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Complete invariant of a family of strongly pseudoconvex domains in A_1-singularity: Bergman function

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ABSTRACT. Recently, Yau has introduced Bergman function for any strongly pseudoconvex variety with only isolated singularities. This is a global biholomorphic invariant for strongly pseudoconvex variety. The purpose of this paper is to show that Yau's Bergman function is a complete invariant of a family of strongly pseudoconvex domains in A_1-singularity. A_1 singularity can be viewed as the quotient of $\mathbb{C}^2$ by the action of cyclic group of order two. Each strongly pseudoconvex domain in A_1 singularity corresponds to a strongly pseudoconvex domain in $\mathbb{C}^2$. Hopefully our technique will shed some light on understanding strongly pseudoconvex domains in $\mathbb{C}^2$.

1. Introduction

Given two projective manifolds, complex and algebraic geometors have developed various techniques and invariants to tell whether these two projective manifolds are biholomorphically equivalent. If we take the cones over these projective manifolds in complex Euclidean space, then we get two isolated singularities. The above problem becomes the problem of determining when two isolated singularities are biholomorphically equivalent. In the case of isolated hypersurface singularities, Mather and Yau [**Ma-Ya**] proved that these singularities are biholomorphically equivalent if and only if their moduli algebras (finite dimensional commutative local algebras) are isomorphic. In [**Ya2**], Yau introduced finite dimensional Lie algebras to these isolated singularities via their moduli algebras and showed that these Lie algebras are solvable ([**Ya3**], [**Ya4**]). Seeley and Yau [**Se-Ya**] studied these Lie algebras for two families of singularities $\widetilde{E}_7$ and $\widetilde{E}_8$. They defined a nilpotent Lie algebra N_t for each of these singularities and showed that $N_s \cong N_t$ if and only if the singularity V_s is analytically equivalent to the V_t. They did this without reference to the resolution of the singularity. Although these methods are quite successful, they are not useful for the following situation. Consider a variety V with an isolated singularity at the origin in $\mathbb{C}^N$. Let V_a and V_b be the varieties obtained by intersecting V with balls of radii a and b centered at the origin respectively. How can one tell whether V_a is biholomorphically equivalent to V_b?

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Recently, Yau [**Ya5**] has introduced Bergman function for any strongly pseudoconvex variety with only isolated singularities. This Bergman function is a global biholomorphically invariant. Recall that A_1 singularity $V = \{(x,y,z) \in \mathbb{C}^3 : xy = z^2\}$ can be viewed as the quotient of $\mathbb{C}^2$ by the action of cyclic group of order two. Each strongly pseudoconvex domain in A_1 singularity has a 2-fold branched cover (with branch locus at the origin) strongly pseudoconvex domain in $\mathbb{C}^2$. Understanding the structures of strongly pseudoconvex domains in A_1 singularity may shed some light on the structures of strongly pseudoconvex domains in $\mathbb{C}^2$. The purpose of this paper is to show that Yau's Bergman function is a completely invariant of a family of strongly pseudoconvex domains in A_1 singularity.

MAIN THEOREM. *Let $a > 0$ and $V_a = \{(x,y,z) \in \mathbb{C}^3 : xy = z^2$ and $a|x|^2 + |y|^2 + |z|^2 < \epsilon_0\}$. Let B_a be the Bergman function of V_a. Then B_a is a complete biholomorphic invariant of V_a within this one parameter family of strongly pseudoconvex varieties.*

As the corollary of the proof, we have the following result.

COROLLARY. *The moduli space of the CR manifold $X_a = \{(x,y,z) \in \mathbb{C}^3 : xy = z^2$ and $a|x|^2 + |y|^2 + |z|^2 = \epsilon_0\}$ contains the positive real half line.*

In section 2, we first recall the construction of Bergman function on strongly pseudoconvex manifold and variety. In section 3, we write down the Bergman function B_a for the strongly pseudoconvex variety V_a explicitly. We construct from the Bergman function B_a a continuous numerical invariant ν_a of the one parameter family of strongly pseudoconvex varieties V_a. In section 4, we show that ν_a is an increasing function of a and thus our main theorem follows.

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2. Bergman function on strongly pseudoconvex manifold and variety

Let V be a Stein variety of dimension $n \geq 2$ in $\mathbb{C}^N$ with only irreducible isolated singularities. We assume that ∂V is a strongly pseudoconvex CR manifold. Let $\pi : M \to V$ be a resolution of singularity with E as a exceptional set. We shall define a Bergman function $B_M(z)$ on M which is a biholomorphic invariant of M.

