b}(5b+4) - 2b - 4 - c - 56 \\ &= \tfrac{1}{9b^2}(6ab^3c + 2ab^3 + 5ab^2c + 2b^3c - 13ab^2 + 8abc - 18b^3 + 23b^2c \\ & \quad -16ab + 8ac - 540b^2 + 20bc) = \tfrac{1}{9b^2}\Delta_2 \end{aligned}$$

Δ_2 is an increasing function of $a,\ b,\ and\ c$ if $a \geq 1+\frac{3}{\beta}, b \geq 1+\frac{3}{\beta}, c \geq 3+\beta$. The minimum of Δ_2 is at $a = 1+\frac{3}{\beta}, b = 1+\frac{3}{\beta}, c = 3+\beta$ for this subcase.

$$\begin{aligned}
&\Delta_2 \mid_{a=1+\frac{3}{\beta}, b=1+\frac{3}{\beta}, c=3+\beta} \\
&= 6(1+\tfrac{3}{\beta})^4(\beta+3) + 2(1+\tfrac{3}{\beta})^4 + 7(1+\tfrac{3}{\beta})^3(\beta+3) \\
&\ - 31(1+\tfrac{3}{\beta})^3 + 31(1+\tfrac{3}{\beta})^2(\beta+3) - 556(1+\tfrac{3}{\beta})^2 + 28(1+\tfrac{3}{\beta})(\beta+3) \\
&= \tfrac{36}{\beta^4}(\beta+3)^2(\beta+0.5)(\beta-1)(\beta-5) > 0 \quad for\ 0<\beta<1
\end{aligned}$$

Hence we prove that (2.3) holds for $3 < d < 4$

Case(b2): $d > 4$

In this range

$$\frac{\partial \Delta}{\partial B} > 0 \quad for\ A \geq 1,\ B \geq 1,\ d > 4$$

Therefore Δ is an increasing function of A , B and C for $A \geq B \geq C \geq 1$ and $d > 4$. Combining this result with (2.8) we conclude that $\Delta > 0 \quad for\ a \geq b \geq c \geq d > 4,\ d = [d] + \frac{d}{b}$.

Finally, combining the results of case (b1) and the results of case (b2) we conclude that $\Delta > 0 \quad for\ a \geq b \geq c \geq d > 3 \quad where\ d = [d] + \frac{d}{b}$

Case(c): $\beta = \frac{d}{a}$

At level $w = d - \beta - 1$,

$$\frac{x}{\frac{a}{d}(\beta+1)} + \frac{y}{\frac{b}{d}(\beta+1)} + \frac{z}{\frac{c}{d}(\beta+1)} = \frac{x}{1+\frac{a}{d}} + \frac{y}{\frac{b}{a}+\frac{b}{d}} + \frac{z}{\frac{c}{a}+\frac{c}{d}} \leq 1$$

L_1, the number of positive integral solutions at level $w = d - \beta - 1$ satisfies

$$6L_1 \leq \frac{a}{d}(\frac{b}{a}+\frac{b}{d}-1)(\frac{c}{a}+\frac{c}{d}-1) - (\frac{c}{a}+\frac{c}{d}-1) = (\frac{c}{a}+\frac{c}{d}-1)(\frac{b}{d}-\frac{a}{d}+\frac{ab}{d^2}-1)$$

by Theorem 2.5. Since $\frac{c}{d} - 1 \geq 0, \frac{b}{d} - 1 \geq 0, \frac{ab}{d^2} - \frac{a}{d} = \frac{a}{d}(\frac{b}{d} - 1) \geq 0,\ \delta = 0$. In (2.4) when $\delta = 0$, Δ is symmetric with respect to a and b, and note that we need only the condition $a > d, b > d$ in the proof of case (b). Hence we also have

$$\Delta > 0 \quad for\ a \geq b \geq c \geq d > 3 \quad where\ d = [d] + \frac{d}{a}$$

This completes the proof. Q.E.D.