DEFINITION 2.1. Let F be the set of all L^2 integrable holomorphic n-forms ψ on M. Let $\{\omega_j\}$ be a complete orthonormal basis of F. The Bergman kernel is defined to be $K(z,\bar{z}) = \sum_j \omega_j(z) \wedge \overline{\omega_j(z)}$. Similarly by requiring ω_j vanishing on the exceptional set E of M, we can define the Bergman kernel $K_0(z,\bar{z})$ vanishing on the exceptional set.

LEMMA 2.2. *F/F_0 is a finite dimensional vector space.*

LEMMA 2.3. *Bergman kernel vanishing on the exceptional set $K_0(z,\bar{z})$ is independent of the choice of the complete orthonormal basis of F_0 and $K_0(z,\bar{z})$ is invariant under biholomorphic maps.*

DEFINITION 2.4. Let M be a resolution of a strongly pseudoconvex variety V of $\dim n \geq 2$ in $\mathbb{C}^N$ with only irreducible isolated singularity at the origin. The Bergman function B_M on M is defined to be $K_{M,0}/K_M$.

THEOREM 2.5. *B_M is a global function defined on M which is invariant under biholomorphic maps. Moreover, B_M is nowhere vanishing outside the exceptional set of M.*

PROOF. Let $\Phi : M' \longrightarrow M$ be a biholomorphic map. Then

$$\Phi^*(B_M) = \frac{\Phi^* K_{M,0}}{\Phi^* K_M} = \frac{K_{M',0}}{K_{M'}} = B_{M'}.$$

Let $\pi : M \longrightarrow V$ be the blowing down map. To finish the proof, we need to show that $K_{M,0}$ does not vanish outside the exceptional set. For this purpose, given $p \in M - E$, it suffices to produce a holomorphic n-form vanishing along E but not a p. Let $\Omega^n(-E)$ be the sheaf of germs of holomorphic n-forms on M vanishing along E. Since $\Omega^n(-E)$ is coherent and π is proper, $\pi_*\Omega^n(-E)$ is a coherent sheaf on V by Grauert's direct image theorem. As V is a Stein variety, we can find ω in $\Gamma(V, \pi_*\Omega^n(-E))$ which does not vanish at $\pi(p)$. Then $\pi^*\omega$ is a holomorphic n-form vanishing along the exceptional set E but not at p. □

The same argument of the proof of Theorem 1 in [**L-Y-Y**] will prove the following theorem.

THEOREM 2.6. *Let M be a strongly pseudoconvex manifold of dimension $n \geq 2$ with exceptional set E. Let A be compact submanifold containing in E. Let $\pi : M_1 \to M$ be the blow up of M along A. Then we have $K_{M_1}(z, \overline{z}) = \pi^* K_M(z, \overline{z})$ and $K_{M_1,0}(z, \overline{z}) = \pi^* K_{M,0}(z, \overline{z})$. Consequently $B_{M_1}(z) = \pi^* B_M(z)$.*

Let $\pi_i : M_i \longrightarrow V, i = 1, 2$ be two resolutions of singularities of V. By Hironaka's theorem [**Hi**], there exists a resolution $\widetilde{\pi} : \widetilde{M} \longrightarrow V$ of singularities of V such that $\widetilde{M}$ can be obtained from M_i, $i = 1, 2$, by successive blowing up along submanifolds in exceptional set. In view of Theorem 2.5 and Theorem 2.6, the following definition is well defined.

DEFINITION 2.7. Let V be a strongly pseudoconvex variety of $\mathbb{C}^N$ with only irreducible isolated singularities. Let $\pi : M \longrightarrow V$ be a resolution of singularities of V. Define the Bergman function B_V on V to be the push forward of the Bergman function B_M by the map π.

THEOREM 2.8. *Let V be a strongly pseudoconvex variety in $\mathbb{C}^N$ with only irreducible isolated singularities. Then the Bergman function B_V on V is invariant under biholomorphic maps and B_V vanishes precisely on the singular set of V.*

THEOREM 2.9. *Let V be a strongly pseudoconvex variety in $\mathbb{C}^N$ with only isolated normal singularities of dimension $n \geq 2$. Let E_V be the set of all L^2-integrable holomorphic n-forms on $V - S$, where S is the singular part of V. Let $F_{V,0} = \{\omega \in F_V : \omega$ vanishes on $S\}$. Let $K_V(z, \overline{z})$ and $K_{V,0}(z, \overline{z})$ be defined in the usual manner* (cf. Definition 2.1). *Then $B_V = \frac{K_V(z,\overline{z})}{K_{V,0}(z,\overline{z})}$ and B_V is a biholomorphical invariant of V.*

3. Continuous numerical invariant constructed from Bergman function

The purpose of this section is to show that our global invariant Bergman function defined in section 2 can be used to construct continuous numerical invariant for a one parameter family of strongly pseudoconvex varieties.

Let $\widetilde{V} = \{(x,y,z) \in \mathbb{C}^3 : xy - z^2 = 0\}$. Consider the one parameter family of $\{V_a : a > 0\}$ of strongly pseudoconvex varieties lying inside $\widetilde{V}$, where $V_a = \{(x,y,z) \in \mathbb{C}^3 : xy - z^2 = 0,\ a|x|^2 + |y|^2 + |z|^2 < \epsilon_0\}$. An explicit resolution $\widetilde{\pi} : \widetilde{M} \longrightarrow \widetilde{V}$ can be given in terms of coordinate charts and transition functions as follows:

$$\text{Coordinate charts} : \widetilde{w}_k = \mathbb{C}^2 = \{(u_k, v_k)\},\ k = 0, 1$$

$$\text{Transition functions} : \begin{cases} u_1 = \frac{1}{v_0} \\ v_1 = u_0 v_0^2 \end{cases} \quad \text{or} \quad \begin{cases} u_0 = u_1^2 v_1 \\ v_0 = \frac{1}{u_1} \end{cases}$$

$$\text{Resolution map} : \widetilde{\pi}(u_k, v_k) = (u_k^{k+1} v_k^k, u_k^{1-k} v_k^{2-k}, u_k v_k) \text{ or}$$

$$(x,y,z) = (u_0, u_0 v_0^2, u_0 v_0) = (u_1^2 v_1, v_1, u_1 v_1)$$

$$\text{Exceptional set} : E = \pi^{-1}(0) = C_1 = \{u_0 = 0\} \cup \{v_1 = 0\}.$$

Then $M_a = \widetilde{\pi}^{-1}(V_a)$ is given by the coordinate charts:

$$W_k = \{(u_k, v_k) : a|u_k^{k+1} v_k^k|^2 + |u_k^{1-k} v_k^{2-k}|^2 + |u_k v_k|^2 < \epsilon_0\},\ k = 0, 1.$$

Observe that under $\pi : M_a \longrightarrow V_1$, $W_0 \backslash C_1$ is mapped biholomorphically onto $V_a \backslash y$-axis. In particular, $M_a \backslash W_0$ is of measure zero in the obvious sense. Hence, we may compute integrals on M using the (u_0, v_0) coordinate on the chart W_0 alone. The following proposition can be found in Proposition 8 of [**L-Y-Y**].

PROPOSITION 3.1. *In the above notations, let* $\phi_{\alpha\beta} = u_0^\alpha v_0^\beta du_0 \wedge dv_0$, $\alpha, \beta = 0, 1, 2 \ldots$ *Then* $\{\frac{\phi_{\alpha\beta}}{\|\phi_{\alpha\beta}\|_{M_a}} : \alpha \geq \frac{1}{2}\beta\}$ *is a complete orthonormal base of* F. *In other words, a complete orthonormal base of* F *is of the form:*

$$\left\{\begin{array}{llll} \frac{1}{\|\phi_{00}\|_{M_a}} du_0 \wedge dv_0, & \frac{u_0}{\|\phi_{10}\|_{M_a}} du_0 \wedge dv_0, & \frac{u_0 v_0}{\|\phi_{11}\|_{M_a}} du_0 \wedge dv_0, & \\ \frac{u_0 v_0^2}{\|\phi_{12}\|_{M_a}} du_0 \wedge dv_0, & \frac{u_0^2}{\|\phi_{20}\|_{M_a}} du_0 \wedge dv_0, & \frac{u_0^2 v_0}{\|\phi_{21}\|_{M_a}} du_0 \wedge dv_0, & \frac{u_0^2 v_0^2}{\|\phi_{22}\|_{M_a}} du_0 \wedge dv_0, \\ \frac{u_0^2 v_0^3}{\|\phi_{23}\|_{M_a}} du_0 \wedge dv_0, & \frac{u_0^2 v_0^4}{\|\phi_{24}\|_{M_a}} du_0 \wedge dv_0, & \frac{u_0^3}{\|\phi_{30}\|_{M_a}} du_0 \wedge dv_0, & \frac{u_0^3 v_0}{\|\phi_{31}\|_{M_a}} du_0 \wedge dv_0, \\ \frac{u_0^3 v_0^2}{\|\phi_{32}\|_{M_a}} du_0 \wedge dv_0, & \frac{u_0^3 v_0^3}{\|\phi_{33}\|_{M_a}} du_0 \wedge dv_0, & \frac{u_0^3 v_0^4}{\|\phi_{34}\|_{M_a}} du_0 \wedge dv_0, & \frac{u_0^3 v_0^5}{\|\phi_{35}\|_{M_a}} du_0 \wedge dv_0, \\ \frac{u_0^3 v_0^6}{\|\phi_{36}\|_{M_a}} du_0 \wedge dv_0, & \ldots & & \end{array}\right\}$$