Theorem 2.8. *Let* $a \geq b \geq c \geq d \geq 2$ *be real number. Consider* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leq 1$. *Let* P_4 *be the number of positive integral solutions of the above equation; i.e* $P_4 = \sharp\{(x,y,z,w) \in Z_+^4 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leq 1\}$ *Suppose* $P_4 > 0$

and d is not an integer and $d=[d]+\beta$ where β is either $\frac{d}{c}$ or $\frac{d}{b}$ or $\frac{d}{a}$. Define $\mu=(a-1)(b-1)(c-1)(d-1)$, then

$$\begin{aligned} 24P_4 < \;& \mu-(2\nu^3-5\nu^2+2\nu+1)\,|_{\nu=d-\beta+1} \\ = \;& abcd-(abc+abd+acd+bcd)+(ab+ac+ad+bc+bd+cd) \\ & -(a+b+c)-(2d^3+d^2-d-1)+2\beta^3-\beta^2(6d+1) \\ & +\beta(6d^2+2d-2) \end{aligned} \tag{2.9}$$

Proof. We need to consider only the case when $2<d<3$. There are three cases to be considered.
Case (a). $\beta=\frac{d}{a}$, Case (b). $\beta=\frac{d}{b}$, and Case (c). $\beta=\frac{d}{c}$.
Case (a). $\beta=\frac{d}{a}$
In this case $d=2+\beta, 0<\beta=\frac{d}{a}<1$. Also note that $a\beta=2+\beta$, *and* $a=1+\frac{2}{\beta}=\frac{d}{d-2}>3$. For level $w=2$, there is no positive integral points because $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{2}{d}=\frac{\beta}{\beta+2}+\frac{1}{b}+\frac{1}{c}+\frac{2}{2+\beta}>1$. Hence in this case we need to consider only the level $w=1$ in computing P_4. For $w=1$, we consider the inequality

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{1}{2+\beta}\le 1$$

$$\frac{x}{a(\frac{1+\beta}{2+\beta})}+\frac{y}{b(\frac{1+\beta}{2+\beta})}+\frac{z}{c(\frac{1+\beta}{2+\beta})}\le 1$$

Let L_1 be the number of positive integral solutions of the above inequality. In view of Lemma 2.4 if $b(\frac{1+\beta}{2+\beta})<2$, then $L_1=0$. If we assume $P_4>0$ then $b\ge 2+\frac{2}{1+\beta}$, and

$$24L_1\le 4\big[abc(\frac{1+\beta}{2+\beta})^3-(ab+ac+bc)(\frac{1+\beta}{2+\beta})^2+(a+b)(\frac{1+\beta}{2+\beta})\big]$$

On the other hand for $d=2+\beta$, $\nu=d-\beta+1=3$, and $a\beta=2+\beta$ in this subcase. Therefore we get

$$\begin{aligned} \mu-(2\nu^3-5\nu^2+2\nu+1) &= (a-1)(b-1)(c-1)(1+\beta)-16 \\ &= (abc-ac-bc+c-ab+a+b-1)(1+\beta)-16 \\ &= abc-ac-bc+c-ab+a+b-1 \\ &\quad +bc(2+\beta)-bc\beta-c(2+\beta) \\ &\quad +c\beta-b(2+\beta)+b\beta+(2+\beta)-\beta-16 \\ &= abc-ab-ac+bc-b-c+a-15 \end{aligned} \tag{2.10}$$

Define Δ as following:

$$\begin{aligned}\Delta = \ & [abc - ab - ac + bc - b - c + a - 15] \\ & -4\big[abc(\tfrac{1+\beta}{2+\beta})^3 - (ab+ac+bc)(\tfrac{1+\beta}{2+\beta})^2 + (a+b)(\tfrac{1+\beta}{2+\beta})\big] := \tfrac{1}{2a^2}\Delta_1\end{aligned} \tag{2.11}$$

Δ_1 is an increasing function of a, b, and c. The minimum of Δ_1 is at $a = 1+\frac{2}{\beta}, b = 2+\frac{2}{1+\beta}, c = 2+\beta$ for this subcase.

$$\begin{aligned}&\Delta_1 \mid_{a=1+\frac{2}{\beta}, b=2+\frac{2}{1+\beta}, c=2+\beta} \\ &= \tfrac{12}{\beta^3}(\beta+2)^2(\beta-1)^2 > 0\end{aligned}$$

Therefore $\Delta > 0$ for this subcase.