Observe that except for $\frac{1}{\|\phi_{00}\|_{M_a}} du_0 \wedge dv_0$, all the other holomorphic 2-forms above are vanishing at the exceptional set. Therefore the Bergman kernel vanishing on the exceptional set $K_{M_a,0}$ and Bergman kernel are given respectively by:

$$K_{M_a,0}((u_0, v_0), (\overline{u}_0, \overline{v}_0)) = \Theta_{M_a} du_0 \wedge dv_0 \wedge d\overline{u_0} \wedge d\overline{v_0}$$

$$K_{M_a}((u_0, v_0), (\overline{u_0}, \overline{v_0})) = \left(\frac{1}{\|\phi_{00}\|_{M_a}^2} + \Theta_{M_a}\right) du_0 \wedge dv_0 \wedge d\overline{u_0} \wedge d\overline{v_0}$$

where

$$\begin{aligned}\Theta_{M_a} = &\frac{|u_0|^2}{\|\phi_{10}\|^2_{M_a}} + \frac{|u_0|^2|v_0|^2}{\|\phi_{11}\|^2_{M_a}} + \frac{|u_0|^2|v_0|^4}{\|\phi_{12}\|^2_{M_a}} + \frac{|u_0|^4}{\|\phi_{20}\|^2_{M_a}} + \frac{|u_0|^4|v_0|^2}{\|\phi_{21}\|^2_{M_a}} \\ &+ \frac{|u_0|^4|v_0|^4}{\|\phi_{22}\|^2_{M_a}} + \frac{|u_0|^4|v_0|^6}{\|\phi_{23}\|^2_{M_a}} + \frac{|u_0|^4|v_0|^8}{\|\phi_{24}\|^2_{M_a}} + \frac{|u_0|^6}{\|\phi_{30}\|^2_{M_a}} + \frac{|u_0|^6|v_0|^2}{\|\phi_{31}\|^2_{M_a}} \\ &+ \frac{|u_0|^6|v_0|^4}{\|\phi_{32}\|^2_{M_a}} + \frac{|u_0|^6|v_0|^6}{\|\phi_{33}\|^2_{M_a}} + \frac{|u_0|^6|v_0|^8}{\|\phi_{34}\|^2_{M_a}} + \frac{|u_0|^6|v_0|^{10}}{\|\phi_{35}\|^2_{M_a}} + \frac{|u_0|^6|v_0|^{12}}{\|\phi_{36}\|^2_{M_a}} + \cdots\end{aligned} \tag{3.1}$$

THEOREM 3.2. *The Bergman function for the strongly pseudoconvex manifold M_a is given by*

$$\begin{aligned}B_{M_a}((u_0,v_0),(\overline{u_0,v_0})) = \|\phi_{00}\|^2_{M_a}\Theta_{M_a}\Big[&1 - \|\phi_{00}\|^2_{M_a}\Theta_{M_a} + (\|\phi_{00}\|^2_{M_a}\Theta_{M_a})^2 \\ &- (\|\phi_{00}\|^2_{M_a}\Theta_{M_a})^3 + (\|\phi_{00}\|^2_{M_a}\Theta_{M_a})^4 - \cdots\Big].\end{aligned} \tag{3.2}$$

The Bergman function for the strongly pseudoconvex variety V_a is given by

$$\begin{aligned}B_{V_a}((x,y,z),(\overline{x,y,z})) = \|\phi_{00}\|^2_{M_a}\Theta_{V_a}\Big[&1 - \|\phi_{00}\|^2_{M_a}\Theta_{V_a} + (\|\phi_{00}\|^2_{M_a}\Theta_{V_a})^2 \\ &- (\|\phi_{00}\|^2_{M_a}\Theta_{V_a})^3 + (\|\phi_{00}\|^2_{M_a}\Theta_{V_a})^4 - \cdots\Big]\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}\Theta_{V_a} = &\frac{|x|^2}{\|\phi_{10}\|^2_{M_a}} + \frac{|z|^2}{\|\phi_{11}\|^2_{M_a}} + \frac{|y|^2}{\|\phi_{12}\|^2_{M_a}} + \frac{|x|^4}{\|\phi_{20}\|^2_{M_a}} + \frac{|x|^2|z|^2}{\|\phi_{21}\|^2_{M_a}} + \frac{|z|^4}{\|\phi_{22}\|^2_{M_a}} \\ &+ \frac{|y|^2|z|^2}{\|\phi_{23}\|^2_{M_a}} + \frac{|y|^4}{\|\phi_{24}\|^2_{M_a}} + \frac{|x|^6}{\|\phi_{30}\|^2_{M_a}} + \frac{|x|^5|y|}{\|\phi_{31}\|^2_{M_a}} + \frac{|x|^4|y|^2}{\|\phi_{32}\|^2_{M_a}} + \frac{|z|^6}{\|\phi_{33}\|^2_{M_a}} \\ &+ \frac{|x|^2|y|^4}{\|\phi_{34}\|^2_{M_a}} + \frac{|x|\,|y|^5}{\|\phi_{35}\|^2_{M_a}} + \frac{|y|^6}{\|\phi_{36}\|^2_{M_a}} + \cdots\end{aligned} \tag{3.4}$$

The following theorem which gives a continuous numerical invariant for one parameter family of strongly pseudoconvex varieties V_a lying in $\widetilde{V} = \{(x,y,z) \in \mathbb{C}^3 : xy = z^2\}$ can be found in [**Ya5**].

THEOREM 3.3. *For $a > 0$, let $V_a = \{(x,y,z) \in \mathbb{C}^3 : xy = z^2$ and $a|x|^2 + |y|^2 + |z|^2 < \epsilon_0\}$. With the notations as above, let $M_a = \widetilde{\pi}^{-1}(V_a)$ be the resolution of singularity of V_a. Then $\nu_a := \frac{\|\phi_{11}\|^2_{M_a}}{\|\phi_{10}\|_{M_a}\|\phi_{12}\|_{M_a}}$ is a biholomorphic invariant of the one parameter family $\{V_a : a \in \mathbb{R}_+\}$, i.e., if V_a and V_b are two such strongly pseudoconvex varieties in this family which are biholomorphically equivalent, then $\frac{\|\Phi_{11}\|^2_{M_a}}{\|\phi_{10}\|_{M_a}\|\phi_{12}\|_{M_a}} = \frac{\|\phi_{11}\|^2_{M_b}}{\|\phi_{10}\|_{M_b}\|\phi_{12}\|_{M_b}}$.*

4. Explicit computation of invariant ν_a

In this section, we shall recall the explicit computation of ν_a in [**Ya5**]. Let a be a positive real number. We shall follow the notations in our previous section. Let $V_a = \{(x,y,z) \in \mathbb{C}^3 : xy = z^2,\ a|x|^2 + |y|^2 + |z|^2 < \epsilon_0\}$. Recall that $(x,y,z) = (u_0, u_0v_0^2, u_0v_0)$. Then M_a is the resolution of V_a with coordinate chart $W_0 =$

$\{(u_0, v_0) : a|u_0|^2+|u_0|^2|v_0|^4+|u_0|^2|v_0|^2 < \epsilon_0\}$. Next wrote $u_0 = re^{i\theta}$ and $v_0 = \rho e^{i\phi}$. Then

$$\begin{aligned}\|\phi_{\alpha\beta}\|^2_{M_a} &= \int_{M_a} \phi_{\alpha\beta} \wedge \overline{\phi}_{\alpha\beta} = \int_{W_0} |u_0^\alpha|^2|v_0^\beta|^2 du_0 \wedge dv_0 \wedge d\overline{u}_0 \wedge d\overline{v}_0 \\ &= 2\pi \int_0^{2\pi} \iint_D r^{2\alpha+1}\rho^{2\beta+1} dr\, d\rho\, d\theta,\end{aligned}$$

where $D = \{(r, \rho) : r \geq 0, \rho \geq 0, ar^2 + r^2\rho^4 + r^2\rho^2 < \epsilon_0\}$. In particular

$$\begin{aligned}\|\phi_{1\beta}\|^2_{M_a} &= 2\pi \int_0^{2\pi} \int_0^\infty \int_0^{\frac{\sqrt{\epsilon_0}}{\sqrt{a+\rho^2+\rho^4}}} r^3\rho^{2\beta+1} dr\, d\rho\, d\theta \\ &= 4\pi \int_0^\infty \frac{\epsilon_0^2\rho^{2\beta+1}}{(a+\rho^2+\rho^4)^2} d\rho\, d\theta.\end{aligned}$$

Therefore we have

$$\begin{aligned}\nu_a &:= \frac{\|\phi_{11}\|^2_{M_a}}{\|\phi_{10}\|_{M_a}\|\phi_{12}\|_{M_a}} \\ &= \frac{\displaystyle\int_0^\infty \frac{\rho^3}{(a+\rho^2+\rho^4)^2} d\rho}{\displaystyle(\int_0^\infty \frac{\rho}{(a+\rho^2+\rho^4)^2} d\rho)^{\frac{1}{2}} (\int_0^\infty \frac{\rho^5}{(a+\rho^2+\rho^4)^2} d\rho)^{\frac{1}{2}}} \\ &= \frac{I_2}{\sqrt{I_1 I_3}},\end{aligned}$$

where $I_k = \displaystyle\int_0^\infty \frac{x^{k-1}}{(x^2+x+a)^2} dx$ for $k = 1, 2, 3$.