Case (b). $\beta = \frac{d}{b}$

In this case $d = 2+\beta, 0 < \beta = \frac{d}{b} < 1$. Also note that $b\beta = 2+\beta$, *and* $b = 1+\frac{2}{\beta} = \frac{d}{d-2} > 3$. Hence we need to consider only $a \geq b > 3, c \geq d > 2, d = 2+\frac{d}{b}$. For level $w = 2$, there is no positive integral solution because $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{2}{d} = \frac{1}{a}+\frac{\beta}{2+\beta}+\frac{1}{c}+\frac{2}{2+\beta} > 1$. Hence in this case we only need to consider the level $w = 1$ in computing P_4. For $w = 1$, we consider the inequality

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{1}{2+\beta} \leq 1$$

Write this inequality as

$$\frac{x}{a(\frac{1+\beta}{2+\beta})}+\frac{y}{b(\frac{1+\beta}{2+\beta})}+\frac{z}{c(\frac{1+\beta}{2+\beta})} \leq 1$$

Let L_1 be the number of positive integral solutions of the above inequality. Since $b(\frac{1+\beta}{2+\beta}) = 1+\frac{1}{\beta} > 2$, we can use Theorem 2.5 to count P_4 without assuming $P_4 > 0$. The rest of the proof is similar to those in case (a).

Case (c). $\beta = \frac{d}{c}$

In this case $d = 2+\beta, 0 < \beta = \frac{d}{c} < 1$. Also note that $c\beta = 2+\beta$, *and* $c = 1+\frac{2}{\beta} = \frac{d}{d-2} > 3$. Hence we need to consider only $a \geq b \geq c > 3,\ d > 2, d = 2+\frac{d}{c}$. For level $w = 2$, there is no positive integral solution because $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{2}{d} = \frac{1}{a}+\frac{1}{b}+\frac{\beta}{2+\beta}+\frac{2}{2+\beta} > 1$. Hence in this case we need to consider only the level $w = 1$ in computing P_4. For $w = 1$, we consider the inequality

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{1}{2+\beta} \leq 1$$

Write this inequality as

$$\frac{x}{a(\frac{1+\beta}{2+\beta})}+\frac{y}{b(\frac{1+\beta}{2+\beta})}+\frac{z}{c(\frac{1+\beta}{2+\beta})} \leq 1$$

Observe that $b(\frac{1+\beta}{2+\beta}) = b(\frac{1+\beta}{d}) = \frac{b}{d} + \frac{b}{c} \geq 2$. we can use Theorem 2.5 to count P_4 without assuming $P_4 > 0$. The rest of the proof is similar to those in case (a). Q.E.D.

From the proof of Theorem 2.8 we have the following corollary.

Corollary 2.9. *Let $a \geq b \geq c \geq d \geq 2$ be real number. Consider $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leq 1$. Let P_4 be the number of positive integral solutions of the above equation; i.e $P_4 = \sharp\{(x,y,z,w) \in Z_+^4 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leq 1\}$ Suppose d is not an integer and $d = [d] + \beta$ where β is either $\frac{d}{c}$ or $\frac{d}{b}$. Define $\mu = (a-1)(b-1)(c-1)(d-1)$, then*

$$\begin{aligned} 24P_4 < \; & \mu - (2\nu^3 - 5\nu^2 + 2\nu + 1)\mid_{\nu=d-\beta+1} \\ = \; & abcd - (abc + abd + acd + bcd) + (ab + ac + ad + bc + bd + cd) \\ & -(a+b+c) - (2d^3 + d^2 - d - 1) + 2\beta^3 - \beta^2(6d+1) \\ & +\beta(6d^2 + 2d - 2) \end{aligned} \tag{2.12}$$

3. Application to Yau conjecture for weighted homogeneous hypersurface singularities and coordinate free characterization of homogeneous hypersurface singularities.