Observe that when $a = \frac{1}{4}$, $\nu_a = \frac{1}{2}$.

5. Bergman functions as complete biholomorphic invariants for one parameter family of strongly pseudoconvex varieties

In section 3, we have shown that ν_a is a biholomorphic invariant of the one parameter family of strongly pseudoconvex varieties $\{V_a : a \in \mathbb{R}_+\}$. We are going to show in this section that ν_a is in fact a complete biholomorphic invariant of $\{V_a : a \in \mathbb{R}_+\}$, i.e., $\nu_a = \nu_b$ if and only if V_a is biholomorphically equivalent of V_b.

THEOREM 5.1. *For $a > 0$, let $V_a = \{(x, y, z) \in \mathbb{C}^3 : xy = z^2$ and $a|x|^2 + |y|^2 + |z|^2 < \epsilon_0\}$. With notation as in section 3, let $\nu_a = \frac{\|\phi_{11}\|^2_{M_a}}{\|\phi_{10}\|_{M_a}\|\phi_{12}\|_{M_a}}$. Then V_a is biholomorphically equivalent to V_b if and only if $\nu_a = \nu_b$.*

PROOF. In view of Theorem 3.5, it is sufficient to prove that ν_a as a function of a is an increasing positive function. For this purpose it is enough to prove that $\nu_a' > 0$. Since $[(\nu_a)^2]' = 2\nu_a\nu_a'$ and $\nu_a > 0$, we only need to prove that $[(\nu_a)^2]' > 0$.

Case 1: $a > \dfrac{1}{4}$

Let $p = \frac{1}{\sqrt{4a-1}}$ i.e. $a = \frac{p^2+1}{4p^2}$. Then $0 < p < \infty$ and

$$\begin{aligned}
I_1 &= \int_0^\infty \frac{dx}{(x^2+x+a)^2} \\
&= \frac{2\pi}{(4a-1)^{3/2}} - \frac{1}{a(4a-1)} - \frac{4}{(4a-1)^{3/2}} \arctan\left(\frac{1}{\sqrt{4a-1}}\right) \\
&= 2\pi p^3 - \frac{4p^4}{p^2+1} - 4p^3 \arctan(p) \\
I_2 &= \int_0^\infty \frac{xdx}{(x^2+x+a)^2} \\
&= \frac{2}{4a-1} - \frac{\pi}{(4a-1)^{3/2}} + \frac{2}{(4a-1)^{3/2}} \arctan\left(\frac{1}{\sqrt{4a-1}}\right) \\
&= 2p^2 - \pi p^3 + 2p^3 \arctan(p) \\
I_3 &= \int_0^\infty \frac{x^2dx}{(x^2+x+a)^2} \\
&= \frac{2a\pi}{(4a-1)^{3/2}} - \frac{1}{4a-1} - \frac{4a}{(4a-1)^{3/2}} \arctan\left(\frac{1}{\sqrt{4a-1}}\right) \\
&= \frac{1}{2}\pi p(p^2+1) - p^2 - p(p^2+1)\arctan(p) \\
(\nu_a)^2 &= \frac{I_2^2}{I_1 I_2}.
\end{aligned}$$

□

By using MATLAB, we get

$$\frac{d(\nu_a)^2}{dp} = \frac{2AB}{C^3}, \tag{5.1}$$

where

$$\begin{aligned}
A &= -2 + \pi p - 2p\arctan(p) = -2 + p(\pi - 2\arctan(p)), \\
B &= -8 - 4\pi\arctan(p) - 4\pi p^2 \arctan(p) + 2\pi p - 4p\arctan(p) \\
&\quad + 4p^2 \arctan^2(p) + 4\arctan^2(p) + \pi^2 + \pi^2 p^2 \\
&= [\pi - 2\arctan(p)]^2(p^2+1) + 2p[\pi - 2\arctan(p)] - 8 \\
C &= \pi p^2 + \pi - 2p - 2p^2\arctan(p) - 2\arctan(p) \\
&= (p^2+1)(\pi - 2\arctan(p)) - 2p.
\end{aligned}$$

LEMMA 5.2. *$\frac{2p}{p^2+1} < \pi - 2\arctan(\rho) < \frac{2}{\rho}$ for $p > 0$.*

PROOF. When $p > 0$, we have $\arctan(p) = \frac{\pi}{2} - \operatorname{arccot}(p)$. Euler first gave a convergent power series of $\operatorname{arccot}(p)$ as follows

$$\operatorname{arccot}(p) = p\left[\frac{1}{p^2+1} + \frac{2}{3(p^2+1)^2} + \frac{2\cdot 4}{3\cdot 5(p^2+1)^3} + \cdots\right].$$

Combining it with $\arctan(p) = \frac{\pi}{2} - \operatorname{arccot}(p)$, we have

$$\pi - 2\arctan(p) = 2\operatorname{arccot}(p) > \frac{2p}{p^2+1}.$$

On the other hand $\arctan(p) = \frac{\pi}{2} - \arctan(\frac{1}{p})$. Thus we have

$$\pi - 2\arctan(p) = 2\arctan\left(\frac{1}{p}\right) < \frac{2}{p}.$$

The last inequality comes form the fact that $\arctan(x) < x$ for all $x > 0$. Indeed, set $f(x) = \arctan(x) - x$. Then $f'(x) = \frac{1}{1+x^2} - 1 < 0$. Hence $f(x) < 0$ for $x > 0$ since $f(0) = 0$. □

LEMMA 5.3. *$A < 0$, $C > 0$, $B > 0$ and* $\frac{d[(\nu_a)^2]}{dp} < 0$.

PROOF. The fact that $A < 0$ and $C > 0$ comes directly from Lemma 5.2. Recall that $\pi - 2\arctan(p) = 2\arctan(\frac{1}{p})$ when $p > 0$. Let $q = \arctan(\frac{1}{p})$. Then $0 < q < \frac{\pi}{2}$, $p = \frac{1}{\tan(q)}$ and we have

$$\begin{aligned} B &= [\pi - 2\arctan(p)]^2(p^2+1) + 2p[\pi - 2\arctan(p)] - 8 \\ &= (2q)^2\left(\frac{1}{\tan^2 q} + 1\right) + \frac{2}{\tan q}(2q) - 8 \\ &= 4\left[\frac{q^2}{\sin^2 q} + \frac{q\cos q}{\sin q} - 2\right] \\ &= \frac{4}{\sin^2 q}\left[q^2 + \frac{1}{2}q\sin 2q + \cos 2q - 1\right]. \end{aligned}$$

To show $B > 0$, it is sufficient to show

$$F(q) = q^2 + \frac{1}{2}q\sin 2q + \cos 2q - 1 > 0 \text{ for } 0 < q < \frac{\pi}{2}.$$

Observe that

$$\begin{aligned} F'(q) &= 2q - \frac{3}{2}\sin 2q + q\cos 2q \\ F''(q) &= 2 - 2\cos 2q - 2q\sin 2q \\ F'''(q) &= 2(\sin 2q - 2q\cos 2q) \\ F^{(4)}(q) &= 8q\sin 2q > 0 \text{ for } 0 < q < \frac{\pi}{2}, \end{aligned}$$

and $F(0) = F'(0) = F''(0) = F'''(0) = 0$. We conclude that $B > 0$. □

PROOF. Now return to the proof of Theorem 5.1 Case 1.

$$\begin{aligned} \frac{d[(\nu_a)^2]}{da} &= \frac{d[(\nu_a)^2]}{dp}\frac{dp}{da} \\ &= -\frac{d[(\nu_a)^2]}{dp}\frac{2}{(4a-1)^{3/2}} < 0. \end{aligned}$$

This implies $(\nu_a)^2$ is an increasing function of a. Hence ν_a is also increasing too.

Case 2: $0 < a < \frac{1}{4}$

Let $p = \frac{1}{\sqrt{1-4a}}$. Then $p > 1$ and $a = \frac{p^2-1}{4p^2}$. We have

$$\begin{aligned}
I_1 &= \int_0^\infty \frac{dx}{(x^2+x+a)^2} \\
&= \frac{1}{a(1-4a)} + \frac{2}{(1-4a)^{3/2}} \ln\left(\frac{1-\sqrt{1-4a}}{1+\sqrt{1-4a}}\right) \\
&= \frac{4p^4}{p^2-1} + 2p^3 \ln\left(\frac{p-1}{p+1}\right) \\
I_2 &= \int_0^\infty \frac{x}{(x^2+x+a)^2} dx \\
&= \quad -\frac{2}{1-4a} - \frac{1}{(1-4a)^{3/2}} \ln\left(\frac{1-\sqrt{1-4a}}{1+\sqrt{1-4a}}\right) = -2p^2 - p^3 \ln\left(\frac{p-1}{1+p}\right) \\
I_3 &= \int_0^\infty \frac{x^2}{(x^2+x+a)^2} dx \\
&= \frac{1}{2}\left[\frac{1}{1-4a} + \frac{2a}{(1-4a)^{3/2}} \ln\left(\frac{1-\sqrt{1-4a}}{1+\sqrt{1-4a}}\right)\right] \\
&= \frac{1}{2}\left[p^2 + \frac{1}{2}(p^2-1)p \ln\left(\frac{p-1}{1+p}\right)\right] \\
(\nu_a)^2 &= \frac{I_2^2}{I_1 I_3}.
\end{aligned}$$