Theorem 3.1. *Let $(V,0)$ be a three dimensional isolated singularity defined by a weighted homogeneous polynomial $f(x,y,z,w) = 0$. Let a, b, c and d be the weights of x, y, z and w respectively and d be an integer. Suppose $a \geq b \geq c \geq d \geq 3$ or $a \geq b \geq c \geq d = 2$ i.e. consider the special case when $d = 2$. Let p_g be the geometric genus and μ be the Milnor number of the singularity. Then for $a \geq b \geq c \geq d \geq 3$ and for $a \geq b \geq c \geq d = 2$ we have*

$$\mu \geq 24p_g + 2d^3 - 5d^2 + 2d + 1 \tag{3.1}$$

with equality if and only if $a = b = c = d =$ integer i.e. $(V,0)$ is defined by homogeneous polynomial.

Proof. Recall that p_g is precisely the number of positive integral solutions of $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leq 1$, i.e $p_g = \sharp\{(x,y,z,w) \in Z_+^4 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leq 1\}$. In view of Theorem 2.1 and Theorem 2.3, we have

$$\begin{aligned} 24p_g \;\leq\; & abcd - \frac{3}{2}(abc + abd + acd + bcd) + \frac{11}{3}(ab + ac + bc) \\ & -2(a+b+c) \end{aligned}$$

On the other hand, a result of Milnor and Orlik says that

$$\mu = (a-1)(b-1)(c-1)(d-1)$$

We will consider the following two cases: Case 1. $a \geq b \geq c \geq d \geq 3$, and Case 2. $a \geq b \geq c \geq d = 2$.

Case 1. $a \geq b \geq c \geq d \geq 3$

First we consider the case for $p_g = 0$

$$\begin{aligned}&\mu - 24p_g - (2d^3 - 5d^2 + 2d + 1)\\&= \mu - (2d^3 - 5d^2 + 2d + 1) \geq (d-1)^4 - (2d^3 - 5d^2 + 2d + 1)\\&= d(d-1)(d-2)(d-3) \geq 0 \quad for\ d \geq 3\end{aligned}$$

So if $p_g = 0$, then (3.1) holds for $a \geq b \geq c \geq d \geq 3$, and $\mu = 24p_g + (2d^3 - 5d^2 + 2d + 1)$ if and only if $a = b = c = d = 3$.

Next we consider the case for $p_g > 0$.

$$\begin{aligned}&\mu - 24p_g - (2d^3 - 5d^2 + 2d + 1)\\&\geq \Delta := \{abcd - (abc + abd + acd + bcd) + (ab + ac + ad + bc + bd + cd)\\&-(a + b + c + d) + 1\} - \{abcd - \tfrac{3}{2}(abc + abd + acd + bcd)\\&+\tfrac{11}{3}(ab + ac + bc) - 2(a + b + c)\} - (2d^3 - 5d^2 + 2d + 1)\\&= \tfrac{1}{2}(abc + abd + acd + bcd) - 2d^3 - \tfrac{8}{3}(ab + ac + bc)\\&+ad + bd + cd + 5d^2 + a + b + c - 3d\end{aligned}$$

Let $A = \frac{a}{d}$, $B = \frac{b}{d}$, $C = \frac{c}{d}$, then $A \geq B \geq C \geq 1$ and $d \geq 3$. Rewrite Δ as

$$\begin{aligned}\Delta = \;& \tfrac{d^3}{2}ABC + \tfrac{d^3}{2}(AB + AC + BC) - 2d^3 - \tfrac{8}{3}d^2(AB + BC + AC)\\&+d^2(A + B + C) + 5d^2 + d(A + B + C - 3)\end{aligned}$$

$$\Delta\,|_{A=B=C=1} = \frac{d^3}{2} + \frac{3}{2}d^3 - 2d^3 - 8d^2 + 3d^2 + 5d^2 + d(3-3) = 0 \qquad (3.2)$$

Δ is an increasing function of A, B, and C for $A \geq 1$, $B \geq 1$, $C \geq 1$ and $d \geq \frac{8}{3}$. In particular $\Delta \geq 0$ for $A \geq B \geq C \geq 1$, $d \geq \frac{8}{3}$. Combining this fact with (3.2) we prove that

$$\mu - 24p_g - (2d^3 - 5d^2 + 2d + 1) \geq \Delta \geq 0 \quad for\ A \geq B \geq C \geq 1, d \geq \frac{8}{3}$$

$\mu - 24p_g - (2d^3 - 5d^2 + 2d + 1) = 0$ implies $\mu - 24p_g - (2d^3 - 5d^2 + 2d + 1) = \Delta$ which implies that equality holds in (2.1). Hence $\mu - 24p_g - (2d^3 - 5d^2 + 2d + 1) = 0$ implies $a = b = c = d = integer$. Conversely if $a = b = c = d = integer$, then $\mu - 24p_g - (2d^3 - 5d^2 + 2d + 1) = (d-1)^4 - d(d-1)(d-2)(d-$

$3) - (2d^3 - 5d^2 + 2d + 1) = 0$.