By using MATLAB, we get

$$\frac{d[(\nu_a)^2]}{dp} = \frac{-4AB}{C^4} \tag{5.2}$$

where

$$\begin{aligned}
A &= L^2p^2 - 2Lp - L^2 - 8 \\
B &= 2 + Lp \\
C &= 2p + Lp^2 - L \\
L &= \log\left(\frac{p-1}{p+1}\right).
\end{aligned}$$

Let $t = \frac{1}{p}$. Then $0 < t < 1$ and the above four expression becomes

$$\begin{aligned}
L &= \log(1-t) - \log(1+t) = -2\tanh^{-1}(t), \\
A &= \frac{4(\tanh^{-1}(t))^2}{t^2} + \frac{4\tanh^{-1}(t)}{t} - 4(\tanh^{-1}(t))^2 - 8, \\
B &= 2\left(1 - \frac{\tanh^{-1}(t)}{t}\right), \\
C &= \frac{2}{t} - 2\left(\frac{1}{t^2} - 1\right)\tanh^{-1}(t) \\
&= \frac{2}{t^2}[t - (1-t^2)\tanh^{-1}(t)].
\end{aligned}$$

Hence $\tanh^{-1}$ is the inverse hyperbolic tangent, which is the inverse function of

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

$\tanh^{-1}(t)$ can be written as

$$\begin{aligned}\tanh^{-1}(t) &= \frac{1}{2}[\ln(1+t) - \ln(1-t)] \\ &= t + \frac{1}{3}t^3 + \frac{1}{5}t^5 + \frac{1}{7}t^6 + \cdots \text{ for } \quad -1 < t < 1.\end{aligned}$$

In particular for $0 < t < 1$, we have

$$\frac{\tanh^{-1}(t)}{t} > 1 \Rightarrow B < 0 \text{ for } \; 0 < t < 1.$$

To show that $C > 0$, notice that for $0 < t < 1$, we have

$$\begin{aligned}\frac{t}{1-t^2} &= t + t^3 + t^5 + t^7 + \cdots \\ &> t + \frac{1}{3}t^3 + \frac{1}{5}t^5 + \frac{1}{7}t^7 + \cdots \\ &= \tanh^{-1}(t).\end{aligned}$$

Therefore $t > (1-t^2)\tanh^{-1}(t)$ which implies $C > 0$.

Finally we shall show that $A > 0$. Let $q = \tanh^{-1}(t)$. Since $0 < t < 1$, we have $q > 0$.

$$\begin{aligned}A &= 4\left[\frac{q^2}{\tanh^2 q} + \frac{q}{\tanh q} - q^2 - 2\right] \\ &= \frac{4}{\sinh^2 q}[q^2(\cosh^2 q - \sinh^2 q) + q(\cosh q)(\sinh q) - 2\sinh^2 q] \\ &= \frac{4}{\sinh^2 q}\left[q^2 + \frac{1}{2}q\sinh(2q) + 1 - \cosh(2q)\right].\end{aligned}$$

To show $A(q) > 0$ for $q > 0$, it is sufficient to show

$$H(q) := q^2 + \frac{1}{2}q\sinh(2q) + 1 - \cosh(2q) > 0 \text{ for } \; q > 0.$$

Observe that

$$\begin{aligned}H'(q) &= 2q - \frac{3}{2}\sinh(2q) + q\cosh(2q), \\ H''(q) &= 2 - 2\cosh(2q) + 2q\sinh(2q), \\ H'''(q) &= -2\sinh(2q) + 4q\cosh(2a), \\ H(0) = 0 &= H'(0) = H''(0) = H'''(0) \text{ and} \\ H^{(4)}(q) &= 8q\sinh(2q) > 0 \text{ for } \; q > 0.\end{aligned}$$

It follows that $A(q) > 0$ for $q > 0$ and hence $\frac{d[(\nu_a)^2]}{dp} > 0$. Observe that

$$\frac{d(\nu_a)^2}{da} = \frac{d(\nu_a)^2}{dp}\frac{2}{(1-4a)^{3/2}} > 0.$$

Hence $(\nu_a)^2$ and ν_a are increasing functions of a. □

THEOREM 5.2. *Let $a > 0$ and $V_a = \{(x, y, z) \in \mathbb{C}^3 : xy = z^2$ and $a|x|^2 + |y|^2 + |z|^2 < \epsilon_0\}$. Let B_a be the Bergman function of V_a. Then B_a is a complete biholomorphic invariant of V_a within this one parameter family of strongly pseudoconvex varieties.*

PROOF. Since B_a is a biholomorphic invariant of V_a and ν_a is constructed from B_a, Theorem 5.2 follows immediately from Theorem 5.1. □

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