Case 2. $a \geq b \geq c \geq d = 2$

In this case we need to consider only the level $w = 1$ in computing the number of positive integral solutions of $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leq 1$. Let p_g be the number of positive integral solutions at the level $w = 1$. Consider first the case $p_g > 0$. Since $p_g > 0$, in view of Lemma 2.4 $\frac{b}{2} > 2$. Hence $a \geq b > 4$, and

$$24p_g \leq \frac{abc}{2} - (ab + ac + bc) + 2(a + b)$$

On the other hand, since $d = 2$, $\mu = (a-1)(b-1)(c-1) = abc - (ab + ac + bc) - (a + b + c) - 1$. Therefore, we get

$$\mu - 24p_g - (2d^3 - 5d^2 + 2d + 1) \geq \Delta_1 := \frac{1}{2}abc - (a + b) + c - 2$$

Δ_1 is an increasing function of a , b and c, for $a \geq b \geq 4$ and $c \geq 2$. The minimum of Δ_1 is at $a = 4, b = 4, c = 2$, and $\Delta_1 \mid_{a=4,b=4c=2}= 8 > 0$. Therefore we conclude that $\mu - 24p_g - (2d^3 - 5d^2 + 2d + 1) > 0$ for $d = 2$ and $p_g > 0$. Now we consider when $d = 2$ and $p_g = 0$. In this case $(2d^3 - 5d^2 + 2d + 1) = 1$ and $\mu = (a-1)(b-1)(c-1) \geq 1$. Hence $\mu - 24p_g - (2d^3 - 5d^2 + 2d + 1) \geq 0$ and the equality holds if and only if $a = b = c = d = 2$. Q.E.D.

A polynomial $f(z_0, z_1, z_2, z_3)$ is weighted homogeneous of type (w_0, w_1, w_2, w_3), where (w_0, w_1, w_2, w_3) are fixed positive rational numbers, if it can be expressed as a linear combination of monomials $z_0^{i_0} z_1^{i_1} z_2^{i_2} z_3^{i_3}$ for which $\frac{i_0}{w_0} + \frac{i_1}{w_1} + \frac{i_2}{w_2} + \frac{i_3}{w_3} = 1$. (w_0, w_1, w_2, w_3) is called the weights of f. We also denote (w_0, w_1, w_2, w_3) as w.

Lemma 3.2. *Let $w_0, w_1, w_2,$ and w_3 be the weights of a weighted homogeneous polynomial $f(z_0, z_1, z_2, z_3)$ such that $f(z_0, z_1, z_2, z_3)$ has an isolated critical point at the origin. Suppose $w_{i_0} \geq w_{i_1} \geq w_{i_2} \geq w_{i_3}$ and w_{i_3} is not an integer, where $\{i_0, i_1, i_2, i_3\} = \{0, 1, 2, 3\}$. Let $w_{i_3} = [w_{i_3}] + \beta$ with $0 < \beta < 1$. Then β is either $\frac{w_{i_3}}{w_{i_0}}$ or $\frac{w_{i_3}}{w_{i_1}}$ or $\frac{w_{i_3}}{w_{i_2}}$.*

Proof. It is well known that $f(z_0, z_1, z_2, z_3)$ can be deformed to one of the following nineteen classes without changing the weights at all.(cf. [Ka], [Or-Ra]) In the following, $w = (w_0, w_1, w_2, w_3)$ is the weights of $f(z_0, z_1, z_2, z_3)$, but we do not assume any ordering for w_0, w_1, w_2, and w_3. Also note that $a_0, a_1, a_2, a_3, p, q, r, and\ s$, are positive integer.

The difficult part of the proof is to observe the fact that we need to prove that for the following nineteen classes if the smallest weight is not an integer then the nonintegral part of the smallest weight is the ratio of this smallest weight with some other weights. Since we know the form of the weights in the following nineteen classes, we just need to write the possible nonintegral weight as a sum of an integer and an nonintegral part. Then we assume this is the smallest weight and write the nonintegral part as the ratio of the smallest weight and some other weight. We only write the details of the proof for class 2, for the other classes we can use the same idea to do the proof.

class 1. $f(z_0, z_1, z_2, z_3) = z_0^{a_0} + z_1^{a_1} + z_2^{a_2} + z_3^{a_3}$
Clearly this case can not happen under the hypothesis of the weights of f

class 2. $f(z_0, z_1, z_2, z_3) = z_0^{a_0} + z_1^{a_1} + z_2^{a_2} + z_2 z_3^{a_3}$, $w = (a_0, a_1, a_2, \frac{a_2 a_3}{a_2-1})$.
We only need to consider when $\frac{a_2 a_3}{a_2-1}$ is minimum. Without loss of generality we can assume $a_0 \geq a_1$. Observe that $\frac{a_2 a_3}{a_2-1} = a_3 + \frac{a_3}{a_2-1}$. Since $a_2 \geq \frac{a_2 a_3}{a_2-1}$ i.e. $1 \geq \frac{a_3}{a_2-1}$, we need only to consider $1 > \frac{a_3}{a_2-1} > 0$. Hence we deduce that $w_3 = [w_3] + \frac{w_3}{w_2}$ where $[w_3] = a_3$ and $\beta = \frac{a_3}{a_2-1} = \frac{w_3}{w_2}$.

class 3. $f(z_0, z_1, z_2, z_3) = z_0^{a_0} + z_1^{a_1} + z_2^{a_2} z_3 + z_2 z_3^{a_3}$, $w = (a_0, a_1, \frac{a_2 a_3-1}{a_3-1}, \frac{a_2 a_3-1}{a_2-1})$.

class 4. $f(z_0, z_1, z_2, z_3) = z_0^{a_0} + z_0 z_1^{a_1} + z_2^{a_2} + z_2 z_3^{a_3}$, $w = (a_0, \frac{a_0 a_1}{a_0-1}, a_2, \frac{a_2 a_3}{a_2-1})$.

class 5. $f(z_0, z_1, z_2, z_3) = z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_2^{a_2} + z_2 z_3^{a_3}$, $w = (\frac{a_0 a_1-1}{a_1-1}, \frac{a_0 a_1-1}{a_0-1}, a_2, \frac{a_2 a_3}{a_2-1})$.

class 6. $f(z_0, z_1, z_2, z_3) = z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_2^{a_2} z_3 + z_2 z_3^{a_3}$, $w = (\frac{a_0 a_1-1}{a_1-1}, \frac{a_0 a_1-1}{a_0-1}, \frac{a_2 a_3-1}{a_3-1}, \frac{a_2 a_3-1}{a_2-1})$.

class 7. $f(z_0, z_1, z_2, z_3) = z_0^{a_0} + z_1^{a_1} + z_1 z_2^{a_2} + z_2 z_3^{a_3}$, $w = (a_0, a_1, \frac{a_1 a_2}{a_1-1}, \frac{a_1 a_2 a_3}{a_1(a_2-1)+1})$.

class 8. $f(z_0, z_1, z_2, z_3) = z_0^{a_0} + z_1^{a_1} + z_1 z_2^{a_2} + z_1 z_3^{a_3} + z_2^p z_3^q$ with the condition $\frac{p(a_1-1)}{a_1 a_2} + \frac{q(a_1-1)}{a_1 a_3} = 1$, $w = (a_0, a_1, \frac{a_1 a_2}{a_1-1}, \frac{a_1 a_3}{a_1-1})$.

class 9. $f(z_0, z_1, z_2, z_3) = z_0^{a_0} + z_1^{a_1} z_3 + z_2^{a_2} z_3 + z_1 z_3^{a_3} + z_1^p z_2^q$ with the condition $\frac{p(a_3-1)}{(a_1 a_3-1)} + \frac{q a_1(a_3-1)}{a_2(a_1 a_3-1)} = 1$, $w = (a_0, \frac{a_1 a_3-1}{a_3-1}, \frac{a_2(a_1 a_3-1)}{a_1(a_3-1)}, \frac{a_1 a_3-1}{a_1-1})$.

class 10. $f(z_0, z_1, z_2, z_3) = z_0^{a_0} + z_1^{a_1} z_2 + z_2^{a_2} z_3 + z_1 z_3^{a_3}$, $w = (a_0, \frac{a_1 a_2 a_3+1}{a_3(a_2-1)+1}, \frac{a_1 a_2 a_3+1}{a_1(a_3-1)+1}, \frac{a_1 a_2 a_3+1}{a_2(a_1-1)+1})$.

class 11. $f(z_0, z_1, z_2, z_3) = z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{a_2} + z_2 z_3^{a_3}$, $w = (a_0, \frac{a_0 a_1}{a_0-1}, \frac{a_0 a_1 a_2}{a_0(a_1-1)+1}, \frac{a_0 a_1 a_2 a_3}{a_0 a_1(a_2-1)+(a_0-1)})$.

class 12. $f(z_0, z_1, z_2, z_3) = z_0^{a_0} + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_2 z_3^{a_3} + z_1^p z_2^q$ with the condition $\frac{p(a_0-1)}{a_0 a_1} + \frac{q(a_0-1)}{a_0 a_2} = 1$, $w = (a_0, \frac{a_0 a_1}{a_0-1}, \frac{a_0 a_2}{a_0-1}, \frac{a_0 a_1 a_3}{a_0(a_1-1)+1})$.

class 13. $f(z_0, z_1, z_2, z_3) = z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{a_2} + z_1 z_3^{a_3} + z_2^p z_3^q$ with the condition $\frac{p(a_0 a_1-a_0+1)}{a_0 a_1 a_2} + \frac{q(a_0 a_1-a_0+1)}{a_0 a_1 a_3} = 1$, $w = (a_0, \frac{a_0 a_1}{a_0-1}, \frac{a_0 a_1 a_2}{a_0(a_1-1)+1}, \frac{a_0 a_1 a_3}{a_0(a_1-1)+1})$.

class 14. $f(z_0, z_1, z_2, z_3) = z_0^{a_0} + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_0 z_3^{a_3} + z_1^p z_2^q + z_2^r z_3^s$ with the condition $\frac{p(a_0-1)}{a_0 a_1} + \frac{q(a_0-1)}{a_0 a_2} = 1$ and $\frac{r(a_0-1)}{a_0 a_2} + \frac{s(a_0-1)}{a_0 a_3} = 1$, $w = (a_0, \frac{a_0 a_1}{a_0-1}, \frac{a_0 a_2}{a_0-1}, \frac{a_0 a_3}{a_0-1})$.

class 15. $f(z_0,z_1,z_2,z_3)=z_0^{a_0}z_1+z_0z_1^{a_1}+z_0z_2^{a_2}+z_2z_3^{a_3}+z_1^pz_2^q$ with the condition $\frac{p(a_0-1)}{a_0a_1-1}+\frac{qa_1(a_0-1)}{a_2(a_0a_1-1)}=1$, $w=(\frac{a_0a_1-1}{a_1-1},\frac{a_0a_1-1}{a_0-1},\frac{a_0a_1a_2-a_2}{a_0a_1-a_1},\frac{a_2a_3(a_0a_1-1)}{a_2(a_0a_1-1)-a_1(a_0-1)})$.
class 16. $f(z_0,z_1,z_2,z_3)=z_0^{a_0}z_1+z_0z_1^{a_1}+z_0z_2^{a_2}+z_0z_3^{a_3}+z_1^pz_2^q+z_2^rz_3^s$ with the condition $\frac{p(a_0-1)}{a_0a_1-1}+\frac{qa_1(a_0-1)}{a_2(a_0a_1-1)}=1$ and $\frac{ra_1(a_0-1)}{a_2(a_0a_1-1)}+\frac{sa_1(a_0-1)}{a_3(a_0a_1-1)}=1$, $w=(\frac{a_0a_1-1}{a_1-1},\frac{a_0a_1-1}{a_0-1},\frac{a_2(a_0a_1-1)}{a_1(a_0-1)},\frac{a_3(a_0a_1-1)}{a_1(a_0-1)})$.
class 17. $f(z_0,z_1,z_2,z_3)=z_0^{a_0}z_1+z_0z_1^{a_1}+z_1z_2^{a_2}+z_0z_3^{a_3}+z_1^pz_3^q+z_0^rz_2^s$ with the condition $\frac{p(a_0-1)}{a_0a_1-1}+\frac{qa_1(a_0-1)}{a_3(a_0a_1-1)}=1$ and $\frac{r(a_1-1)}{(a_0a_1-1)}+\frac{sa_0(a_1-1)}{a_2(a_0a_1-1)}=1$, $w=(\frac{a_0a_1-1}{a_1-1},\frac{a_0a_1-1}{a_0-1},\frac{a_2(a_0a_1-1)}{a_0(a_1-1)},\frac{a_3(a_0a_1-1)}{a_1(a_0-1)})$.
class 18. $f(z_0,z_1,z_2,z_3)=z_0^{a_0}z_2+z_0z_1^{a_1}+z_1z_2^{a_2}+z_1z_3^{a_3}+z_2^pz_3^q$ with the condition $\frac{p(a_0a_1-a_0+1)}{a_0a_1a_2+1}+\frac{qa_2(a_0a_1-a_0+1)}{a_3(a_0a_1a_2+1)}=1$, $w=(\frac{a_0a_1a_2+1}{a_1(a_2-1)+1},\frac{a_0a_1a_2+1}{a_2(a_0-1)+1},\frac{a_0a_1a_2+1}{a_0(a_1-1)+1},\frac{a_3(a_0a_1a_2+1)}{a_2(a_0(a_1-1)+1)})$.
class 19. $f(z_0,z_1,z_2,z_3)=z_0^{a_0}z_2+z_0z_1^{a_1}+z_3z_2^{a_2}+z_1z_3^{a_3}$,
$w=(\frac{a_0a_1a_2a_3-1}{a_1(a_3(a_2-1)+1)-1},\frac{a_0a_1a_2a_3-1}{a_3(a_2(a_0-1)+1)-1},\frac{a_0a_1a_2a_3-1}{a_0(a_1(a_3-1)+1)-1},\frac{a_0a_1a_2a_3-1}{a_2(a_0(a_1-1)+1)-1})$.
Q.E.D.

Theorem 3.3. *Let $(V,0)$ be a three dimensional isolated singularity defined by a weighted homogeneous polynomial $f(x,y,z,w)=0$. Let p_g be the geometric genus, ν be the multiplicity and μ be the Milnor number of the singularity. Suppose $p_g>0$. Then we have*

$$\mu \geq 24p_g+2\nu^3-5\nu^2+2\nu+1 \tag{3.3}$$

with equality if and only if $(V,0)$ is defined by homogeneous polynomial.

Proof. Let a,b,c, and d be the weights of x,y,z, and w respectively so that $f(x,y,z,w)$ is a weighted homogeneous polynomial. We may assume without loss of generality that $a\geq b\geq c\geq d$. Note that p_g is precisely the number of positive integral solutions of $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{u}{d}\leq 1$, i.e $p_g=\sharp\{(x,y,z,w)\in Z_+^4:\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{w}{d}\leq 1\}$. On the other hand $\mu=(a-1)(b-1)(c-1)(d-1)$. Recall that $\nu=inf\{n\in Z_+:n\geq inf(a,b,c,d)\}$. If d is an integer, then Theorem 3.3 follows from Theorem 3.1. If d is not an integer, then write $d=[d]+\beta$ with $0<\beta<1$. It follows that the multiplicity $\nu=[d]+1=d-\beta+1$. By Lemma 3.2, β is either $\frac{d}{c}$ or $\frac{d}{b}$ or$\frac{d}{a}$. Suppose $a\geq b\geq c\geq d\geq 3$. Theorem 3.3 with a strict inequality in (3.3) follows from Theorem 2.7. For the case $a\geq b\geq c\geq d\geq 2$, in view of Theorem 2.8, Theorem 3.3 with a strict inequality in (3.3) is true for $p_g>0$. Q.E.D.

Remark: The Main Theorem is true even without the hypothesis $p_g>0$. The proof is quite technical and will appear elsewhere.

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